REDUCED GW INVARIANTS FROM CUSPIDAL CURVES

L.BATTISTELLA, F.CAROCCI, C.MANOLACHE

Abstract. We prove that the 1-stabilisation morphism, replacing elliptic tails with cusps, is well-defined at the level of genus 1 weighted-stable curves. We use this to endow a substack of $\overline{\mathcal{M}}_1(\mathbb{P}^4, d)$ with a virtual class that ignores the boundary contribution of maps with a contracted elliptic tail. We thus prove that Vakil-Zinger's reduced invariants of the quintic threefold can be computed as Smyth-Viscardi's 1-stable (i.e. cuspidal) invariants. To do so we adapt Jun Li's techniques of p-fields and local equations for the moduli space.

Introduction

The moduli stack of stable maps to the projective space $\mathbb{P}^r$ is of fundamental importance in Gromov-Witten theory, since any moduli space of maps to a projective variety $X \subseteq \mathbb{P}^r$ is cut inside the former by a number of equations induced by generators of $\mathcal{I}_{X/\mathbb{P}^r}$. Yet its structure in higher genus is still largely mysterious.

Recall that in genus zero $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ is a smooth stack of the expected dimension $r - 3 + d(r+1) + n$. Furthermore the rational Gromov-Witten theory of a complete intersection $i\colon X \hookrightarrow \mathbb{P}^r$ of degree $(l_1, \dots, l_k)$ can be determined as a twisted theory for $\mathbb{P}^r$ and thus computed via localisation: for any split vector bundle $E = \oplus_i \mathscr{O}_{\mathbb{P}^r}(l_i)$ the sheaf $\pi_* f^* E$ is a vector bundle on $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$, where we have denoted by π the projection from the universal curve and by f the universal stable map. It is a matter of functoriality of virtual fundamental classes [Kon95] [CKL01] [KKP03] that the following *hyperplane property* holds

$$j_*[\bigsqcup_{\beta \in A_1(X)\colon i_*(\beta) = d} \overline{\mathcal{M}}_{0,n}(X, \beta)]^{\mathrm{vir}} = c_{\mathrm{top}}(E) \cap [\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)].$$

For $g \geq 1$ the moduli space $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ is neither irreducible, nor pure dimensional, and the sheaf $\pi_* f^* \mathscr{O}_{\mathbb{P}^r}(l)$ is not a vector bundle. The enumerative meaning of the invariants is spoilt by degenerate contributions from curves of lower genera, represented by boundary components of the moduli space where a subcurve of positive genus is contracted.

The situation in genus 1 is better understood: the geometry of $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$ has been widely studied since [Vak00]. Contrary to higher genus, the closure of the locus of maps the source of which is a smooth elliptic curve is

Date: January 6, 2018.

an irreducible component of $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r,d)$, which is usually called the *main component*. Its boundary points were characterised in [Vak00] as follows: $[f\colon C\to\mathbb{P}^r]\in\overline{\mathcal{M}}_1(\mathbb{P}^r,d)$ is smoothable, i.e. it can be made into the special fiber of a 1-dimensional family the generic fiber of which is a non-constant map from a smooth elliptic curve, if and only if either f is not constant on the minimal genus 1 subcurve E of C, or the image of the tangent vectors to the rational tails are linearly dependent in $T_{f(E)}\mathbb{P}^r$. In Section 1 we give a new proof of this fact 1.2 using Hu-Li's local equations for the moduli space; furthermore we provide a different characterisation of the smoothable maps as those that factor through a non-degenerate map from a Cohen-Macaulay genus 1 singularity 1.7, which we classify extending an argument due to Smyth.

From the enumerative point of view, it would be ideal to isolate the contribution to the invariants given by the main component. In an attempt to extract this information hidden within Kontsevich's moduli space, Vakil and Zinger managed to desingularise the main component by performing a sequence of blow-ups on an underlying space of weighted curves [VZ07] [VZ08] (see also [HL10]). The end result $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d)^{\mathrm{main}}$ is a smooth and irreducible stack on which $\tilde{\pi}_*\tilde{f}^*\mathscr{O}_{\mathbb{P}^r}(l)$ is a vector bundle. This fact allowed them to define a notion of *reduced invariants* for complete intersections in $\mathbb{P}^r$ by mimicking the hyperplane property of the genus 0 theory.

Definition 0.1. Let X be a complete intersection of degree $(l_1,\dots,l_k)$ in $\mathbb{P}^r$. The genus 1 reduced invariants of X are defined by integrating against:

$$c_{\mathrm{top}}(\tilde{\pi}_*\tilde{f}^*(\oplus_{i=1}^k\mathscr{O}_{\mathbb{P}^r}(l_i)))\cap[\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d)^{\mathrm{main}}].$$

Recently there has been a different attempt to discard some of the boundary components. Based on Smyth's work on the birational geometry of $\overline{\mathcal{M}}_{1,n}$ [Smy11a, Smy11b], Viscardi [Vis12] has introduced a series of *alternate compactifications* of the moduli space of maps from smooth elliptic curves. Compared to Kontsevich's space of stable maps, in Viscardi's space $\overline{\mathcal{M}}^{(m)}_{1,n}(X,\beta)$ maps that contract a genus 1 curve with at most m rational tails are destabilised, and replaced by maps from l-elliptic singularities with $l\leq m$.

The main result of the present paper is that reduced invariants of the quintic threefold can be recovered from Viscardi's alternate compactifications.

Theorem 0.2. *Let $X_5\subseteq\mathbb{P}^4$ be a generic smooth quintic threefold. Then:*

$$N_1^{\mathrm{red}}(X_5,d)=N_1^{\mathrm{cusp}}(X_5,d),$$

where:

$$N_1^{\mathrm{red}}(X_5,d):=\deg\left(c_{\mathrm{top}}(\tilde{\pi}_*\tilde{f}^*(\mathscr{O}_{\mathbb{P}^4}(5)))\cap[\widetilde{\mathcal{M}}_1(\mathbb{P}^4,d)^{\mathrm{main}}]\right)$$

and

$$N_1^{\mathrm{cusp}}(X_5,d):=\deg[\overline{\mathcal{M}}^{(1)}_1(X_5,d)]^{\mathrm{vir}}.$$

Below we will suppress the subscript 5 from X_5. The insight for this theorem comes from the following formula, the proof of which was first given by J. Li and A. Zinger [LZ07] in the symplectic category:

$$N_1(X,d) = N_1^{\mathrm{red}}(X,d) + \frac{1}{12}N_0(X,d). \tag{1}$$

The first proof of Li-Zinger's formula in the algebro-geometric setting is due to H.-W. Chang and J. Li in [CL15]. Their splitting of the intrinsic normal cone makes it patent that the genus 0 contribution to the right hand side only comes from the boundary component with a single rational tail of degree d. This suggests that removing such a component would provide a more direct approach to reduced invariants for the quintic threefold. This is precisely what Viscardi's space of 1-stable maps does.

The proof of our main theorem follows from adapting the techniques of Chang and Li to Viscardi's moduli spaces. Here is an outline of the proof.

We first consider a moduli space of 1-stable maps with p-fields $\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4,d)^p$. In the work of Chang and Li, p-fields provide a way around the fact that we cannot control the geometry of $\overline{\mathcal{M}}_1(X,d)$ and there is no compatible perfect obstruction theory relative to the inclusion $\overline{\mathcal{M}}_1(X,d) \hookrightarrow \overline{\mathcal{M}}_1(\mathbb{P}^4,d)$. By introducing the moduli space of maps with p-fields the authors manage to include the unwanted part of the obstruction theory into the moduli space, and to endow a well-understood space (indeed very similar to $\overline{\mathcal{M}}_1(\mathbb{P}^4,d)$) with an obstruction theory counting genus 1 curves in the quintic. Paralleling their proof, i.e. using cosection localised virtual cycles [KL13] and the deformation to the normal cone $(X_5 \subseteq \mathbb{P}^4) \rightsquigarrow (X_5 \subseteq N_{X/\mathbb{P}^4})$, we see that:

Proposition 0.3. $\deg[\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4,d)^p]^{\mathrm{vir}}_{\mathrm{loc}} = \deg[\overline{\mathcal{M}}_1^{(1)}(X,d)]^{\mathrm{vir}}$.

Indeed it is enough to notice that all the constructions in [CL12] carry over to the setting of a family of Gorenstein (not necessarily nodal) curves.

We then need the following comparison morphism.

Proposition 0.4. *There exists a well-defined* 1*-stabilisation morphism at the level of weighted-stable curves:*

$$\mathfrak{M}_{1,n}^{\mathrm{wt,st}} \to \mathfrak{M}_{1,n}^{\mathrm{wt,st}}(1)$$

The fiber product

$$\mathcal{Z} := \mathfrak{M}_{1,n}^{\mathrm{wt,st}} \times_{\mathfrak{M}_{1,n}^{\mathrm{wt,st}}} \overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4,d)$$

is a substack of $\overline{\mathcal{M}}_1(\mathbb{P}^4,d)$ avoiding the boundary component with one rational tail of degree d.

It is harder to compare the fiber product

$$\mathcal{Z}^p := \mathfrak{M}_{1,n}^{\mathrm{wt,st}} \times_{\mathfrak{M}_{1,n}^{\mathrm{wt,st}}} \overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4,d)$$

with $\overline{\mathcal{M}}_1(\mathbb{P}^4,d)^p$. Yet $\mathcal{Z}^p$ is endowed with a cosection-localised virtual class counting 1-stable maps to the quintic threefold (by cosection-localised virtual pullback). The same splitting arguments as in [CL12] allow us to evince a main contribution which coincides with Vakil-Zinger's reduced invariants. On the other hand a dimensional computation shows that this is the only enumeratively meaningful component.

Future directions.

Outline of the paper.

Notations and conventions. We work over an algebraically closed field $\mathbf{k}$ of characteristic 0.

Throughout the paper we shall have fixed a homogeneous polynomial $\mathbf{w} \in \mathbf{k}[x_0, \ldots, x_4]_5$ of degree 5 such that $X = V(\mathbf{w}) \subseteq \mathbb{P}^4$ is a generic quintic threefold. Furthermore let us consider a fixed positive integer d determining all the homology classes $d[\ell] \in A_1(\mathbb{P}^4)$, line bundle weights, etc. that we shall take into account.

Let $\mathfrak{M}_{1,n}$ denote the algebraic stack of prestable n-marked curves of arithmetic genus 1. In the following definitions, by *special point* we mean the preimage of a node or a marking in the normalisation.

Definition 0.5. Let $\mathfrak{M}_{1,n}^{\mathrm{wt=d,st}}$ be the stack of *weighted-stable* curves of genus 1 with n markings and total weight d: every geometric point of $\mathfrak{M}_{1,n}^{\mathrm{wt=d,st}}$ represents a connected reduced nodal projective curve of arithmetic genus 1 with n distinct smooth markings and an integer-valued function on the dual graph, such that it is compatible with specialisation maps and the sum of the values on all the vertices is d; furthermore we require that it takes nonnegative values, and every $p_a = 0$ component of weight 0 has at least three special points (every $p_a = 1$ component of weight 0 has at least one special point).

There is an étale non-separated morphism $\mathfrak{M}_{1,n}^{\mathrm{wt=d,st}} \to \mathfrak{M}_{1,n}$ and the stability condition is such that the forgetful map $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d) \to \mathfrak{M}_{1,n}$ factors through $\mathfrak{M}_{1,n}^{\mathrm{wt=d,st}}$, the weight assignment coming from the degree of the map to $\mathbb{P}^r$.

We shall denote by $\pi\colon \mathscr{C} \to \mathfrak{M}_{1,n}$ the universal curve (and use similar notation for all its pullbacks). We shall also consider the universal Picard stack $\mathfrak{Pic}_{1,n}^{\mathrm{totdeg=d,st}}$ of $\mathscr{C} \to \mathfrak{M}_{1,n}^{\mathrm{wt=d,st}}$ with universal line bundle $\mathscr{L}$. We shall often use shorthand notation $\mathfrak{M}$ for $\mathfrak{M}_{1,n}^{\mathrm{wt=d,st}}$ and $\mathfrak{P}$ for $\mathfrak{Pic}_{1,n}^{\mathrm{totdeg=d,st}}$. Let λ denote the projection $\mathfrak{P} \to \mathfrak{M}$.

Definition 0.6. Let $\mathfrak{M}_{1,n}^{\mathrm{wt}=d,\mathrm{st}}(1)$ be the stack of *at worst cuspidal* projective reduced connected n-marked curves of arithmetic genus 1 that are weighted-stable with total weight d, i.e. the weight is nonnegative, and every $p_a = 0$ component of weight 0 has at least three special points (every $p_a = 1$ component of weight 0 has at least *two* special points).

We shall denote by $\hat{\pi}\colon \hat{\mathscr{C}} \to \mathfrak{M}_{1,n}(1)$ the universal curve (and use similar notation for all its pullbacks). We shall also consider the universal Picard stack $\mathfrak{Pic}_{1,n}^{\text{totdeg=d,st}}(1)$ of $\hat{\mathscr{C}} \to \mathfrak{M}_{1,n}^{\text{wt=d,st}}(1)$ with universal line bundle $\hat{\mathscr{L}}$. We shall often use shorthand notation $\hat{\mathfrak{M}}$ for $\mathfrak{M}_{1,n}^{\text{wt=d,st}}(1)$ and $\hat{\mathfrak{P}}$ for $\mathfrak{Pic}_{1,n}^{\text{totdeg=d,st}}(1)$. Let $\hat{\lambda}$ denote the projection $\hat{\mathfrak{P}} \to \hat{\mathfrak{M}}$.

T shall often the spectrum of a discrete valuation ring (DVR) with closed point 0 and generic point η.

Subcurves are always connected.

1. $\overline{\mathcal{M}}_1(\mathbb{P}^r, d)$ - Components, equations and alternate compactifications

1.1. Local equations and components. We start by recalling a description of the global geometry of $\overline{\mathcal{M}}_1(\mathbb{P}^r, d)$. Besides the main component, which was defined in the Introduction, for every positive integer k and partition $\lambda \vdash d$ into k parts such that no $\lambda_i = 0$, there is an irreducible *boundary component* $D_\lambda(\mathbb{P}^r, d)$ defined to be the closure of the locus where:

i: the source curve is obtained by gluing a smooth elliptic curve E with k many $\mathbb{P}^1$'s as rational tails R_i,
ii: the map contracts the elliptic curve E to a point, and
iii: the map has degree λ_i on the rational tail R_i.

Indeed $D_\lambda(\mathbb{P}^r, d)$ is the image of the gluing morphism from the fiber product:

$$\overline{\mathcal{M}}_{1,k}(\mathbb{P}^r, 0) \times_{\mathbb{P}^r} \prod_{i=1}^{k} \overline{\mathcal{M}}_{0,1}(\mathbb{P}^r, \lambda_i).$$

The *minimal* arithmetic genus 1 subcurve is called the *core* or the *circuit*.

Proposition 1.1. (1) *All the irreducible components of $\overline{\mathcal{M}}_1(\mathbb{P}^r, d)$ have been described above:*

$$\overline{\mathcal{M}}_1(\mathbb{P}^r, d) = \overline{\mathcal{M}}_1(\mathbb{P}^r, d)^{\text{main}} \cup \bigcup_{\lambda} D_\lambda(\mathbb{P}^r, d).$$

(2) *A map $[f]$ lies in* the boundary of the main component *if and only if:*

- *either f is non-constant on at least one irreducible component of the core,*
- *or if $(E\ {}_{\mathbf{p}}\sqcup_{\mathbf{q}} \bigsqcup_{i=1}^k R_i, f)$ is a stable map with k rational tails, where E is the* maximal *contracted genus* 1 *subcurve, then $\{\mathrm{d}f(T_{q_i} R_i)\}_{i=1}^k$ are* linearly dependent *in $T_{f(E)}\mathbb{P}^r$.*

This is due to R. Vakil and A. Zinger in [Vak00, VZ07]. We shall later discuss a different proof of this fact based on local equations for the moduli space.

We now review Hu-Li's procedure for finding local equations of $\overline{\mathcal{M}}_1(\mathbb{P}^r, d)$ in a smooth ambient space [HL10]. They will be useful when describing the structure of the intrinsic normal cone and its splitting.

Recall that a map $C \to \mathbb{P}^r$ is given by a line bundle L on C together with $r+1$ sections in $H^0(C,L)$ that generate the line bundle. It is therefore natural to embed the space of stable maps as an open inside $\pi_*\mathscr{L}^{\oplus r+1}$ on the universal Picard stack $\mathfrak{P}_1 = \mathfrak{Pic}(\mathscr{C}_1 \to \mathfrak{M}_1)$. Hu and Li actually work over a local chart of the algebraic stack $\mathfrak{M}_1^{\mathrm{div}}$, parametrising families of prestable curves with a relative Cartier divisor (of total degree d).

Let $[f\colon C \to \mathbb{P}^r]$ be a point of $\overline{\mathcal{M}}_1(\mathbb{P}^r,d)$: we may fix homogeneous coordinates on $\mathbb{P}^r$ in such a way that $D_0 := f^{-1}\{x_0=0\}$ is a simple divisor contained supported on the smooth locus of C (i.e. a d-uple of smooth points); this property will then hold in a neighbourhood of $[f]$. This gives a morphism from an étale chart $\mathcal{U}$ of $\overline{\mathcal{M}}_1(\mathbb{P}^r,d)$ to an étale chart $\mathcal{V}$ of $\mathfrak{M}_1^{\mathrm{div}}$, that we may assume to land in the locus where the divisor consists of d distinct smooth points. Notice that this locus is smooth over $\mathfrak{M}_1$ by the deformation theory of smooth subschemes. A map to $\mathbb{P}^r$ shall now be thought of as a curve-divisor pair (C,D) together with r sections of $\mathscr{O}_C(D)$: the map can be written as $[1 : u_1 : \ldots : u_r]$, where 1 denotes the given section of $\mathscr{O}_C \to \mathscr{O}_C(D)$.

Furthermore, étale locally on $\overline{\mathcal{M}}_1(\mathbb{P}^r,d)$, we may pick extra sections $\mathcal{A}$ and $\mathcal{B}$ of $\mathscr{C} := \mathscr{C}_1(\mathbb{P}^r,d) \to \overline{\mathcal{M}}_1(\mathbb{P}^r,d)$ such that:

(1) they pass through the core genus 1 curve;
(2) they are distinct smooth points disjoint from the support of $\mathcal{D}_0$.

Now $\pi_*\mathscr{O}_{\mathscr{C}}(\mathcal{D}+\mathcal{A})$ is a vector bundle on $\mathcal{V}$ and $\pi_*\mathscr{O}_{\mathscr{C}}(\mathcal{D})$ is carved inside it by requiring that the restriction map $\pi_*\mathscr{O}_{\mathscr{C}}(\mathcal{D}+\mathcal{A}) \to \pi_*\mathscr{O}_{\mathscr{C}}(\mathcal{D}+\mathcal{A})|_{\mathcal{A}}$ be zero.

Proposition 1.2. (1) *Étale locally on $\overline{\mathcal{M}}_1(\mathbb{P}^r,d)$, there is a locally closed embedding of $\overline{\mathcal{M}}_1(\mathbb{P}^r,d)$ inside the vector bundle $V := \mathrm{Spec}(\pi_*\mathscr{O}_{\mathscr{C}}(\mathcal{D}+\mathcal{A})^{\oplus r})$ over $\mathcal{V}$:*

$$\begin{tikzcd} \overline{\mathcal{M}}_1(\mathbb{P}^r,d) & \mathcal{U} \arrow[l, "\acute{e}t"'] \arrow[r, hook] \arrow[d] & V \arrow[dl] \\ \mathfrak{M}_1^{\mathrm{div}} & \mathcal{V} \arrow[l, "\acute{e}t"'] & \end{tikzcd}$$

Notice that V is smooth since $\mathcal{V}$ is.

(2) *Assume furthermore that $\mathcal{V}$ is affine; then on $\mathcal{V}$ we have*

$$\pi_*\mathscr{O}_{\mathscr{C}}(\mathcal{D}+\mathcal{A}) \cong \pi_*\mathscr{O}_{\mathscr{C}}(\mathcal{D}+\mathcal{A}-\mathcal{B}) \oplus \pi_*\mathscr{O}_{\mathscr{C}}(\mathcal{D}+\mathcal{A})_{|\mathcal{B}}$$

and the restriction-to-$\mathcal{A}$ map is zero on the second factor.

(3) *Call $\varphi\colon \pi_*\mathscr{O}_{\mathscr{C}}(\mathcal{D}+\mathcal{A}-\mathcal{B}) \to \pi_*\mathscr{O}_{\mathscr{C}}(\mathcal{D}+\mathcal{A})_{|\mathcal{A}}$ the map induced on the first factor by restricting to $\mathcal{A}$. Locally on $\mathcal{V}$, $\pi_*\mathscr{O}_{\mathscr{C}}(\mathcal{D}+\mathcal{A}-\mathcal{B})$ can be written as a sum of line bundles: let $D=\sum_{i=1}^d \delta_i$, then*

$$\varphi = \oplus\varphi_i\colon \oplus_{i=1}^d \pi_*\mathscr{O}_{\mathscr{C}}(\delta_i+\mathcal{A}-\mathcal{B}) \to \pi_*\mathscr{O}_{\mathscr{C}}(\mathcal{D}+\mathcal{A})_{|\mathcal{A}}.$$

After choosing a trivialisation for each of the line bundles above, $\varphi_i\colon \mathscr{O}_{\mathcal{V}} \to \mathscr{O}_{\mathcal{V}}$ is given by multiplication by

$$\prod_{q\in[\mathcal{A},\delta_i]} \zeta_q,$$

where ζ_q is the smoothing coordinate on $\mathfrak{M}_1$ corresponding to the node q, and $[\mathcal{A},\delta_i]$ denotes the set of nodes separating $\mathcal{A}$ (the core) from the point δ_i.

We refer the reader to [HL10, Lemma 4.10, Proposition 4.13] for the details; we are going to review the key ideas in §4, where we find local equations for $\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^r,d)$. We include here a proof of Proposition 1.1-(2) based on Hu-Li's equations.

Proof. Let us start with the easiest degenerate situation: a contracted elliptic curve attached to a $\mathbb{P}^1$ of degree d at a single node q. Equations for the moduli space of maps around such a point look like

$$\zeta_q \sum_{i=1}^d w_i^j = 0, \quad \text{for} \quad j = 1,\dots,r$$

where the w_i are coordinates on the fibers of $\pi_*\mathscr{O}_{\mathscr{C}}(D+\mathcal{A}-\mathcal{B})$ in the basis given by $\pi_*\mathscr{O}_{\mathscr{C}}(\delta_i+\mathcal{A}-\mathcal{B}),\ i=1,\dots,d$.

Our point corresponds to a smoothable map if and only if the equations admit a solution with $\zeta_q \neq 0$, hence it must be $\sum_{i=1}^d w_i^j = 0$ for every j. Taking a coordinate z on $\mathbb{P}^1$ centred at the node q, we see that the i-th basis vector corresponds to a polynomial vanishing at q and at $\delta_j,\ \forall j\neq i$. This can be written as

$$e_i(z) = z\prod_{j\neq i}\frac{(z-\delta_j)}{-\delta_j},$$

where we have chosen a convenient normalisation. So the restriction to the rational tail of the map corresponding to the point of coordinates $(w_i^j)_{i=1,\dots,d}^{j=1,\dots,r}$ can be written as

$$[1 : \sum_{i=1}^d w_i^1 e_i(z) : \dots, \sum_{i=1}^d w_i^r e_i(z)].$$

Differentiating with respect to z we see that the image of the tangent vector at q is given in affine coordinates around $f(E)$ by:

$$(\sum_{i=1}^d w_i^1,\dots,\sum_{i=1}^d w_i^r).$$

Hence smoothability is equivalent to the image of the tangent vector being zero.

More generally we may assume that the dual graph is terminally weighted [HL10, §3.1]. Assume there are k positive-weighted rational tails and denote by $D(h),\ h=1,\dots,k$, the set of indices i s.t. δ_i belongs to the h-th rational tail, by $E(h)$ the set of nodes separating the core from the h-th rational tail. The equations will then take the following form:

$$\sum_{h=1}^{k}\left(\prod_{q\in E(h)}\zeta_q\right)\left(\sum_{i\in D(h)}w_i^j\right)=0,\ j=1,\dots,r,$$

which can be assembled in matrix form as follows:

$$W\cdot\underline{\zeta}:=\left(\sum_{i\in D(h)}w_i^j\right)_{j,h}\cdot\left(\prod_{q\in E(h)}\zeta_q\right)_h=0.$$

We see that smoothability is equivalent to the linear dependence of the rows of the above matrix W. On the other hand, on every positive-weighted rational tail R_h, we can choose a suitable coordinate z_h around the node q_h and write the map as

$$[1:p_h^1(z_h):\ldots:p_h^r(z_h)],$$

where:

$$p_h^j(z_h)=\sum_{i\in D(h)}w_i^je_i^h(z_h)\quad\text{and}\quad e_i^h(z_h)=z_h\prod_{l\in D(h)\setminus\{i\}}\frac{(z_h-\delta_l)}{-\delta_l}.$$

The elliptic curve is contracted to the point $[1:0:\ldots:0]$ and the tangent vector to R_h at q_h is mapped to the h-th row of W (in affine coordinates around $f(E)$). Again we see that the map is smoothable if and only if the image of the tangent vectors to the rational tails at the nodes are linearly dependent in $T_{f(E)}\mathbb{P}^r$. □

1.2. **Smyth-Viscardi's compactifications.** For any homology class $\beta\in H_2(X,\mathbb{Z})$, let $\mathcal{M}_{1,n}(X,\beta)$ be the stack of maps $f\colon C\to X$ from a *smooth* elliptic curve with n marked points, satisfying $f_*[C]=\beta$. The moduli spaces of m-stable maps give alternate compactifications of $\mathcal{M}_{1,n}(X,\beta)$.

Definition 1.3. Let C be a reduced, connected, projective curve of arithmetic genus 1 over $\mathbf{k}$, and let $p_1,\ldots,p_n\in C$ be smooth, distinct points. A map $f\colon C\to X$ is said to be *m-stable* if the following conditions hold:

(1) C has only nodes and elliptic l-fold points, $l\leq m$, as singularities.
(2) For any connected subcurve $E\subset C$ of arithmetic genus 1 on which f is constant,
$$\left|\left\{E\cap\overline{C\setminus E}\right\}\cup\{p_i:p_i\in E\}\right|>m.$$
(3) f has no non-trivial infinitesimal automorphisms.

Recall that a $\mathbf{k}$-rational $p\in C$ is called an *elliptic m-fold point* if:

$$\hat{\mathscr{O}}_{C,p}\cong\begin{cases}\mathbf{k}[[x,y]]/(y^2-x^3) & m=1\\ \mathbf{k}[[x,y]]/(x(x-y^2)) & m=2\\ \mathbf{k}[[x,y]]/I_m & m\geq 3\end{cases}$$

where $I_m = (x_h x_i - x_h x_j : i, j, h \in \{1, \dots, m-1\})$ and i, j, h are distinct.

Viscardi's main result [Vis12, Thm. 3.6] is the following:

Theorem 1.4. *The moduli functor of m-stable maps $\overline{\mathcal{M}}_{1,n}^{(m)}(X, \beta)$ is represented by a proper Deligne-Mumford stack of finite type over $\mathbf{k}$.*

Remark 1.5. We think that the algorithm proposed by Viscardi to prove the properness of his moduli spaces oversees a case. The issue is that, given a map $f \colon \mathscr{C}_T \to \mathbb{P}^r_T$ over a DVR scheme T such that $\mathscr{C}_\eta$ is smooth and f_0 is constant on a genus 1 connected subcurve $E \subseteq \mathscr{C}_0$, it is not always true that f descends to a map $\hat{f} \colon \hat{\mathscr{C}}_T \to \mathbb{P}^r_T$, where $\hat{\mathscr{C}}_0$ has a genus 1 *Gorenstein* singularity.

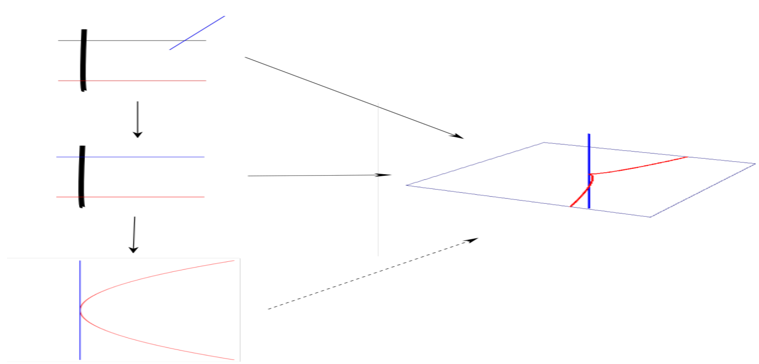
Consider the stable map $[f]$ in $\overline{\mathcal{M}}_1(\mathbb{P}^3, 4)$ from an elliptic bridge $C := R_1 {}_{q_1}\!\sqcup E \sqcup_{q_2} R_2$ to $\mathbb{P}^3$ that maps R_1 to the z-axis, contracts E to the origin, and makes (R_2, q_2) into the normalisation of a cusp in the (x, y)-plane, i.e. its image is the non-Gorenstein singularity:

$$D := \operatorname{Spec}\left(\mathbf{k}[x, y, z]/(x, y)\right) \cup \operatorname{Spec}\left(\mathbf{k}[x, y, z]/(z, y^2 - x^3)\right).$$

Notice that $\mathrm{d}f(T_{q_2} R_2) = 0$, so there is a non-trivial linear relation:

$$0 \cdot \mathrm{d}f(T_{q_1} R_1) + 1 \cdot \mathrm{d}f(T_{q_2} R_2) = 0,$$

and hence the map is smoothable. Viscardi claims that the map factors through the family $\hat{\mathscr{C}}_T$, obtained by contracting E to a tacnode. Notice that the image of $\hat{f}$ would still be D, since there is at most one indeterminacy point of $\hat{\mathscr{C}}_T \dashrightarrow \mathbb{P}^r_T$, located at the singularity. However in our example f cannot factor through the tacnode. Indeed in that case we would have a birational morphism between two singular curves with the same δ invariant and the same normalisation, so $\hat{f} \colon \hat{C} \to D$ would be an isomorphism, which is a contradiction. We suggest that in this case the correct procedure would be to sprout (R_1, q_1).



However Viscardi's argument can be fixed. Let $(\mathscr{C}_\eta, F_\eta)$ be a stable map to $\mathbb{P}^r$, defined on the generic point of a DVR scheme T; we may assume that $\mathscr{C}_\eta$ is smooth [Vis12, Section 3.2.1]. As described in [Vis12, Theorem 3.6, Step 1], after applying nodal reduction we get a map $F\colon \mathscr{C}_T \to \mathbb{P}^r_T$, for which we may suppose that $C := \mathscr{C}_0$ is and nodal and reduced $f := F_0$ is stable.

If f is not constant on the minimal genus 1 subcurve, then it is already m-stable and there is nothing to say. Otherwise let $E \subset C$ be the maximal connected genus 1 subcurve where f is constant and let $R_1 \sqcup \ldots \sqcup R_m = \overline{C/E}$.

By Proposition 1.1 (2) we know there is a non-trivial linear relation among the $\mathrm{d}f(T_{q_i}R_i)$'s. Consider a maximal one with all non-zero coefficients. Possibly after relabelling, such a relation looks like:

$$\alpha_1 \,\mathrm{d}f(T_{q_1}R_1) + \ldots + \alpha_j \,\mathrm{d}f(T_{q_j}R_j) = 0.$$

In this case we blow-up $\mathscr{C}_T$ in $q_{j+1}, \ldots, q_m$. The induced map $\tilde{F}_0$ is constant on the exceptional divisors $G_{j+1}, \ldots, G_m$ and we can complete the above linear relation to

$$\alpha_1 \,\mathrm{d}\tilde{f}(T_{q_1}\tilde{R}_1) + \ldots + \alpha_j \,\mathrm{d}\tilde{f}(T_{q_j}\tilde{R}_j) + \beta_{j+1} \,\mathrm{d}\tilde{f}(T_{q_{j+1}}E_{j+1}) + \ldots + \beta_m \,\mathrm{d}\tilde{f}(T_{q_m}E_m) = 0$$

with any choice of non-zero coefficients β. Now this *sprouting* [Smy11b, Section 2.3] ensures that the sections descend to the corresponding elliptic m-fold singularity. Notice that, for every $m_1 \le m_2$, there is a morphism from an m_2-elliptic singularity to an m_1-elliptic point that is birational on the target and contracts $m_2 - m_1$ branches of the source to the singular point. At this point proceed with Step 2 of Viscardi's algorithm.

The irreducible components of Viscardi's moduli space $\overline{\mathcal{M}}^{(m)}_{1,n}(\mathbb{P}^r, d)$ are also well understood [Vis12, Thm. 5.9]; indeed they have a similar description to the ones of Kontsevich's space. The main advantage of the m-stable compactification is that the number of components drops as m increases.

In particular the space $\overline{\mathcal{M}}^{(1)}_1(\mathbb{P}^r, d)$ we will consider in the next sections has not got the boundary component $D_{(d)}(\mathbb{P}^r, d)$.

1.3. **A different characterisation of smoothability.** Inspired by Viscardi's alternate compactification, we give yet another characterisation of smoothable maps in genus 1.

Definition 1.6. Let C be a reduced curve over $\mathbf{k}$ and $p \in C$ a singular point. We define the *genus of the singularity* in p to be the quantity:

$$g(p) = \delta(p) - b(p) + 1,$$

where $\delta(p) = \dim_{\mathbf{k}}(\nu_* \mathscr{O}_{\overline{C}} / \mathscr{O}_C) \otimes k(p)$, $\nu\colon \overline{C} \to C$ is the normalisation of C at p, and $b(p)$ is the number of branches of C at p.

Proposition 1.7. *Let $[f\colon C \to \mathbb{P}^r]$ be a stable map from a genus 1 curve $C = E_{\mathbf{p}} \sqcup_{\mathbf{q}} \bigsqcup_{i=1}^m R_i$, where E is the maximal genus 1 subcurve contracted by f, and the R_i's are rational tails on which f has positive degree.*

Then f is smoothable if and only if it factors through a Cohen-Macaulay genus 1 singularity with positive degree on at least one of its branches.

Lemma 1.8. *(Classification of Cohen-Macaulay genus* 1 *singularities) The CM (i.e. reduced) genus* 1 *singularities with* m *branches are obtained by gluing a genus* 0 *singularity with* k *branches together with a Smyth's* $(m-k)$*-elliptic fold. Notice that* k *may be* 0 *(i.e. a point) or* 1 *(i.e. a smooth rational curve).*

Proof. We extend the argument given by Smyth [Smy11a, Appendix A] to classify the Goreinstein genus 1 singularities. We study the analytic germ of the singularity and we adopt Smyth's notation: R denotes the completion of the local ring of C at the singularity; $\widetilde{R} = \mathbf{k}[[t_1]] \oplus \ldots \oplus \mathbf{k}[[t_m]]$ its integral closure; $\mathfrak{m}_R$ the maximal ideal of R and $\mathfrak{m}_{\widetilde{R}}$ that of $\widetilde{R}$.

Let us recall that to describe R as a quotient polynomial ring, it is enough to find a $\mathbf{k}$-basis for $\mathfrak{m}_R/\mathfrak{m}_R^2 = \langle e_1, \ldots, e_s\rangle_{\mathbf{k}}$ where the e_i's are some elements in $\widetilde{R}$. Indeed, once given such a basis, it is a consequence of completeness that R can be recognized as $\mathbf{k}[[x_1, \ldots, x_s]]/I$, where I is the kernel of the ring homomorphism

$$\begin{array}{ccc} \mathbf{k}[[x_1, \ldots, x_s]] & \to & R \subset \mathbf{k}[[t_1]] \oplus \ldots \oplus \mathbf{k}[[t_m]] \\ x_i & \mapsto & e_i \end{array}$$

Smyth observes that $\widetilde{R}/R$ is graded by:

$$(\widetilde{R}/R)_i := \mathfrak{m}^i_{\widetilde{R}}/(\mathfrak{m}^i_{\widetilde{R}} \cap R) + \mathfrak{m}^{i+1}_{\widetilde{R}}$$

Furthermore:

(1) $m = \delta(p) = \sum_{i\geq 0} \dim_{\mathbf{k}}(\widetilde{R}/R)_i$;
(2) $1 = g(p) = \sum_{i\geq 1} \dim_{\mathbf{k}}(\widetilde{R}/R)_i$;
(3) if $(\widetilde{R}/R)_i = (\widetilde{R}/R)_j = 0$ then $(\widetilde{R}/R)_{i+j} = 0$.

From (3) and (2) it follows that $\dim_{\mathbf{k}}(\widetilde{R}/R)_1 = 1$ and $\dim_{\mathbf{k}}(\widetilde{R}/R)_i = 0$ for $i \geq 2$. Then the exact sequence:

$$0 \to \frac{\mathfrak{m}^2_{\widetilde{R}}}{\mathfrak{m}^2_{\widetilde{R}} \cap R} \to \frac{\mathfrak{m}_{\widetilde{R}}}{\mathfrak{m}_{\widetilde{R}} \cap R} \to (\widetilde{R}/R)_1 \to 0$$

entails that:

$$\mathfrak{m}^2_{\widetilde{R}} \subseteq \mathfrak{m}_R, \qquad \mathfrak{m}_R/\mathfrak{m}^2_{\widetilde{R}} \subseteq \mathfrak{m}_{\widetilde{R}}/\mathfrak{m}^2_{\widetilde{R}} \text{ is a codimension 1 } \mathbf{k}\text{-subspace.}$$

Obviously $\mathfrak{m}_R^2 \subseteq \mathfrak{m}^2_{\widetilde{R}}$. Hence s is at least $m-1$. After relabelling we may assume $e_1, \ldots e_{m-1}$ generate $\mathfrak{m}_R/\mathfrak{m}^2_{\widetilde{R}}$, and after Gaussian elimination they take the following form:

$$\begin{aligned} e_1 &= (t_1, 0, \ldots, 0, a_1 t_m) \\ e_2 &= (0, t_2, \ldots, 0, a_2 t_m) \\ &\ldots \\ e_{m-1} &= (0, 0, \ldots, t_{m-1}, a_{m-1} t_m) \end{aligned}$$

with $a_1, \dots, a_{m-1} \in \mathbf{k}$. At this point Smyth restricts his attention to Gorenstein singularities and shows that under this assumption he can choose all the a_i to be 1; furthermore $\mathfrak{m}_{\widetilde{R}}^2 = \mathfrak{m}_R^2$ holds if $m \geq 3$, thus the above are generators for $\mathfrak{m}_R/\mathfrak{m}_R^2$ and the goal is reached. For $m = 1$ (resp. 2) he finds extra generators and shows they satisfy the equation of a cusp (resp. a tacnode).

Removing the Gorenstein restriction we have three possibilities for $m \geq 3$:

(1) At least two of the a_i's are non-zero, say $i = 1, 2$: then $a_1 a_2 t_m^2 = e_1 \cdot e_2$ so $\mathfrak{m}_{\widetilde{R}}^2 = \mathfrak{m}_R^2$ and we are done as above. It is easily seen that $e_1, \dots, e_{m-1}$ satisfy the equations of a $(k, m-k)$ singularity where k is the number of a_i's that are zero.

(2) Only one of the a_i's is non-zero, say $a_1 = 1$: then $\mathfrak{m}_{\widetilde{R}}^2 = \mathfrak{m}_R^2 + (t_m^2)$ and by adjoining $e_m = t_m^2$ to the e_i's we see that they generate $\mathfrak{m}_R/\mathfrak{m}_R^2$ and they satisfy the equation of a tacnode $(e_1^2 - e_m)e_m = 0$ and $e_i e_j = 0$ for $i \neq j$ and $(i, j) \neq (1, m)$, so this is an $(m-2, 2)$ singularity.

(3) Finally all the a_i's are zero: then we have to add t_m^2, t_m^3 to generate $\mathfrak{m}_R/\mathfrak{m}_R^2$ and we obviously find an $(m-1, 1)$ singularity.

Similarly for $m = 2$ there are two possibilities: the tacnode (corresponding to $a_1 \neq 0$) and the union of a cusp with a non-coplanar line (for $a_1 = 0$), with equations:

$$\mathbf{k}[[x, y, z]]/(xz, yz, y^2 - x^3)$$

For $m = 1$ the only possible singularity is the cusp: indeed it can be proven that $\mathfrak{m}_{\widetilde{R}}^2 = \mathfrak{m}_R$. □

Remark 1.9. The genus and number of branches are not sufficient to tell these singularities apart anymore, neither is the embedding dimension. A numerical invariant that distinguishes them is:

$$\dim_{\mathbf{k}} R/I_p$$

where

$$I_p = \mathrm{Ann}_R(\widetilde{R}/R) = \left\{ f \in R \mid f \cdot \widetilde{R} \subseteq R \right\}$$

is the conductor ideal at the singular point. Using the explicit description of R given in the lemma above, we can easily find generators of I_p and check that for a $(k, m-k)$ singularity we have:

$$\dim_{\mathbf{k}} R/I_p = m - k.$$

Lemma 1.10. *Every Cohen-Macaulay genus* 1 *singularity is smoothable.*

Proof. We exhibit explicit smoothings for the above singularities. Start with two families $(\mathscr{C}^0; \sigma_0, p_1, \dots, p_k) \to T$ and $(\mathscr{C}^1; \sigma_1, q_{k+1}, \dots, q_m) \to T$ respectively of smooth $(k+1)$-pointed rational curves and smooth $(m-k+1)$-pointed elliptic curves over a DVR scheme T, e.g. consider two trivial families.

Then perform a blow-up along the trace of $p_1, \dots p_k$ and $q_{k+1}, \dots q_m$ on the central fibers $(\mathscr{C}^0)_0$ and $(\mathscr{C}^1)_0$ to get families $\widetilde{\mathscr{C}}^0 \xrightarrow{\pi_0} \mathscr{C}^0$ and $\widetilde{\mathscr{C}}^1 \xrightarrow{\pi_1} \mathscr{C}^1$ respectively.

We can now find semi-ample line bundles which are trivial on the strict transform of the central fiber and ample everywhere else, namely

$$\mathscr{L}_0 = \pi_0^* \mathscr{O}_{\mathscr{C}^0}(p_1 + \dots + p_k)(-F_1 - \dots - F_k),$$
$$\mathscr{L}_1 = \pi_1^* \mathscr{O}_{\mathscr{C}^1}(q_{k+1} + \dots + q_m)(-F_{k+1} - \dots - F_m)$$

where we denoted by F_j the exceptional divisors of the blow-ups.

We can then use them to give contraction maps. The contracted families $\hat{\mathscr{C}}^0 \to T$ and $\hat{\mathscr{C}}^1 \to T$ still have smooth generic fibers of genus 0 and 1 respectively and, by construction, $(\hat{\mathscr{C}}^0)_0$ has a genus 0 singularity with k branches and $(\hat{\mathscr{C}}^1)_0$ has a Smyth's $(m-k)$-elliptic point [Smy11a, Lemma 2.12].

Observe now that σ_0 and σ_1 induce sections of the contracted curves $\hat{\mathscr{C}}^0 \to T$ and $\hat{\mathscr{C}}^1 \to T$ passing through the singular points of the central fiber. Let $\hat{\mathscr{C}} \to T$ the family of curves obtained by gluing $\hat{\mathscr{C}}^0 \to T$ and $\hat{\mathscr{C}}^1 \to T$ along σ_0 and σ_1 (pushout). Then $\hat{\mathscr{C}}$ has a nodal generic fiber which clearly can be further smoothed. □

Proof. 1.7 The argument that if f is smoothable then it has to factor through a genus 1 singularity was given by Vakil when he showed that the tangent vectors have to be linearly dependent in the image [Vak00, Lemma 5.9].

Viceversa, let us suppose that $f\colon C \to \mathbb{P}^r$ factors through $\hat{f}\colon \hat{C} \to \mathbb{P}^r$ with $\hat{C}$ a genus 1 singularity in a way that at least one branch is not contracted.

We first show that:

Claim 1. *We can produce smoothings $\mathscr{C}$ of C and $\hat{\mathscr{C}}$ of $\hat{C}$ with a contraction map $\mathscr{C} \to \hat{\mathscr{C}}$ compatible with the given $C \to \hat{C}$. See Remark 1.11 below*

Proof. Assume that $\hat{C}$ has a $(k, m-k)$-singularity. Pick a smoothing $\mathscr{C}$ of C such that after resolving the A_n singularities at the nodes, i.e. when looking at the semistable model with regular total space $\mathscr{C}^{\mathrm{ss}}$, the following holds: call E the maximal *balanced* genus 1 subcurve [Smy11a, Proposition 2.11] contracted by f^{ss}, then there are $m-k$ rational tails of positive degree directly adjacent to E and the other k rational tails are further away, possibly at different distances.

We may then perform a two-step contraction: by the work of Smyth we can find a line bundle contracting E, namely:

$$\mathscr{L} = \omega_{\mathscr{C}^{\mathrm{ss}}/T}\left(\sum_{F \subseteq E} (l + 1 - l(F, Z)F\right),$$

where Z is the core, $l(\cdot,\cdot)$ is the distance function on the dual graph and l is the maximum value attained by $l(\cdot, Z)$ on E. Let us call $\bar{\mathscr{C}}$ the family of curves obtained from this contraction; notice that f^{ss} descends to a $\bar{f}\colon \bar{\mathscr{C}} \to$

PP^r. The central fiber $\bar{C}$ has an $(m-k+h)$-fold elliptic point with $h \leq k$, and it still has a contraction map ρ to $\hat{C}$.

If necessary we may add markings $p_1 + \ldots + p_N = \Sigma$ on all the f-contracted rational components away from the genus 1 singularity, and on the branches of the $(m-k+h)$-fold that are not contracted by ρ. Now use $\hat{f}^*\mathscr{O}_{\mathbb{P}^r}(1)(\Sigma)$ to perform a second contraction. Since it has no H^1, it satisfies cohomology and base change, and we can check that the result of this second contraction is a $(k, m-k)$-singularity simply by looking at the vanishing orders of the sections on $\bar{C}$.

□

Let $\hat{\mathscr{L}}$ an extension of $\hat{L} := \hat{f}^*\mathscr{O}_{\mathbb{P}^r}(1)$ on $\hat{\mathscr{C}}$, which exists because deforming line bundles on curves is unobstructed. In order to extend $\hat{f}$ to $\hat{F}\colon \hat{\mathscr{C}} \to \mathbb{P}^r$ all we have to show is that the $r+1$ sections $\hat{s}_0, \ldots, \hat{s}_{r+1}$ representing the map $\hat{f}$ extend to sections of $\hat{\mathscr{L}}$. Thus it is enough to verify that $H^1(\hat{C}, \hat{L}) = 0$ [Wan12]. Once this is done we are going to obtain the smoothing of the original map by precomposing with the contraction $F\colon \mathscr{C} \to \hat{\mathscr{C}} \xrightarrow{\hat{F}} \mathbb{P}^r$.

Since all the curves we are considering are Cohen-Maculay, they have a dualising sheaf $\omega_{\hat{C}}$ such that for any line bundle $\hat{L}$

$$H^1(\hat{C}, \hat{L}) = (H^0(\hat{C}, \hat{L}^{-1} \otimes \omega_{\hat{C}}))^\vee.$$

It is known [Ser00, IV, § 3] that $\omega_{\hat{C}}$ can be described as the subsheaf of $\nu_*\omega_{\overline{C}} \otimes K(\overline{C})$ (where $\nu\colon \bar{C} \to \hat{C}$ is the normalisation) satisfying: for every open $U \subseteq \hat{C}$ a rational 1-form $\omega \in \nu_*\left(\omega_{\overline{C}} \otimes K(\overline{C})\right)(U)$ is a section of $\omega_{\hat{C}}(U)$ iff

$$\sum_{q \in \nu^{-1}(0)} \mathrm{Res}(\nu^*(\mathrm{f})\omega) = 0 \qquad \forall \mathrm{f} \in \mathscr{O}_{\mathrm{C}}(\mathrm{U})$$

From this description it is patent that the pullback of $\omega_{\hat{C}}$ to the normalisation restricts to a line bundle of degree -1 or 0 on every branch, according to it being one of the k or $m-k$ components. Hence it can be seen from the normalisation exact sequence and Serre duality that $H^1(\hat{C}, \hat{L}) = 0$ as soon as $\hat{L}$ has positive degree on one of the branches. □

Remark 1.11. In general is not possible to find compatible smoothings as above: pick any genus 2 curve with a rational tail and map it to the ramphoid cusp, a planar singularity with local equation $y^2 - x^5 = 0$. Then a compatible smoothing can be found only if the rational tail is attached to a Weierstrass point of the genus 2 curve, as can be seen by computing the semi-stable model of the ramphoid cusp. This example was kindly suggested to us by Prof. D.I. Smyth.

Remark 1.12. It is not hard to adapt Vakil's argument [Vak00] to show that factoring through a genus 1 singularity is equivalent to the tangent vectors being linearly dependent in the image.

Remark 1.13. The ideas above are similar to those employed in [RSW17, Theorem 4.5.1] to show that $\mathcal{VZ}_{1,n}(\mathbb{P}^r, d)$ is smooth, where the latter is defined by requiring the factorisation property for the map through a Smyth's singularity after sprouting.

2. From p-fields to the quintic three-fold

We introduce the moduli space $\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4, d)^p$ of 1-stable curves with p-fields, we endow it with a 0-dimensional virtual class and show that its degree coincides with the genus 1 cuspidal invariants of the quintic three-fold X up to a sign. This is a word-by-word repetition of the arguments in [CL12] once noticed that they carry over to the situation where we work over a family of Gorenstein (not necessarily nodal) curves. Our aim in this section is to improve the legibility of our paper by providing the non-expert reader with a résumé of some of the key ideas contained in [CL12]; it could otherwise be skipped.

2.1. Moduli of sections.

Definition 2.1. Let B be an algebraic stack and let $\pi\colon \mathscr{C} \to B$ be a flat proper morphism of finite presentation which is representable by algebraic spaces and whose geometric fibers are reduced l.c.i. curves. Let $\mathscr{Z}$ be an algebraic stack smooth representable and quasi-projective over $\mathscr{C}$. The *cone of sections* of $\mathscr{Z}$ over $\mathscr{C}$ is a B-stack $\mathfrak{S}$ defined by:

$$\mathfrak{S}(S \to B) = \{\text{sections of } \mathscr{Z}_S \to \mathscr{C}_S\}\,;$$

The groupoid $\mathfrak{S}$ is an algebraic stack representable and quasi-projective over B [CL15, Proposition 2.3].

To fix notation, let us write down the following diagram

$$\begin{tikzcd}
 & \mathscr{Z} \arrow[d] \\
\mathscr{C}_{\mathfrak{S}} \arrow[ur, bend left, "\mathfrak{e}"] \arrow[r] \arrow[d, "\pi_{\mathfrak{S}}"] & \mathscr{C} \arrow[d, "\pi"] \\
\mathfrak{S} \arrow[r] & B
\end{tikzcd}$$

where $\mathfrak{e}$ denotes the universal section. Let us see some examples of the above construction we will be interested in.

Direct image cones. When $\mathscr{Z} = \mathrm{Vb}(\mathscr{L})$, then the algebraic stack $\mathfrak{S} \to B$ representing sections of $\mathscr{L}$, can be described as

$$C(\pi_*\mathscr{L}) = \mathrm{Spec}_B(\mathrm{R}^1\,\pi_*\mathscr{L}).$$

This is essentially because $\mathrm{R}^1\,\pi_*\mathscr{L}$ commutes with pullback, and it has the desired modular interpretation by Serre duality.

Moduli of stable maps and 1*-stable maps.* Let $X \hookrightarrow \mathbb{P}^r$ be a projective variety and

$$\mathscr{Z}_X = (\mathrm{Tot}(\mathscr{L}_{\mathfrak{P}}^{\oplus r+1}) - 0_{\mathscr{C}_{\mathfrak{P}}}) \times_{\mathbb{P}^r} X.$$

Then the moduli space of stable maps $\overline{\mathcal{M}}_{1,n}(X,\beta)$ is an open substack of the moduli space $\mathfrak{S}_X$ of sections of $\mathscr{Z}_X \to \mathscr{C}_{\mathfrak{P}}$: the morphism $\overline{\mathcal{M}}_{1,n}(X,\beta) \to \mathfrak{P}$ is given by the line bundle $f^*\mathscr{O}_{\mathbb{P}^r}(1)$ on the universal curve, that to $\mathfrak{S}_X$ by its sections $[s_0 : \ldots : s_r]$, and the stability condition is open. This is the point of view we have already taken for $X = \mathbb{P}^n$ in § 4Proposition 1.2 to describe the local equations of the moduli space of maps relative to $\mathfrak{M}^{\mathrm{div}}$.

Analogously, the moduli space of 1-stable maps, e.g. $\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^n,d)$, can be thought as an open inside the moduli of sections of

$$\hat{\mathscr{Z}} = (\mathrm{Tot}(\hat{\mathscr{L}}_{\hat{\mathfrak{P}}}^{\oplus n+1}) - 0_{\mathscr{C}_{\hat{\mathfrak{P}}}})$$

over $\hat{\mathscr{C}}_{\hat{\mathfrak{P}}}$.

Moduli space of 1-Stable Maps with fields. Denote with M the moduli space of 1-stable maps $\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4,d)$ and we write $\mathscr{P}_{\mathrm{M}} = \mathscr{L}_{\mathrm{M}}^{\otimes -5} \otimes \omega_{\hat{\pi}_{\mathrm{M}}}$. Moreover, $(\hat{\pi}_{\mathrm{M}}, f_{\mathrm{M}})\colon \hat{\mathscr{C}}_{\mathrm{M}} \to \mathrm{M} \times \mathbb{P}^4$ are the universal 1-stable curve and map over it. We define the moduli space of p-fields as the cone of sections of the line bundle $\mathscr{P}_{\mathrm{M}}$ over $\hat{\mathscr{C}}_{\mathrm{M}}$.

Definition 2.2. The moduli space of 1-stable maps with p-fields

$$\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4,d)^p := C(\hat{\pi}_{\mathrm{M},*}(\mathscr{P}_{\mathrm{M}}))$$

parametrises 1-stable maps

$$\begin{tikzcd} \hat{\mathscr{C}}_S \arrow[r, "f_S"] \arrow[d, "\hat{\pi}_S"] & \mathbb{P}^4 \\ S & \end{tikzcd}$$

with a p-field $\hat{\psi} \in H^0(\hat{\mathscr{C}}_S, f_S^*\mathscr{O}_{\mathbb{P}^4}(-5) \otimes \omega_{\hat{\pi}_S})$.

We abbreviate $\mathcal{P} := \overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4,d)^p$. This should be thought of as the space of sections of the line bundle $\mathscr{P}_{\mathrm{M}}$ over $\hat{\mathscr{C}}_{\mathrm{M}}$:

Otherwise, in parallel with the above description on the moduli space of 1-stable maps, $\mathcal{P}$ can be thought as a (open in a) moduli space of sections of the bundle $\mathrm{Vb}(\hat{\mathscr{L}}_{\hat{\mathfrak{P}}}^{\oplus 5} \oplus \mathscr{P}_{\hat{\mathfrak{P}}})$.

2.2. **Obstruction theories.** In the notation of the previous section, the morphism $\mathfrak{S} \to B$ admits a relative dual perfect obstruction theory:

$$\phi_{\mathfrak{S}/B}\colon \mathbb{T}_{\mathfrak{S}/B} \to \mathbb{E}_{\mathfrak{S}/B} := \mathrm{R}^\bullet \pi_{\mathfrak{S},*}\mathfrak{e}^* T_{\mathscr{Z}/\mathscr{C}}$$

The proof of this fact is given in [CL12, Proposition 2.5], and it is enough to notice that it relies on general properties of obstruction theories and the cotangent complex and the fact that $\mathscr{Z} \to \mathscr{C}$ is smooth, but never on the specification that $\mathscr{C} \to B$ be a family of *nodal* curves, to conclude that the same results apply when the fibers of $\mathscr{C} \to B$ are l.c.i curves.

In the examples considered above we get: in the case of a line bundle we recover $\mathbb{E}_{C(\pi_*\mathscr{L})/B} = \mathrm{R}^\bullet\, \pi_*\mathscr{L}$;

in the case of the moduli space $\mathrm{M} = \overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4, \mathrm{d})$ we get an obstruction theory relative to $\hat{\mathfrak{P}}$ given by $\mathrm{R}^\bullet\, \hat{\pi}_{\mathrm{M},*}(\bigoplus_0^4 \mathscr{L}_{\mathrm{M}})$;

in the case of 1- stable maps with fields $\mathcal{P}$ we get $\mathrm{R}^\bullet\, \hat{\pi}_{\mathcal{P},*}(\mathscr{L}_{\mathcal{P}}^{\oplus 5} \oplus \mathscr{P}_{\mathcal{P}})$.

On the moduli space of 1-stable map we have another natural perfect obstruction theory relative to $\hat{\mathfrak{M}}$, which is the standard one defined by Behrend-Fantechi [**?**, Proposition 6.3]. The next result explain how to compare the two.

Lemma 2.3. *$\hat{\mathfrak{P}}$ is a smooth Artin stack of dimension 0. Furthermore there is a compatible triple of dual perfect obstruction theories:*

$$\begin{tikzcd}
\lambda^*\mathbb{T}_{\hat{\mathfrak{P}}/\hat{\mathfrak{M}}}[-1] \arrow[r] \arrow[d, "\wr"] & \mathbb{T}_{\mathrm{M}/\hat{\mathfrak{P}}} \arrow[r] \arrow[d] & \mathbb{T}_{\mathrm{M}/\hat{\mathfrak{M}}} \arrow[r, "{[1]}"] \arrow[d] & {} \\
\mathrm{R}^\bullet\, \hat{\pi}_{\mathrm{M},*}(\mathscr{O}_{\hat{\mathscr{C}}}) \arrow[r] & \mathrm{R}^\bullet\, \hat{\pi}_{\mathrm{M},*}(\bigoplus_0^4 \mathscr{L}_{\mathrm{M}}) \arrow[r] & \mathrm{R}^\bullet\, \hat{\pi}_{\mathrm{M},*}(f_{\mathrm{M}}^* T_{\mathbb{P}^4}) \arrow[r, "{[1]}"] & {}
\end{tikzcd}$$

implying that $\mathbb{T}_{\mathrm{M}/\hat{\mathfrak{P}}} \to \mathbb{E}_{\mathrm{M}/\hat{\mathfrak{P}}} := \mathrm{R}^\bullet\, \hat{\pi}_{\mathrm{M},*}(\bigoplus_0^4 \mathscr{L}_{\mathrm{M}})$ gives the standard Behrend-Fantechi-Viscardi virtual class on M.

Proof. The first statement follows from deformation theory: the projection $\lambda\colon \hat{\mathfrak{P}} \to \hat{\mathfrak{M}}$ is unobstructed of relative dimension 0 and $\hat{\mathfrak{M}}$ is a smooth Artin stack of dimension 0, since both nodal and cuspidal singularities are l.c.i., so obstructions to their deformations are contained in $\mathrm{Ext}^2_{\mathscr{O}_{\hat{C}}}(\Omega_{\hat{C}}, \mathscr{O}_{\hat{C}}) = 0$.

The fact that $\mathbb{T}_{\mathrm{M}/\hat{\mathfrak{M}}} \to \mathbb{E}_{\mathrm{M}/\hat{\mathfrak{M}}} := \mathrm{R}^\bullet\, \hat{\pi}_{\mathrm{M},*}(f_{\mathrm{M}}^* T_{\mathbb{P}^4})$ is a perfect obstruction theory when $\hat{\mathscr{C}} \to \mathrm{M}$ is a family of Gorenstein curves is proved in [**?**, Proposition 6.3].

The lower row in the above diagram is induced by the Euler sequence of $\mathbb{P}^4$. The middle column comes from identifying the space of stable maps as an open substack of the cone of sections (see above) of $\mathrm{Vb}(\bigoplus_0^4 \mathscr{L})$ over $\hat{\mathfrak{P}}$. The existence of such a commutative diagram is [CL12, Lemma 2.8].

Finally, the fact that $\mathbb{T}_{\mathrm{M}/\hat{\mathfrak{P}}} \to \mathbb{E}_{\mathrm{M}/\hat{\mathfrak{P}}} := \mathrm{R}^\bullet\, \hat{\pi}_{\mathrm{M},*}(\bigoplus_0^4 \mathscr{L}_{\mathrm{M}})$ gives the standard virtual class for the moduli space of maps follows from functoriality of virtual pullback [**?**]. □

Finally notice that the relative perfect obstruction theory for $\mathcal{P}$ fits in the following exact triangle:

Lemma 2.4. *There is a compatible triple of dual perfect obstruction theories*

$$(\mathrm{R}^\bullet\, \hat{\pi}_{\mathcal{P},*}(\mathscr{P}_{\mathcal{P}}), \mathrm{R}^\bullet\, \hat{\pi}_{\mathcal{P},*}(\mathscr{L}_{\mathcal{P}}^{\oplus 5} \oplus \mathscr{P}_{\mathcal{P}}), \mathrm{R}^\bullet\, \hat{\pi}_{\mathcal{P},*}(\mathscr{L}_{\mathcal{P}}^{\oplus 5}))$$

for the triangle:

$$\begin{tikzcd} \mathcal{P} \arrow[rr, "\rho"] \arrow[dr] & & \mathrm{M} \arrow[dl] \\ & \hat{\mathfrak{P}} & \end{tikzcd}$$

This follows from the general theory of moduli spaces of sections recalled above.

Notice that the virtual rank of $\mathbb{E}_{\mathcal{P}/\hat{\mathfrak{P}}} := \mathrm{R}^\bullet\,\hat{\pi}_{\mathcal{P},*}(\mathscr{L}_{\mathcal{P}}^{\oplus 5} \oplus \mathscr{P}_{\mathcal{P}})$ is 0, hence it endows the moduli space of 1-stable maps with p-fields with a cycle class of dimension 0. In the next section we show that this cycles is actually supported on the proper sub-stack $\overline{\mathcal{M}}_1^{(1)}(X,d)$ where $X = V(\mathbf{w}) \subseteq \mathbb{P}^4$ is the smooth quintic three-fold defined by a homogeneous polynomial $\mathbf{w} \in k[x_0,\dots,x_4]_5$.

2.3. **Cosection Localization.** We show the existence of a cosection of the obstruction bundle whose degeneracy locus is the proper substack $\overline{\mathcal{M}}_1^{(1)}(X,d)$. The machinery of cosection localised virtual classes [KL13]. Recall their Theorem 1.1:

Theorem (Localization by cosection). *Let $\mathcal{M}$ be a Deligne-Mumford stack endowed with a perfect obstruction theory. Suppose the obstruction sheaf $\mathrm{Ob}_{\mathcal{M}}$ admits a surjective homomorphism $\sigma : \mathrm{Ob}_{\mathcal{M}}|_{\mathrm{U}} \to \mathscr{O}_{\mathrm{U}}$ over an open $U \subseteq \mathcal{M}$. Let $\mathcal{M}(\sigma) = \mathcal{M} \setminus U$. Then $(\mathcal{M},\sigma)$ has a localized virtual cycle*

$$[\mathcal{M}]^{\mathrm{virt}}_{\mathrm{loc}} \in A_*\mathcal{M}(\sigma).$$

This cycle enjoys the usual properties of the virtual cycles; it relates to the usual virtual cycle $[\mathcal{M}]^{\mathrm{virt}}$ via $[\mathcal{M}]^{\mathrm{virt}} = \iota_*[\mathcal{M}]^{\mathrm{virt}}_{\mathrm{loc}} \in A_*\mathcal{M}$, where $\iota : \mathcal{M}(\sigma) \to \mathcal{M}$ is the inclusion.

We are now going to construct the cosection as in [CL12, §§3.2-3.4]. There is a morphism of vector bundles on $\hat{\mathfrak{P}}$ induced by tensoring of line bundles:

$$h_1 \colon \mathrm{Vb}(\mathscr{L}_{\hat{\mathfrak{P}}}^{\oplus 5} \oplus \mathscr{P}_{\hat{\mathfrak{P}}}) \to \mathrm{Vb}(\omega_{\hat{\pi}_{\hat{\mathfrak{P}}}}), \quad \mathrm{h}_1(\mathrm{z},\mathrm{p}) = \mathrm{p}\mathbf{w}(\mathrm{z}_0,\dots,\mathrm{z}_4)$$

By differentiating it and pulling it back along the universal evaluation

$$\begin{tikzcd} & \mathrm{Vb}(\mathscr{L}_{\hat{\mathfrak{P}}}^{\oplus 5}) \setminus \{0\} \oplus \mathrm{Vb}(\mathscr{P}_{\hat{\mathfrak{P}}}) \arrow[d] \\ \hat{\mathscr{C}}_{\mathcal{P}} \arrow[ur, bend left, "\mathfrak{e}"] \arrow[r] \arrow[d, "\hat{\pi}_{\mathcal{P}}"] & \hat{\mathscr{C}}_{\hat{\mathfrak{P}}} \arrow[d, "\hat{\pi}_{\hat{\mathfrak{P}}}"] \\ \mathcal{P} \arrow[r] & \hat{\mathfrak{P}} \end{tikzcd}$$

we obtain a cosection of the relative obstruction sheaf

$$\sigma_1\colon \mathrm{Ob}_{\mathcal{P}/\hat{\mathfrak{P}}} = \mathrm{R}^1\,\hat{\pi}_{\mathcal{P},*}(\mathscr{L}_{\mathcal{P}}^{\oplus 5} \oplus \mathscr{P}_{\mathcal{P}}) \to \mathrm{R}^1\,\hat{\pi}_{\mathcal{P},*}(\omega_{\hat{\pi}_{\mathcal{P}}}) \simeq \mathscr{O}_{\mathcal{P}}$$
$$\sigma_{1|(u,\psi)}(x,p) = p\mathbf{w}(u) + \psi\sum_{i=0}^{4}\partial_i\mathbf{w}(u)x_i \tag{2}$$

The degeneracy locus of this cosection is $\overline{\mathcal{M}}_1^{(1)}(X,d)$: by Serre duality if $\mathbf{w}(u)\neq 0$ then we can find a p such that the cosection does not vanish; similarly we can do if $\psi\sum_{i=0}^4\partial_i\mathbf{w}(u)\neq 0$, but then $\psi=0$ by smoothness of $\mathbf{w}$.

Moreover, Chang-Li prove that σ_1 lifts to a cosection of the absolute obstruction bundle $\mathrm{Ob}_{\mathcal{P}}\to\mathscr{O}_{\mathcal{P}}$ with the same degeneracy locus. Why is this needed

We may thus endow $\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4,d)^p$ with a localised virtual cycle:

$$[\mathcal{P}]^{\mathrm{vir}}_{\sigma_1} = 0^!_{\sigma_1,\mathrm{loc}}[\mathfrak{C}_{\mathcal{P}/\hat{\mathfrak{P}}}] \;\in\; A_0(\overline{\mathcal{M}}_1^{(1)}(X,d)).$$

We we want to show that it gives the same numerical invariants as the cuspidal Gromov-Witten theory of X, up to a sign:

Theorem 2.5.

$$\deg[\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4,d)^p]^{\mathrm{vir}}_{\mathrm{loc}} = (-1)^{5d}\deg[\overline{\mathcal{M}}_1^{(1)}(X,d)]^{\mathrm{vir}}$$

2.4. **Sketch of the proof.** This is done in two steps: first we compare the invariants coming from $\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4,d)^p$ with the ones coming from $\overline{\mathcal{M}}_1^{(1)}(N_{X/\mathbb{P}^4},d)^p$, where $N_{X/\mathbb{P}^4}$ is the normal of the quintic in $\mathbb{P}^4$.

Second we compare the latter invariants with $\deg[\overline{\mathcal{M}}_1^{(1)}(X,d)]^{\mathrm{vir}}$.

Step 1. This is achieved in [CL12, §§4-5] by a family version of the p-fields construction applied to the deformation to the normal cone of $X\subseteq\mathbb{P}^4$; let us denote the latter by $V\to\mathbb{A}^1_t$, so that $V_{t\neq 0}=\mathbb{P}^4$ and $V_0=N_{X/\mathbb{P}^4}$. The idea is: construct a perfect obstruction theory for $\overline{\mathcal{M}}_1^{(1)}(V,d)^p$ relative to $\hat{\mathfrak{P}}$ which is compatible with the obstruction theories of the fibers $\overline{\mathcal{M}}_1^{(1)}(V_t,d)^p$; then construct a family version of the cosection and the fact that for compatible obstruction theories taking cosection localized cycles commutes with Gysin pull-back to conclude this Step.

Lemma 2.6. *The deformation to the normal cone V is cut inside* $\mathrm{Vb}(\mathscr{O}_{\mathbb{P}^4}(5))\times\mathbb{A}^1_{\mathrm{t}}$ *with basis coordinates $[x_0:\ldots:x_4]$ and fiber coordinate y by the equation $\mathbf{w}(x)-ty=0$. If $C(V)$ denotes the affine cone over V, then its tangent bundle is determined by the following exact sequences:*

$$0\to T_{C(V)/\mathbb{A}^1_t}\to\mathscr{O}_{C(V)}^{\oplus 5}\oplus\mathscr{O}_{C(V)}\xrightarrow{\sum_i\partial_i\mathbf{w}(x)\mathring{x}_i-t\mathring{y}}\mathscr{O}_{C(V)}\to 0$$
$$0\to T_{C(V)}\to\mathscr{O}_{C(V)}^{\oplus 5}\oplus\mathscr{O}_{C(V)}\oplus\mathscr{O}_{C(V)}\xrightarrow{\sum_i\partial_i\mathbf{w}(x)\mathring{x}_i-t\mathring{y}-y\mathring{t}}\mathscr{O}_{C(V)}\to 0$$

This is just a computation in local coordinates. This allows a description of the moduli space of maps to V as the cone of sections of a certain smooth object $\mathscr{Z}'$ over $\hat{\mathscr{C}}_{\hat{\mathfrak{P}}\times\mathbb{A}^1}$:

$$\begin{tikzcd}
 & \mathscr{Z}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V \arrow[d] \\
 & \mathrm{Vb}(\mathscr{L}_{\hat{\mathfrak{P}}}^{\oplus 5})\setminus\{0\}\oplus\mathrm{Vb}(\mathscr{L}_{\hat{\mathfrak{P}}}^{\otimes 5}) \arrow[r] \arrow[d] & \mathrm{Vb}(\mathscr{O}_{\mathbb{P}^4}(5))\times\mathbb{A}^1_{\mathrm{t}} \\
\hat{\mathscr{C}}_{\overline{\mathcal{M}}_1^{(1)}(V)} \arrow[r] \arrow[d, "\hat{\pi}_{\overline{\mathcal{M}}_1^{(1)}(V)}"] \arrow[uur, bend left, "\mathfrak{e}"] & \hat{\mathscr{C}}_{\hat{\mathfrak{P}}\times\mathbb{A}^1_t} \arrow[d, "\hat{\pi}_{\hat{\mathfrak{P}}\times\mathbb{A}^1_t}"] & \\
\overline{\mathcal{M}}_1^{(1)}(V,(d,0)) \arrow[r] & \hat{\mathfrak{P}}\times\mathbb{A}^1_{\mathrm{t}} &
\end{tikzcd}$$

Similarly $\mathcal{V} := \overline{\mathcal{M}}_1^{(1)}(V,(d,0))^p$ can be defined as the cone of sections of $\mathscr{Z} := \mathscr{Z}'\oplus\mathrm{Vb}(\mathscr{P}_{\hat{\mathfrak{P}}})$. The general theory explained above provides an obstruction theory for $\mathcal{V}\to\hat{\mathfrak{P}}\times\mathbb{A}^1_t$ [CL12, Proposition 4.2]:

Lemma 2.7. *A dual perfect obstruction theory is given by*

$$\phi_{\mathcal{V}/\hat{\mathfrak{P}}\times\mathbb{A}^1_t}\colon \mathbb{T}_{\mathcal{V}/\hat{\mathfrak{P}}\times\mathbb{A}^1_t}\to\mathbb{E}_{\mathcal{V}/\hat{\mathfrak{P}}\times\mathbb{A}^1_t} := \mathrm{R}^\bullet\,\hat{\pi}_{\mathcal{V}}(f_{\mathcal{V}}^*\mathscr{H}\oplus\mathscr{P}_{\mathcal{V}})$$

where $f_{\mathcal{V}}\colon\hat{\mathscr{C}}_{\mathcal{V}}\to V$ is the universal map and $\mathscr{H}$ is the vector bundle on V defined by

$$0\to\mathscr{H}\to\mathrm{pr}_{\mathbb{P}^4}^*\left(\mathscr{O}_{\mathbb{P}^4}(1)^{\oplus 5}\oplus\mathscr{O}_{\mathbb{P}^4}(5)\right)\xrightarrow{\sum_{\mathrm{i}}\partial_{\mathrm{i}}\mathbf{w}(\mathrm{x})\mathring{\mathrm{x}}_{\mathrm{i}}-\mathrm{t}\mathring{\mathrm{y}}}\mathrm{pr}_{\mathbb{P}^4}^*\mathscr{O}_{\mathbb{P}^4}(5)\to 0$$

The restriction of $\phi_{\mathcal{V}/\hat{\mathfrak{P}}\times\mathbb{A}^1_t}$ to a fiber

$$\mathcal{V}_t=\begin{cases}\mathcal{P} & t\neq 0\\ \overline{\mathcal{M}}_1^{(1)}(N_{X/\mathbb{P}^4},d)^p & t=0\end{cases}$$

gives the standard obstruction theory of $\mathcal{V}_t\to\hat{\mathfrak{P}}$.

We would like to conclude that the restriction of the virtual cycle to the fibers is the standard virtual cycle on the fiber. The techniques of functoriality in intersection theory teach us that we should look for a triple of compatible obstruction theories for the triangle:

$$\begin{tikzcd}
\mathcal{V}_t \arrow[rr, hook, "\iota_t"] \arrow[dr] & & \mathcal{V} \arrow[dl] \\
 & \hat{\mathfrak{P}} &
\end{tikzcd}$$

The cone of sections interpretation provides us with an obstruction theory relative to $\mathcal{V}\to\hat{\mathfrak{P}}$ given by:

$$\mathbb{E}'_{\mathcal{V}/\hat{\mathfrak{P}}} := \mathrm{R}^\bullet\,\hat{\pi}_{\mathcal{V}}(f_{\mathcal{V}}^*\mathscr{K}\oplus\mathscr{P}_{\mathcal{V}})$$

where $\mathscr{K}$ is determined by the following exact sequence on V:

$$0 \to \mathscr{K} \to \mathrm{pr}_{\mathbb{P}^4}^* \left(\mathscr{O}_{\mathbb{P}^4}(1)^{\oplus 5} \oplus \mathscr{O}_{\mathbb{P}^4}(5) \oplus \mathscr{O}_{\mathbb{P}^4}\right) \xrightarrow{\sum_i \partial_i \mathbf{w}(x)\mathring{x}_i - t\mathring{y} - y\mathring{t}} \mathrm{pr}_{\mathbb{P}^4}^* \mathscr{O}_{\mathbb{P}^4}(5) \to 0$$

Of course $h^0(\mathbb{E}'_{\mathcal{V}/\hat{\mathfrak{P}}}) \simeq \mathbb{T}_{\mathcal{V}/\hat{\mathfrak{P}}}$, but the previous lemma hints at the fact that the obstruction sheaf $h^1(\mathbb{E}'_{\mathcal{V}/\hat{\mathfrak{P}}})$ contains one factor $\mathrm{R}^1\,\hat{\pi}_{\mathcal{V},*}\mathscr{O}_{\hat{\mathscr{C}}_{\mathcal{V}}}$ too many, so that restricting to the fibers we would find a different obstruction theory with respect to the standard one. A confirmation of this fact is that $\mathbb{E}'_{\mathcal{V}/\hat{\mathfrak{P}}}$ equips $\mathcal{V}$ with a 0-dimensional cycle, while we are looking for a 1-dimensional cycle such that restricting to any fiber, i.e. applying $\iota_t^!$, we get $[\mathcal{V}_t]^{\mathrm{vir}} \in A_0(\mathcal{V}_t)$.

This issue is solved in [CL12, §§4.5-6] by lifting the obstruction theory to

$$\phi_{\mathcal{V}/\hat{\mathfrak{P}}} \colon \mathbb{T}_{\mathcal{V}/\hat{\mathfrak{P}}} \to \mathbb{E}_{\mathcal{V}/\hat{\mathfrak{P}}},$$

where the latter fits into an exact triangle:

$$\mathrm{R}^1\,\hat{\pi}_{\mathcal{V},*}\mathscr{O}_{\hat{\mathscr{C}}_{\mathcal{V}}}[-2] \to \mathbb{E}_{\mathcal{V}/\hat{\mathfrak{P}}} \xrightarrow{\nu} \mathbb{E}_{\mathcal{V}/\hat{\mathfrak{P}}} \xrightarrow{[1]} .$$

Furthermore the lifted obstruction theory is compatible with the one the fibers,

Lemma 2.8. *There is the following commutative diagram of exact triangles:*

$$\begin{tikzcd}
\hat{\pi}_{\mathcal{V}_c *}\mathscr{O}_{\hat{\mathscr{C}}_{\mathcal{V}_c}}[-1] \arrow[r] & \mathbb{E}_{\mathcal{V}_c/\hat{\mathfrak{P}}} \arrow[r] & \mathbb{E}_{\mathcal{V}/\hat{\mathfrak{P}}}|_{\mathcal{V}_c} \\
\mathbb{T}^{\geq 1}_{\mathcal{V}_c/\mathcal{V}} \arrow[r] \arrow[u] & \mathbb{T}^{\geq 1}_{\mathcal{V}_c/\hat{\mathfrak{P}}} \arrow[r] \arrow[u] & \mathbb{T}^{\geq 1}_{\mathcal{V}/\hat{\mathfrak{P}}}|_{\mathcal{V}_c} \arrow[u]
\end{tikzcd}$$

The proof is exactly the same given in [CL15, § 4.6].

Since we are working with cosection localised cycles, this is not yet enough to conclude. A family version of the cosection is induced by differentiating the following vector bundle morphism on $\hat{\mathfrak{P}} \times \mathbb{A}^1_t$:

$$\mathrm{Vb}(\mathscr{L}_{\hat{\mathfrak{P}}}^{\oplus 5} \oplus \mathscr{L}_{\hat{\mathfrak{P}}}^{\otimes 5} \oplus \mathscr{P}_{\hat{\mathfrak{P}}}) \xrightarrow{(\mathrm{pr}_2, \mathrm{pr}_3)} \mathrm{Vb}(\mathscr{L}_{\hat{\mathfrak{P}}}^{\otimes 5} \oplus \mathscr{P}_{\hat{\mathfrak{P}}}) \xrightarrow{\cdot} \mathrm{Vb}(\omega_{\hat{\pi}_{\hat{\mathfrak{P}}}})$$

The cosection takes then the following form:

$$\bar{\sigma}_{1|(u,v,\psi)}(\mathring{x}, \mathring{y}, \mathring{p}) = \psi\mathring{y} + v\mathring{p}$$

It is showed in [CL12, §4.7] that $\bar{\sigma}_1$ lifts to a cosection $\bar{\sigma} \colon \mathrm{Ob}_{\mathcal{V}} \to \mathscr{O}_{\mathcal{V}}$ and that the degeneracy locus of $\bar{\sigma}$ is

$$\overline{\mathcal{M}}_1^{(1)}(X, d) \times \mathbb{A}^1_t.$$

Recall that the sections (x, y) are required to satisfy $\mathbf{w}(x) - ty = 0$. So $\bar{\sigma}_1$ coincides up to a non-zero scalar with the above defined cosection σ_1 for $\mathcal{P}$ when $t \neq 0$. It is proved in [KL13, Theorem 5.2] that, given compatible perfect obstruction theories, the construction of a cosection localised virtual cycle is compatible with Gysin pullbacks, so that [CL12, Proposition 4.9]:

$$\iota_{t\neq 0}^![\mathcal{V}]^{\mathrm{vir}}_{\bar{\sigma}} = [\mathcal{P}]^{\mathrm{vir}}_{\sigma} \in A_0(\mathcal{Q}), \quad \iota_0^![\mathcal{V}]^{\mathrm{vir}}_{\bar{\sigma}} = [\overline{\mathcal{M}}_1^{(1)}(N_{X/\mathbb{P}^4}, d)^p]^{\mathrm{vir}}_{\bar{\sigma}_0} \in A_0(\mathcal{Q})$$

where we have denoted by $\mathcal{Q} := \overline{\mathcal{M}}_1^{(1)}(X, d)$.

Step 2. We are left with proving that $[\overline{\mathcal{M}}_1^{(1)}(N_{X/\mathbb{P}^4}, d)^p]^{\mathrm{vir}}_{\bar{\sigma}_0}$ coincides up to sign with $[\overline{\mathcal{M}}_1^{(1)}(X, d)]^{\mathrm{vir}}$. In the rest of the section we use the following notation $\mathcal{N} := \overline{\mathcal{M}}_1^{(1)}(N_{X/\mathbb{P}^4}, d)^p$, and $v\colon \mathcal{N} \to \mathcal{Q}$. The idea is now to conclude finding a perfect obstruction theory for $v\colon \mathcal{N} \to \mathcal{Q}$ compatible with the ones of $\mathcal{N} \to \hat{\mathfrak{P}}$ and $\mathcal{Q} \to \hat{\mathfrak{P}}$ and then prove a cosection localized version of Manolache's virtual pull-back construction [?].

Chang and Li proved indeed that there is a perfect obstruction theory

$$\mathbb{E}_{\mathcal{N}/\mathcal{Q}} := \mathrm{R}^\bullet\, \hat{\pi}_{\mathcal{N},*}(\mathscr{L}_{\mathcal{N}}^{\otimes 5} \oplus \mathscr{P}_{\mathcal{N}})$$

compatible with $\mathbb{E}_{\mathcal{N}/\hat{\mathfrak{P}}}$ and $v^*\mathbb{E}_{\mathcal{Q}/\hat{\mathfrak{P}}}$, so that $\mathbb{E}_{\mathcal{N}/\mathcal{Q}}$ inherits a cosection σ_0' with degeneracy locus $D(\sigma_0') = \mathcal{Q}$. Then [CL12, Lemma 5.5] they show by using the techniques of [KKP03] that $\mathfrak{C}_{\mathcal{N}/\mathcal{Q}}$ is supported inside

$$h^1/h^0(\mathbb{E}_{\mathcal{N}/\mathcal{Q}})_{\sigma_0'} := \mathrm{Ker}(\sigma_0') \cup h^1/h^0(\mathbb{E}_{\mathcal{N}/\mathcal{Q}})_{|D(\sigma_0')}$$

so that they may combine the techniques of virtual pullback [?] and localisation by cosection [KL13] to define:

$$v^!_{\mathbb{E}_{\mathcal{N}/\mathcal{Q}},\mathrm{loc}}\colon A_*(\mathcal{Q}) \to A_*(D(\sigma_0') = \mathcal{Q})$$

sending $[\mathcal{Q}]^{\mathrm{vir}}$ to $[\overline{\mathcal{M}}_1^{(1)}(N_{X/\mathbb{P}^4}, d)^p]^{\mathrm{vir}}_{\bar{\sigma}_0}$.

To conclude it is now enough to compute the degree of $v^!_{\mathbb{E}_{\mathcal{N}/\mathcal{Q}},\mathrm{loc}}$. Since we are just interested in what happens on $A_0(\mathcal{Q})$, we only have to compute $\deg(v^!_{\mathbb{E}_{\mathcal{N}/\mathcal{Q}},\mathrm{loc}}(\zeta)$ for ζ a closed point. This is done in [CL12, Theorem 5.7] and the same considerations work in our case since the construction of the localized pull-back is completely analogous.

Hopefully we have managed to convince the reader that the subtle intersection theory perpetuated in [CL12] does not rely at all on working with families of *nodal* curves but every step generalizes automatically in our setting; let us recall that in the end, the moduli space of 1-stable maps should be though as the moduli space of stable maps with the boundary divisor of elliptic tails contracted.

3. The comparison morphism

In this section we shall:

- argue in two different ways that there is a natural fashion to associate a weighted 1-stable curve to a weighted-stable nodal curve of genus 1, essentially by replacing elliptic tails of weight 0 by cusps, which results in a morphism:

  $$\mathfrak{M}_{1,n}^{\mathrm{wt,st}} \to \mathfrak{M}_{1,n}^{\mathrm{wt,st}}(1)$$

  extending the identity on the locus of smooth elliptic curves;

- introduce the fiber product

$$\mathcal{Z} := \mathfrak{M}_{1,n}^{\mathrm{wt,st}} \times_{\mathfrak{M}_{1,n}^{\mathrm{wt,st}}(1)} \overline{\mathcal{M}}_{1,n}^{(1)}(\mathbb{P}^4, d)$$

  and show that it is a closed substack of $\overline{\mathcal{M}}_1(\mathbb{P}^4, d)$, isomorphic to it outside the boundary divisor D^1;
- introduce the fiber product

$$\mathcal{Z}^p := \mathfrak{M}_{1,n}^{\mathrm{wt,st}} \times_{\mathfrak{M}_{1,n}^{\mathrm{wt,st}}(1)} \overline{\mathcal{M}}_{1,n}^{(1)}(\mathbb{P}^4, d)^p$$

  and endow it with an obstruction theory and a cosection thereof, which are pulled back from $\mathcal{M}_{1,n}^{(1)}(\mathbb{P}^4, d)^p$; unfortunately we will not be able to compare them directly with $\overline{\mathcal{M}}_1(\mathbb{P}^4, d)^p$;
- explain along the way all the notation one needs in order to understand the previous sentences.

Note that the only type of nodal curves that we are getting rid of are the ones with an elliptic tail of weight zero.

Theorem 3.1. *There exists a morphism $\mathfrak{M}_{1,n}^{\mathrm{wt}=d,st} \to \mathfrak{M}_{1,n}^{\mathrm{wt}=d,st}(1)$ which extends the identity on the smooth locus.*

As anticipated we explain two different approaches to the proof:

(1) we adopt the strategy of constructing the graph of such a morphism within the product $\mathfrak{M}_{1,n}^{\mathrm{wt}=d,\mathrm{st}} \times \mathfrak{M}_{1,n}^{\mathrm{wt}=d,\mathrm{st}}(1)$ and prove that the projection onto the first factor is an isomorphism;

(2) we prove that the 1-stabilisation exists at the level of curves with a divisor constructing the contraction directly, then argue that it descends to a morphism between moduli spaces of weighted curves.

3.1. **First approach: the graph.** Let $\mathscr{C}$ and $\mathscr{C}'$ be the universal curves over $\mathfrak{M} := \mathfrak{M}_{1,n}^{\mathrm{wt}=d,\mathrm{st}}$ and $\mathfrak{M}' := \mathfrak{M}_{1,n}^{\mathrm{wt}=d,\mathrm{st}}(1)$ respectively. Abusing notation, we will still write $\mathscr{C}$ and $\mathscr{C}'$ for their pullbacks to the product $\mathfrak{M} \times \mathfrak{M}'$ along the two projections. The proof of the theorem follows from two Lemmas:

Lemma 3.2. *There is a locally closed substack $\mathscr{X} \subseteq \mathrm{Mor}_{\mathfrak{M}\times\mathfrak{M}'}(\mathscr{C}, \mathscr{C}')$ representing morphisms $C \to C'$ that contract weight-zero elliptic tails to cusps and are weight-preserving isomorphisms everywhere else.*

Lemma 3.3. *The first projection $\mathrm{pr}_1\colon \mathrm{Mor}_{\mathfrak{M}\times\mathfrak{M}'}(\mathscr{C}, \mathscr{C}') \to \mathfrak{M}$ restricted to $\mathscr{X}$ is an isomorphism with $\mathfrak{M}$.*

Proof. 3.2 Recall that $\mathrm{Mor}_{\mathfrak{M}\times\mathfrak{M}'}(\mathscr{C}, \mathscr{C}')$ is an algebraic stack; in fact the map to $\mathfrak{M} \times \mathfrak{M}'$ is representable (by algebraic spaces) [Ols06]. We now proceed to construct $\mathscr{X}$ as a locally closed substack in the space of morphisms.

Step 1: Consider

$$\pi\colon \mathfrak{P} = \mathfrak{Pic}_{1,n}^{totdeg=d,st} \to \mathfrak{M},\ \pi'\colon \mathfrak{P}' = \mathfrak{Pic}_{1,n}^{totdeg=d,st}(1) \to \mathfrak{M}'$$

the Picard stacks of $\mathscr{C} \to \mathfrak{M}$ and $\mathscr{C}' \to \mathfrak{M}'$ with universal line bundles $\mathfrak{L}$ and $\mathfrak{L}'$, where π and π' are defined by taking the multi-degree of line bundles. We can now look at the algebraic stack $\mathrm{Mor}_{\mathfrak{P}\times\mathfrak{P}'}(\mathscr{C},\mathscr{C}')$ with universal morphism Φ and natural projection Π to $\mathrm{Mor}_{\mathfrak{M}\times\mathfrak{M}'}(\mathscr{C},\mathscr{C}')$. We claim that there exists a locally closed substack $\mathscr{Y}' \subseteq \mathrm{Mor}_{\mathfrak{P}\times\mathfrak{P}'}(\mathscr{C},\mathscr{C}')$ representing those morphisms that *preserve the line bundles*. Indeed, given a chart

$$S \to \mathrm{Mor}_{\mathfrak{P}\times\mathfrak{P}'}(\mathscr{C},\mathscr{C}'),$$

the locus of $s \in S$ where $\Phi_s^*\mathfrak{L}'_s \cong \mathfrak{L}_s$ is nothing else than the locus T where the two sections $\mathfrak{L}_S$ and $\Phi_S^*\mathfrak{L}'_S$ of $\mathfrak{P}(S) \to \mathfrak{M}(S)$ coincide. In other words, we are looking at the fiber product

$$\begin{tikzcd} T \arrow[r] \arrow[d] & \mathfrak{P} \arrow[d, "\Delta"] \\ S \arrow[r] & \mathfrak{P}\times_{\mathfrak{M}}\mathfrak{P} \end{tikzcd}$$

Being $\mathfrak{P} \to \mathfrak{M}$ representable by locally separated algebraic spaces [BLR12, Theorem 8.3.1], Δ is a quasi-compact locally closed immersion [Sta16, Tag 04YU], so in particular $T \subseteq S$ is locally closed.

Step 2: Furthermore there is a closed substack $\mathscr{Y} \subseteq \mathscr{Y}'$ representing *surjective morphisms that preserve the markings.*

Given a chart $S \to \mathrm{Mor}_{\mathfrak{P}\times\mathfrak{P}'}(\mathscr{C},\mathscr{C}')$, the locus of $s \in S$ where Φ_s is marking-preserving is the equaliser of the two sections

$$S \underset{\times\Phi\circ\sigma_i}{\overset{\times\sigma'_i}{\rightrightarrows}} \mathscr{C}'_S \times_S \ldots \times_S \mathscr{C}'_S$$

This defines a closed subscheme of S, since $\mathscr{C}'_S \to S$ is separated.

As regards surjectivity, since Φ is proper and the dimension of the fiber is upper semicontinuous [Sta16, Tag 0D4I], the locus in $\mathscr{C}'_S$ where the fiber of Φ is empty is open. Its image in S is open by flatness of $\mathscr{C}' \to S$ [Sta16, Tag 01UA], and the complement of it is the locus we need.

Step 3: Let $\mathscr{X}'$ be the image of $\mathscr{Y}$ under Π. This is a constructible substack of $\mathrm{Mor}_{\mathfrak{M}\times\mathfrak{M}'}(\mathscr{C},\mathscr{C}')$ by Chevalley's theorem [LMB00, Theorem 5.9.4]. Recall that to show that a constructible set is open (respectively closed) it is enough to check that it contains all the generisations of its points (respectively all the specialisations) [Sta16, Tag 0DQNTag 0903]. Finally, under Noetherian assumptions, two points related by specialisation/generisation are contained in the image of a DVR scheme [Sta16, Tag 054F].

It is clear that being surjective and marking-preserving are closed conditions, as above. The requirement that ϕ can be covered by a line bundle-preserving map can be translated into the following combinatorial conditions:

(1) *ϕ contracts only weight zero components.* We show that this is open. Assume that S is a DVR scheme with closed point 0 and generic point η, and we are given $S \to \mathscr{X}'$ such that $\phi_0 \colon \mathscr{C}_0 \to \mathscr{C}'_0$ does not contract

any positive weight component. Suppose there exists an irreducible component $D_\eta \subseteq C_\eta$ of positive weight d_W which is contracted by ϕ_η. The contracted locus, i.e. $\left\{c \in \mathscr{C}_S | \dim_c \phi^{-1}(\phi(c)) \geq 1\right\}$, is closed by semicontinuity of fiber dimension, hence it contains all the components $D_i \subseteq C_0$ to which D_η specialises. At least one of them has positive weight, since the sum of their weights is d_W, which is a contradiction.

(2) ϕ *has degree 1 on every non contracted component* or, equivalently, there is an S-dense open in $\mathscr{C}'_S$ such that the restriction of ϕ_S to its preimage is an isomorphism. This is an open and closed condition; we show it is open. Let S be a DVR scheme as above and assume that ϕ_0 satisfies the property. Since ϕ_S is proper, we may consider

$$\phi_{S,*}[\mathscr{C}_S] = \sum n_i[\mathscr{C}'_{S,i}] \in A_2(\mathscr{C}'_S)$$

. Applying Gysin pull-back to 0 (which is a regular closed point of the base) [Ful98, Prop. 10.1(a)], we see that all the n_i's are 1 for those $\mathscr{C}'_i$'s such that $0^![\mathscr{C}'_i] \neq 0$. On the other hand there is no irreducible component of $\mathscr{C}'$ supported on $\mathscr{C}'_\eta$.

(3) ϕ *is weight-preserving.* This is again an open condition, as we can see from the weighted dual graphs. Let $\tilde{\phi}$ be the map induced at the level of weighted dual graphs $\Gamma(\mathscr{C}) \to \Gamma(\mathscr{C}')$. It is compatible with the specialisation maps:

$$\begin{array}{ccc} \Gamma(\mathscr{C}_0) & \xrightarrow{\tilde{\phi}_0} & \Gamma(\mathscr{C}'_0) \\ \Big\downarrow{\scriptstyle \mathrm{sp}} & & \Big\downarrow{\scriptstyle \mathrm{sp}} \\ \Gamma(\mathscr{C}_\eta) & \xrightarrow{\tilde{\phi}_\eta} & \Gamma(\mathscr{C}'_\eta) \end{array}$$

Since the weight of a component of the generic fiber is determined by those of the components to which it specialises

$$\deg(v) = \sum_{w \in \mathrm{sp}^{-1}(\mathrm{v})} \deg(w)$$

$\tilde{\phi}_\eta$ has to be weight-preserving as well.

Step 4: We have seen that, if ϕ contracts a connected subcurve E of the fiber, it must have zero weight. Since the target only has nodes and cusps as singularities, and the markings are required to be smooth points, we observe that E must be of arithmetic genus one by weighted stability and $|\overline{C \setminus E} \cap E| \leq 2$, i.e. E is either an elliptic tail or an elliptic bridge. There are basically two possibilities:

(1) ϕ contracts an elliptic tail to a cusp and is an isomorphism everywhere else, or there is no elliptic tail to start with and ϕ is an isomorphism;
(2) the elliptic tail/bridge is contracted to a smooth point/node, then a non-separating node or a cusp must be created somewhere else in order to preserve the arithmetic genus.

We want to avoid the second scenario, so we define the open substack $\mathscr{X} \subseteq \mathscr{X}'$ as follows. Given $\mathscr{C}_S \to \mathscr{C}'_S \in \mathscr{X}'(S)$, let $U' \subseteq \mathscr{C}'$ be the *maximal* S-dense open subset such that $\phi_S|_{\phi_S^{-1}(U')}\colon \phi_S^{-1}(U') \to U'$ is an isomorphism and Z' its closed complement in $\mathscr{C}'_S$. Then $\mathscr{X}$ is the open locus [Sta16, Tag 055G] where the fibers of $\pi|_{\phi^{-1}(Z')}\colon \phi^{-1}(Z') \to S$ are geometrically connected.

□

Proof. 3.3 This result will follow from an application of Zariski's Main Theorem for algebraic spaces. First we claim that the projection is *representable by algebraic spaces*: by [Sta16, Tag 04Y5] we only need to check that it is faithful, and by [Con07, Theorem 2.2.5] it is enough to look at geometric points. Hence we need to say that, given $\phi\colon C \to C'$ a k-point of $\mathcal{X}$, we have $\mathrm{Aut}(\phi) \subseteq \mathrm{Aut}(C)$. Recall that automorphisms of ϕ are commutative diagrams:

$$\begin{tikzcd} C \arrow[r, "\phi"] \arrow[d, "\psi"] & C' \arrow[d, "\psi'"] \\ C \arrow[r, "\phi"] & C' \end{tikzcd}$$

Now ψ' is determined by ψ due to our description of ϕ.

Secondly $\mathrm{pr}_{1|\mathscr{X}}$ is *proper*: this can be seen using the valuative criterion

$$\begin{tikzcd} \eta' = \mathrm{Spec}(K') \arrow[r] \arrow[d] & \eta = \mathrm{Spec}(K) \arrow[r] \arrow[d, "\exists ?"'] & \mathscr{X} \arrow[d] \\ S' = \mathrm{Spec}(R') \arrow[r] \arrow[urr, dashed] & S = \mathrm{Spec}(R) \arrow[r] & \mathfrak{M} \end{tikzcd}$$

Let $\pi\colon \mathscr{C}_S \to S$ be the family of nodal curves on S; there are three cases to consider:

(1) the central fiber contains no elliptic tail, then the same is true for $\mathscr{C}_\eta$, hence ϕ_η is an isomorphism. We can extend ϕ_η as follows:

$$\begin{tikzcd} \mathscr{C}_\eta \arrow[r, "\sim", "\phi_\eta"'] \arrow[d, hook, "\iota"] & \mathscr{C}'_\eta \arrow[d, hook, "\iota\circ\phi_\eta^{-1}"] \\ \mathscr{C}_S \arrow[r, "\mathrm{id}_{\mathscr{C}}"] & \mathscr{C}_S =: \mathscr{C}'_S \end{tikzcd}$$

Another extension $\phi'\colon \mathscr{C}_S \cong \mathscr{C}'_S$ would be isomorphic to the previous one via:

$$\begin{tikzcd} \mathscr{C}_S \arrow[r, "\phi'"] \arrow[d, "\mathrm{id}"] & \mathscr{C}'_S \arrow[d, "(\phi')^{-1}"] \\ \mathscr{C}_S \arrow[r, "\mathrm{id}"] & \mathscr{C}_S \end{tikzcd}$$

If instead $\mathscr{C}_0$ has got an elliptic tail, then we have two possibilities:

(2) $\mathscr{C}_\eta$ has an elliptic tail as well; that is the image of $S \to \mathfrak{M}$ is contained in the boundary, so we can find a lift

$$\begin{tikzcd} & \mathfrak{M}_{1,1} \times \mathfrak{M}^{\mathrm{wt}}_{0,1+n} \arrow[d] & \\ S \arrow[ur, dashed] \arrow[r] & \mathfrak{D}_{\{1,\emptyset\},\{0,n\}} \arrow[r, hook] & \mathfrak{M}. \end{tikzcd}$$

Then $\mathscr{C}_S$ is the push-out of a family of rational curves $\mathscr{R}_S$ and a family of genus one curves $\mathscr{Z}_S$:

$$\begin{tikzcd} S \arrow[r] \arrow[d] & \mathscr{R}_S \arrow[d] \\ \mathscr{Z}_S \arrow[r] & \mathscr{C}_S \end{tikzcd} \tag{3}$$

Recall that the cuspidal curve $\mathscr{C}'_K$ can be characterised as the push-out of the following diagram:

$$\begin{tikzcd} 2K \arrow[r] \arrow[d] & \mathscr{R}_K \arrow[d] \\ K \arrow[r] & \mathscr{C}'_K. \end{tikzcd}$$

Since the smooth section $S \to \mathscr{R}_S$ defines a Cartier divisor, it makes sense to take its double and we can thus define $\mathscr{C}'_S$ by means of the similar diagram:

$$\begin{tikzcd} 2S \arrow[r] \arrow[d] & \mathscr{R}_S \arrow[d] \\ S \arrow[r] & \mathscr{C}'_S \end{tikzcd}$$

The morphism $\phi_S \colon \mathscr{C}_S \to \mathscr{C}'_S$ extending ϕ_η is then defined by exploiting the description of $\mathscr{C}_S$ as a push-out (3), and the morphisms $\mathrm{id}\colon \mathscr{R}_S \to \mathscr{R}_S$ and $\mathrm{pr}_{\mathscr{Z}_\mathrm{S}} \colon \mathscr{Z}_\mathrm{S} \to \mathrm{S}$.

(3) If $\mathscr{C}_S$ smoothens the elliptic tail, then ϕ_η is an isomorphism. We may assume that S is the spectrum of a complete DVR with algebraically closed residue field [LMB00, Theorem 7.10]. Then we may pick one smooth section for each rational component of $\mathscr{C}_0$ and extend them to sections of $\mathscr{C}_S \to S$ by Grothendieck's existence theorem; let us denote by Σ the Cartier divisor that is the sum of all such sections. Let Z be the elliptic tail in the central fiber; then we claim that $\omega_{\mathscr{C}_S/S}(Z) \otimes \mathscr{O}_{\mathscr{C}_S}(\Sigma)$ is π_S semi-ample, ample on the generic fiber, and gives the contraction of the elliptic tail to the cusp in the central fiber. We shall not prove the claim here, since this is the core of the second approach.

Finally, observe that the map is bijective by construction and $\mathfrak{M}$ is normal, hence $\pi_{|\mathscr{X}}\colon \mathscr{X} \to \mathfrak{M}$ is an isomorphism by Zariski's main theorem (as in [Sta16, Tag 082I]). $\square$

3.2. **Second approach: constructing the contraction.** The idea behind this construction is essentially due to Hassett [HH09, §2] and it has recently been reviewed and simplified in [RSW17, §3.7].

Let $\mathfrak{P} = \mathfrak{Pic}^{d,rmst}_{\mathscr{C}/\mathfrak{M}_1}$ with the stability condition that the line bundle $\omega_\pi \otimes \mathscr{L}^{\otimes 3}$ on the universal curve is π-relatively ample.

We work over $\mathfrak{M}_1^{\mathrm{div}}$, parametrising families of nodal curves with a relative Cartier divisor. More precisely, this can be thought of as the open inside $C(\pi_*\mathscr{L}) = \mathrm{Spec}_{\mathfrak{P}}(\mathrm{R}^1\,\pi_*\mathscr{L})$ (see [CL12] and Section **??** above), where the section of $\mathscr{L}$ is not 0 on any irreducible component of the curve. Alternatively this is the moduli functor of a prestable curve with a line bundle and a section up to scalar, which can be thought of as the hom-stack $\mathrm{Hom}_{\mathfrak{M}_1}(\mathscr{C}, [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}])$; then again one picks the connected component where the line bundle has total degree d, and the open substack obtained by requiring weighted stability and the section not to vanish on any irreducible component of the curve.

We shall construct the contraction over $\mathfrak{M}_1^{\mathrm{div}}$ first, and then show that it descends to $\mathfrak{M}_1^{\mathrm{wt}}$.

Let E be the locus inside the universal curve spanned by elliptic tails of weight 0; this is a Cartier divisor in the universal curve over $\mathfrak{M}_1^{\mathrm{wt}}$, which we shall freely pullback and keep denoting by E. We shall also denote by $\mathfrak{D}^1$ its image in $\mathfrak{M}_1^{\mathrm{wt}}$ (and its various pullbacks), which is a Cartier divisor as well.

Consider the following line bundle on the universal curve over $\mathfrak{M}_1^{\mathrm{div}}$:

$$\mathscr{N} := \omega_\pi(E) \otimes \mathscr{O}_{\mathscr{C}}(3D), \tag{4}$$

where D is the universal Cartier divisor over $\mathfrak{M}_1^{\mathrm{div}}$ (so $\mathscr{O}_{\mathscr{C}}(D)$ is the pullback of the universal line bundle on $\mathfrak{P}$). Notice that $\mathscr{N}$ is trivial on the locus of elliptic tails, so the Proj construction applied to it will contract this locus.

Proposition 3.4. *In the following diagram $\hat{\mathscr{C}}$ is a family of weighted 1-stable curves and ϕ is a regular morphism:*

$$\begin{tikzcd}
(\mathscr{C}, D) \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & (\hat{\mathscr{C}} = \mathrm{Proj}_{\mathfrak{M}_1^{\mathrm{div}}}(\bigoplus_{n\geq 0} \pi_*\mathscr{N}^{\otimes n}), \phi(D)) \arrow[dl, "\hat{\pi}"] \\
& \mathfrak{M}_1^{\mathrm{div}} &
\end{tikzcd}$$

This defines the 1-stabilisation morphism $\mathfrak{M}_1^{\mathrm{div}} \to \mathfrak{M}_1^{(1),\mathrm{div}}$.

We need to prove that $\mathscr{N}$ is π-semi-ample (regularity of ϕ) and that $\pi_*\mathscr{N}$ is locally free (flatness of $\hat{\pi}$). Both these facts are clear generically, but less so on points of E and $\mathfrak{D}^1$. We shall check this by exploiting the next lemma, which is a nice technical gadget drawn from [RSW17]: it implies that the Proj construction we perform commutes with base-change to generic curves inside $\mathfrak{M}_1^{\mathrm{div}}$, allowing us to work with a smoothing of the elliptic tail over a DVR scheme T.

Lemma 3.5 (pullback with a boundary)**.** *Let $\pi\colon \mathscr{C} \to S$ be a proper family of curves over a smooth basis, and let $\mathscr{N}$ be a line bundle on $\mathscr{C}$ such that $\mathrm{R}^1\pi_*\mathscr{N}$ is a line bundle supported on a Cartier divisor $\mathfrak{D} \subseteq S$. Then for every spectrum of a DVR T with closed point 0 and generic point η, and for every morphism $f\colon T \to S$ such that $f(0) \in \mathfrak{D}$ and $f(\eta) \in S \setminus \mathfrak{D}$ we have*

$$f^*\pi_*\mathscr{N} \cong \pi_* f^*\mathscr{N}.$$

Proof. The argument can be found in [RSW17, Lemmma 3.7.2.2]. Let $K^\bullet$ the complex of locally free sheaves on S which satisfy cohomology and base change, i.e. such that for any $T \xrightarrow{f} S$ we get $H^i(f^*K^\bullet) = \mathrm{R}^i\pi_* f^*\mathscr{M}$. The construction of $K^\bullet$ is standard, (see e.g. [Har77, Proposition 12.2] and since the cohomology is concentrated in degree 0,1, we can assume $K^\bullet = K_0 \to K_1$. Let f be the map defined in the statement of the Lemma. Then we have the following exact sequences:

$$0 \to \pi_*\mathscr{M} \to K^0 \to K^1 \to \mathrm{R}^1\pi_*\mathscr{M} \to 0, \tag{5}$$

$$0 \to \pi_* f^*\mathscr{M} \to f^*K^0 \to f^*K^1 \to \mathrm{R}^1\pi_* f^*\mathscr{M} \to 0. \tag{6}$$

The derived pull-back Lf^* of the complex (5) is zero since the complex is exact. Writing down explicitly the spectral sequence which compute the derived pull-back, we find that its convergence to zero implies that:

$$\operatorname{coker}(f^*\pi_*\mathscr{M} \to \pi_* f^*\mathscr{M}) \cong \mathrm{L_1f^*}\,\mathrm{R}^1\pi_*\mathscr{M}$$
$$\ker(f^*\pi_*\mathscr{M} \to \pi_* f^*\mathscr{M}) \cong \mathrm{L_2f^*}\,\mathrm{R}^1\pi_*\mathscr{M}.$$

To prove that $\mathrm{L_if^*}\,\mathrm{R}^1\pi_*\mathscr{M} = 0$ for $i = 1, 2$ it is enough to work locally on S, where we have a free resolution for of the form $\mathrm{R}^1\pi_*\mathscr{M}$ of the form:

$$0 \to \mathscr{O}_S \xrightarrow{g} \mathscr{O}_S \to \mathrm{R}^1\pi_*\mathscr{M} \to 0$$

and thus pulling back along f

$$0 \to \mathrm{L_1f^*}\,\mathrm{R}^1\pi_*\mathscr{M} \to \mathscr{O}_\mathrm{T} \xrightarrow{\mathrm{f^*g}} \mathscr{O}_\mathrm{T} \to \mathrm{f^*}\,\mathrm{R}^1\pi_*\mathscr{M} \to 0.$$

It immediately follows that $\mathrm{L_2f^*}\,\mathrm{R}^1\pi_*\mathscr{M} = 0$. Finally notice that since the image of f is not contained in the boundary divisor $f^*g \neq 0$ and thus injective of sheaves as T is a DVR. We can then conclude $\mathrm{L_1f^*}\,\mathrm{R}^1\pi_*\mathscr{M} = 0$ as well which implies the Lemma. □

Now recall that in our case $\mathscr{N}$ is trivial on elliptic tails and of positive degree everywhere else. The rank of $\mathrm{R}^1\pi_*\mathscr{N}$ can be checked on the fibers [Har77, Theorem III.12.11], so we see that it is 0 outside $\mathfrak{D}^1$ and 1 on it.

Lemma 3.6. *The line bundle $\mathscr{N}$ is π-semi-ample, i.e. the natural map*

$$\pi^*\pi_*\mathscr{N}^{\otimes n} \to \mathscr{N}^{\otimes n}$$

is surjective for $n \gg 0$ (in fact $n = 1$ suffices).

Proof. Outside the locus of elliptic tails $\mathscr{N}$ is π-ample. We are left to check on points belonging to an elliptic tail; thanks to the above Lemma we can reduce to the case that C is the central fiber of a one-parameter smoothing of the elliptic tail. This has been dealt with by Smyth [Smy11a, Lemma 2.12]. □

Lemma 3.7. *$\pi_*\mathscr{N}$ is locally free on $\mathfrak{M}_1^{\mathrm{div}}$.*

Proof. [RSW17, Proposition 3.7.2.1] We have to check that $\pi_*\mathscr{N}$ has constant rank. On $\mathfrak{M}_1^{\mathrm{div}} \setminus \mathfrak{D}^1$ we see that $\mathrm{R}^1\,\pi_*\mathscr{N} = 0$, so $\pi_*\mathscr{N}$ satisfies Cohomology and Base Change and its rank is determined by Riemann-Roch. Given a point x on the boundary $\mathfrak{D}^1$, we can pick a DVR scheme T whose closed point maps to x and whose generic point maps to $\mathfrak{M}_1^{\mathrm{div}} \setminus \mathfrak{D}^1$. Then we are in the hypotheses of Lemma 3.5 and we can check the rank at x by looking at $\pi_*f^*\mathscr{N}$ over T. Now $f^*\mathscr{N}$ is flat over T, so $\pi_*f^*\mathscr{N}$ is as well, which implies torsion-free (constant rank). □

Proof. 3.4 Let $S \to \mathfrak{M}_1^{\mathrm{div}}$ be a smooth atlas, then we have:

$$\begin{tikzcd}
(\mathscr{C}_S, D_S) \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & (\hat{\mathscr{C}}_S = \mathrm{Proj}_S(\bigoplus_{n\geq 0} \pi_{S,*}\mathscr{N}^{\otimes n}), \phi(D_S)) \arrow[dl, "\hat{\pi}"] \\
& S &
\end{tikzcd}$$

where ϕ is a proper and birational morphism since $\mathscr{N}$ is π-semi-ample and $\hat{\pi}$ is flat since $\pi_*\mathscr{N}$ is locally free. To verify that this defines a morphism $S \to \mathfrak{M}_1^{\mathrm{div}}(1)$ we have to argue that $\hat{\mathscr{C}}_S$ has reduced fibers and only nodes and cusps as singularities.

Since these properties only concern the fibers of $\hat{\pi}$ we can verify them after base change to a DVR scheme T chosen as in Lemma 3.5, so that the construction commutes with base change. Furthermore we can pick $f\colon T \to S$ such that the total space $\mathscr{C}_T$ is regular, so we may just apply Smyth's Contraction Lemma [Smy11a, Lemma 2.13].

Finally to conclude that this defines a morphism

$$\mathfrak{M}_1^{\mathrm{div}} \to \mathfrak{M}_1^{\mathrm{div}}(1)$$

it is enough to verify that there is an isomorphism $\mathrm{pr}_1^*\hat{\mathscr{C}}_\mathrm{S} \cong \mathrm{pr}_2^*\hat{\mathscr{C}}_\mathrm{S}$ satisfying the cocycle condition, where $\mathrm{pr_i}\colon \mathrm{S}' = \mathrm{S} \times_{\mathfrak{M}_1^{\mathrm{div}}} \mathrm{S} \rightrightarrows \mathrm{S}$.

This follows from the fact that $\mathrm{pr}_\mathrm{i}^*\hat{\mathscr{C}}_\mathrm{S}$ are obtained from applying the Proj construction to $\mathrm{pr}_\mathrm{i}^*\pi_{\mathrm{S},*}\mathscr{N} \cong \pi_{\mathrm{S}',*}\mathrm{pr}_\mathrm{i}^*\mathscr{N}$, by flatness of $S' \to S$. Thus it is enough to show that

$$\mathrm{pr}_1^*\mathscr{N} \cong \mathrm{pr}_2^*\mathscr{N}.$$

But $\mathscr{N}$ is the pullback of a line bundle on $\mathfrak{M}_1^{\mathrm{div}}$, thus the desired isomorphism follows from the commutativity of the diagram

$$\begin{tikzcd} S \times_{\mathfrak{M}_1^{\mathrm{div}}} S \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] & S \arrow[d] \\ S \arrow[r] & \mathfrak{M}_1^{\mathrm{div}} \end{tikzcd}$$

The cocycle condition is derived similarly. □

Proposition 3.8. *The* 1*-stabilisation just defined for curves with a divisor induces an analogue morphism on weighted curves:*

$$\begin{tikzcd} \mathfrak{M}_1^{\mathrm{div}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\curvearrowright"] & \mathfrak{M}_1^{\mathrm{div}}(1) \arrow[d] \\ \mathfrak{M}_1^{\mathrm{wt}} \arrow[r, "\exists"] & \mathfrak{M}_1^{\mathrm{wt}}(1) \end{tikzcd}$$

Proof. Étale locally on $\mathfrak{M}_1^{\mathrm{wt}}$ we can choose smooth sections s_i of the universal curve in a way that their sum as a relative Cartier divisor D has degree compatible with the weight function, so in particular it makes $\omega_\pi(E) \otimes \mathscr{O}_{\mathscr{C}}(3D)$ trivial on the elliptic tails and π-ample elsewhere. If $S \to \mathfrak{M}_1^{\mathrm{wt}}$ is a smooth atlas, up to taking an étale cover we can assume there are sections s_i of $\mathscr{C}_S \to S$ that define a lifting $S \to \mathfrak{M}_1^{\mathrm{div}}$, and thus a morphism $\xi \colon S \to \mathfrak{M}_1^{\mathrm{wt}}(1)$ through the above construction.

In order to show that this descends to a morphism $\mathfrak{M}_1^{\mathrm{wt}} \to \mathfrak{M}_1^{\mathrm{wt}}(1)$ we need to verify that $\mathrm{pr}_1^*(\xi) \cong \mathrm{pr}_2^*(\xi)$ and the cocycle condition is satisfied, where $\mathrm{pr_i} \colon \mathrm{S}' = \mathrm{S} \times_{\mathfrak{M}_1^{\mathrm{wt}}} \mathrm{S} \rightrightarrows \mathrm{S}$.

This boils down to checking that for two different choices of a lifting $D_1, D_2 \colon S \to \mathfrak{M}_1^{\mathrm{div}}$ there exists a unique isomorphism

$$\hat{\mathscr{C}}_1 = \mathrm{Proj}_S\left((\omega_\pi(E) \otimes \mathscr{O}_{\mathscr{C}}(3D_1))^{\otimes n}\right) \cong \mathrm{Proj}_S\left((\omega_\pi(E) \otimes \mathscr{O}_{\mathscr{C}}(3D_2))^{\otimes n}\right) = \hat{\mathscr{C}}_2.$$

By construction there is a birational map ψ

$$\begin{tikzcd} & \mathscr{C}_S \arrow[dl, "\phi_1"'] \arrow[dr, "\phi_2"] & \\ \hat{\mathscr{C}}_1 \arrow[rr, dashed, "\psi"] & & \hat{\mathscr{C}}_2. \end{tikzcd}$$

We want to show that ψ extends to a regular morphism. Notice that $\hat{\mathscr{C}}_i$ is normal, $i = 1, 2$. Indeed since S is smooth and the singularities of the fibers are in codimension 1, $\hat{\mathscr{C}}_i$ is regular in codimension 1. Moreover since both S (smooth) and the fibers (Cohen-Macaulay) satisfy Serre's condition S_2, so does the total space of $\hat{\mathscr{C}}_i$ by [Mat89, Thorem 23.9]. Then, since the geometric fibers of ϕ_i are either points or $p_a = 1$ subcurves, and in either case connected, Zariski's connectedness theorem implies that

$$\phi_{i,*}\mathscr{O}_{\mathscr{C}_S} \cong \mathscr{O}_{\hat{\mathscr{C}}_i}.$$

Moreover by construction $\mathrm{Exc}(\phi_1) = \mathrm{Exc}(\phi_2)$ is the locus of elliptic tails of weight 0, so in particular ϕ_2 contracts all fibers of ϕ_1. Then [Deb13, Lemma

1.15] implies that ϕ_2 factors through ϕ_1, and viceversa. This proves the regularity of ψ and its inverse. Notice that ψ is unique as it is the only extension of $\phi_2 \circ \phi_1^{-1}$. $\square$

3.3. **Fiber products and induced obstruction theories.** Let $\mathcal{Z}$ be defined by the pullback diagram:

$$\begin{tikzcd}
\mathcal{Z} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{\mathscr{M}}_1^{(1)}(\mathbb{P}^4, d) \arrow[d] \\
\mathfrak{M}_1^{\mathrm{wt}} \arrow[r] & \mathfrak{M}_1^{\mathrm{wt}}(1)
\end{tikzcd}$$

Objects of $\mathcal{Z}$ over S consist of diagrams

$$\begin{tikzcd}
C \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & \hat{C} \arrow[r, "f"] \arrow[dl, "\hat{\pi}"] & \mathbb{P}^4 \\
& S & &
\end{tikzcd}$$

where f is a 1-stable map and ϕ is the weighted 1-stabilisationdefined above (i.e. contraction of elliptic tails of weight 0); arrows over id_S are commutative diagrams

$$\begin{tikzcd}
C \arrow[r, "\phi"] \arrow[d, "\psi"] & \hat{C} \arrow[r, "f"] \arrow[d, "\hat{\psi}"] & \mathbb{P}^4 \arrow[d, "\mathrm{id}_{\mathbb{P}}"] \\
C' \arrow[r, "\phi'"] & \hat{C}' \arrow[r, "f'"] & \mathbb{P}^4
\end{tikzcd}$$

where ψ and $\hat{\psi}$ are isomorphisms. Recall that $\hat{\psi}$ is determined by ψ.

Forgetting $\hat{C}$ and keeping $f \circ \phi \colon C \to \mathbb{P}^4$, we obtain a morphism $i \colon \mathcal{Z} \to \overline{\mathcal{M}}_1(\mathbb{P}^4, d)$.

Lemma 3.9. *The morphism $i \colon \mathcal{Z} \hookrightarrow \overline{\mathcal{M}}_1(\mathbb{P}^4, d)$ is a closed immersion. In particular $\mathcal{Z}$ is a proper DM stack.*

Proof. From the above description of arrows in $\mathcal{Z}$, i is representable (i.e. faithful) and a monomorphism (i.e. full).

We can check properness using the valuative criterion. We argue as in [RSW17, Theorem 4.3]. Let T be a DVR scheme with generic point η; consider a diagram:

$$\begin{tikzcd}
\mathscr{C}_\eta \arrow[r] \arrow[d, "\phi_\eta"'] & \mathscr{C}_T \arrow[r, "f"] \arrow[d, "\phi_T"] & \mathbb{P}^4 \\
\hat{\mathscr{C}}_\eta \arrow[r, "j"'] \arrow[urr] & \hat{\mathscr{C}}_T \arrow[ur, dashed, "g"'] &
\end{tikzcd}$$

Notice that there is an open dense substack of $\mathcal{Z}$ where ϕ is an isomorphism. Indeed the generic point of either the main component or any boundary component is already 1-stable. Thus we can assume that ϕ_η in the above diagram is an isomorphism.

Observe that f is constant on the fibers of ϕ_T, so it factors topologically through $\hat{\mathscr{C}}$. We can conclude as in [RSW17] or appeal to [Deb13, Lemma 1.15] using $\phi_*\mathscr{O}_{\mathscr{C}_T} \cong \mathscr{O}_{\hat{\mathscr{C}}_T}$. To see this consider the exact sequence:

$$0 \to \mathscr{O}_{\hat{\mathscr{C}}_T} \to \phi_*\mathscr{O}_{\mathscr{C}_T} \to \phi_*\mathscr{O}_{\mathscr{C}_T}/\mathscr{O}_{\hat{\mathscr{C}}_T} \to 0$$

Since ϕ is an isomorphism away from the cuspidal point, the cokernel is supported in dimension 0. However $\chi(\mathscr{O}_{\hat{\mathscr{C}}_\eta}) = \chi(\phi_*\mathscr{O}_{\mathscr{C}_\eta})$ implies the same equality holds on the whole of T, since the Euler characteristic is constant in flat families. So $\chi(\phi_*\mathscr{O}_{\mathscr{C}_T}/\mathscr{O}_{\hat{\mathscr{C}}_T}) = \text{length}(\phi_*\mathscr{O}_{\mathscr{C}_T}/\mathscr{O}_{\hat{\mathscr{C}}_T}) = 0.$ $\square$

So we may add the commutative diagram to the left:

$$\begin{tikzcd}
\overline{\mathcal{M}}_1(\mathbb{P}^4,d) \arrow[d] & \mathcal{Z} \arrow[l, hook', "i"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{\mathscr{M}}_1^{(1)}(\mathbb{P}^4,d) \arrow[d] \\
\mathfrak{M}_1^{\mathrm{wt}} & \mathcal{X} \arrow[l, "\sim"'] \arrow[r] & \mathfrak{M}_1^{\mathrm{wt}}(1)
\end{tikzcd}$$

From the inclusion i we easily obtain a description of the irreducible components of $\mathcal{Z}$: there is a main component $\mathcal{Z}^{\text{main}}$ which is the closure of the locus of maps from a smooth elliptic curve, and for every $k \geq 2$ a boundary component $D^k\mathcal{Z}$, whose general point represents a contracted elliptic curve with k-many rational tails of positive degree.

Notice that the above lemma means that each and every component of $\mathcal{Z}$ is isomorphic to the corresponding one in $\overline{\mathcal{M}}_1(\mathbb{P}^4,d)$. Indeed given any stable map there is at most one factorisation through the weighted 1-stabilisation of the curve: objects of $\overline{\mathcal{M}}_1(\mathbb{P}^4,d) \setminus D^1$ are 1-stable already; objects of $D^1 \cap \overline{\mathcal{M}}_1(\mathbb{P}^4,d)^{\text{main}}$ do factor through the cusp thanks to Vakil's criterion, which implies the node joining the elliptic tail to the rational tail of positive degree is a ramification point of the map; and objects of $D^1 \cap D^k$ ($k \geq 2$) do factor through a map which is constant on the cusp. On the other hand, objects of $D^{1,\circ} = D^1 \setminus (\overline{\mathcal{M}}_1(\mathbb{P}^4,d)^{\text{main}} \cup \bigcup_{k\geq 2} D^k)$ do not admit any factorisation, so $\mathcal{Z}$ has no corresponding component.

insert figure of a typical element of $D^1 \cap D^2$.

We introduce two more relevant spaces: let $\mathcal{XP}$ and $\mathcal{Z}^p$ be the algebraic stacks defined by the following cartesian diagram:

$$\begin{tikzcd}
\mathcal{Z}^p \arrow[r, "\alpha"] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{\mathscr{M}}_1^{(1)}(\mathbb{P}^4,d)^p \arrow[d] \\
\mathcal{XP} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{Pic}_1(1) \arrow[d] \\
\mathcal{X} \arrow[r] & \mathfrak{M}_1^{\mathrm{wt}}(1)
\end{tikzcd}$$

The obstruction theory $\mathrm{R}^\bullet\,\hat{\pi}_*(\hat{\mathscr{L}}^{\oplus 5} \oplus \hat{\mathscr{P}})$ for the morphism $\overline{\mathscr{M}}_1^{(1)}(\mathbb{P}^4,d)^p \to \mathfrak{Pic}_1(1)$ together with its cosection induce a localised virtual class on $\mathcal{Z}^p$ by virtual pullback of $[\mathcal{XP}]$. In Section 5 we shall conduct on this a similar analysis to that in [CL15] and compare the contribution of the main component to Vakil-Zinger's reduced invariants.

Remark 3.10. The stack $\mathcal{XP}$ parametrises

$$\begin{tikzcd} C \arrow[rr, "\phi"] \arrow[dr] & & \hat{C} \arrow[dl] \\ & S & \end{tikzcd}$$

with a line bundle $\hat{\mathscr{L}}$ on $\hat{C}$. Notice that by pulling back $\hat{\mathscr{L}}$ via ϕ we obtain a line bundle on C, hence a morphism $\mathcal{XP} \to \mathfrak{Pic}_1$. This is generically an isomorphism, but has 1-dimensional fibers over the locus of elliptic tails, due to the fact that $\operatorname{Pic}(\hat{C}) \to \operatorname{Pic}(C)$ has kernel $\mathbb{G}_{\mathrm{a}}$ when $\hat{C}$ has a cusp.

On the other hand notice that

$$\mathcal{X} \times_{\mathfrak{M}_1^{\mathrm{wt}}(1)} \mathfrak{M}_1^{\mathrm{div}}(1) \to \mathfrak{M}_1^{\mathrm{div}}$$

I am not sure, CHECK ME.

is an isomorphism. We just check on points.

Remark 3.11. Similarly the stack $\mathcal{Z}^p$ parametrises

$$\begin{tikzcd} C \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & \hat{C} \arrow[dl, "\hat{\pi}"] \arrow[r, "f"] & \mathbb{P}^4 \\ & S & & \end{tikzcd}$$

with a p-field $\psi \in H^0(\hat{C}, f^*\mathscr{O}_{\mathbb{P}^4}(-5) \otimes \omega_{\hat{\pi}})$.

We were not able to compare $\mathcal{Z}^p$ with $\overline{\mathcal{M}}_1(\mathbb{P}^4, d)^p$ directly, since denoting by $\hat{\mathscr{L}} = f^*\mathscr{O}_{\mathbb{P}^4}(1)$ and by $\mathscr{L} = \phi^*\hat{\mathscr{L}}$ we only have a map $\mathrm{R}^1\,\hat{\pi}_*\hat{\mathscr{L}} \to \mathrm{R}^1\,\pi_*\mathscr{L}$ on $\mathcal{Z}^p$ which is not an isomorphism (dually, since $\phi^*\omega_{\hat{\pi}} = \omega_\pi(E)$, the p-fields on C that we get by pulling back from $\hat{C}$ vanish on elliptic tails).

As anticipated we may endow $\mathcal{Z}^p$ with a localised virtual cycle by "localised virtual pullback". Indeed, consider on $\mathcal{Z}^p$ the perfect obstruction theory $\mathbb{E}_{\mathcal{Z}^p/\mathcal{XP}} := \alpha^*\left(\mathbb{E}_{\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4,d)^p/\mathfrak{Pic}_1(1)}\right)$ and the induces cosection

$$\alpha^*\sigma\colon \mathscr{O}b_{\mathcal{Z}^p/\mathcal{XP}} \to \mathcal{Z}^p$$

whose degeneracy locus is

$$\mathcal{D}(\alpha^*\sigma) = \mathcal{D}(\sigma) \times_{\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4,d)^p} \mathcal{Z}^p.$$

Then we have that

$$\begin{aligned} \mathfrak{C}_{\mathcal{Z}^p/\mathcal{XP}} &\subseteq \alpha^*\mathfrak{C}_{\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4,d)^p/\mathfrak{Pic}_1(1)} \\ &\subseteq \alpha^* h^1/h^0\left(\mathbb{E}_{\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4,d)^p/\mathfrak{Pic}_1(1)}\right)_\sigma \\ &\cong h^1/h^0\left(\mathbb{E}_{\mathcal{Z}^p/\mathcal{XP}}\right)_{\alpha^*\sigma} \end{aligned}$$

and we can thus define $[\mathcal{Z}^p]^{\mathrm{vir}}_{\alpha^*\sigma,\mathrm{loc}}$.

Lemma 3.12. *localised virtual pullback commutes with pushforward. In particular,*

$$\alpha_*[\mathcal{Z}^p]^{\mathrm{vir}}_{\alpha^*\sigma,\mathrm{loc}} = [\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4,d)^p]^{\mathrm{vir}}_{\sigma,\mathrm{loc}}$$

check [KL13], probably needs the proper morphism to be surjective

Corollary 3.13. $\deg[\mathcal{Z}^p]^{\mathrm{vir}}_{\mathrm{loc}} = \deg[\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4,d)^p]^{\mathrm{vir}}_{\mathrm{loc}}$

4. Local equations and desingularisation

4.1. **Equations for $\mathcal{Z}^p$ relative to $\mathcal{XP}$.** We are going to need a description of the normal cone $\mathfrak{C}_{\mathcal{Z}^p/\mathcal{XP}}$ in order to perform a splitting.

Since $\mathcal{Z}^p$ simply is a line bundle over the boundary of $\mathcal{Z}$, we may instead find equations for the latter.

Recall that $\mathcal{Z}$ can be embedded as an open inside $C(\hat{\pi}_*\mathscr{L}^{\oplus 5})$ over $\mathcal{XP}$. We are going to find an embedding of $C(\hat{\pi}_*\mathscr{L}^{\oplus 5})$ in a smooth ambient space over $\mathcal{XP}$, that will be a vector bundle obtained by suitably twisting $\mathscr{L}$.

Following [HL10], we work locally on $\mathcal{Z}$: start with a point $\xi = [(C \xrightarrow{\phi} \hat{C} \xrightarrow{f} \mathbb{P}^4] \in \mathcal{Z}$ and choose coordinates on $\mathbb{P}^4$ such that $f^{-1}\{x_0 = 0\}$ is a simple smooth divisor $D = \sum_{i=1}^d \delta_i$ on $\hat{C}$. This continues to be true on a neighbourhood U.

Locally the morphism $\mathcal{Z} \to \mathcal{XP}$ can be written as $\xi \mapsto [C \to \hat{C}, \mathscr{O}_{\hat{C}}(D)]$, which admits a local lifting $U \to \mathfrak{M}_1^{(1),\mathrm{div}}$ and in fact hits the smooth locus of the latter.

Notice though that the projection $\mathfrak{M}_1^{(1),\mathrm{div}} \to \mathcal{XP}$ (or to $\mathrm{Pic}_1^{(1)}$ for what it is worth) is *not* smooth. In fact, when the line bundle is trivial on the minimal elliptic subcurve E, it may be deformed to a degree 0, non-effective line bundle on such a subcurve, so that sections of $\mathscr{O}_{\hat{C}}(D)$ which are constant and non-zero on E are obstructed.

There is a way around this: in a neighbourhood $V \subseteq \mathcal{XP}$ of $[C \to \hat{C}, \mathscr{O}_{\hat{C}}(D)]$ we can write the universal line bundle $\mathscr{L}_V$ as $\mathscr{O}_{\hat{\mathscr{C}}_V}(\mathcal{D} + p - p_0)$. Indeed we can pick a local section p_0 through the minimal genus 1 subcurve, so that $\mathscr{L}_V(p_0)$ becomes effective. We should then think of p as a local coordinate on $\mathcal{XP}$ relative to $\mathcal{X}$.

Locally on V we can pick another smooth section $\mathcal{A}$ of the minimal genus 1 subcurve not intersecting p_0, neither the support of $\mathcal{D} + p$.

Lemma 4.1. *$C(\hat{\pi}_*\mathscr{L})$ is the kernel of the vector bundle map:*

$$\hat{\pi}_*\mathscr{O}_{\hat{\mathscr{C}}}(\mathcal{A} + \mathcal{D} + p - p_0) \xrightarrow{\varphi} \hat{\pi}_*\mathscr{O}_{\mathcal{A}}(\mathcal{A})$$

Up to shrinking V we may write:

$$\hat{\pi}_*\mathscr{O}_{\hat{\mathscr{C}}}(\mathcal{A} + \mathcal{D} + p - p_0) \cong \bigoplus_{i=1}^d \hat{\pi}_*\mathscr{O}_{\hat{\mathscr{C}}}(\mathcal{A} + \delta_i + p - p_0) \oplus \hat{\pi}_*\mathscr{O}_{\hat{\mathscr{C}}}(\mathcal{A} + p - p_0)$$

Compare with [HL10, Lemma 4.10]. Denote by

$$\varphi_i \colon \hat{\pi}_*\mathscr{O}_{\hat{\mathscr{C}}}(\mathcal{A} + \delta_i + p - p_0) \to \hat{\pi}_*\mathscr{O}_{\mathcal{A}}(\mathcal{A})$$

(and similarly φ_p) the composite of the inclusion with φ.

Let us introduce some more notation: around a point $[\hat{C}] \in \mathfrak{M}_1^{(1),\mathrm{wt}}$, for every node q of $\hat{C}$ there is a coordinate ζ_q whose vanishing locus is the divisor where such a node is *not* smoothed. These functions can be pulled back to V. Denote by

$$\zeta_{[\delta_i,a]} = \prod \zeta_q$$

where the product runs over all the nodes separating δ_i from the minimal genus 1 curve.

Lemma 4.2. *We may find an explicit local expression for φ_i and φ_p after trivialising the relevant line bundles:*

$$\varphi_i = \zeta_{[\delta_i,a]}, \quad \varphi_p = (p - p_0)$$

Compare with [HL10, Proposition 4.13].

Remark 4.3. The vanishing locus of $(p-p_0)$ on the boundary means that the line bundle restricts to the trivial one on the minimal genus 1 subcurve. The meaning of it is not clear outside the boundary.

Lemma 4.4. *A local chart U for $\mathcal{Z}$ can be embedded as an open inside:*

$$(F_0 = \ldots = F_4 = 0) \subseteq \mathrm{Vb}(\hat{\pi}_* \mathscr{L}^{\oplus 5})$$

where

$$F_j = \sum_{i=1}^{d} \zeta_{[\delta_i,a]} w_i^j + (p-p_0) w_{d+1}^j$$

and w_i^j are coordinates on the fiber of the j-th copy of $\mathrm{Vb}(\hat{\pi}_*\mathscr{L})$ over $\mathcal{XP}$.

Compare with [HL10, Theorems 2.17-19].

4.2. **Hu-Li blow-up and desingularisation.** We perform a modular blow-up of $\mathfrak{M}_1^{1,\mathrm{wt}}$: we successively blow up $\hat{\Theta}_k,\ k \geq 2$ defined as the closure of the loci where the minimal elliptic subcurve E has weight 0 and $|\overline{C \setminus E} \cap E| = k$.

Notice that after the k-th blow-up, the strict transform of Θ_{k+1} is smooth, so the final result $\widetilde{\mathfrak{M}}_1^{1,\mathrm{wt}}$ is smooth as well.

Remark 4.5. The fiber product

$$\widetilde{\mathfrak{M}}_1^{1,\mathrm{wt}} \times_{\mathfrak{M}_1^{1,\mathrm{wt}}} \mathfrak{M}_1^{\mathrm{wt}}$$

recovers the Hu-Li blow-up $\widetilde{\mathfrak{M}}_1^{\mathrm{wt}}$. The key observation is that Θ_1 is already a Cartier divisor and the pullback of $\hat{\Theta}_k$ is precisely Θ_k.

you can use the universal property of the blow up to show there are maps in both directions. Check the statement about the Θ_k

Remark 4.6. After blowing up, the equations in 4.4 are simplified and assume the following form:

$$\tilde{\zeta}\tilde{w} + (p - p_0) w_{d+1} = 0$$

where $\tilde{\zeta}$ is one of the newly created boundary divisors $\widetilde{\Theta_k}$ from $\widetilde{\mathfrak{M}}_1^{\mathrm{wt}}$ (i.e. one of the exceptional divisors produced by the blow-up process), and $\tilde{w}$ is a suitably defined coordinate on the fiber of $\mathrm{Vb}(\hat{\pi}_*\mathscr{L}) \times_{\mathcal{XP}} \widetilde{\mathcal{XP}}$.

Summing up, we get:

$$\begin{tikzcd}
 & \widetilde{\mathcal{Z}}^p \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \widetilde{\mathcal{M}}_1^{(1)}(\mathbb{P}^4,d)^p \arrow[d] \\
\widetilde{\mathcal{M}}_1(\mathbb{P}^4,d) \arrow[d] & \widetilde{\mathcal{Z}} \arrow[l, hook', "i"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \widetilde{\mathcal{M}}_1^{(1)}(\mathbb{P}^4,d) \arrow[d] \\
\widetilde{\mathfrak{Pic}}_1 \arrow[d] & \widetilde{\mathcal{XP}} \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \widetilde{\mathfrak{Pic}}_1^{(1)} \arrow[d] \\
\widetilde{\mathfrak{M}}_1^{\mathrm{wt}} & \widetilde{\mathcal{X}} \arrow[l, "\sim"'] \arrow[r] & \widetilde{\mathfrak{M}}_1^{(1),\mathrm{wt}}
\end{tikzcd}$$

We conclude this brief section remarking that the blow up procedure does not affect the invariants:

Lemma 4.7. *We have the identity:*

$$\deg[\widetilde{\mathcal{Z}}^p]^{\mathrm{vir}}_{\mathrm{loc}} = \deg[\mathcal{Z}^p]^{\mathrm{vir}}_{\mathrm{loc}}$$

Compare with [CL15, Proposition 2.5].

5. Splitting the cone and proof of the Main Theorem

We are finally in a position to study the cone $\mathfrak{C}_{\widetilde{\mathcal{Z}}^p/\widetilde{\mathcal{XP}}}$. This is essentially going to be a word-by-word repetition of the arguments in [CL12].

Lemma 5.1. *Let* $\tilde\rho\colon \widetilde{\mathcal{Z}}^p \to \widetilde{\mathcal{XP}}$ *be the natural map and* $\mathbb{E}_{\widetilde{\mathcal{Z}}^p/\widetilde{\mathcal{XP}}} = \mathrm{R}\,\hat\pi_*(\mathscr{L}^{\oplus 5}\oplus \mathscr{P})$ *the relative perfect obstruction theory for* $\tilde\rho$*, then* $\mathfrak{C}_{\widetilde{\mathcal{Z}}^p/\widetilde{\mathcal{XP}}}$ *has the following properties:*

1. *Its restriction to* $\widetilde{\mathcal{Z}}^{p,\circ} = \widetilde{\mathcal{Z}}^{p,\mathrm{main}} \setminus \bigcup_{k\geq 2} D^k\widetilde{\mathcal{Z}}^p$ *can be described as the zero section of* $h^1/h^0(\mathbb{E}_{\widetilde{\mathcal{Z}}^p/\widetilde{\mathcal{XP}}})|_{\widetilde{\mathcal{Z}}^{p,\circ}}$
2. *Its restriction to* $\widetilde{\mathcal{Z}}^{p,gst,\circ} = \widetilde{\mathcal{Z}}^p - \widetilde{\mathcal{Z}}^{p,\mathrm{main}}$ *is a rank 2 subbundle stack of* $h^1/h^0(\mathbb{E}_{\widetilde{\mathcal{Z}}^p/\widetilde{\mathcal{XP}}})|_{\widetilde{\mathcal{Z}}^{p,gst,\circ}}$

Proof. Compare with [CL12, Lemma 4.3]

1. Observe that $\widetilde{\mathcal{Z}}^{p,\circ} \cong \widetilde{\mathcal{Z}}^\circ$ because here $H^0(\hat C, L^{\otimes -5}\otimes\omega_{\hat C}) = 0$. Moreover $\widetilde{\mathcal{Z}}^\circ$ is unobstructed on its image, which is an open of $\widetilde{\mathcal{XP}}$, because $\mathrm{R}^1\,\hat\pi_*\mathscr{L} = 0$. So the normal cone is $[\widetilde{\mathcal{Z}}^{p,\circ}/\hat\pi_*\mathscr{L}^{\oplus 5}]$, which is the zero section of $h^1/h^0(\mathbb{E}_{\widetilde{\mathcal{Z}}^p/\widetilde{\mathcal{XP}}})|_{\widetilde{\mathcal{Z}}^{p,\circ}} = [0\oplus \mathrm{R}^1\,\hat\pi_*\mathscr{P}/\hat\pi_*\mathscr{L}^{\oplus 5}\oplus 0]$.
2. We know that $\widetilde{\mathcal{Z}}^{p,gst,\circ}$ is a line bundle over $\widetilde{\mathcal{Z}}^{gst,\circ}$. From the equations 4.6 we see that the latter is smooth over its image $\mathcal{W}$ in $\widetilde{\mathcal{XP}}$, which is the codimension 2 locus where the minimal genus 1 subcurve has weight 0 and the line bundle is trivial on it. Recall that every smooth morphism $A\to B$ factors as $A \xrightarrow{\acute{e}t} B\times\mathbb{A}^n \xrightarrow{\mathrm{pr}_1} B$. So we have

$$\begin{tikzcd}
\widetilde{\mathcal{Z}}^{p,gst,\circ} \arrow[r, "\acute{e}t"] & \mathcal{W} \times \mathbb{A}^{5d+5} \arrow[r, hook] \arrow[d, "q"] & \widetilde{\mathcal{XP}} \times \mathbb{A}^{5d+5} \arrow[d] \\
& \mathcal{W} \arrow[r, hook] & \widetilde{\mathcal{XP}}
\end{tikzcd}$$

where the bottom horizontal arrow is a codimension 2 regular embedding. Thus

$$\mathfrak{C}_{\widetilde{\mathcal{Z}}^p/\widetilde{\mathcal{XP}}}|_{\widetilde{\mathcal{Z}}^{gst,\circ}} \cong \left[q^* C_{\mathcal{W}/\widetilde{\mathcal{XP}}}/\hat{\pi}_*\mathscr{L}^{\oplus 5}\right]$$

is a rank 2 subbundle stack of $h^1/h^0(\mathbb{E}_{\widetilde{\mathcal{Z}}^p/\widetilde{\mathcal{XP}}})|_{\widetilde{\mathcal{Z}}^{p,gst,\circ}}$.

□

Notice that the image of $\widetilde{\mathcal{Z}}^\circ$ in $\widetilde{\mathcal{M}}_1(\mathbb{P}^4)$ contains $\widetilde{\mathcal{M}}_1(\mathbb{P}^4)^{main} \cap \tilde{D}^1$.

Recall the definition of the *closure of the zero section of a vector bundle stack*: let B be an integral algebraic stack and let $[F_0 \xrightarrow{d} F_1]$ be a complex of locally free sheaves on B. The zero section is $0_{\mathbf{F}} \colon [F_0/F_0] \to \mathbf{F} = [F_1/F_0]$ (notice that it is not in general a closed embedding); the closure of the zero section is then defined as:

$$\overline{0}_{\mathbf{F}} = [\mathrm{cl}(dF_0)/F_0]$$

$\overline{0}_{\mathbf{F}}$ is an integral stack.

Example 5.2. When $h^0(F_\bullet) = 0$, the closure of the zero section looks like B with some stacky structure on the vanishing locus of d. Consider for example $B = \mathbb{P}^1$ and $F_\bullet = [\mathscr{O}_{\mathbb{P}^1} \xrightarrow{x} \mathscr{O}_{\mathbb{P}^1}(1)]$. Then the action of $e \in F_0$ on F_1 is by $f \mapsto f + xe$. Clearly $\mathrm{cl}(dF_0)$ is the whole line bundle F_1; the F_0-action is transitive on the fibers over $\{x \neq 0\}$ and trivial on the $\{x = 0\}$-fiber. Hence $\overline{0}_{\mathbf{F}}$ is isomorphic to $\mathbb{P}^1 \setminus \{x = 0\}$ with a gerbe $[\mathbb{A}^1/\mathbb{G}_{\mathrm{a}}]$ replacing the point $\{x = 0\}$.

We may now split the cone $\mathfrak{C}_{\widetilde{\mathcal{Z}}^p/\widetilde{\mathcal{XP}}}$ in the following manner: we denote by $\mathfrak{C}^{\mathrm{main}}$ the closure of the zero section of $h^1/h^0(\mathbb{E}_{\widetilde{\mathcal{Z}}^p/\widetilde{\mathcal{XP}}})|_{\widetilde{\mathcal{Z}}^{p,\mathrm{main}}}$, which is an irreducible cone supported on the main component; all the rest is supported on the boundary components, possibly on their intersection with the main one, and we are going to pack all the components supported on $D^k\widetilde{\mathcal{Z}}^p$ together and label them $\mathfrak{C}^k$ accordingly, so in the end we obtain a splitting:

$$\mathfrak{C}_{\widetilde{\mathcal{Z}}^p/\widetilde{\mathcal{XP}}} = \mathfrak{C}^{\mathrm{main}} + \sum_{k \geq 2} \mathfrak{C}^k$$

We are going to show that:

(1) the contribution of $\mathfrak{C}^{\mathrm{main}}$ is exactly the reduced invariants of X;
(2) the other cones $\mathfrak{C}^k,\ k \geq 2$, are enumeratively meaningless.

In order to prove the first we proceed as in [CL12, §5]: notice that the obstruction theory $\mathbb{E}$ splits as $\mathbb{E}_1 \oplus \mathbb{E}_2$ where $\mathbb{E}_1 = \mathrm{R}\,\hat{\pi}_*(\mathscr{L}^{\oplus 5})$ and $\mathbb{E}_2 =$

$\mathrm{R}\,\hat{\pi}_*(\mathscr{P})$. When we restrict ourselves to $\widetilde{\mathcal{Z}}^{p,\mathrm{main}}$ we see that $\mathbb{E}_1$ is the closure of its own zero section; it follows that:

$$\mathfrak{C}^{\mathrm{main}} = \overline{0}_{\mathbb{E}|_{\widetilde{\mathcal{Z}}^{p,\mathrm{main}}}} = \mathbb{E}_1|_{\widetilde{\mathcal{Z}}^{p,\mathrm{main}}} \oplus \overline{0}_{\mathbb{E}_2|_{\widetilde{\mathcal{Z}}^{p,\mathrm{main}}}}$$

Then by excess intersection:

$$0^!_{\mathbb{E}}[\mathfrak{C}^{\mathrm{main}}] = 0^!_{\mathbb{E}_2}[\overline{0}_{\mathbb{E}_2|_{\widetilde{\mathcal{Z}}^{p,\mathrm{main}}}}]$$

At this point we recall the following [CL12, Lemma 5.3]:

Lemma 5.3. *Let $\mathbb{E} = [E_0 \to E_1]$ be a complex of locally free sheaves on an integral Deligne-Mumford stack B such that $H^1(\mathbb{E})$ is a torsion sheaf on B and the image sheaf of $E_0 \to E_1$ is locally free. Let $U \subseteq B$ be the complement of the support of $H^1(\mathbb{E})$, and let $\mathbf{B} \subseteq h^1/h^0(\mathbb{E}^\vee[-1])$ be the closure of the zero section of the vector bundle $h^1/h^0(\mathbb{E}^\vee[-1]|_U) = H^0(\mathbb{E}|_U)^\vee$. Then*

$$0^![\mathbf{B}] = e(H^0(\mathbb{E})^\vee) \in A_*(B).$$

We apply this lemma to $\mathbb{E} = \mathrm{R}\,\hat{\pi}_*\mathscr{L}^{\otimes 5}$. Notice that it satisfies the hypotheses by virtue of the equations in Remark 4.6: indeed the question may be addressed locally; looking at the resolution of $\mathbb{E}$:

$$\hat{\pi}_*\mathscr{L}^{\otimes 5}(\mathcal{A}) \to \hat{\pi}_*\mathscr{L}^{\otimes 5}(\mathcal{A})|_{\mathcal{A}}$$

we deduce from the equation that the image sheaf of this map is $\pi_*\mathscr{L}^{\otimes 5}(\mathcal{A})|_{\mathcal{A}} \otimes \mathscr{O}_{\widetilde{\mathcal{Z}}^{p,\mathrm{main}}}(-\Delta)$, where Δ denotes the boundary of the main component $\Delta = \widetilde{\mathcal{Z}}^{p,\mathrm{main}} \cap \left(\bigcup_{k\geq 2} D^k\widetilde{\mathcal{Z}}^p\right)$. Then $\hat{\pi}_*\mathscr{L}^{\otimes 5}$ is a vector bundle.

Lemma 5.4. *If we let i be the inclusion of $\widetilde{\mathcal{Z}}$ in $\widetilde{\mathcal{M}}_1(\mathbb{P}^4, d)$, then:*

$$i_*(c_{\mathrm{top}}(\hat{\pi}_*\hat{\mathscr{L}}^{\otimes 5}) \cap [\widetilde{\mathcal{Z}}^{p,\mathrm{main}}]) = c_{\mathrm{top}}(\pi_*\mathscr{L}^{\otimes 5}) \cap [\widetilde{\mathcal{M}}_1(\mathbb{P}^4, d)^{p,\mathrm{main}}]$$

Proof. Obviously $i_*[\widetilde{\mathcal{Z}}^{p,\mathrm{main}}] = [\widetilde{\mathcal{M}}_1(\mathbb{P}^4, d)^{p,\mathrm{main}}]$. On the other hand notice that on $\widetilde{\mathcal{Z}}^{p,\mathrm{main}}$ we have:

$$\pi_*\mathscr{L} = \hat{\pi}_*\phi_*\phi^*\hat{\mathscr{L}} = \hat{\pi}_*\hat{\mathscr{L}}$$

by projection formula and since $\phi_*\mathscr{O}_{\mathscr{C}_{\widetilde{\mathcal{Z}}^{p,\mathrm{main}}}} = \mathscr{O}_{\hat{\mathscr{C}}_{\widetilde{\mathcal{Z}}^{p,\mathrm{main}}}}$. The result follows then from the projection formula for Chern classes. □

We are left with showing that the rest of the $\mathfrak{C}^k$'s do not contribute to the invariants. This is essentially a dimensional computation. The arguments of [CL12, SS6-8] may be directly applied; we shall outline them for the benefit of the reader. We introduce the notation $\widetilde{\mathcal{Z}}^{p,\mathrm{gst}} := \bigcup_{k\geq 2} D^k\widetilde{\mathcal{Z}}^p$ for the union of the boundary components, and $\mathfrak{C}^{\mathrm{gst}} = \bigcup_{k\geq 2}\mathfrak{C}^k$.

Step I: reduction to the case of a cone inside a vector bundle and homogeneity. This section deals with removing the technicalities of working with a cone stack inside a vector bundle stack.

The key point is that $\mathbb{E}|_{\widetilde{\mathcal{Z}}^{p,\mathrm{gst}}}$ has locally free h^0 *and* h^1, as can be seen from the equations. Then, when we pick a resolution by locally free sheaves $\mathbb{E} = [F_0 \xrightarrow{d} F_1]$, the image sheaf $d(F_0)$, which is the kernel of $F_1 \to h^1(\mathbb{E}|_{\widetilde{\mathcal{Z}}^{p,\mathrm{gst}}})$, is really a sub-vector bundle of F_1. Then the projections:

$$\beta\colon F_1 \to h^1/h^0(\mathbb{E}) \quad \text{and} \quad \beta'\colon F_1 \to \tilde{V} := \mathrm{R}^1\,\hat{\pi}_*(\hat{\mathscr{L}}^{\oplus 5} \oplus \mathscr{P})$$

are both flat. The cone stack $\mathfrak{C}^{\mathrm{gst}}$ can be descended to a cone C^{gst} by means of taking the quotient of $\beta^{-1}\mathfrak{C}^{\mathrm{gst}}$ by the free action of $d(F_0)$; C^{gst} should then be thought of as the coarse moduli of $\mathfrak{C}^{\mathrm{gst}}$. Recall that the cosection σ is induced by a cosection σ_1 of $\tilde{V}$ (see Eq. (2)). It follows from the commutativity of localised Gysin pullback with flat pullback that:

rewrite what they are

$$0^!_{h^1/h^0(\mathbb{E}),\sigma}(\mathfrak{C}^{\mathrm{gst}}) = 0^!_{\tilde{V},\sigma_1}(C^{\mathrm{gst}})$$

See [CL12, Proposition 6.3].

They then introduce the notion of *homogeneity* for substacks of $\tilde{V}$: write

$$\tilde{V} = \tilde{V}_1 \oplus \tilde{V}_2 \quad \text{with} \quad \tilde{V}_1 = \mathrm{R}^1\,\hat{\pi}_*(\hat{\mathscr{L}}^{\oplus 5}) \text{ and } \tilde{V}_2 = \mathrm{R}^1\,\hat{\pi}_*(\mathscr{P}),$$

and $\gamma\colon \widetilde{\mathcal{Z}}^{p,\mathrm{gst}} \to \widetilde{\mathcal{Z}}^{\mathrm{gst}}$ the projection. Since $\widetilde{\mathcal{Z}}^{p,\mathrm{gst}}$ is the total space of a line bundle over $\widetilde{\mathcal{Z}}^{\mathrm{gst}}$, it comes with a natural $\mathbb{G}_{\mathrm{m}}$-action on the fibers of γ. Moreover, since

$$\tilde{V}_i = \gamma^* V_i$$

say few words, that is not the definition of $\tilde{V}_i$ but a property (Lemma 6.1)

for the corresponding vector bundles V_i on $\widetilde{\mathcal{Z}}^{\mathrm{gst}}$, the total space of $\tilde{V}_i$ can be endowed with a $\mathbb{G}_{\mathrm{m}}$-action that makes the projection to $\widetilde{\mathcal{Z}}^{p,\mathrm{gst}}$ equivariant. A closed substack of $\tilde{V}$ is 0-homogeneous if it is the pullback of a closed substack of V along γ. In facts the $\tilde{V}_i$'s can be endowed with different $\mathbb{G}_{\mathrm{m}}$ actions that make the projection to $\widetilde{\mathcal{Z}}^{p,\mathrm{gst}}$ equivariant by twisting with two characters of $\mathbb{G}_{\mathrm{m}}$, one for each $\tilde{V}_i$; this defines the concept of (l_1, l_2)-homogeneous substacks of $\tilde{V}$. Here is how we are going to use the homogeneity:

Lemma 5.5. *Let $\tilde{C} \subseteq \tilde{V}$ be an (l_1, l_2)-homogeneous sub*cone *of $\tilde{V}$; then the cone $\tilde{C} \cap (0 \oplus \tilde{V}_2)$ is pulled back from a cone in V_2.*

Proof. Locally we may pick coordinates t on the fibers of γ, $x_1, \ldots, x_5$ on the fibers of $\tilde{V}_1$, and $y_1, \ldots, y_{5d+5}$ on the fibers of $\tilde{V}_2$, such that the ideal of $\tilde{C}$ is generated by homogeneous polynomials p_j in $t^{-l_1}x_i$ and $t^{-l_2}y_i$. The ideal of $\tilde{C} \cap (0 \oplus \tilde{V}_2)$ is then given by $\langle x_1, \ldots, x_5, p_j(0, t^{-l_2}y)\rangle_j$, where $p_j(0, t^{-l_2}y)$ results from setting $x_i = 0$ in p_j; notice now that $\tilde{C}$ being a cone, it is invariant by scalar multiplication on the fibers of $\tilde{V}$, so we may as well say that $\tilde{C} \cap (0 \oplus \tilde{V}_2)$ is cut by the ideal $\langle x_1, \ldots, x_5, p_j(0, y)\rangle_j$. This makes it clear that $\tilde{C} \cap (0 \oplus \tilde{V}_2)$ is pulled back from $(0 \oplus V_2)$ on $\widetilde{\mathcal{Z}}^{\mathrm{gst}}$. $\square$

Finally Chang and Li point out that the coarse moduli cone C^{gst} is $(0,1)$-homogeneous [CL12, Proposition 6.7]: recalling the interpretation of $\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4,d)^p$ as an open inside the cone of sections of $\mathscr{V} = \mathrm{Vb}(\mathscr{L}^{\oplus 5} \oplus \mathscr{P})$ over $\hat{\mathscr{C}}_{\mathfrak{Pic}_1^{(1)}}$, the desired $\mathbb{G}_{\mathrm{m}}$-action on the fibers of $\gamma\colon \widetilde{\mathcal{Z}}^{p,\mathrm{gst}} \to \widetilde{\mathcal{Z}}^{\mathrm{gst}}$ is induced by endowing $\mathrm{Vb}(\mathscr{L}^{\oplus 5} \oplus \mathscr{P})$ with the $\mathbb{G}_{\mathrm{m}}$-action that is trivial on the fibers of every copy of $\mathrm{Vb}(\mathscr{L})$, and has character 1 on those of $\mathrm{Vb}(\mathscr{P})$, with an underlying trivial action on $\mathfrak{Pic}_1^{(1)}$. This influences the obstruction theory $\mathrm{R}^\bullet\,\hat{\pi}_*\left(\mathfrak{e}^* T_{\mathscr{V}/\hat{\mathscr{C}}_{\mathfrak{Pic}_1^{(1)}}}\right)$ of $\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4,d)^p$ in the obvious way, and consequently $\mathbb{E}$ on $\widetilde{\mathcal{Z}}^p$.

Step II: from cosection localized to standard Gysin map. Recall that when we are working on a proper DM stack $\mathcal{M}$ endowed with a perfect obstruction theory and admitting a cosection σ, then the cosection localized Gysin map can be reduced to the ordinary Gysin map, i.e.

$$\iota_* \circ 0^!_{E,\sigma} = 0^!_E \circ \tilde{\iota} \quad \text{where} \quad \iota\colon D(\sigma) \hookrightarrow \mathcal{M}, \;\; \tilde{\iota}\colon E(\sigma) \hookrightarrow E$$

and E is the vector bundle containing the cone.

Then to compute $0^!_{\tilde{V},\sigma_1}(C^{\mathrm{gst}})$ we plan to compactify $\widetilde{\mathcal{Z}}^{p,\mathrm{gst}}$, extend the cone, the vector bundle and the section over it and then make use of the fact we just recalled.

Since $\widetilde{\mathcal{Z}}^{p,\mathrm{gst}} \cong \mathrm{Tot}_{\widetilde{\mathcal{Z}}^{\mathrm{gst}}}\left(\mathrm{R}^1\,\hat{\pi}_*\mathscr{P}\right)$ we compactify considering

$$\overline{\gamma}\colon \overline{\widetilde{\mathcal{Z}}^{p,\mathrm{gst}}} := \mathbb{P}\left(\mathrm{R}^1\,\hat{\pi}_*\mathscr{P} \oplus \mathscr{O}_{\widetilde{\mathcal{Z}}^{\mathrm{gst}}}\right) \to \widetilde{\mathcal{Z}}^{\mathrm{gst}}$$

and extend $\tilde{V}_1, \tilde{V}_2$ via:

$$\tilde{V}_1^{\mathrm{cpt}} = \overline{\gamma} V_1(-D_\infty), \qquad \tilde{V}_2^{\mathrm{cpt}} = \overline{\gamma} V_2.$$

Finally, let us extend the cosection as well. From the cosection σ defined by 2 we clearly get a cosection on $\mathcal{Z}^p$ by pull-back and again a

$$\widetilde{\sigma}\colon \mathcal{O}b_{\widetilde{\mathcal{Z}}^p} \to \mathscr{O}_{\widetilde{\mathcal{Z}}^p}$$

on the blow-up. Let us denote by $\widetilde{\sigma}_{\mathrm{gst}}$ its restriction to $\widetilde{\mathcal{Z}}^{\mathrm{gst}}$. Writing down what $\widetilde{\sigma}_{\mathrm{gst}}$ looks like on points[CL15, Lemma 6.2], we see that it extend to a homomorphism of vector bundles:

$$\overline{\sigma}\colon \tilde{V}^{\mathrm{cpt}} = \tilde{V}_1^{\mathrm{cpt}} \oplus \tilde{V}_2^{\mathrm{cpt}} \to \overline{\widetilde{\mathcal{Z}}^{p,\mathrm{gst}}}.$$

Proposition 5.6. *Let $\iota_!\colon Z_*(\tilde{V}(\widetilde{\sigma}_{\mathrm{gst}})) \to Z_*(\tilde{V}^{\mathrm{cpt}})$ be defined by $\iota_![C] = [\overline{C}]$. And $i\colon D(\widetilde{\sigma}_{\mathrm{gst}}) \to \tilde{Z}_{\mathrm{gst}}$ the inclusion. Then*

$$\overline{\gamma}_* \circ 0^!_{\tilde{V}^{\mathrm{cpt}}} \circ \iota_! = i \circ 0^!_{\widetilde{\sigma}_{\mathrm{gst}},\mathrm{loc}}\colon Z_*(\tilde{V}(\widetilde{\sigma}_{\mathrm{gst}})) \to A_*(\tilde{Z}_{\mathrm{gst}}).$$

This follows essentially from the property of cosection localized cycles recalled at the beginning of the section; see [CL15, Proposition 6.4] for full details.

Step III: reduction of the support of the cone. We now explain a key technical lemma which will enable us to show that C^{gst} pushes forward to zero under a suitably defined morphism. It is basically reducing the support of $C^{\mathrm{gst}} \cap 0 \oplus \tilde{V}_2$ to a manageable substack of $\tilde{V}_2$, that is the union of the zero-section (i.e. $\widetilde{\mathcal{Z}}^p$) and a sub-line bundle of $\tilde{V}_2$ supported on $\tilde{\Delta} = \widetilde{\mathcal{Z}}^{p,\mathrm{main}} \cap \widetilde{\mathcal{Z}}^{p,\mathrm{gst}}$. Even better, using the homogeneity we can show that such a line bundle comes from $\Delta = \widetilde{\mathcal{Z}}^{\mathrm{main}} \cap \widetilde{\mathcal{Z}}^{\mathrm{gst}}$. This is [CL12, Proposition 7.1].

Lemma 5.7. *There is a sub-line bundle F of $V_2|_\Delta$ such that:*

$$C^{\mathrm{gst}} \cap 0 \oplus \tilde{V}_2 \subseteq 0_{\tilde{V}_2} \cup \tilde{F} := \widetilde{\mathcal{Z}}^{p,\mathrm{gst}} \cup \gamma^* F$$

First they use the fact that there is a triple of compatible obstruction theories for the triangle:

$$\begin{tikzcd} \widetilde{\mathcal{Z}}^p \arrow[rr] \arrow[dr, "\gamma"] & & \widetilde{\mathcal{Z}} \arrow[dl] \\ & \mathcal{XP} & \end{tikzcd}$$

such that their restrictions to $\widetilde{\mathcal{Z}}^{p,\mathrm{gst}}$ have locally free h^0 and h^1. To the effect that by taking h^1 of the dual obstruction theories we obtain a commutative diagram:

$$\begin{tikzcd} h^1(\mathbb{L}^\vee_{\widetilde{\mathcal{Z}}^p/\widetilde{\mathcal{Z}}}|_{\widetilde{\mathcal{Z}}^{p,\mathrm{gst}}}) \arrow[r] \arrow[d] & h^1(\mathbb{L}^\vee_{\widetilde{\mathcal{Z}}^p/\mathcal{XP}}|_{\widetilde{\mathcal{Z}}^{p,\mathrm{gst}}}) \arrow[r] \arrow[d] & h^1(\gamma^*\mathbb{L}^\vee_{\widetilde{\mathcal{Z}}/\mathcal{XP}}|_{\widetilde{\mathcal{Z}}^{p,\mathrm{gst}}}) \arrow[d] \\ \tilde{V}_2 \arrow[r, "i_2"] & \tilde{V}_1 \oplus \tilde{V}_2 \arrow[r, "\mathrm{pr}_1"] & \tilde{V}_1 \end{tikzcd}$$

The vertical arrows are injective by the definition of an obstruction theory, and the bottom triangle is exact. Notice that $0 \oplus \tilde{V}_2$ is precisely the kernel of pr_1. It follows that, in order to understand the support of $C^{\mathrm{gst}} \cap 0 \oplus \tilde{V}_2$, it is enough to study that of N, where N is the coarse moduli cone of $\mathfrak{C}_{\widetilde{\mathcal{Z}}^p/\widetilde{\mathcal{Z}}}$, living in the upper left corner of the above diagram.

This is an easier task, since we know that $\widetilde{\mathcal{Z}}^p/\widetilde{\mathcal{Z}}$ is a line bundle on $\widetilde{\mathcal{Z}}^{\mathrm{gst}}$ and an isomorphism on $\widetilde{\mathcal{Z}}^{\mathrm{main},\circ}$. Hence we can always find a local chart $S \to \widetilde{\mathcal{Z}}$ such that the following diagram holds:

$$\begin{tikzcd} \widetilde{\mathcal{Z}}^p \arrow[d] & T \arrow[l] \arrow[r, hook, "V(\zeta t)"] \arrow[d] \arrow[dl, phantom, "\square"] & S \times \mathbb{A}^1_t \arrow[dl] \\ \widetilde{\mathcal{Z}} & S \arrow[l, "\acute{e}t"] & \end{tikzcd}$$

where ζ is a local equation for the boundary. Then $\tau^{\geq -1}\mathbb{L}_{\widetilde{\mathcal{Z}}^p/\widetilde{\mathcal{Z}}}|_T = [I/I^2 \xrightarrow{\delta} \Omega_{\mathbb{A}^1_S/S}]$; I is generated by ζt, whose image under δ is $\zeta \mathrm{dt}$, which restricts to 0 on $\widetilde{\mathcal{Z}}^{p,\mathrm{gst}} \times_{\widetilde{\mathcal{Z}}^p} T = \{\zeta = 0\}$. So the action is trivial, and in fact the restriction of the coarse moduli cone is precisely $\mathrm{Spec}_{T^{\mathrm{gst}}} \mathrm{Sym}^\bullet I/I^2$, which is a line bundle supported on $\tilde{\Delta} \times_{\widetilde{\mathcal{Z}}^p} T$ and trivial otherwise. By gluing different charts we get the line bundle $\tilde{F}$ on $\tilde{\Delta}$.

The last part of the statement, namely that $\tilde{F}$ descends to a line bundle F on Δ is proved by homogeneity: the normal cone of $\widetilde{\mathcal{Z}}^p/\widetilde{\mathcal{Z}}$ is homogeneous with respect to the $\mathbb{G}_{\mathrm{m}}$-action with character 1 on the fibers of $\tilde{V}_2 \to \widetilde{\mathcal{Z}}^{p,\mathrm{gst}}$, but being a cone it is 0-homogeneous as well (see the above discussion of homogeneity), so it is γ^*F for some line bundle F on $\Delta \subseteq \widetilde{\mathcal{Z}}$.

Step IV: pushing forward to zero. Recall from the previous sections that we need to show that the degree of the following class is 0:

$$0^!_{\overline{V}}[\overline{C}^{\mathrm{gst}}] = 0^!_{\overline{V}_2}(0^!_{\overline{V}_1}[\overline{C}^{\mathrm{gst}}]) = 0^!_{\overline{V}_2}([N_{\overline{C}^{\mathrm{gst}} \cap (0 \oplus \overline{V}_2)} \overline{C}^{\mathrm{gst}}])$$

$$\begin{tikzcd}
? \arrow[r] \arrow[d] & X \arrow[d] & \\
\overline{C}^{\mathrm{gst}} \cap (0 \oplus \overline{V}_2) \arrow[r] \arrow[d] & V_2 \arrow[r] \arrow[d] & X \arrow[d] \\
\overline{C}^{\mathrm{gst}} \arrow[r] & \overline{V} \arrow[r] & \overline{V}_1
\end{tikzcd}$$

We know from the previous section that the cycle in the last bracket can be split into two parts, one (call it N_1) supported on a sub-line bundle of $\overline{V}_2$ on $\tilde{\Delta}$, the other one (call it N_2) supported on the zero section of $\overline{V}_2$.

Lemma 5.8. *Both N_1 and N_2 are $5d+1$-dimensional cycles, and* $\deg(0^!_{\overline{V}_2}[N_i]) = 0$ *for $i = 1, 2$.*

Proof. Compare with [CL12, Lemma 8.1]. The dimension of $\widetilde{\mathcal{Z}}^{\mathrm{gst}}$ is $5d+3$, being locally a $5(d+1)$ vector bundle over a dimension -2 stack; so $\widetilde{\mathcal{Z}}^{p,\mathrm{gst}}$, which is a line bundle on the former, has dimension $5d+4$. The coarse moduli cone has then dimension $5d+6$, as can be argued from Lemma 5.1. $\overline{V}_1|_{\widetilde{\mathcal{Z}}^{p,\mathrm{gst}}}$ has rank 5, so $0^!_{\overline{V_1}}[\overline{C}^{\mathrm{gst}}]$ is represented by a cycle of dimension $5d+1$. We shall exploit the commutativity of Gysin pullback with proper pushforward.

For N_1 we conclude easily from the above facts, since $\deg(0^!_{\overline{V}_2}[N_1]) = \deg(0^!_{V_2}\gamma_*[N_1])$, but $\gamma_*[N_1] \in A_{5d+1}(F)$ must be trivial, since F has dimension $5d$, being a line bundle on Δ which is a divisor in $\mathcal{Z}^{\mathrm{main}}$.

On the other hand $N_2 \subseteq \overline{V}_2$ admits a further splitting into $N_{2,\mu} \subseteq \overline{V}_{2,\mu}$ according to the number k of rational tails of the generic element of the boundary component on which they are supported and a partition of the degree $\mu \vdash d$ in k many parts. There exists a comparison morphism:

$$\beta_\mu \colon D^\mu \widetilde{\mathcal{Z}}^p \to D^\mu \mathcal{Z} = D^\mu \overline{\mathcal{M}}_1(\mathbb{P}^4, d) \to \mathcal{W}_\mu$$

where $\mathcal{W}_\mu \subseteq \prod_{i=1}^k \overline{\mathcal{M}}_{0,1}(\mathbb{P}^4, d_i)$ is defined by requiring that the maps do attain the same value at the markings. β_μ is obtained by forgetting the p-field, the Vakil-Zinger blow-up and the k-pointed elliptic curve contracted by the map to $\mathbb{P}^4$. This has the nice property that $\overline{V}_2$ is the pullback along β_μ of a vector bundle on $\mathcal{W}_\mu$. Namely the universal curve over each factor can be glued to the other ones along the given sections, producing a genus 0 non-Gorenstein (unless $k=2$) singularity to which the universal maps to $\mathbb{P}^4$ descend by the property of pushouts:

$$\begin{tikzcd} \overline{\mathscr{C}}_\mu \arrow[r, "\bar{f}"] \arrow[d, "\bar{\pi}"] & \mathbb{P}^4 \\ \mathcal{W}_\mu & \end{tikzcd}$$

The sheaf $V_\mu := \bar{\pi}_* \bar{f}^* \mathscr{O}_{\mathbb{P}^4}(5)$ is indeed a vector bundle of rank $5d+1$ on $\mathcal{W}_\mu$, as can be checked by Riemann-Roch and the normalisation sequence (the topological type of the curve is constant near the singularity), and $\overline{V}_{2,\mu} = \beta_\mu^* V_\mu$. Finally the actual dimension of $\mathcal{W}_\mu$ is $5d+4-2k < 5d+1$ for $k \geq 2$ by Kleiman-Bertini theorem, so $\beta_{\mu,*}[N_{2,\mu}] = 0$.

□

References

[BLR12] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. *Néron models*, volume 21. Springer Science & Business Media, 2012.

[CKL01] David A. Cox, Sheldon Katz, and Yuan-Pin Lee. Virtual fundamental classes of zero loci. In *Advances in algebraic geometry motivated by physics (Lowell, MA, 2000)*, volume 276 of *Contemp. Math.*, pages 157–166. Amer. Math. Soc., Providence, RI, 2001.

[CL12] Huai-Liang Chang and Jun Li. Gromov-Witten invariants of stable maps with fields. *Int. Math. Res. Not. IMRN*, (18):4163–4217, 2012.

[CL15] Huai-Liang Chang and Jun Li. An algebraic proof of the hyperplane property of the genus one GW-invariants of quintics. *J. Differential Geom.*, 100(2):251–299, 2015.

[Con07] Brian Conrad. Arithmetic moduli of generalized elliptic curves. *J. Inst. Math. Jussieu*, 6(2):209–278, 2007.

[Deb13] Olivier Debarre. *Higher-dimensional algebraic geometry*. Springer Science & Business Media, 2013.

[Ful98] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998.

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[HH09] Brendan Hassett and Donghoon Hyeon. Log canonical models for the moduli space of curves: the first divisorial contraction. *Trans. Amer. Math. Soc.*, 361(8):4471–4489, 2009.

[HL10] Yi Hu and Jun Li. Genus-one stable maps, local equations, and Vakil-Zinger's desingularization. *Math. Ann.*, 348(4):929–963, 2010.

[KKP03] Bumsig Kim, Andrew Kresch, and Tony Pantev. Functoriality in intersection theory and a conjecture of Cox, Katz, and Lee. *J. Pure Appl. Algebra*, 179(1-2):127–136, 2003.

[KL13] Young-Hoon Kiem and Jun Li. Localizing virtual cycles by cosections. *J. Amer. Math. Soc.*, 26(4):1025–1050, 2013.

[Kon95] Maxim Kontsevich. Enumeration of rational curves via torus actions. In *The moduli space of curves (Texel Island, 1994)*, volume 129 of *Progr. Math.*, pages 335–368. Birkhäuser Boston, Boston, MA, 1995.

[LMB00] Gérard Laumon and Laurent Moret-Bailly. *Champs algébriques*, volume 39 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2000.

[LZ07] Jun Li and Aleksey Zinger. Gromov-Witten invariants of a quintic threefold and a rigidity conjecture. *Pacific J. Math.*, 233(2):417–480, 2007.

[Mat89] Hideyuki Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1989. Translated from the Japanese by M. Reid.

[Ols06] Martin C. Olsson. $\underline{\mathrm{Hom}}$-stacks and restriction of scalars. *Duke Math. J.*, 134(1):139–164, 2006.

[RSW17] D. Ranganathan, K. Santos-Parker, and J. Wise. Moduli of stable maps in genus one and logarithmic geometry I. *ArXiv e-prints*, August 2017.

[Ser00] Jean-Pierre Serre. *Local algebra*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2000. Translated from the French by CheeWhye Chin and revised by the author.

[Smy11a] David Ishii Smyth. Modular compactifications of the space of pointed elliptic curves I. *Compos. Math.*, 147(3):877–913, 2011.

[Smy11b] David Ishii Smyth. Modular compactifications of the space of pointed elliptic curves II. *Compos. Math.*, 147(6):1843–1884, 2011.

[Sta16] The Stacks Project Authors. *Stacks Project*. `http://stacks.math.columbia.edu`, 2016.

[Vak00] Ravi Vakil. The enumerative geometry of rational and elliptic curves in projective space. *J. Reine Angew. Math.*, 529:101–153, 2000.

[Vis12] Michael Viscardi. Alternate compactifications of the moduli space of genus one maps. *Manuscripta Math.*, 139(1-2):201–236, 2012.

[VZ07] Ravi Vakil and Aleksey Zinger. A natural smooth compactification of the space of elliptic curves in projective space. *Electron. Res. Announc. Amer. Math. Soc.*, 13:53–59, 2007.

[VZ08] Ravi Vakil and Aleksey Zinger. A desingularization of the main component of the moduli space of genus-one stable maps into $\mathbb{P}^n$. *Geom. Topol.*, 12(1):1–95, 2008.

[Wan12] Jie Wang. Deformations of pairs (X, L) when X is singular. *Proc. Amer. Math. Soc.*, 140(9):2953–2966, 2012.