REDUCED GW INVARIANTS FROM CUSPIDAL CURVES

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Abstract. abstract

The moduli stack of stable maps to projective space is a fundamental object in Gromov-Witten theory, since any moduli space of maps to a (smooth) projective variety is cut inside that of the ambient projective space by a set of induced equations. Yet its structure in higher genus is still largely mysterious.

Recall that, in genus zero, $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)$ is a smooth stack of the expected dimension $\mathrm{vdim} = r-3+d(r+1)+n$ (we see that it is unobstructed from the fact that $H^1(\mathbb{P}^1, f^*T_{\mathbb{P}^r}) = 0$ for any map $f\colon \mathbb{P}^1 \to \mathbb{P}^r$). Furthermore, for any split vector bundle $E = \oplus_i \mathcal{O}_{\mathbb{P}^r}(l_i)$, the sheaf $\pi_* f^* E$ (where π is the structure map of the universal curve and f the universal stable map) is actually a vector bundle on $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)$.

$$\begin{tikzcd} \mathcal{C}_{0,n}(\mathbb{P}^r,d) \arrow[r, "f"] \arrow[d, "\pi"] & \mathbb{P}^r \times \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d) \\ \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d) & \end{tikzcd}$$

It is a matter of functoriality of virtual fundamental classes [Kon95] [CKL01] [KKP03] that, if $i\colon X \to \mathbb{P}^r$ is a complete intersection of degree $(l_1,\ldots,l_k)$ (if X is cut by a generic section $s \in H^0(\mathbb{P}^r, E)$, then the induced section $\tilde{s} = \pi_* f^*(s)$ is also generic) and we denote by j the corresponding inclusion

$$\bigsqcup_{\beta \in A_1(X)\colon\, i_*(\beta)=d} \overline{\mathcal{M}}_{0,n}(X,\beta) \hookrightarrow \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)$$

then the following equality, expected by the interpretation of Chern classes as zero loci, holds:

$$j_*[\bigsqcup_{\beta \in A_1(X)\colon\, i_*(\beta)=d} \overline{\mathcal{M}}_{0,n}(X,\beta)]^{\mathrm{vir}} = c_{\mathrm{top}}(E) \cap [\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)].$$

This so-called "hyperplane property" allows us to compute GW invariants of complete intersections as twisted invariants on projective space. The situation in higher genera is much more intricated: the moduli space $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)$ is neither irreducible, nor pure dimensional; the sheaf $\pi_* f^* \mathcal{O}_{\mathbb{P}^r}(l)$ is not a vector bundle. One of the reason for this (in fact the only reason in genus one) is that there are maps from reducible curves with a genus g core that is

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contracted and some rational tails that carry all the degree; if $f\colon C \to \mathbb{P}^r$ is such a map, then $H^1(C, f^*\mathcal{O}_{\mathbb{P}^r}(1)) \neq 0$, as can be easily deduced from the normalisation sequence.

1. Genus one stable maps to $\mathbb{P}^r$ - Equations, components and alternate compactifications

1.1. **Local equations and components.** The geometry of the genus one moduli space of Kontsevich' stable maps to projective space $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$ has been widely studied since [Vak00] and is by now well understood. Assume $d > 0$. Let E be a smooth genus one curve with a non-constant morphism $f\colon E \to \mathbb{P}^r$; then $H^1(E, f^*T_{\mathbb{P}^r}) = 0$ by Riemann-Roch, hence we may define the *main component* of $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$ as the closure of the locus of maps the source of which is a smooth genus one curve. It is irreducible of the expected dimension $\mathrm{vdim} = d(r+1)+n$. On the other hand, for every positive integer k and partitions λ of d into k (positive) parts and μ of n into k (positive) parts, consider the boundary component $D_{\lambda,\mu}(\mathbb{P}^r, d)$ that is the closure of the locus where (i) the source curve is obtained by gluing a smooth elliptic curve with k many $\mathbb{P}^1$'s (as rational tails), (ii) the map contracts the core elliptic curve to a point, and (iii) the rational tail R_i has degree λ_i and μ_i many marked points. In fact $D_{\lambda,\mu}(\mathbb{P}^r, d)$ is the image of the gluing morphism from the fiber product

$$\overline{\mathcal{M}}_{1,k}(\mathbb{P}^r, 0) \times_{\mathbb{P}^r} \prod_{i=1}^{k} \overline{\mathcal{M}}_{0,\mu_i+1}(\mathbb{P}^r, \lambda_i).$$

Proposition 1.1. (1) *The set of irreducible components of $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$ is in bijection with $\{(\lambda \vdash d, \mu \vdash n) |\ \#(parts)(\lambda) = \#(parts)(\mu)\} \cup \{\circ = \mathrm{main}\}$, i.e.*

$$\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d) = \overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)^\circ \cup \bigcup_{\lambda,\mu} D_{\lambda,\mu}(\mathbb{P}^r, d).$$

(2) *Let $(E_{p_1,\dots,p_k} \sqcup_{q_1,\dots,q_k} \bigsqcup_{i=1}^k R_i, f)$ be a degenerate stable map with k rational tails; then it lies in the boundary of the main component if and only if $\mathrm{d}f(T_{q_i}R_i)$ are* linearly dependent *in $T_{f(E)}\mathbb{P}^r$.*

In order to understand how the Gromov-Witten invariants differ from the naive expectation of counting smooth embedded genus g curves in the given homology class, one needs to have a good understanding of the virtual fundamental class and of its splitting on the various components. This has led to the study of explicit local equation for the moduli space of stable maps in a smooth (over $\mathfrak{M}_{1,n}$) ambient space [HL10], which we shall now briefly explain. Recall that a map from a scheme C to projective r-space is the same as the datum of a line bundle L on C together with $r+1$ global sections in $H^0(C, L)$ that generate the line bundle at every point of C. Then it seems natural to embed the space of stable maps as an open inside $\pi_*\mathcal{L}^{\oplus r+1}$ on the universal Picard stack $\mathfrak{P}_{1,n}$. The problem is that this cone is not smooth, due to the jumps in fiber dimension at the boundary.

Let $[f\colon C\to\mathbb{P}^r]$ be a point of $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r,d)$; we may fix homogeneous coordinates on $\mathbb{P}^r$ in a way that $D_0:=f^*\{X_0=0\}$ is a d-uple of smooth points on C (i.e. étale on the base) (this property will hold in a neighbourhood of $[f]$). This gives a map from (an étale chart of) $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r,d)$ to (an étale chart of) the moduli space of pre-stable curves with a degree d divisor $\mathfrak{M}_{1,n}^{\mathrm{div}=d}$, that we may assume to be landing in the locus where the divisor is composed by d smooth distinct points (notice this locus is smooth over $\mathfrak{M}_{1,n}$ by the deformation theory of smooth subschemes). A morphism to $\mathbb{P}^r$ shall now be thought of as a curve-divisor pair (C,D) together with r global sections of $\mathcal{O}_C(D)$: the morphism can be written as $[1:u_1:\ldots:u_r]$, where 1 is the image of the constant global section along the given map $\mathcal{O}_C\to\mathcal{O}_C(D)$ (i.e. we have taken dishomogenised local coordinates on $\mathbb{P}^r$).

Furthermore, étale locally on $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r,d)$, we may pick extra sections $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{C}_{1,n}(\mathbb{P}^r,d)$ such that (i) they pass through the core elliptic curve, and (ii) they are disjoint smooth points away from $\mathcal{D}_0$. Then $\pi_*\mathcal{L}(\mathcal{A})$ *is* a vector bundle on $\mathfrak{P}_{1,n}$ and $\pi_*\mathcal{L}$ is carved inside it by requiring that the restriction (residue) map $\pi_*\mathcal{L}(\mathcal{A})\to\pi_*\mathcal{L}(\mathcal{A})_{|\mathcal{A}}$ is zero.

Proposition 1.2. (1) *Étale locally, there is a locally closed imbedding of $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r,d)$ inside the vector bundle $V_1:=\operatorname{Spec}(\pi_*\mathcal{L}(\mathcal{A})^{\oplus r})$ over $\mathfrak{M}_{1,n}^{\mathrm{div}=d}$. It is obtained by imposing the nondegeneracy and stability conditions* (open) *on the vanishing locus of the restriction of the universal section of $\pi_*\mathcal{L}(\mathcal{A})^{\oplus r}$ to $\mathcal{A}$* (closed).

(2) *Let $\mathcal{V}$ be an (affine local) étale chart around a smooth point of $\mathfrak{M}_{1,n}^{\mathrm{div}=d}$. Then $\pi_*\mathcal{L}(\mathcal{A})\cong\pi_*\mathcal{L}(\mathcal{A}-\mathcal{B})\oplus\pi_*L(\mathcal{A})_{|\mathcal{B}}$ (trivially, $\mathcal{O}_{\mathcal{V}}^{\oplus d+1}\cong\mathcal{O}_{\mathcal{V}}^{\oplus d}\oplus\mathcal{O}_{\mathcal{V}}$) and the restriction-to-$\mathcal{A}$ map is zero on the second factor. Call $\varphi\colon\pi_*\mathcal{L}(\mathcal{A}-\mathcal{B})\to\pi_*\mathcal{L}(\mathcal{A})_{|\mathcal{A}}$ the induced restriction to $\mathcal{A}$.*

(3) *If we choose a suitable basis of $\pi_*\mathcal{L}(\mathcal{A}-\mathcal{B})$, the restriction-to-$\mathcal{A}$ map can be written in a very explicit form: let $D=\sum_{i=1}^d\delta_i$, then $\pi_*\mathcal{O}_{\mathfrak{C}}(\delta_i+\mathcal{A}-\mathcal{B})$ is a sub-line bundle of $\pi_*\mathcal{L}(\mathcal{A}-\mathcal{B})$ and $\varphi=\oplus\varphi_i\colon\ \oplus_{i=1}^d\pi_*\mathcal{O}_{\mathfrak{C}}(\delta_i+\mathcal{A}-\mathcal{B})\to\pi_*\mathcal{L}(\mathcal{A})_{|\mathcal{A}}$. Furthermore $\varphi_i\colon\mathcal{O}_{\mathcal{V}}\to\mathcal{O}_{\mathcal{V}}$ is given (up to invertibles) by multiplication by $\prod_{q\in[\mathcal{A},\delta_i]}\zeta_q$, where ζ_q is the smoothening variable on $\mathfrak{M}_1$ corresponding to the node q, and $q\in[a,\delta_i]$ stands for all the nodes separating $\mathcal{A}$ (i.e. the core elliptic curve) from the point δ_i.*

Given these explicit equations and by carefully examining the proof of [HL10, Prop. 4.13], we can look back at the problem of determining the boundary of the main component of the space of stable maps. The following characterisation of smoothability was already known to (and partly proven in) [Vak00, Lem. 5.9].

Proposition 1.3. *Let $[f\colon C\to\mathbb{P}^r]$ be a degenerate stable map from a (n-marked) genus one curve of the form $(E_{p_1,\ldots,p_m}\sqcup_{q_1,\ldots,q_m}\bigsqcup_{i=1}^m R_i,f)$, where E is the maximal connected contracted arithmetic genus one subcurve, and R_i*

are the rational tails (chains of $\mathbb{P}^1$ along which f has positive degree). The following are equivalent:

(1) *$[f]$ is smoothable;*
(2) *$\mathrm{d}f(T_{q_i}R_i)$ are linearly dependent in $T_{f(E)}\mathbb{P}^r$;*

Proof. Let us start with the easiest degenerate situation: a contracted elliptic curve joined with a rational tail of degree d at a single node q. Equations for the moduli space of maps around such a point look like $\zeta_q \sum_{i=1}^d w_i^j = 0$, for $j = 1, \ldots, r$, the w_i's being coordinates for $\pi_*\mathcal{O}_{\mathfrak{C}}(D+\mathcal{A}-\mathcal{B})$ in the basis given by $\pi_*\mathcal{O}_{\mathfrak{C}}(\delta_i+\mathcal{A}-\mathcal{B}),\ i = 1, \ldots, d$. Our point corresponds to a smoothable map if and only if the equations admit a solution with $\zeta_q \neq 0$, hence it must be $\sum_{i=1}^d w_i^j = 0$ for every j. Taking a coordinate z on the rational tail around the node q (i.e. $\{z = 0\}$ corresponds to the node), we see that the i-th vector in the chosen basis corresponds to a polynomial vanishing at the node and at every other point $\delta_j,\ j \neq i$; hence that can be written as $e_i(z) = z \prod_{j\neq i} \frac{(z-\delta_j)}{-\delta_j}$ (we chose a convenient normalisation). So the map corresponding to the point of coordinates $(w_i^j)_{i=1,\ldots,d;j=1,\ldots,r}$ can be represented as $[1 : \sum_{i=1}^d w_i^1 e_i(z) : \ldots, \sum_{i=1}^d w_i^r e_i(z)]$ on the rational tail; differentiating we see that the tangent vector at the node is sent to $(\sum_{i=1}^d w_i^1, \ldots, \sum_{i=1}^d w_i^r)$ (in affine coordinates), hence smoothability is equivalent to the image of the tangent vector being zero.

More generally, we may assume the dual graph is terminally weighted. Let us say there are m positive-weight rational tails and denote by $D(k),\ k = 1, \ldots, m$, the set of indices j s.t. δ_j belongs to the k-th rational tail, by $E(k)$ the set of nodes separating the core genus one curve from the k-th rational tail. The equations have then the following form

$$\sum_{k=1}^m \Big(\prod_{q\in E(k)} \zeta_q\Big)\Big(\sum_{i\in D(k)} w_i^j\Big) = 0,\ j = 1, \ldots, r$$

which can be assembled in matrix form

$$W \cdot \underline{\zeta} := \Big(\sum_{i\in D(k)} w_i^j\Big)_{j,k} \cdot \Big(\prod_{q\in E(k)} \zeta_q\Big)_k = 0.$$

We see that smoothability is equivalent to linear dependence of the rows of the above matrix W. On the other hand, on every positive-weight rational tail R_k, we can choose a suitable coordinate z_k around the node and write the map as $[1 : p_k^1(z_k) : \ldots : p_k^r(z_k)]$, where $p_k^j(z_k) = \sum_{i\in D(k)} w_i^j e_i^k(z_k)$ and $e_i^k(z_k) = z_k \prod_{h\in D(k)\setminus\{i\}} \frac{(z_k-\delta_h)}{-\delta_h}$. The elliptic curve is contracted to the point $[1 : 0 : \ldots : 0]$ and the tangent vector to R_k at the node is mapped to the k-th row of W (in affine coordinates around that point). Again we see that the map is smoothable if and only if the image of the tangent vectors to the rational tails at the nodes are linearly dependent in $T_{f(E)}\mathbb{P}^r$. $\square$

1.2. **Viscardi's compactifications.** For any homology class $\beta \in H_2(X, \mathbb{Z})$, let $\mathcal{M}_{1,n}(X, \beta)$ be the stack of maps $f\colon C \to X$ from a smooth genus one curve with n marked points, satisfying $f_*[C] = \beta$. The m-stable maps give compactifications of such a moduli space.

Definition 1.4. Let C be a reduced, connected, proper curve of arithmetic genus one, and let $p_1, \dots, p_n \in C$ be smooth, distinct points. A map $f\colon C \to X$ is said to be *m-stable* if the following conditions hold:

(1) C has only nodes and elliptic l-fold points, $l \leq m$, as singularities.
(2) For any connected subcurve $E \subset C$ of arithmetic genus one on which f is constant,
$$|\left\{E \cap \overline{C \setminus E}\right\} \cup \{p_i : p_i \in E\}| > m.$$
(3) f has no non-trivial infinitesimal automorphisms.

Recall that a k-rational $p \in C$ is called an *elliptic m-fold point* if
$$\hat{\mathcal{O}}_{C,p} \cong \begin{cases} k[[x,y]]/(y^2 - x^3) & \text{if } m = 1 \\ k[[x,y]]/(x(x-y^2)) & \text{if } m = 2 \\ k[[x,y]]/I_m & \text{if } m \geq 3 \end{cases}$$
where $I_m = (x_h x_i - x_h x_j : i, j, h \in \{1, \dots, m-1\})$ and i, j, h are distinct.

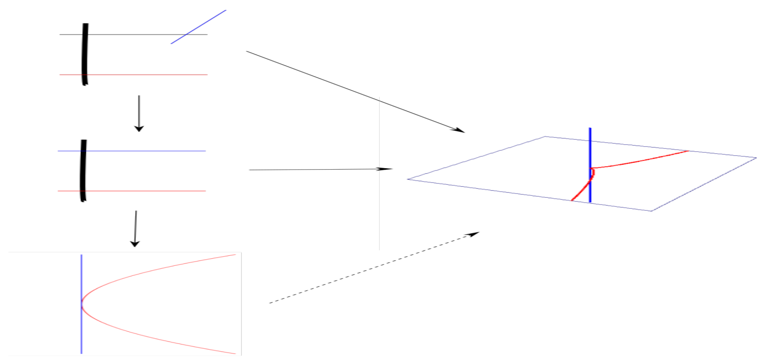
The definition of a family of m-stable maps is then the natural one and Viscardi's main result [Vis12, Thm. 3.6] is that the associated moduli functor gives a compactification of $\mathcal{M}_{1,n}(X, \beta)$ (alternative to Kontsevich' moduli space).

Theorem 1.5. *The moduli functor of m-stable maps in a fixed homology class $\overline{\mathcal{M}}^m_{1,n}(X, \beta)$ is represented by a proper DM stack of finite type.*

Remark 1.6. It seems to us that the algorithm presented by Viscardi to prove properness of the moduli space oversees a case. The problem is that, given a map $f\colon C \to \mathbb{P}^r$ with f constant on a genus one connected sub-curve $E \subseteq C$, it is not always true that f descends to a map $f'\colon C' \to \mathbb{P}^r$, where C' is the singularity obtained from C by contracting E. Indeed, it is likely that descending to C' imposes differential conditions on f that go beyond just order zero. A counterexample is the following.

Consider the stable map $[f]$ in $\overline{\mathcal{M}}_{1,0}(\mathbb{P}^3, 4)$ from an elliptic bridge $R_1 \sqcup_{q_1} E \sqcup_{q_2} R_2$ to $\mathbb{P}^3$ that maps R_1 to the z-axis, contracts E to the origin, and makes (R_2, q_2) into the normalisation of a cusp in the (x,y)-plane, i.e. its image is the non-Gorenstein singularity $C := \mathbb{C}[x,y,z]/(x,y) \cap (z, y^2 - x^3)$. Notice that $\mathrm{d}f(T_{q_2}R_2) = 0$, so there is a non-trivial linear relation
$$0 \cdot \mathrm{d}f(T_{q_1}R_1 + 1 \cdot \mathrm{d}f(T_{q_2}R_2) = 0$$
and the map is smoothable; however, it is impossible to make the first coefficient non-zero and the map *does not factorise* through an elliptic m-fold singularity. Indeed, the only chance would be to factorise through a birational map $f'\colon C' \to C$ from a tacnodal singularity C'. This is impossible

since C and C' have the same δ invariant and so f' would actually be an isomorphism. Observe that Viscardi suggests that the map should extend to the contraction, in a way that the image should still be C, since there is only one indeterminacy point (the singularity). We suggest that in this case the correct procedure would be to sprout (R_1, q_1), but then there is actually no factorisation.



However, the argument can easily be fixed. Let $(\mathcal{C}_\eta, F_\eta)$ be a stable map to $\mathbb{P}^r$, defined on the generic point of a DVR scheme S; we may assume that $\mathcal{C}_\eta$ is smooth [Vis12, Section 3.2.1]. As described in [Vis12, Step 1, Theorem 3.6], after applying nodal reduction we get a map $F\colon \mathcal{C}_S \to \mathbb{P}^r_S$, for which we may suppose that C_0 is nodal and $f := F_0$ is stable.

If f is not constant on the minimal genus one sub-curve [Smy11a], then it is also m-stable and there is nothing to say. Otherwise, let $E \subset C$ be the maximal connected genus one sub-curve where f is constant and let $R_1 \sqcup \dots \sqcup R_m = \overline{C/E}$. By Proposition 1.1(2) we know there is a non-trivial linear relation among the $\mathrm{d}f(T_{q_i}R_i)$'s. There are two possible situations:

(1) $\mathrm{d}f(T_{q_i}R_i) \neq 0$ for every $i = 1, \dots, m$. Possibly after relabelling, this non-trivial relation looks like

$$\alpha_1 \,\mathrm{d}f(T_{q_1}R_1) + \ldots + \alpha_j \,\mathrm{d}f(T_{q_j}R_j) = 0$$

with all the coefficients being non-zero. Then, we may blow-up $\mathcal{C}$ in $q_{j+1}, \dots, q_m$. The induced map $\tilde{F}_0$ is constant on the exceptional divisors $E_{j+1}, \dots, E_m$, so we can complete the above linear relation to

$$\alpha_1 \,\mathrm{d}\tilde{f}(T_{q_1}\tilde{R}_1) + \ldots + \alpha_j \,\mathrm{d}\tilde{f}(T_{q_j}\tilde{R}_j) + \beta_{j+1} \,\mathrm{d}\tilde{f}(T_{q_{j+1}}E_{j+1}) + \ldots + \beta_m \,\mathrm{d}\tilde{f}(T_{q_m}E_m) = 0$$

with any choice of non-zero coefficients β. Now this *sprouting* [Smy11b, Section 2.3] ensures that the sections descend to the corresponding elliptic m-fold singularity. At this point, go to Step 2 of Viscardi's algorithm and follow it.

(2) $\mathrm{d}f(T_{q_i}R_i) = 0$ for some i, say for $i \in \{1, \dots, j\}$. Then blow-up $\mathcal{C}$ at $q_{j+1}, \dots, q_m$. Observe that all $\mathrm{d}\tilde{f}(T_{q_i}R_i)$, $i = 1, \dots, j$ and $\mathrm{d}\tilde{f}(T_{q_i}E_i)$, $i = j+1, \dots, m$ vanish, hence they satisfy any linear

relation with non-zero coefficients, and they factorise through the corresponding singularity. The map has positive degree on the first $i = 1, \ldots, j$ branches.

The irreducible components of Viscardi's moduli space $\overline{\mathcal{M}}^m_{1,n}(\mathbb{P}^r, \beta)$ are well understood too [Vis12, Thm. 5.9]; indeed they have a similar description to the ones of Kontsevich' space. The main advantage of the m-stable compactification is that the number of components drops as m increases.

Corollary 1.7. [Vis12, Cor. 5.10] *For $m \geq \min(r, d) + n$, $\overline{\mathcal{M}}^m_{1,n}(\mathbb{P}^r, \beta)$ is irreducible.*

Moreover, inspired by Viscardi's alternate compactification, we express a sufficient condition for smoothability in yet another way.

Proposition 1.8. *Let $[f : C \to \mathbb{P}^r]$ be a degenerate stable map from a (n-marked) genus one curve of the form $(E_{p_1,\ldots,p_m} \sqcup_{q_1,\ldots,q_m} \bigsqcup_{i=1}^m R_i, f)$, where E is the maximal connected contracted arithmetic genus one subcurve, and R_i are the rational tails (chains of $\mathbb{P}^1$ along which f has positive degree). If f factorises through a* non-degenerate *(i.e. having positive degree along at least one of the branches of the singularity) map $f' : C' \to \mathbb{P}^r$ from an elliptic k-fold singularity, with $k \leq m$, then $\mathrm{d}f(T_{q_i} R_i)$ are linearly dependent in $T_{f(E)}\mathbb{P}^r$.*

Proof. Let $C \xrightarrow{\pi} C' \xrightarrow{f'} \mathbb{P}^r$ be the factorization of f. Then $E' = \mathrm{Exc}(\pi) \subseteq E$ and $\overline{C \setminus E'} = (S_1, q'_1) \sqcup \cdots \sqcup (S_k, q'_k) \to (C', p)$ is the normalization of the elliptic k-fold point. By hypothesis, $f = f' \circ \pi$ and f is non-constant on $R_1, \ldots R_m$. Being f' non-degenerate implies that there exists a subset $\{S_1, \ldots S_j\} \subseteq \{S_1, \ldots S_k\}$ such that

$$(S_1, q'_1) \cong (R_{i_1}, q_{i_1}), \ldots, (S_j, q'_j) \cong (R_{i_j}, q_{i_j}), \quad \{i_1, \ldots, i_j\} \subseteq \{1, \ldots, m\}\,.$$

By Smyth's characterisation of elliptic k-fold singularities, [Smy11a, Lem. 2.2], we obtain a non trivial linear relation

$$\alpha_1\, \mathrm{d}\pi(T_{q_{i_1}} R_{i_1}) + \ldots + \alpha_j\, \mathrm{d}\pi(T_{q_{i_j}} R_{i_j}) + \alpha_{j+1}\, \mathrm{d}\pi(T_{q_{j+1}} S_{j+1}) + \ldots + \alpha_k\, \mathrm{d}\pi(T_{q_k} S_k) = 0$$

since all the coefficients are non-zero, $j \neq 0$, and the second half of the formula can be ignored, we've got what we wanted. □

Remark. The viceversa is not true, as Remark 1.6 shows.

In order to achieve the original goal of Gromov-Witten theory, i.e. the enumeration of smooth curves of genus g in a target variety, mathematicians have been trying either to eliminate the unwanted components of Kontsevich's space (e.g. degenerate contributions from lower genus curves and multiple covers of lower degree curves) by defining alternate compactifications, or to concentrate the attention on the main component (i.e. find a suitable fundamental class supported on it). In genus one, Vakil and Zinger managed to do so by performing a sequence of blow-ups on an underlying space of weighted curves [VZ07] [VZ08] (see also [HL10]). The idea is that blowing up successively the locus of degree zero elliptic curve with one rational tail (*elliptic tails*), then with two (*elliptic bridges*), etc. and taking the fiber product

$$\begin{array}{ccc} \widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d) & \longrightarrow & \overline{\mathcal{M}}_{1,n}(\mathbb{P}^r,d) \\ \downarrow & & \downarrow \\ \widetilde{\mathfrak{M}}_{1,n}^{\mathrm{wt}=d} & \longrightarrow & \mathfrak{M}_{1,n}^{\mathrm{wt}=d} \end{array}$$

the strict transform of the main component becomes smooth and meets the rest of the components in normal crossing fashion (the étale local model being the union of two linear subspaces of different dimensions in affine space). Furthermore, the sheaves $\tilde{\pi}_*\tilde{f}^*\mathcal{O}_{\mathbb{P}^r}(l)$ contain a vector subbundle of rank dl supported on the strict transform of the main component. This paved the way to the definition of reduced invariants.

Definition 1.9. Let X be a complete intersection of degree $(l_1,\ldots,l_k)$ in $\mathbb{P}^r$. The genus one *reduced* Gromov-Witten invariants of X are defined by integration on

$$[\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r,d)^\circ]\cap c_{\mathrm{top}}(\tilde{\pi}_*\tilde{f}^*(\oplus_{i=1}^k\mathcal{O}_{\mathbb{P}^r}(l_i))^\circ).$$

It is a matter of fact that the virtual dimension of the moduli space of maps to a three-fold X is independent of the genus. In this context it was plausible to test the better enumerative properties of reduced invariants; in fact, Li-Zinger [LZ07] in the symplectic category and, later on, Chang-Li [CL15] in the algebraic one, proved a comparison formula for the quintic three-fold $X_5\subseteq\mathbb{P}^4$ (or, more generally, for a complete intersection three-fold X of degree $(l_1,\ldots,l_k)$ in $\mathbb{P}^r$ with $\sum l_i\leq r+1$) of the form

$$GW_1(X)=\widetilde{GW}_1(X)+c\cdot GW_0(X).$$

We notice that the genus zero contribution to this formula comes from the boundary component with a single rational tail of degree d, while elliptic bridges and degenerate contributions with more rational tails do not matter to the genus one Gromov-Witten count. This suggests that removing the component with elliptic tails would provide a more direct and efficient definition of reduced invariants. We plan to do so by comparing the ordinary space of stable maps with Viscardi's space of 1-stable maps.

2. Cuspidal curves and maps

In this section we start investigating the Gromov- Witten type invariant obtained from Viscardi compactification studying the case $m=1$: the source curve is at worst cuspidal. We will see how in this case we can give a comparison with the GW-invariant and explain why the same technique does not work for higher m. The first thing we do is the following: we introduce a moduli space of stably-weighted, prestable, at worst cuspidal, marked curves.

We show that there is a birational morphism from a correspondingly decorated moduli space of prestable nodal curves. Taking the fiber product with Viscardi's moduli space of stable cuspidal maps, we show that the resulting space includes in the usual Kontsevich's moduli space, precisely avoiding the component with elliptic tails $D_{\{d\},\{n\}}(X,d)$.

Why should such a morphism on decorated prestable curves exist? Consider an elliptic tail C; a one-dimensional family $\mathcal{C}$ with central fiber $\mathcal{C}_0 \cong C$ will either be locally constant or smoothen the node. In the first case we can associate to this the locally constant family of cusps obtained by forgetting the elliptic tail and contracting a length-two infinitesimal neighborhood of the node in the rational component; in the second case, we claim that the line bundle $\mathcal{O}_{\mathcal{C}}(3q)$ gives a morphism to a smoothing of the cuspidal $\mathbb{P}^1$. This heuristics (i.e. being able to substitute an elliptic tail with a cusp over points or DVR schemes) motivates us to believe we can define a morphism from suitably defined moduli spaces of nodal genus one curves to cuspidal curves (cfr. [Smy11b, Lemma 4.2]). In fact we shall extend [Smy11b, Corollary 4.5] to an appropriate setup of moduli stacks of weighted prestable curves. Assume $d>0$ throughout the following.

Definition 2.1. Let $\mathfrak{M}_{1,n}^{\mathrm{wt}=d,\mathrm{st}}$ be the stack of *prestable* (projective, nodal and reduced), connected, arithmetic genus one, n-marked curves that are *stably weighted* with total weight d, i.e. for every geometric point there is an integer-valued function on the set of irreducible components of the corresponding curve, such that it is compatible with specialisation maps and the sum of the integers is d; furthermore we require that all integers are nonnegative and every $p_a=0$ weight-0 component has at least three special points (every $p_a=1$ weight-0 component has at least one special point).

Remark that there is an étale, non-separated morphism $\mathfrak{M}_{1,n}^{\mathrm{wt}} \to \mathfrak{M}_{1,n}$ and the weight stability condition is such that the forgetful map $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r,d) \to \mathfrak{M}_{1,n}$ factorises through $\mathfrak{M}_{1,n}^{\mathrm{wt}=d,\mathrm{st}}$, the weight assignment coming from the degree of the map to $\mathbb{P}^r$.

Definition 2.2. Let $\mathfrak{M}_{1,n}^{\mathrm{wt}=d,\mathrm{st}}(1)$ be the stack of *at worst cuspidal* projective, reduced, connected, arithmetic genus one, n-marked curves that are stably weighted with total weight d, i.e. as above with (stricter) stability condition: every $p_a=0$ weight-0 component has at least three special points (every $p_a=1$ weight-0 component has at least *two* special point) *and* every weight-0 cusp has at least two further special points.

Note that the only type of nodal curves that we are getting rid of is the one with an elliptic tail of weight zero.

Theorem 2.3. *There exists a morphism* $\mathfrak{M}_{1,n}^{\mathrm{wt}=d,st} \to \mathfrak{M}_{1,n}^{\mathrm{wt}=d,st}(1)$ *which extends the identity on the smooth locus.*

Since it is hard to define such a map on an arbitrary base, we proceed as follows. Let $\mathcal{C}$ and $\mathcal{C}'$ be the universal curves over $\mathfrak{M} := \mathfrak{M}_{1,n}^{\mathrm{wt}=d,\mathrm{st}}$ and

$\mathfrak{M}' := \mathfrak{M}_{1,n}^{\mathrm{wt}=d,\mathrm{st}}(1)$ respectively. Abusing notation slightly, we will still write $\mathcal{C}$ and $\mathcal{C}'$ for their pullbacks to the product $\mathfrak{M}\times\mathfrak{M}'$ along the two projections. The proof of the theorem follows from two Lemmas:

Lemma 2.4. *There is a locally closed substack $\mathcal{X} \subseteq \mathrm{Mor}_{\mathfrak{M}\times\mathfrak{M}'}(\mathcal{C},\mathcal{C}')$ representing morphisms $C \to C'$ that contract weight-zero elliptic tails to cusps and are weight-preserving isomorphisms everywhere else.*

Lemma 2.5. *The first projection $\mathrm{pr}_1\colon \mathrm{Mor}_{\mathfrak{M}\times\mathfrak{M}'}(\mathcal{C},\mathcal{C}') \to \mathfrak{M}$ restricted to $\mathcal{X}$ is an isomorphism with $\mathfrak{M}$.*

Proof. 2.4 Recall that $\mathrm{Mor}_{\mathfrak{M}\times\mathfrak{M}'}(\mathcal{C},\mathcal{C}')$ is an algebraic stack; in fact the map to $\mathfrak{M}\times\mathfrak{M}'$ is representable by Grothendieck's theory of the Hilbert scheme for projective morphisms. We now proceed to construct $\mathcal{X}$ as a locally closed substack in the space of morphisms.

Step 1: Consider

$$\pi\colon \mathfrak{P} = \mathfrak{Pic}_{1,n}^{totdeg=d,st} \to \mathfrak{M},\ \pi'\colon \mathfrak{P}' = \mathfrak{Pic}_{1,n}^{totdeg=d,st}(1) \to \mathfrak{M}'$$

the Picard stacks on $\mathfrak{M}$ and $\mathfrak{M}'$ with universal line bundles $\mathfrak{L}$ and $\mathfrak{L}'$, where π and π' are defined by taking the multi-degree of line bundles. We can now look at the algebraic stack $\mathrm{Mor}_{\mathfrak{P}\times\mathfrak{P}'}(\mathcal{C},\mathcal{C}')$ with universal morphism Φ and natural projection Π to $\mathrm{Mor}_{\mathfrak{M}\times\mathfrak{M}'}(\mathcal{C},\mathcal{C}')$. We claim that there exists a locally closed substack $\mathcal{Y}' \subseteq \mathrm{Mor}_{\mathfrak{P}\times\mathfrak{P}'}(\mathcal{C},\mathcal{C}')$ representing those morphisms that *preserve the line bundles*. Indeed, given a chart

$$S \to \mathrm{Mor}_{\mathfrak{P}\times\mathfrak{P}'}(\mathcal{C},\mathcal{C}'), \tag{1}$$

the locus of $s \in S$ where $\Phi_s^*\mathfrak{L}'_s \cong \mathfrak{L}_s$ is nothing else than the locus T where the two sections $\mathfrak{L}_S$ and $\Phi_S^*\mathfrak{L}'_S$ of $\mathfrak{P}(S) \to \mathfrak{M}(S)$ induced by 1 coincide. In other words, we are looking for the fiber product

$$\begin{tikzcd} T \arrow[r] \arrow[d] & \mathfrak{P} \arrow[d, "\Delta"] \\ S \arrow[r] & \mathfrak{P}\times_{\mathfrak{M}}\mathfrak{P} \end{tikzcd}$$

Being $\mathfrak{P} \to \mathfrak{M}$ representable by schemes, it follows that $U \to S$ is a locally closed immersion.

Step 2: Furthermore, there is a closed substack $\mathcal{Y} \subseteq \mathcal{Y}'$ representing *surjective morphisms that preserve the markings.*

Given a chart $S \to \mathrm{Mor}_{\mathfrak{P}\times\mathfrak{P}'}(\mathcal{C},\mathcal{C}')$, the locus of $s \in S$ where Φ_s is marking-preserving is the equaliser of the two sections

$$S \underset{\times\Phi\circ\sigma_i}{\overset{\times\sigma'_i}{\rightrightarrows}} \mathcal{C}'_S \times_S \ldots \times_S \mathcal{C}'_S$$

This defines a closed subscheme of S, since $\mathcal{C}'_S \to S$ is separated.

As regards surjectivity, since Φ is proper, and by upper semicontinuity of the dimension of the fiber [Sta16, Tag 0D4I], the locus in $\mathcal{C}'_S$ where the fiber of Φ is empty is open. The complement of its image in S is the locus we were after.

Step 3: Let $\mathcal{X}'$ be the image of $\mathcal{Y}$ under Π. This is a constructible substack of $\mathrm{Mor}_{\mathfrak{M}\times\mathfrak{M}'}(\mathcal{C},\mathcal{C}')$ by Chevalley's theorem [Sta16, Tag 054K]. Recall that, to show that a constructioble set is open (respectively closed), it is enough to check that it contains all the generisations of its points (respectively all the specialisations) [Sta16, Tag 0903]. Finally, under Noetherian assumptions, two points related by specialisation/generisation are contained in the image of a DVR scheme [Sta16, Tag 054F].

It is clear that being surjective and marking-preserving are closed conditions, as above. The requirement that ϕ can be covered by a line bundle-preserving map can be translated into the following combinatorial conditions:

(1) *ϕ contracts only weight zero components.* We show that this is open. Assume that S is a DVR scheme with closed point 0 and generic point η, and we are given $S \to \mathcal{X}'$ such that $\phi_0\colon \mathcal{C}_0 \to \mathcal{C}'_0$ does not contract any positive weight component. Suppose, by contradiction, that there exists an irreducible component $W \subseteq C_\eta$ of positive weight d_W which is contracted by ϕ_η. The contracted locus, i.e. $\left\{c \in \mathcal{C}_S \,|\, \dim_c \phi^{-1}(\phi(c)) = 1\right\}$, is closed by semicontinuity of fiber dimension and contains all the components $D_i \subseteq C_0$ to which D specialises; at least one of them must have positive weight, since the sum of their weights is d_W, which is a contradiction.

(2) *ϕ has degree 1 on every non contracted component*, or, equivalently, there is an S-dense open in $\mathcal{C}'_S$ such that the restriction of ϕ_S to its preimage is an isomorphism. This is an open and closed condition; we show it is open. Let S be a DVR scheme as above and assume that ϕ_0 satisfies the property. We may assume that S is the local ring at 0 of a smooth projective curve $\bar{S}$ [Har77, Thm. I.6.9] and that $S \to \mathrm{Mor}_{\mathfrak{M}\times\mathfrak{M}'}(\mathcal{C},\mathcal{C}')$ extends to $\bar{S}$ (by the valuative criterion of properness). Let us now write $\phi_{\bar{S},*}[\mathcal{C}_{\bar{S}}] = \sum n_i[\mathcal{C}'_{\bar{S},i}]$; by Gysin pull-back to 0 and [Ful98, Prop. 10.1(a)], we see that all the n_i's are 1, apart from some that involve components of $\mathcal{C}'_{\bar{S}}$ away from 0. Discarding these, we find an open neighborhood of 0 such that the restriction of the morphism to every closed point in the base is birational in our sense, i.e. we find a $\bar{S}$-dense (around 0) open in $\mathcal{C}'_{\bar{S}}$ on which ϕ gives an isomorphism, hence ϕ_η also satisfies the property.

FIX ME: properness seems to fail, but we don't need to extend to $\bar{S}$ to use Fulton's intersection theory

(3) *ϕ is weight-preserving.* This is again an open condition: assume that ϕ_0 is weight-preserving; the existence of ϕ_S implies that the maps $\tilde{\phi}_0$ and $\tilde{\phi}_\eta$ (induced at the level of weighted dual graphs) are compatible with the specialisation maps. Since the weight of a component of the generic fiber is determined by those of the components to which it specialises, this proves that ϕ_η must be weight-preserving as well.

Step 4: We have seen that, if ϕ contracts a component E of the fiber, it must have zero weight. Then it has to be elliptic by the stability condition. Since the target has only nodes and cusps as singularities, it must be $|\overline{C\setminus E}\cap E|\leq 2$, i.e. E is either an elliptic tail or an elliptic bridge. There are basically two possibilities: (a) the elliptic tail gets contracted to a cusp, or ϕ is an isomorphism everywhere; (b) the elliptic tail/bridge gets contracted to a smooth point or node, then a non-separating node or a cusp must be created somewhere else in order to preserve the genus. We want to get rid of case (b), so we define the open substack $\mathcal{X}\subseteq\mathcal{X}'$ as follows. Given $\mathcal{C}_S\to\mathcal{C}'_S\in\mathcal{X}'(S)$, let $U'\subseteq\mathcal{C}'$ be the *maximal* S-dense open subset such that $\phi_S|_{\phi_S^{-1}(U')}\colon\phi_S^{-1}(U')\to U'$ is an isomorphism and Z' its closed complement in $\mathcal{C}'_S$. Then $\mathcal{X}$ is the open locus [Sta16, Tag 055D, Tag 055G] where the fibers of $\pi|_{\phi^{-1}(Z')}\colon\phi^{-1}(Z')\to S$ are geometrically connected.

□

Proof. 2.5 This result will follow from an application of Zariski's Main Theorem for algebraic spaces. First of all, we claim that the projection is *representable by algebraic spaces*: by [Sta16, Tag 04Y5] we only need to check that it is faithful, and by [Con07, Theorem 2.2.5] it is enough to look at geometric points; hence we are just saying that $\mathrm{Aut}(\phi)\subseteq\mathrm{Aut}(C)$. This follows from the fact that, if $\phi\in\mathcal{X}$, then it is dominant and of degree one. Secondly, $\mathrm{pr}_{1|\mathcal{X}}$ is *proper*: this can be seen using the valuative criterion

$$\begin{tikzcd}
\eta=\mathrm{Spec}(K) \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\
S=\mathrm{Spec}(R) \arrow[r] & \mathfrak{M}
\end{tikzcd}$$

Let $\pi\colon\mathcal{C}_S\to S$ be the family of nodal curves on S; there are three cases to consider:

(1) the central fiber contains no elliptic tail, then so does $\mathcal{C}_\eta$, hence ϕ_η is an isomorphism and admits a unique extension to an isomorphism $\phi\colon\mathcal{C}_S\cong\mathcal{C}'_S$.

It's right but confusing! Write it better!!!

If instead $\mathcal{C}_0$ has got an elliptic tail, then we have two possibilities:

(2) $\mathcal{C}_\eta$ still has an elliptic tail, i.e. the node separating the weight zero, genus one curve from the rest of $\mathcal{C}_S$ is not smoothened. Then $\mathcal{C}_S$ is the push-out of a family of rational curves $\mathcal{R}_S$ and a family of genus one curves $\mathcal{Z}_S$.

$$\begin{tikzcd}
S \arrow[r] \arrow[d] & \mathcal{R}_S \arrow[d] \\
\mathcal{Z}_S \arrow[r] & \mathcal{C}_S
\end{tikzcd}$$

Before writing the push down diagram we might need to change base. Write down the double push-out diagram

Locally around the node, ϕ_η is the contraction of the elliptic curve to the cusp, and, on the adjacent rational component, it is the normalisation of the cusp; by the universal property of push-outs, it can be uniquely extended to the same map on the central fiber (far from this node, ϕ_η is an isomorphism).

(3) If $\mathcal{C}_S$ smoothens the elliptic tail, then the map is given on the generic fiber by a π-ample line bundle $\mathcal{L}_\eta$. Then the extension to the central fiber is determined by the unique extension $\mathcal{L}_0$ of $\mathcal{L}_\eta$ which is trivial on the minimal subcurve $Z \subseteq \mathcal{C}_0$ of arithmetic genus 1 and, possibly after twisting, is ample elsewhere.

Finally, observe that the map is bijective by construction and $\mathfrak{M}$ is normal, hence $\pi_{|\mathcal{X}}\colon \mathcal{X} \to \mathfrak{M}$ is an isomorphism by an application of Zariski's main theorem (as in [Sta16, Tag 082I]). $\square$

Why only cusps?

Alternative constructions.

- adapt the ideas of [HH09]: check out section 3 of this paper;
- check [RSW17, §3].

3. Comparison with p-fields and splitting

3.1. **Notation.** Let $\mathcal{Z}$ be defined by the pullback diagram:

$$\begin{tikzcd} \mathcal{Z} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4, d) \arrow[d] \\ \mathcal{X} \arrow[r] & \mathfrak{M}_1^{(1)} \end{tikzcd}$$

Objects of $\mathcal{Z}$ over S consist of diagrams

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & \hat{\mathcal{C}} \arrow[r, "f"] \arrow[dl, "\hat{\pi}"] & \mathbb{P}^4 \\ & S & & \end{tikzcd}$$

where f is a 1-stable map and ϕ is the weighted 1-stabilisation (i.e. contraction of elliptic tails of weight 0); arrows over id_S are commutative diagrams

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\phi"] \arrow[d, "\psi"] & \hat{\mathcal{C}} \arrow[r, "f"] \arrow[d, "\hat{\psi}"] & \mathbb{P}^4 \arrow[d, "\mathrm{id}_{\mathbb{P}}"] \\ \mathcal{C}' \arrow[r, "\phi'"] & \hat{\mathcal{C}}' \arrow[r, "f'"] & \mathbb{P}^4 \end{tikzcd}$$

where ψ and $\hat{\psi}$ are isomorphisms; note once again that $\hat{\psi}$ is determined by ψ, due to the required commutativity with the dominant "birational" morphism ϕ.

Forgetting $\hat{\mathcal{C}}$ and keeping $f \circ \phi\colon \mathcal{C} \to \mathbb{P}^4$, we obtain a morphism $i\colon \mathcal{Z} \to \overline{\mathcal{M}}_1(\mathbb{P}^4, d)$, which we claim is a *closed embedding*: in fact, from the above description of arrows in $\mathcal{Z}$, it is representable (i.e. faithful) and a proper (check the valuative criterion: use the fact that $\mathcal{X} \to \mathfrak{M}_1$ is proper, and the map to $\mathbb{P}$ descends to the 1-stabilisation because we are only contracting weight 0 elliptic tails; compare with [RSW17, Theorem 4.3]) monomorphism (i.e. full). So we may add the commutative diagram to the left:

$$\begin{tikzcd}
\overline{\mathcal{M}}_1(\mathbb{P}^4,d) \arrow[d] & \mathcal{Z} \arrow[l, hook', "i"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4,d) \arrow[d] \\
\mathfrak{M}_1 & \mathcal{X} \arrow[l, "\sim"'] \arrow[r] & \mathfrak{M}_1^{(1)}
\end{tikzcd}$$

Lemma 3.1. *There is a closed embedding $i\colon \mathcal{Z} \hookrightarrow \overline{\mathcal{M}}_1(\mathbb{P}^4,d)$. In particular $\mathcal{Z}$ is a proper DM stack.*

Furthermore we inherit a description of the irreducible components of $\mathcal{Z}$: there is a main component $\mathcal{Z}^{\text{main}}$ which is the closure of the locus of maps from a smooth elliptic curve, and for every $k \geq 2$ a boundary component $D^k\mathcal{Z}$, whose general point represents a contracted elliptic curve with k-many rational tails of positive degree. Notice that the above lemma signifies that each and every component of $\mathcal{Z}$ is isomorphic to the corresponding one in $\overline{\mathcal{M}}_1(\mathbb{P}^4,d)$. The point is that given any stable map there is at most one factorisation through the weighted 1-stabilisation of the curve: objects of $\overline{\mathcal{M}}_1(\mathbb{P}^4,d)\setminus D^1$ are 1-stable already; objects of $D^1\cap\overline{\mathcal{M}}_1(\mathbb{P}^4,d)^{\text{main}}$ do factor thanks to Vakil's criterion and objects of $D^1\cap D^k$ ($k\geq 2$) do factor through a map which is constant on the cusp. On the other hand, objects of $D^{1,\circ} = D^1\setminus(\overline{\mathcal{M}}_1(\mathbb{P}^4,d)^{\text{main}}\cup\bigcup_{k\geq 2}D^k)$ do not admit any factorisation, so that $\mathcal{Z}$ has no corresponding component.

insert figure of a typical element of $D^1\cap D^2$.

Introduce a Hu-Li type blow-up of $\mathfrak{M}_1^{(1)}$.

We can push this reasoning further to the level of p-fields:

$$\begin{tikzcd}
\overline{\mathcal{M}}_1(\mathbb{P}^4,d)^p \arrow[d] & \mathcal{Z}^p \arrow[l, hook', "j"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4,d)^p \arrow[d] \\
\overline{\mathcal{M}}_1(\mathbb{P}^4,d) \arrow[d] & \mathcal{Z} \arrow[l, hook', "i"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4,d) \arrow[d] \\
\mathfrak{M}_1 & \mathcal{X} \arrow[l, "\sim"'] \arrow[r] & \mathfrak{M}_1^{(1)}
\end{tikzcd}$$

where this time an object of $\mathcal{Z}^p$ over S is given by

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\phi"] \arrow[dr, "\pi"'] & \hat{\mathcal{C}} \arrow[r, "f"] \arrow[d, "\hat{\pi}"] & \mathbb{P}^4 \\
& S &
\end{tikzcd} \qquad \text{and} \qquad \xi \in \Gamma(\hat{\mathcal{C}}, f^*\mathcal{O}_{\mathbb{P}}(-5)\otimes\omega_{\hat{\mathcal{C}}/S})$$

(I think to satisfy the stack condition ξ should be really thought of as a section of $\hat{\pi}_*(f^*\mathcal{O}_{\mathbb{P}}(-5)\otimes\omega_{\hat{\mathcal{C}}/S})$.)

For the existence of j we are claiming that, pulling back along ϕ, ξ induces a section of $\pi_*((f\circ\phi)^*\mathcal{O}_{\mathbb{P}}(-5)\otimes\omega_{\mathcal{C}/S})$. We may work on an atlas for $\mathcal{Z}^p$, which we shall (maybe abusing notation slightly) denote by S, as the base scheme in the above diagram; let us also denote by $\hat{\mathcal{L}}=f^*\mathcal{O}(1)$ and $\mathcal{L}=\phi^*\hat{\mathcal{L}}$.

First we claim that the adjunction map $\mathcal{O}_{\hat{\mathcal{C}}}\to\phi_*\mathcal{O}_{\mathcal{C}}$ is an isomorphism: it is injective by dominance of ϕ; to check the cokernel is trivial it is enough to do so after restricting to each and every irreducible component of $\mathcal{Z}^p$. Then $\hat{\mathcal{C}}$ is normal (so is the base[1]; R1 follows from the fact that singularities are isolated in the fiber, S2 [2] follows from reducedness of the fibers) and we conclude by Zariski's main theorem (geometric fibers of ϕ are either points or elliptic curves, in either case connected).

NO, $\hat{\mathcal{C}}_{|D^k}$ has multiple components

The equality $\phi^*\omega_{\hat{\mathcal{C}}/S}=\omega_{\mathcal{C}/S}(\mathcal{E})$, where $\mathcal{E}$ is the locus spanned by elliptic tails, can be checked on $\mathcal{X}$ and then pulled back. Notice that $\mathcal{E}$ is a Cartier divisor; ϕ is generically an isomorphism, so $\phi^*\omega_{\hat{\mathcal{C}}/S}$ and $\omega_{\mathcal{C}/S}$ may differ only on the locus where ϕ is not an isomorphism. The statement may now be proved by an intersection-theoretic argument by reducing to smoothings over the spectrum of a DVR, see[Smy11a, Lemma 2.13].

It is clear now that $\omega_{\mathcal{C}/S}\otimes\mathcal{L}^{\otimes -5}\hookrightarrow\phi^*(\omega_{\hat{\mathcal{C}}/S}\otimes\hat{\mathcal{L}}^{\otimes -5})$. Pushing down along $\pi=\hat{\pi}\circ\phi$, we get a map $\pi_*(\omega_{\mathcal{C}/S}\otimes\mathcal{L}^{\otimes -5})\hookrightarrow\hat{\pi}_*(\omega_{\hat{\mathcal{C}}/S}\otimes\hat{\mathcal{L}}^{\otimes -5})$ (using the projection formula and $\phi_*\mathcal{O}_{\mathcal{C}}=\mathcal{O}_{\hat{\mathcal{C}}}$). These are both 0 on $\mathcal{Z}^{\mathrm{main},\circ}=\mathcal{Z}^{\mathrm{main}}\setminus(\bigcup_{k\geq 2}D^k\mathcal{Z})$; on the other hand they are both line bundles on $D^k\mathcal{Z}$...

Fix me

Lemma 3.2. *There is a close embedding $j\colon\mathcal{Z}^p\hookrightarrow\overline{\mathcal{M}}_1(\mathbb{P}^4,d)^p$.*

Notice that the p-field is forced to vanish on $\mathcal{Z}^{\mathrm{main},\circ}$, so $(\mathcal{Z}^p)^{\mathrm{main},\circ}\simeq\mathcal{Z}^{\mathrm{main},\circ}$ is the zero-section of the cone $\mathcal{Z}^p\to\mathcal{Z}$ (even though $\overline{\mathcal{M}}_1(\mathbb{P}^4,d)^p\to\overline{\mathcal{M}}_1(\mathbb{P}^4,d)$ is non-trivial on $\overline{\mathcal{M}}_1(\mathbb{P}^4,d)^{\mathrm{main}}\cap D^1$).

Finally it will be useful to introduce

$$\begin{tikzcd} \mathcal{XP} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{Pic}_1^{(1)} \arrow[d] \\ \mathcal{X} \arrow[r] & \mathfrak{M}_1^{(1)} \end{tikzcd}$$

so that objects of $\mathcal{XP}(S)$ are

[1] on VZ here

[2] OK when fibers are Cartier i.e. over a DVR; actually take a look at Thm 23.9 in Matsumura's Commutative Ring Theory, fibers are CM in the situation at hand

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & \hat{\mathcal{C}} \arrow[dl, "\hat{\pi}"] \\ & S & \end{tikzcd} \qquad \text{and} \qquad \hat{\mathcal{L}} \in \operatorname{Pic}_{\hat{\mathcal{C}}/S}(S)$$

Notice that the map $\mathcal{XP} \to \mathfrak{Pic}$ that associates to an object as above $(\mathcal{C} \to S, \phi^*\hat{\mathcal{L}})$ is highly non-trivial on the locus of elliptic tails/cusps, since

- line bundles of the same degree on the cusp are identified (there is a $\mathbb{G}_a$ wealth of them), and
- only line bundles that restrict to effective line bundles on the elliptic tail are in the image of this morphism.

3.2. **Splitting the cone.** We may notice that $(\mathcal{Z}^p)^{\mathrm{main},\circ} \to \mathcal{XP}$ is smooth onto its image, which is an open substack; furthermore on this open the relative tangent is represented by $\hat{\pi}_*\hat{\mathcal{L}}^{\oplus 5}$, and $\mathrm{R}^1\,\hat{\pi}_*\hat{\mathcal{L}}^{\oplus 5} = \hat{\pi}_*(\hat{\omega}\otimes\hat{\mathcal{L}}^{\otimes -5}) = 0$. Let $E^\bullet = \mathrm{R}^\bullet\,\hat{\pi}_*(\hat{\mathcal{L}}^{\oplus 5} \oplus \hat{\omega}\otimes\hat{\mathcal{L}}^{\otimes -5})$ denote the dual perfect obstruction theory of $\mathcal{Z}^p \to \mathcal{XP}$. Then the intrinsic normal cone of $\mathcal{Z}^p \to \mathcal{XP}$ splits into:

- the closure of the zero section of $h^1/h^0(E^\bullet)_{|\mathrm{main},\circ}$, which we shall name $\mathfrak{C}^{\mathrm{main}}$;
- other components supported on the boundary, which we should accordingly name $\mathfrak{C}^k$ $(k \geq 2)$.

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