MODULAR COMPACTIFICATIONS OF $\mathcal{M}_{2,n}$ I

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Abstract.

Contents

1. Introduction

One of the most beautiful and influential results of modern algebraic geometry is the construction of a modular compactification of the stack of smooth pointed curves $\mathcal{M}_{g,n}$, due to P. Deligne and D. Mumford [DM69], through the introduction of *stable* pointed curves.

Definition 1.1. Assume $2g-2+n>0$. A connected, reduced, complete curve C with distinct markings $(p_1,\ldots,p_n)$ is *stable* if:

(1) C has only nodes as singularities, and p_i are smooth points of C;
(2) every rational component of C has at least three special points (markings or nodes).

Theorem 1.2. *Assume $2g-2+n>0$. The moduli stack of stable pointed curves $\overline{\mathcal{M}}_{g,n}$ is a smooth and proper connected DM stack over* $\operatorname{Spec}(\mathbb{Z})$*, with projective coarse moduli space $\overline{\mathbf{M}}_{g,n}$. The boundary $\overline{\mathcal{M}}_{g,n}\setminus\mathcal{M}_{g,n}$, representing nodal curves, is a normal crossing divisor.*

On one hand, the Deligne-Mumford compactification has every desirable property one can hope for; on the other, it is not unique, and it is an interesting deep question to classify all possible modular compactifications of $\mathcal{M}_{g,n}$ - a natural partial answer has been given by D.I. Smyth [Smy13].

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Even though the existence of $\overline{\mathbf{M}}_{g,n}$ can be deduced from nowadays standard general theory on stacks [KM97], this space was first constructed by means of GIT [Gie82, MFK94, BS08]. The study of alternative compactifications of $\mathcal{M}_{g,n}$ is motivated as well by an interest in the birational geometry of $\overline{\mathbf{M}}_{g,n}$, and it is not by chance that the first steps in this direction were moved from a GIT perspective - by changing the invariant theory problem or the stability condition, and realising that the resulting quotient still enjoys a modular interpretation [Sch91, Has05]. The consequent program that goes under the name of B. Hassett and S. Keel aims to describe all the different possible quotients, and to understand whether every step of a minimal model program for $\overline{\mathbf{M}}_{g,n}$ can be given a modular interpretation in terms of curves with worse than nodal singularities. Since the early stages of this study, it has developed into a fascinating playground for testing implementations of ideas originating from GIT into a general structural theory for Artin stacks [AFSvdW17, AFS17a, AFS17b, CTV18].

Only few steps of the Hassett-Keel program have been carried out in full generality. On the other hand, the program has been completed to a larger extent in low genus: with the introduction of weighted pointed curves [Has03] in genus zero, and with Smyth's pioneering work in genus one [Smy11a, Smy11b, Smy18].

Smyth extended earlier work of Schubert by following the philosophy that an alternative compactification can be defined by allowing a reasonably larger class of singularities (*local condition*), studying their stable models, and disallowing the latter by imposing a stronger stability condition (*global condition*); this ensures that the resulting moduli problem be again separated and universally closed, by the valuative criterion.

A useful notion in this respect is that of the *genus* of an isolated curve singularity: let (C,x) be (the germ of) a reduced curve over an algebraically closed field $\mathbf{k}$ at its unique singular point x, with normalisation $\nu\colon \widetilde{C}\to C$.

Definition 1.3. Let C have m branches at x, and δ the $\mathbf{k}$-dimension of $\nu_*\mathscr{O}_{\widetilde{C}}/\mathscr{O}_C$, which is a skyscraper sheaf supported at x. The genus of (C,x) is then

$$g=\delta-m+1.$$

It can be thought of as the number of conditions that a function must satisfy in order to descend from the seminormalisation to C. It is a notion adapted to work in families, in the sense that a complete, reduced curve C with only one singular point at x and normalisation a disjoint union of m copies of $\mathbb{P}^1$ will appear in a family of curves of arithmetic genus g.

Smyth found that, for every fixed number of branches m, there is a unique germ of Gorenstein singularity of genus one up to isomorphism, namely:

$m=1$: the cusp, $V(y^2-x^3)\subseteq\mathbb{A}^2_{x,y}$;
$m=2$: the tacnode, $V(y^2-yx^2)\subseteq\mathbb{A}^2_{x,y}$;
$m\geq 3$: the union of m general lines in $\mathbb{A}^{m-1}$.

Singularities of this kind, with up to m branches, together with nodes, form a deformation-open class of singularities. Furthermore, the elliptic m-fold point can be obtained by contracting a smooth elliptic curve with m rational tails in a one-parameter smoothing, and, roughly speaking, all stable models look like this.

Definition 1.4. [Smy11a] For $m < n$, a connected, reduced, complete curve C of arithmetic genus one with smooth distinct markings $(p_1, \ldots, p_n)$ is m-stable if:

(1) it has only nodes and elliptic l-fold points, $l \leq m$, as singularities;
(2) for every connected subcurve $E \subseteq C$ of arithmetic genus one, its *level* $|E \cap \overline{C \setminus E}| + |\{i \colon p_i \in E\}|$ is strictly bigger than m;
(3) $H^0(C, \Omega_C^\vee(-\sum_i p_i)) = 0$.

The latter can be thought of as a decency condition on the moduli stack. The second one is instead essential in guaranteeing the uniqueness of m-stable limits, seen the discussion above. Smyth's main result is the following.

Theorem 1.5. [Smy11a, Smy11b] *The moduli stack of m-stable curves $\overline{\mathcal{M}}_{1,n}(m)$ is a proper irreducible DM stack over* $\operatorname{Spec} \mathbb{Z}[1/6]$. *It is not smooth for $m \geq 6$.*

1.1. **Outline of results.** Here are the main novelties of our construction. It is not a semistable compactification (see [Smy13, Definition 1.2]), in that we allow dangling singularities of type *II* (see ??? below for a definition), whose normalisation has a one-pointed rational component. The necessity to do so was prefigured in [AFS16] already.

The dualising line bundle of a minimal (with no separating nodes) Gorenstein curve of genus one is trivial [Smy11a, Lemma 3.3]. This plays a fundamental role in the study of semistable tails, as well as in the boundedness of the moduli functor. On the other hand the dualising line bundle of a minimal Gorenstein curve of genus two Z has degree two. For this reason we had to desymmetrise the problem and use the first marking as a reference point, in order to write $\omega_Z = \mathscr{O}_Z(q_1 + \bar{q}_1)$ whenever needed, where q_1 is the point of Z closest to p_1, and $\bar{q}_1$ is conjugate to q_1 (it is not always defined uniquely).

1.2. **Future directions.**

(1) Resolve the indeterminacy of the rational map $\overline{\mathcal{M}}_{2,n}(m_1) \dashrightarrow \overline{\mathcal{M}}_{2,n}(m_2)$; we expect the construction to rely on a semistable compactification of the crimping spaces of the genus two singularities, as in [vdW10, §1.10] and [Smy18]. It would be interesting to put this work in the context of the Hassett-Keel program, as in [Smy11b]. More generally, a question outstanding to our knowledge is whether the whole program fits in the theoretical framework developed in [Hal14].
(2) Our main motivation to start this project arose from Gromov-Witten theory. In genus one, the link between reduced Gromov-Witten invariants (see for example [VZ08, Zin09, LZ09]) and maps from singular curves (see [Vis12]) was partially uncovered in [BCM18], and brought in plain light

in [RSW17a, RSW17b]. With F. Carocci we are investigating whether similar techniques may serve to desingularise the main component of the space of genus two maps to projective space. If this is the case, they will provide a clear definition of reduced invariants, and hopefully ease the access to comparison (standard vs. reduced) results. We expect there will be a(n iso)morphism to the modular blow-up constructed in [HLN12]. This might lead to a mathematical definition of all-genera Gopakumar-Vafa invariants, one day.

2. Gorenstein curve singularities of genus two

In this and the next sections we work over an algebraically closed field of characteristic different from $2, 3, 5$. We provide an algebraic classification of the (complete) local rings of Gorenstein curve singularities of genus two.

Let (C, x) be the germ of a curve singularity, and let $(R, \mathfrak{m})$ denote $(\hat{\mathscr{O}}_{C,x}, \mathfrak{m}_x)$, with normalisation $(\widetilde{R}, \widetilde{\mathfrak{m}}) \simeq (\mathbf{k}[[t_1]] \oplus \ldots \oplus \mathbf{k}[[t_m]], \langle t_1, \ldots, t_m \rangle)$. Here m is the number of branches. Recall from Definition 1.3 that the genus is:

$$g = \delta - m + 1,$$

so, for genus two, $\delta = m + 1$. Following [Smy11a, Appendix A], we consider $\widetilde{R}/R$ as a $\mathbb{Z}$-graded module with:

$$(\widetilde{R}/R)_i := \widetilde{\mathfrak{m}}^i / (\widetilde{\mathfrak{m}}^i \cap R) + \widetilde{\mathfrak{m}}^{i+1};$$

furthermore, Smyth (*loc.cit.*) notices that:

(1) $m + 1 = \delta(p) = \sum_{i \geq 0} \dim_{\mathbf{k}} (\widetilde{R}/R)_i$;
(2) $2 = g = \sum_{i \geq 1} \dim_{\mathbf{k}} (\widetilde{R}/R)_i$;
(3) if $(\widetilde{R}/R)_i = (\widetilde{R}/R)_j = 0$ then $(\widetilde{R}/R)_{i+j} = 0$.

We will also find the following observations to be useful:

(4) $\sum_{i \geq k} (\widetilde{R}/R)_i$ is a grading of $\widetilde{\mathfrak{m}}^k / (\widetilde{\mathfrak{m}}^k \cap R)$;
(5) there is an exact sequence of $R/\mathfrak{m} = \mathbf{k}$-modules:

$$0 \to A_k := \frac{\widetilde{\mathfrak{m}}^k \cap R}{\widetilde{\mathfrak{m}}^{k+1} \cap R} \to \frac{\widetilde{\mathfrak{m}}^k}{\widetilde{\mathfrak{m}}^{k+1}} \to \left(\widetilde{R}/R\right)_k \to 0$$

Lemma 2.1. *There are two unibranch curve singularities of genus two; only one of them is Gorenstein, the A_4-singularity or* ramphoid cusp*: $V(y^2 - x^5) \subseteq \mathbb{A}^2_{x,y}$.*

Proof. In the unibranch case $\dim_{\mathbf{k}} (\widetilde{R}/R)_1 \leq 1$, hence equality must hold (by observation (3) above). We are left with two cases:

- Either $\dim_{\mathbf{k}} (\widetilde{R}/R)_2 = 1$ and $\dim_{\mathbf{k}} (\widetilde{R}/R)_i = 0$ for all $i \geq 3$: in this case $\widetilde{\mathfrak{m}}^3 \subseteq \mathfrak{m}$ by observation (4). From (5) we see that $\widetilde{\mathfrak{m}}^3 = \mathfrak{m}$, hence $R \simeq \mathbf{k}[[t^3, t^4, t^5]]$, a spatial non-Gorenstein singularity.

- Or $\dim_{\mathbf{k}}(\widetilde{R}/R)_3 = 1$ and $\dim_{\mathbf{k}}(\widetilde{R}/R)_i = 0$ for $i = 2$ and for all $i \geq 4$: in this case $\widetilde{\mathfrak{m}}^4 \subseteq \mathfrak{m}$ by observation (4). On the other hand from $\dim_{\mathbf{k}}(\widetilde{\mathfrak{m}}^2 \cap R/\widetilde{\mathfrak{m}}^3 \cap R) = 1$ we deduce that there is a generator of degree 2, and from $\dim_{\mathbf{k}}(\widetilde{\mathfrak{m}}^3 \cap R/\widetilde{\mathfrak{m}}^4 \cap R) = 0$ there is none of degree 3. We may write the generator as $x = t^2 + ct^3$, and $\mathfrak{m} = \langle x \rangle + \widetilde{\mathfrak{m}}^4$. Up to a coordinate change (i.e. automorphism of $\mathbf{k}[[t]]$), we may take $x = t^2$, and
$$\mathfrak{m}/\mathfrak{m}^2 = \langle t^2, t^5 \rangle,$$
so $R \simeq \mathbf{k}[[x, y]]/(x^5 - y^2)$, as anticipated.

□

From now on, we are going to study only Gorenstein singularities. With notation as above, let $I = (R : \tilde{R}) = \mathrm{Ann}_R(\tilde{R}/R)$ be the conductor ideal of the singularity. Recall (see e.g. [Hun99, Theorem 6.4]) that (C, x) is Gorenstein iff :
$$\dim_{\mathbf{k}}(R/I) = \dim_{\mathbf{k}}(\widetilde{R}/R)(= \delta).$$

Recall from [Ste96, Definition 2-1] that a curve singularity (C, p) is *decomposable* if C is the union of two curves C_1 and C_2 that lie in distinct smooth spaces intersecting each other transversely in p. Given a parametrisation $x_i = x_i(t_1, \ldots, t_m),\ i = 1, \ldots, n$, this means that we can find a partition $S \sqcup S' = \{1, \ldots, m\}$ such that x_i only depends on $t_s, s \in S$, or $s \in S'$, for all i. Recall from [AFS16, Proposition 2.1] that, aside from the node, Gorenstein singularities are never decomposable.

Proposition 2.2. *For every fixed integer $m \geq 2$, there are exactly two Gorenstein curve singularities of genus two with m branches.*

Proof. We wish to find a basis for $\mathfrak{m}/\mathfrak{m}^2$, because a map of complete local rings that is surjective on cotangent spaces is surjective. We use the filtrations above.

Case I: $(2, 0, 0)$. We see that $\widetilde{\mathfrak{m}}^2 \subseteq I$, so (by the Gorenstein assumption) $m + 1 = \delta = \dim_{\mathbf{k}}(R/I) \leq \dim_{\mathbf{k}}(R/\widetilde{\mathfrak{m}}^2) = \dim_{\mathbf{k}} A_0 + \dim_{\mathbf{k}} A_1 = m - 1$, contradiction. We note that in this case the singularity is decomposable.

Case II: $(1, 1, 0)$. We have $\widetilde{\mathfrak{m}}^3 \subseteq I$. We are going to write down the $m - 1$ generators of $A_1 \pmod{\widetilde{\mathfrak{m}}^3}$[1]. The first generator, call it, x_1, has a non-trivial linear term in at least one of the variables, wlog t_1. By scaling x_1 and possibly adding a multiple of x_1^2, we can make it into the form: $x_1 = t_1 \oplus p_{1,2}(t_2) \oplus \ldots \oplus p_{1,m}(t_m)$ mod $\widetilde{\mathfrak{m}}^3$. Now we can use x_1 and x_1^2 to make sure the second generator does not involve t_1 at all. It will still have a linear term independent of t_1, say non-trivial in t_2. By scaling and adding a multiple of x_2^2, we can write $x_2 = 0 \oplus t_2 \oplus \ldots \oplus p_{1,m}(t_m)$ mod $\widetilde{\mathfrak{m}}^3$. By taking a linear combination of x_1 with x_2 and x_2^2, we may now reduce

[1]To make them into the simplest possible form, we allow ourselves to perform linear algebra operations at first, and only at the end we may change coodinates - the benefit of this two-step process will become apparent in the next section.

x_1 to the form $t_1 \oplus 0 \oplus p_{1,3}(t_3) \oplus \ldots \oplus p_{1,m}(t_m) \mod \widetilde{\mathfrak{m}}^3$. Therefore, by Gaussian elimination with the generators and their squares, we may assume that

$$\begin{aligned} x_1 =& t_1 \oplus 0 \oplus \ldots \oplus \alpha_{1,m} t_m + \beta_{1,m} t_m^2 \\ x_2 =& 0 \oplus t_2 \oplus \ldots \oplus \alpha_{2,m} t_m + \beta_{2,m} t_m^2 \\ & \ldots \\ x_{m-1} =& 0 \oplus \ldots \oplus t_{m-1} \oplus \alpha_{m-1,m} t_m + \beta_{m-1,m} t_m^2 \mod \widetilde{\mathfrak{m}}^3 \end{aligned}$$

We must have $R/I = \langle 1, x_1, \ldots, x_{m-1}, y \rangle$ by the Gorenstein condition (if $x_i \in I$, then $t_i \in R$, and it is then easy to see that the singularity would be decomposable). Hence $x_i^2 \in I$ for all but at most one i, say $i = 1$. Then $t_i^2 \in R$ for $i = 2, \ldots, m-1$. If $\alpha_{i,m} \neq 0$ for some i in this range, then $t_m^2 \in R$ as well, so $t_1^2 = x_1^2 - O(t_m^2) \in R$, contradicting $\dim_{\mathbf{k}}(\widetilde{R}/R)_2 = 1$. Therefore $\alpha_{i,m} = 0$ for $i \in \{2, \ldots, m-1\}$. If $\alpha_{1,m} = 0$, then we need a further generator of $\mathfrak{m}/\mathfrak{m}^2$, namely $z = 0 \oplus \ldots \oplus t_m^3$. In this case, though, both x_1^2 and z belong to I, so $\dim_k(R/I) = m$, and the singularity cannot be Gorenstein. We have then:

$$\begin{aligned} x_1 =& t_1 \oplus 0 \oplus \ldots \oplus \alpha_{1,m} t_m + \beta_{1,m} t_m^2 \\ x_2 =& 0 \oplus t_2 \oplus \ldots \oplus \beta_{2,m} t_m^2 \\ & \ldots \\ x_{m-1} =& 0 \oplus \ldots \oplus t_{m-1} \oplus \beta_{m-1,m} t_m^2 \mod \widetilde{\mathfrak{m}}^3, \end{aligned} \tag{1}$$

with $\beta_{1,m} \in \mathbf{k}$ and $\alpha_{1,m}, \beta_{i,m} \in \mathbf{k}^\times$, $i = 2, \ldots, m-1$ (by indecomposability). Finally, we may change coordinates in t_m and scale the other t_i to obtain

$$\begin{aligned} x_1 =& t_1 \oplus 0 \oplus \ldots \oplus t_m \\ x_2 =& 0 \oplus t_2 \oplus \ldots \oplus t_m^2 \\ & \ldots \\ x_{m-1} =& 0 \oplus \ldots \oplus t_{m-1} \oplus t_m^2 \mod \widetilde{\mathfrak{m}}^3. \end{aligned} \tag{2}$$

We may check that $R/I = \langle 1, x_1, \ldots, x_{m-1}, x_1^2 \rangle$ and $\widetilde{R}/R$ is of type $(1,1,0)$. In the case $m = 2$, we need an extra generator $y = t_2^3$. Equations are given by

- $y(y - x_1^3)$ if $m = 2$ (A_5-singularity);
- $x_1 x_2 (x_2 - x_1^2)$ if $m = 3$ (D_6-singularity);
- $\langle x_3(x_1^2 - x_2), x_i(x_j - x_k) \rangle_{1 \leq i < j < k \leq m-1 \text{ or } 1 < j < k < i \leq m-1}$ if $m \geq 4$.

Case III: $(1,0,1)$. We have $\widetilde{\mathfrak{m}}^4 \subseteq I$. By an argument similar to the above, we write generators for A_1 as $x_i = \ldots \oplus t_i \oplus \ldots \oplus \alpha_{i,m} t_m + \beta_{i,m} t_m^2 + \gamma_{i,m} t_m^3$, for $i = 1, \ldots, m-1$. Then $R/I = \langle 1, x_1, \ldots, x_{m-1}, y \rangle$. For all but at most one i, $x_i^2 \in I$, but definitely $x_i^3 \in I$ for all i. On the other hand $t_m^3 \notin R$, because otherwise $t_i^3 = x_i^3 - \alpha_{i,m}^3 t_m^3 + O(t_m^4)$ would belong to R as well, contradicting $\dim_{\mathbf{k}}(\widetilde{R}/R)_3 = 1$. From this we deduce that $\alpha_{i,m} = 0$ for all $i = 1, \ldots, m-1$. By

$\dim_{\mathbf{k}}(\widetilde{R}/R)_2 = 0$ there has to be another generator of degree two in t_m, which we may write as $x_m = t_m^2 + \gamma_{m,m} t_m^3$ of $\mathfrak{m}/\mathfrak{m}^2$. We can use x_m to remove all the t_m^2 pieces from $x_1, \ldots, x_{m-1}$, so we are reduced to

$$\begin{aligned} x_1 =& t_1 \oplus 0 \oplus \ldots \oplus \gamma_{1,m} t_m^3 \\ x_2 =& 0 \oplus t_2 \oplus \ldots \oplus \gamma_{2,m} t_m^3 \\ & \ldots \\ x_{m-1} =& 0 \oplus \ldots \oplus t_{m-1} \oplus \gamma_{m-1,m} t_m^3 \\ x_m =& 0 \oplus \ldots \oplus t_m^2 + \gamma_{m,m} t_m^3 \quad \mod \widetilde{\mathfrak{m}}^4, \end{aligned} \tag{3}$$

with $\gamma_{m,m} \in \mathbf{k}$ and $\gamma_{i,m} \in \mathbf{k}^\times,\ i = 1, \ldots, m-1$ (by indecomposability). Finally, we may change coordinates in t_m and scale the other t_i to obtain

$$\begin{aligned} x_1 =& t_1 \oplus 0 \oplus \ldots \oplus t_m^3 \\ x_2 =& 0 \oplus t_2 \oplus \ldots \oplus t_m^3 \\ & \ldots \\ x_{m-1} =& 0 \oplus \ldots \oplus t_{m-1} \oplus t_m^3 \\ x_m =& 0 \oplus \ldots \oplus t_m^2 \quad \mod \widetilde{\mathfrak{m}}^4. \end{aligned} \tag{4}$$

We may check that $R/I = \langle 1, x_1, \ldots, x_{m-1}, x_m \rangle$ and $\widetilde{R}/R$ is of type $(1,0,1)$. For $m = 1$ we recover the unique Gorenstein singularity of Lemma 2.1. Equations are

- $x^5 - y^2$ if $m = 1$ (A_4-singularity or *ramphoid cusp*);
- $y(y^3 - x^2)$ if $m = 2$ (D_5-singularity);
- $\langle x_3(x_1 - x_2), x_3^3 - x_1 x_2 \rangle$ if $m = 3$;
- $\langle x_i(x_j - x_k), x_m(x_i - x_j), x_m^3 - x_1 x_2 \rangle_{i,j,k \in \{1,\ldots,m-1\} \text{ all different}}$ if $m \geq 4$.

□

Remark 2.3. Not-necessarily Gorenstein singularities can be obtained by gluing various Gorenstein singularities of genus ≤ 2 along subschemes of length ≤ 3. Classifying all of them would not necessarily be easy.

Remark 2.4. Singularities of type II do appear in the miniversal family of singularities of type III, and viceversa. For low values of m - which is the playground of al our speculations -, this follows neatly from a beautiful result of A. Grothendieck that I have learnt from [CML13] (see also [Arn72, Dem75]):

Theorem 2.5. *Let (C, p) be a curve singularity of ADE type. Singularities that appear in the miniversal deformation of (C, p) are all and only those ADE, whose Dynkin diagram can be obtained as a full subgraph of the diagram of (C, p).*

Definition 2.6. In case II, we shall call the branches parametrised by t_1 and t_m *twin*; in case III, the branch parametrised by t_m is called the *singular* branch. We shall refer to them as *special* or *distinguished* branches; all other branches are *axes*. *Branch* remains a generic name, indicating any of the previous ones.

3. Tangent sheaf, crimping space, and automorphisms

In this section we analyse the tangent sheaf of a genus two singularity. For a complete Gorenstein curve of genus two with markings, we translate the absence of infinitesimal automorphisms into a (mostly) combinatorial condition. The crimping space naturally makes its appearance in the process.

Lemma 3.1. *Let (C,p) be a Gorenstein curve singularity of genus two, with pointed normalisation $\nu\colon(\tilde{C},\{p_i\}_{i=1,\ldots,m})\to(C,p)$, and assume* $\mathrm{char}(\mathbf{k})\neq 2,3,5$. *There is a diagram of exact sequences of sheaves*

$$\begin{tikzcd}
0 \arrow[r] & \nu_*\Omega^\vee_{\tilde{C}}(-\sum_i 3p_i) \arrow[r] & \nu_*\Omega^\vee_{\tilde{C}}(-\sum_i p_i) \arrow[r] & \nu_*\bigoplus_i\Omega^\vee_{\tilde{C}}(-p_i)_{|2p_i} \arrow[r] & 0 \\
0 \arrow[r] & \nu_*\Omega^\vee_{\tilde{C}}(-\sum_i 3p_i) \arrow[r] \arrow[u, equal] & \Omega^\vee_C \arrow[r] \arrow[u] \arrow[ur, phantom, "\square"] & \mathbf{k}^{\oplus m} \arrow[r] \arrow[u, hook, "\phi"] & 0
\end{tikzcd}$$

The right-most vertical map admits an explicit description in local coordinates.

Proof. Let $K(\tilde{C})$ denote the constant sheaf of rational functions on $\tilde{C}$. A section of $\Omega^\vee_{\tilde{C}}\otimes K(\tilde{C})$ is contained in $\Omega^\vee_C$ iff its image under the push-forward map

$$\nu_*\colon \nu_*\mathscr{H}om(\Omega_{\tilde{C}},K(\tilde{C}))\to\mathscr{H}om(\Omega_C,K(\tilde{C}))$$

lies in the subspace $\mathscr{H}om(\Omega_C,\mathscr{O}_C)$. We may work locally around the singular point in the coordinates studied in the previous section.

A_4: In the coordinates $x=t^2+ct^3, y=t^4, z=t^5$ (they are redundant, but it will not matter in what follows), the section $f(t)\frac{d}{dt}\in\nu_*\Omega^\vee_{\tilde{C}}\otimes K(\tilde{C})$ pushes forward to

$$\nu_*\left(f(t)\frac{d}{dt}\right)=(2t+3ct^2)f(t)\frac{d}{dx}+4t^3f(t)\frac{d}{dy}+5t^4f(t)\frac{d}{dz},$$

from which, writing $f(t)=f_0+f_1t+f_2t^2+O(t^3)$, we see that

$$(2t+3ct^2)f(t),4t^3f(t),5t^4f(t)\in\hat{\mathscr{O}}_{C,p}\Leftrightarrow f_0=0,cf_1+2f_2=0.$$

A_5: In the coordinates $x=t_1\oplus at_2+bt_2^2, y=t_1^3$, the section $f_1(t_1)\frac{d}{dt_1}\oplus f_2(t_2)\frac{d}{dt_2}$ pushes forward to

$$\nu_*\left(f_1(t_1)\frac{d}{dt_1}\oplus f_2(t_2)\frac{d}{dt_2}\right)=\left(f_1(t_1)\oplus(a+2bt_2)f_2(t_2)\right)\frac{d}{dx}+3t_1^2f_1(t_1)\frac{d}{dy},$$

from which, writing $f_i(t_i)=f_{i0}+f_{i1}t_i+f_{i2}t_i^2+O(t_i^3), i=1,2$, we see that

$$f_1(t_1)\oplus(a+2bt_2)f_2(t_2),3t_1^2f_1(t_1)\in\hat{\mathscr{O}}_{C,p}\Leftrightarrow\begin{cases}f_{10}=f_{20}=0,\\ f_{11}=f_{21},\\ 2bf_{21}+af_{22}=a^2f_{12}.\end{cases}$$

$II_{m\geq 3}$: In the coordinates of (1),

$$\nu_*\left(\sum_{i=1}^m f_i(t_i)\frac{d}{dt_i}\right) = (f_1(t_1)\oplus(\alpha_{1,m}+2\beta_{1,m}t_m)f_m(t_m))\frac{d}{dx_1}+ \\ \sum_{i=2}^m (f_i(t_i)\oplus 2\beta_{i,m}t_mf_m(t_m))\frac{d}{dx_i},$$

hence we deduce that

$$\nu_*\left(\sum_{i=1}^m f_i(t_i)\frac{d}{dt_i}\right)\in\Omega_C^\vee\otimes\hat{\mathscr{O}}_{C,p}\Leftrightarrow\begin{cases} f_{i0}=0 & \text{for } i=1,\ldots,m,\\ 2f_{11}=f_{i1}=2f_{m1}, & \text{for } i=2,\ldots,m-1,\\ \beta_{1,m}f_{m1}+\alpha_{1,m}f_{m2}=\alpha_{1,m}^2 f_{12}. \end{cases}$$

$III_{m\geq 2}$: In the coordinates of (3),

$$\nu_*\left(\sum_{i=1}^m f_i(t_i)\frac{d}{dt_i}\right) = \sum_{i=1}^{m-1}\left(f_i(t_i)\oplus 3\gamma_{i,m}t_m^2f_m(t_m)\right)\frac{d}{dx_i}+ \\ (2t_m+3\gamma_{m,m}t_m^2)f_m(t_m)\frac{d}{dx_m},$$

hence we deduce that

$$\nu_*\left(\sum_{i=1}^m f_i(t_i)\frac{d}{dt_i}\right)\in\Omega_C^\vee\otimes\hat{\mathscr{O}}_{C,p}\Leftrightarrow\begin{cases} f_{i0}=0 & \text{for } i=1,\ldots,m,\\ f_{i1}=3f_{m1}, & \text{for } i=1,\ldots,m-1,\\ 3\gamma_{m,m}f_{m1}+2f_{m2}=0. \end{cases}$$

□

From this description we see that the letters α,β and γ will play a role in determining the automorphism group of a complete curve with markings. We recall some key concepts from F. van der Wyck's thesis.

We work over $\mathbf{k}$. We can consider the stack $\mathscr{S}$ of reduced one-dimensional $\mathbf{k}$-algebras R, and the stack $\mathscr{T}$ of reduced 1d algebras with resolution $(R\hookrightarrow(S,J))$, where S is a smooth one-dimensional $\mathbf{k}$-algebra, and J is the radical of the conductor of $R\subseteq S$. Basically, R is the (local) ring of a reduced curve with one singular point, S is its normalisation, and J is the ideal of the reduced fiber over the singular point of $\operatorname{Spec}(R)$. $\mathscr{S}$ and $\mathscr{T}$ are limit-preserving stacks over $\operatorname{Spec}(\mathbf{k})$ [vdW10, Proposition 1.21]. Furthermore, we may fix a reduced 1d algebra with resolution $\tau_0:(R_0\hookrightarrow(S_0,J_0))$, and consider the substack $\mathscr{T}(\tau_0)$ of reduced 1d algebras with singularity type τ_0 (i.e. isomorphic to τ_0 locally on both the base and the curve, see [vdW10, Definition 1.64]; that various notions of "locally" coincide is proved in [vdW10, Proposition 1.50]). There is a forgetful morphism $\mathscr{T}\to\mathscr{S}$, and the *crimping space* of τ_0 is defined to be the fiber over R_0 of the restriction of such morphism to $\mathscr{T}(\tau_0)$. The crimping space is a smooth $\mathbf{k}$-scheme [vdW10, Theorems 1.70 and 1.73]; indeed, it is isomorphic to the quotient of $\mathrm{Aut}_{(S_0,J_0)/\mathbf{k}}$ by $\mathrm{Aut}_{(S_0,J_0)/R_0}$, the latter being the subgroup of automorphisms of the normalisation preserving the subalgebra of the singularity; moreover, the quotient can be computed after modding out the lowest power of J that is contained in R [vdW10, Theorem 1.53]. Crimping spaces should be thought of as moduli for the normalisation map.

Lemma 3.2. *The crimping space of a genus two singularity with m branches is (a number - depending on the type - of copies of) $\mathbb{A}^1 \times (\mathbb{A}^1 \setminus \{0\})^{m-1}$.*

Proof. We resume notation from the previous section. We are going to fix the subalgebra τ_0 given in coordinates by (2) and (4) above respectively.

Type II: recall that in this case $\widetilde{\mathfrak{m}}^3 \subseteq R$. For a $\mathbf{k}$-algebra A, let

$$G_i(A) = \{t_i \mapsto g_{i1}t_i + g_{i2}t_i^2, t_j \mapsto t_j \mid g_{i1} \in A^\times, g_{i2} \in A\},$$

and notice that

$$\mathrm{Aut}_{(\widetilde{R},\widetilde{\mathfrak{m}})}^{\mathrm{mod}\ \widetilde{\mathfrak{m}}^3}(A) = (G_1 \times \ldots \times G_m) \rtimes \mathfrak{S}_m(A).$$

Consider now the action of a group element of the form $(g_1, \ldots, g_m; \mathrm{id}_{\mathfrak{S}_m})$ on the given generators of R:

$$\begin{aligned} x_1 &\mapsto g_{11}t_1 + g_{12}t_1^2 \oplus \ldots \oplus g_{m1}t_m + g_{m2}t_m^2; \\ x_i &\mapsto \ldots \oplus g_{i1}t_i + g_{i2}t_i^2 \oplus \ldots \oplus g_{m1}^2 t_m^2, \quad \text{for } i = 2, \ldots, m-1. \end{aligned}$$

The former belongs to R iff $g_{11} = g_{m1}$ and $g_{12} = g_{m2}$; the latter does iff $g_{i1} = g_{m1}^2$. Thus, such elements span a subgroup isomorphic to $\mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{a}^{m-1}(A)$. On the other hand, all branches are isomorphic to one another, but there is a pair of distinguished ones (parametrised by t_1 and t_m respectively). We conclude that

$$\mathrm{Aut}_{\tau_0}^{\mathrm{mod}\ \widetilde{\mathfrak{m}}^3}(A) = (\mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{a}^{m-1}) \rtimes (\mathfrak{S}_2 \times \mathfrak{S}_{m-2})(A).$$

The quotient is then isomorphic to $\binom{m}{2}$ copies of $\mathbb{A}^1 \times (\mathbb{A}^1 \setminus \{0\})^{m-1}$.

Type III: in this case $\widetilde{\mathfrak{m}}^4 \subseteq R$. For a $\mathbf{k}$-algebra A, let

$$G_i(A) = \{t_i \mapsto g_{i1}t_i + g_{i2}t_i^2 + g_{i3}t_i^3, t_j \mapsto t_j \mid g_{i1} \in A^*, g_{i2}, g_{i3} \in A\},$$

and notice that

$$\mathrm{Aut}_{(\widetilde{R},\widetilde{\mathfrak{m}})}^{\mathrm{mod}\ \widetilde{\mathfrak{m}}^4}(A) = (G_1 \times \ldots \times G_m) \rtimes \mathfrak{S}_m(A).$$

Consider now the action of a group element of the form $(g_1, \ldots, g_m; \mathrm{id}_{\mathfrak{S}_m})$ on the given generators of R:

$$\begin{aligned} x_i &\mapsto \ldots \oplus g_{i1}t_i + g_{i2}t_i^2 + g_{i3}t_i^3 \oplus \ldots \oplus g_{m1}^3 t_m^3, \quad \text{for } i = 1, \ldots, m-1; \\ x_m &\mapsto \ldots \oplus g_{m1}^2 t_m^2 + 2g_{m1}g_{m2}t_m^3. \end{aligned}$$

The former belongs to R iff $g_{i1} = g_{m1}^3$; the latter does iff $g_{m2} = 0$. Thus such elements span a subgroup isomorphic to $\mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{a}^{m-1} \times \mathbb{G}_\mathrm{a}^m(A)$. On the other hand, there is a special (singular) branch, parametrised by t_m. We conclude that

$$\mathrm{Aut}_{\tau_0}^{\mathrm{mod}\ \widetilde{\mathfrak{m}}^3}(A) = (\mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{a}^{m-1} \times \mathbb{G}_\mathrm{a}^m) \rtimes (\mathfrak{S}_{m-1})(A).$$

The quotient is therefore isomorphic to m copies of $\mathbb{A}^1 \times (\mathbb{A}^1 \setminus \{0\})^{m-1}$. $\square$

It is now clear that the benefit of a two-step classification where at first we do not allow ourselves to change coordinates (i.e. act by automorphisms of the normalisation) is that it makes the crimping space apparent already from the expressions (1) and (3) for the generators of the singularity subalgebra.

There is a more geometric way to see the crimping spaces. It is well-known that a cusp can be obtained by collapsing (*push-out*) a generic (i.e. non-zero) tangent vector at $p \in \mathbb{P}^1$. More generally, a Gorenstein singularity of genus one and m branches can be obtained by collapsing a generic (not contained in any coordinate linear subspace) tangent line at an ordinary (i.e. rational) m-fold point [Smy11a, Lemma 2.2]. Hence, we recover the crimping space of the elliptic m-fold point, which is isomorphic to $(\mathbb{A}^1 \setminus \{0\})^{m-1}$, as the maximal torus inside $\mathbb{P}(T_pR_m) \simeq \mathbb{P}^{m-1}$, where (R_m, p) is the rational m-fold point. Besides, this gives a natural compactification of the crimping space supporting a universal family of curves - in fact, two: either we collapse non-generic tangent vectors, obtaining non-Gorenstein singularities along the boundary (this family $\mathscr{C}$ has a common normalisation, that is the trivial family $\widetilde{\mathscr{C}} = R_m \times \mathbb{P}(T_pR_m)$); or we blow-up $\widetilde{\mathscr{C}}$ along the boundary (*sprouting*), and we replace the non-Gorenstein singularities by more elliptic m-fold points, this time with strictly semistable branches [Smy11b, §2.2-3].

Similarly, a Gorenstein singularity of genus two can be obtained by collapsing a generic tangent line to a genus one non-Gorenstein singularity. Indeed, type τ_0^{II} admits a partial normalisation by σ_0^{II}, which is the decomposable union of a tacnode in the (t_1, t_m)-plane together with $m-2$ axes, by adjoining the generator t_m^2; while type τ_0^{III} by σ_0^{III}, which is the decomposable union of a cusp (parametrised by t_m) together with $m-1$ axes, by adjoining t_m^3. These fit together nicely in the following picture: if we restrict $\mathscr{C}$ from the previous paragraph to the union of the coordinate lines in $\mathbb{P}(T_pR_m)$, we obtain m copies of σ_0^{III} over the points, together with $\binom{m}{2}$ copies of the universal curve of type σ_0^{II} over its crimping space - which is isomorphic to $\mathbb{A}^1 \setminus \{0\}$ -, identified with the open lines. Let $P = \mathbb{P}(T_p\mathscr{C}_{|\cup \text{lines}})$ be the projectivised tangent space at the singular point in the fiber of this family of genus one singularities. For each of the $\binom{m}{2}$ coordinate lines, P has one component P_i^{II} that is a $\mathbb{P}^{m-1}$-bundle over such line; besides, P has m components P_j^{III} isomorphic to $\mathbb{P}^m$ and supported over the points. The crimping space of the genus two singularities (of type II and III together) with m branches can be seen to be an open subscheme of P: it is obtained by removing from the $\mathbb{P}^{m-1}$-fibers of P^{II} the $m-1$ hyperplanes generated by (a) the tangent line to the tacnode and the $m-2$ axes, and (b) the plane containing the tacnode and all but one of the $m-2$ axes; and from each component of P^{III} the m planes generated by (a) the tangent cone of the cusp and the $m-1$ axes, and (b) the plane containing the cusp and all but one of the $m-1$ axes.

Remark 3.3. The crimping space is related to the moduli of arrows ϕ as in the diagram of Lemma 3.1, satisfying a number of requirements.

We note that $H^0(\Omega^\vee_{\mathbb{P}^1}(-p)_{|2p})$ is the tangent space to the subgroup of automorphisms of $\mathbb{P}^1$ fixing one point p, so it is isomorphic as a Lie algebra to the only non-abelian Lie algebra of dimension two V. It has a basis $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ with $[e_1, e_2] = -2e_2$. The vector (φ, ψ) therefore corresponds to the infinitesimal automorphism:

$$t \mapsto \frac{1+\epsilon\varphi t}{1-\epsilon(\varphi t + \psi)} = t + \epsilon(2\varphi t - \psi t^2).$$

We require ϕ to be the embedding (i.e. a point of $\mathrm{Gr}(m, V^{\oplus m})$) of a Lie subalgebra, such that the corresponding group of infinitesimal automorphisms fixes uniquely the subalgebra of a singularity of genus two in $\mathbf{k}[[t_1]] \oplus \ldots \oplus \mathbf{k}[[t_m]]$.

We start with some heuristics. The unibranch case goes as follows: the subalgebra of $\mathbf{k}[[t]]$ generated by $x = t^2 + ct^3$ is preserved by (φ, ψ) iff

$$(1+2\varphi)^2 t^2 - 2\psi(1+2\varphi)t^3 + c(1+2\varphi)^3 t^3 \text{ is a multiple of } t^2 + ct^3,$$

which reduces to $\varphi(1+2\varphi)c = \psi$. This furthermore determines c iff $\varphi \neq 0$. Note that in this case (dimension one) the Lie subalgebra condition is automatically satisfied. We have found $(\varphi, \psi) \in \mathbf{k}^\times \times \mathbf{k}$.

The case of type II_2-algebras is more interesting. Let $x = (t_1, \alpha t_2 + \beta t_2^2)$ be the generator of such an algebra. The image of x under $(\varphi_1, \psi_1, \varphi_2, \psi_2)$ is:

$$\left((1+2\varphi_1)t_1 - \psi_1 t_1^2, \alpha(1+2\varphi_2)t_2 + (\beta(1+4\varphi_2) - \alpha\psi_2)t_2^2\right),$$

from which we deduce:

$$\varphi_1 = \varphi_2 \quad \text{and} \quad 2\beta\varphi_2 - \alpha\psi_2 = -\alpha^2\psi_1. \tag{5}$$

Now let $\phi\colon \mathbf{k}^2 \to V^{\oplus 2}$ be given by $\begin{pmatrix} \varphi_{11} & \psi_{11} & \varphi_{12} & \psi_{12} \\ \varphi_{21} & \psi_{21} & \varphi_{22} & \psi_{22} \end{pmatrix}$, with Plücker coordinates w_{ij} for the minor of the i-th and j-th columns. The first condition in (5) immediately implies

$$w_{13} = 0 \quad \text{and} \quad w_{12} = -w_{23}, w_{14} = w_{34}. \tag{6}$$

The second condition in (5) implies

$$\alpha w_{12} = w_{14} \quad \text{and} \quad 2\beta w_{12} = \alpha w_{24} \tag{7}$$

so that $(\alpha, \beta) \in \mathbf{k}^\times \times \mathbf{k}$ is determined as soon as $w_{12}, w_{14} \neq 0$. Notice that (6) implies in particular the Plücker equation

$$w_{12}w_{34} - w_{13}w_{24} + w_{14}w_{23}.$$

It is easy to see that the condition for ϕ to be a sub-Lie algebra is

$$\operatorname{rk}\begin{pmatrix}\varphi_{11} & \psi_{11} & \varphi_{12} & \psi_{12}\\ \varphi_{21} & \psi_{21} & \varphi_{22} & \psi_{22}\\ 0 & w_{12} & 0 & w_{34}\end{pmatrix} = 2,$$

translating into

$$\begin{aligned} w_{12}w_{13} &= 0 & \qquad w_{12}(w_{34}-w_{14}) &= 0\\ w_{13}w_{34} &= 0 & \qquad (w_{12}+w_{23})w_{34} &= 0 \end{aligned}$$

which also are automatically satisfied after (6). We see the various equations above cut inside $\mathbb{P}^5_{[w_{ij}]}$ the locus

$$(\mathbb{A}^1_{w_{14}/w_{12}} \setminus \{0\}) \times \mathbb{A}^1_{w_{24}/w_{12}}.$$

More generally, given a subalgebra $R_{\alpha,\beta}$ of $W = \bigoplus_{i=1}^m \mathbf{k}[[t_i]]/(t_i^2)$, with generators of the form described in (1), the subalgebra of $V^{\oplus m}_{(\varphi_i,\psi_i)_{i=1,\dots,m}}$ preserving $R_{\alpha,\beta}$ is isomorphic to $\mathbf{k}^{\oplus m}$ with equations (see Lemma 3.1):

$$\begin{cases} 2\varphi_1 = \varphi_i = 2\varphi_m, & \text{for } i = 2,\dots,m-1,\\ 2\beta_{1,m}\varphi_m - \alpha_{1,m}\psi_m = -\alpha_{1,m}^2\psi_1; \end{cases}$$

it is easily seen that such a subalgebra of $V^{\oplus m}$ does not determine $R_{\alpha,\beta}$, but it does determine $(\alpha_{1,m},\beta_{1,m})$. The situation of type III is analogous.

We are going to use the preceding discussion in order to study the automorphism group of complete marked curves with a genus two singularity. The only case that actually requires such a discussion is when every component of the normalisation is rational and contains one extra marking, besides the preimage of the singularity: in this case it does make a difference what point of the crimping space we are looking at. The concept has been formalised again in van der Wyck's thesis, see [vdW10, Proposition 1.102, Theorem 1.105 and Corollary 1.106], where he introduces the concept of reduced pointed curve with resolution of type T (encoding the amount and type of the singular points, the distribution of genus and markings among the components of the normalisation, and the adjacency data between components and singular points), and the algebraic stack $\mathscr{N}_T$ of such objects. In the case at hand, such stack is isomorphic to $[\mathbb{A}^1/\mathbb{G}_\mathrm{m}]$ (see also [vdW10, Examples 1.111-112]), and it therefore has two points: one with $\mathbb{G}_\mathrm{m}$, and the other with trivial stabiliser.

Definition 3.4. The *atom* of type II_m is obtained by gluing the subalgebra of $\mathbf{k}[t_1] \oplus \ldots \oplus \mathbf{k}[t_m]$ generated by $x_1,\dots,x_{m-1}$ (and y) as in (2) (and following lines) with m copies of $(\mathbf{k}[s],(s))$ under the identification $s_i = t_i^{-1}$. There is a $\mathbb{G}_\mathrm{m}$-action on the type II atom by $\lambda.t_i = \lambda t_i$ for $i = 1, m$ and $\lambda.t_i = \lambda^2 t_i$ for $i = 2,\dots,m-1$.

Similarly, the atom of type III_m is obtained by gluing the subalgebra of $\mathbf{k}[t_1] \oplus \ldots \oplus \mathbf{k}[t_m]$ generated by $x_1,\dots,x_m$ as in (4) with m copies of $(\mathbf{k}[s],(s))$ under the identification $s_i = t_i^{-1}$. There is a $\mathbb{G}_{\mathrm{m}}$-action on the type III atom by $\lambda.t_i = \lambda t_i$ for $i = m$ and $\lambda.t_i = \lambda^3 t_i$ for $i = 1, \ldots, m-1$.

The curve with a genus two singularity and one marked point for every branch that has trivial automorphism group will be called the *non-atom*.

Following the previous discussion, there is a more geometric way to realise the dicotomy between the atom and the non-atom. The non-Gorenstein genus one singularity of type σ_0^{II} (resp. σ_0^{III}), with every branch rational and one-marked, has automorphism group $\mathbb{G}_{\mathrm{m}}^{m-1}$ (resp. $\mathbb{G}_{\mathrm{m}}^{m}$); the latter group therefore acts on the tangent space at the singular point, and it can be checked that of the lines fixed by this action only one sits inside the open subset corresponding to the crimping space, while all other lines in the open are identified under the group action - by collapsing them, they give rise to the atom and non-atom respectively.

As a third viewpoint, automorphisms can be studied by twisting the exact sequences of Lemma 3.1 by the ideal of the markings and then taking global sections. It appears that the dicotomy arises from the map ϕ: if the last condition imposed on infinitesimal automorphisms interweaves first and second order non-trivially (i.e. when $\beta_{1,m}$, resp. $\gamma_{m,m}$, are non-zero) then it is enough that automorphisms be trivial to second order on all branches for them to be trivial tout-court.

Finally, we shall turn the condition that the automorphism group be finite into a combinatorial one. For this, recall Smyth's description of Gorenstein curves of genus one with no automorphisms [Smy11a, Proposition 2.3, Corollary 2.4].

Definition 3.5. Let $(C, p_1, \ldots, p_n)$ be a pointed reduced curve. A connected subcurve $D \subseteq C$ is said to be *nodally attached* if $D \cap \overline{C \setminus D}$ consists of nodes only. We say that C is *residually DM* (rDM) if every nodal and nodally attached subcurve D of C, marked by $\{p_i \in D\} \cup D \cap \overline{C \setminus D}$, is Deligne-Mumford stable.

Special points are either nodes or markings.

Corollary 3.6. *Let $(C, p_1 \ldots, p_n)$ be a pointed Gorenstein curve of arithmetic genus two. $H^0(C, \Omega_C^\vee(-\sum_{i=1}^n p_i)) = 0$ is equivalent to either of the following:*

(1) C has a singularity of type $II_{m\geq 2}$: either all branches contain exactly one special point and C is not atomic; or at least one of its twin branches contains a special point, each of its axes contains at least one special point, and at least one branch has at least two. Furthermore C is rDM.

(2) C has a singularity of type $III_{m\geq 1}$: either all branches contain exactly one special point and C is not atomic; or each of its axes contains at least one special point, and at least one branch has at least two. Furthermore C is rDM.

(3) C has two elliptic m-fold points: each of their branches contains at least one special point, and either they share a branch, or at least one branch of each singular point contains at least two special points. Furthermore C is rDM.

(4) C has one elliptic m-fold point: one of its branches is a genus one curve, or two of its branches coincide, and each of the other ones contains at least one

special point; otherwise, all branches contain at least one special point, and at least one branch has at least two. Furthermore C is rDM.

(5) C contains only nodes and is Deligne-Mumford stable.

4. Dualising line bundle and semistable tails

Given a family of prestable (pointed) curves of genus two over the spectrum of a discrete valuation ring $\mathscr{C} \to \Delta$, with smooth generic fiber $\mathscr{C}_\eta$ and regular total space, we classify the subcurves of the central fiber $\mathscr{C}_0$ that can be contracted to produce a Gorenstein singularity of genus two. In the genus one case, Smyth answered the analogous question by identifying the class of *balanced* subcurves: subcurves of arithmetic genus one, such that, when breaking them into a *core* (minimal subcurve of genus one, i.e. not containing any separating node) and a number of rational trees (with root corresponding to the component adjacent to the core, and leaves corresponding to the components adjacent to the portion of $\mathscr{C}_0$ that is not contracted), the distance between any leaf and the root for any such tree is constant. In the case at hand, the answer turns out to be slightly more complicated: first, the special branch(es) of a type I (resp. II) singularity correspond through a rational chain to a Weierstrass (resp. two conjugate) point(s) of the core, and the special branches are always the closest to the core. Second, if the core is reducible, the lengths of the rational trees may vary according to where their attaching points lie on the core, but they are determined by the length of the special chains and the configuration of the attaching points on the core.

Remark 4.1. While there are no special points on a smooth curve of genus zero or one, the simplest instance of Brill-Noether theory involves smooth curves of genus two. Every such C is *hyperelliptic*: it admits a unique (up to reparametrisation) $2:1$ cover $\phi\colon C \to \mathbb{P}^1$, induced by the complete canonical linear system, i.e. $|K_C|$ is the unique $\mathfrak{g}^1_2$ on C; said otherwise, there is a unique element $\sigma \in \mathrm{Aut}(C)$, called the *hyperelliptic involution*, such that $C/\langle\sigma\rangle \simeq \mathbb{P}^1$. A point $x \in C$ is called *Weierstrass* if it is a ramification point for ϕ (or, equivalently, a fixed point for σ); from the Riemann-Hurwitz formula it follows that there are six Weierstrass points on every smooth curve of genus two. Two points x_1, x_2 are said to be conjugate (write $x_2 = \overline{x_1}$) if there exists a point $z \in \mathbb{P}^1$ such that $\phi^{-1}(z) = \{x_1, x_2\}$ (or, equivalently, $\sigma(x_1) = x_2$). These notions may be extended to nodal curves by declaring (C, x) to be Weierstrass if its stabilisation lies in the closure of

$$\mathscr{W} = \{(C,x)|\ C \text{ smooth and } x \text{ Weierstrass}\} \subseteq \overline{\mathcal{M}}_{2,1},$$

and similarly for conjugate points. We then need to study the limiting behaviour of Weierstrass points when a smooth curve degenerates to a nodal one. This is a difficult problem when it comes to higher genus curves; it has received considerable attention since the '70s, in work of Arbarello, Eisenbud-Harris, and many others. In our case it boils down to understanding admissible covers [HM82] of degree two with a branch locus of degree six; said otherwise, up to the involution action, the Weierstrass locus is isomorphic to $\overline{\mathcal{M}}_{0,6}/\mathfrak{S}_5$, and the conjugate locus is isomorphic

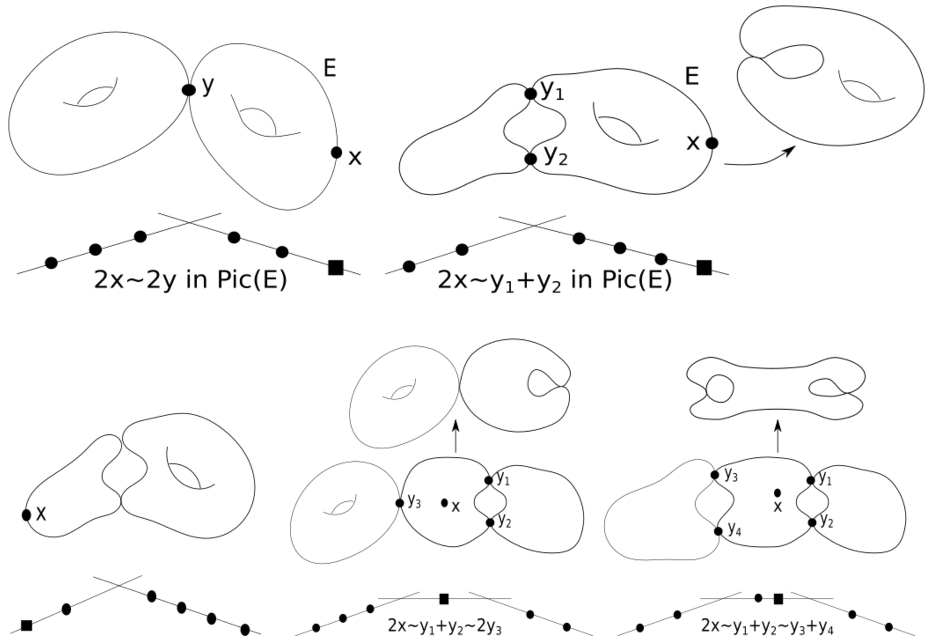


FIGURE 1. Admissible covers and Weierstrass points.

to $\overline{\mathcal{M}}_{0,7}/\mathfrak{S}_6$. We remark that (C, x) being Weierstrass is an intrinsic notion if C is of compact type (or, more generally, tree-like), but it may depend on the smoothing otherwise (i.e. the fiber of $\overline{\mathscr{W}} \to \overline{\mathcal{M}}_2$ may have positive dimension); we have benefitted from the exposition in [Dia85, Appendix 2], [Cuk89, Proposition (3.0.6)], and [HM98, Theorem 5.45].

- If x belongs to a component of genus one E, which is attached to another component of genus one at a node y, then x is Weierstrass iff $2x \sim 2y \in \mathrm{Pic}(E)$; if instead E has a self-node that glues y_1 with y_2, then x is Weierstrass iff $2x \sim y_1 + y_2 \in \mathrm{Pic}(E)$.

  If x is on a rational component R, x is Weierstrass if either R is attached to a genus one curve at two distinct points, or R has a self-node gluing y_1 and y_2 and is attached to a genus one tail at y_3, in which case we require $\phi(y_1) = \phi(y_2)$ for a double cover $\phi\colon R \to \mathbb{P}^1$ ramified at x and y_3, or R has two self-nodes gluing y_1 with y_2, and y_3 with y_4, in which case we require x to be a ramification point for a double cover $\phi\colon R \to \mathbb{P}^1$ such that $\phi(y_1) = \phi(y_2)$ and $\phi(y_3) = \phi(y_4)$. See Figure 1.
- If x_1 and x_2 are conjugate, they have to map to the same component of the target of the admissible cover. The description of the previous point works by replacing every condition on $2x$ by its analogue for $x_1 + x_2$. There are a few more situations to take into account: x_1 and x_2 could belong to a rational component R bubbling off from a Weierstrass point of a genus two curve; or

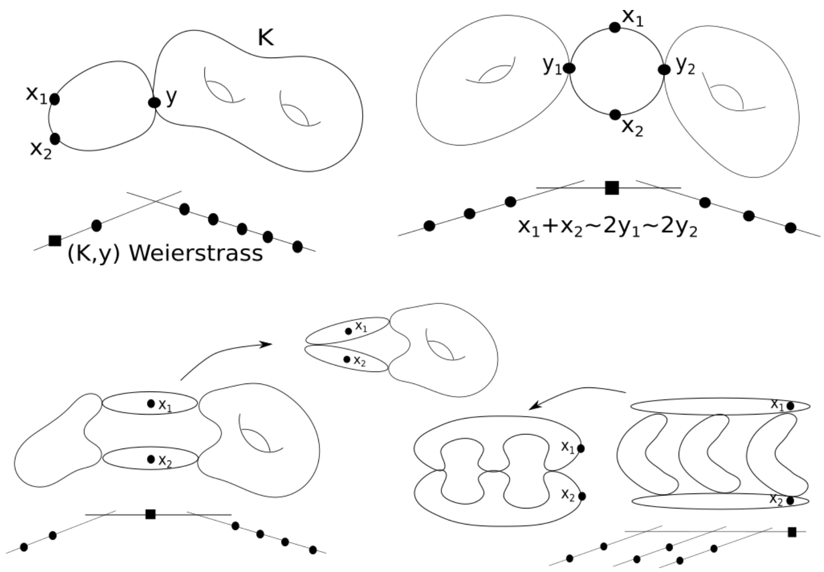


FIGURE 2. Admissible covers and conjugate points.

bridging between two distinct curves of genus one; or x_1 and x_2 could lie on two distinct rational components R_1 and R_2 intersecting at one node and meeting a curve of genus one in two distinct points (†); or R_1 and R_2 intersecting each other in three points. See Figure 2.

We observe here that in case (†), the singularity of the total space of a smoothing $\mathscr{C} \to \Delta$ at the two distinguished nodes (separating the elliptic component from the rational chain) are both A_k for the same k, because they map to the same node of the target in the admissible cover picture. This fact is stable under base change, and it determines a symmetry of the rational chain in the model with regular total space.

Proposition 4.2. *Let $\phi\colon \mathscr{C} \to \overline{\mathscr{C}}$ be a contraction over the spectrum of a discrete valuation ring Δ, where: $\mathscr{C} \to \Delta$ is a family of prestable (reduced, nodal) curves of arithmetic genus two, with regular total space and smooth generic fiber $\mathscr{C}_\eta$; and $\overline{\mathscr{C}} \to \Delta$ is a family of Gorenstein curves of arithmetic genus two, with $\overline{\mathscr{C}}_\eta$ smooth, and $\overline{\mathscr{C}}_0$ containing a genus two singularity of type I_m at q. Denote by $(Z; q_1, \ldots, q_m)$ the exceptional locus $\mathrm{Exc}(\phi) = \phi^{-1}(q)$, marked with $Z \cap \overline{\mathscr{C}_0 \setminus Z}$, where q_m corresponds to the special branch of $\overline{\mathscr{C}}_0$. Then:*

(1) The stabilisation of (Z, q_m) is Weierstrass.
(2) Let x_m be the point of the core of Z closest to q_m, and let k be the length of R_m, the rational chain separating q_m from x_m. With similar notation, for

every $i = 1, \dots, m-1$, R_i has length

$$\begin{cases} 3k + 1 + \operatorname{dist}(x_m, x_i) & \text{if } x_i \neq x_m, \\ k + 2\operatorname{dist}(q_m, r_i) & \text{if } x_i = x_m, \end{cases}$$

where r_i is the point of R_m closest to q_i, and $\operatorname{dist}(a,b)$ is the number of irreducible components between the points a and b (so for example it is 1 if a and b lie on the same irreducible component but $a \neq b$).

Proposition 4.3. *Same as above with $\overline{\mathscr{C}}_0$ containing a genus two singularity of type II_m, and q_1, q_m corresponding to the special branches. Then:*

(1) The stabilisation of (Z, q_1, q_m) is conjugate.
(2) R_1 and R_m have the same length k, and, for $i = 2, \dots, m-1$, R_i has length

$$\begin{cases} 2k + \min_{\epsilon \in \{1,m\}} \operatorname{dist}^*(x_\epsilon, x_i) & \text{if } x_1 \neq x_m, \text{ and } x_i \notin \{x_1, x_m\}, \\ k + \operatorname{dist}(q_1, r_i) & \text{if } x_1 \neq x_m, \text{ and } x_i = x_1 (+\text{symm. } 1 \leftrightarrow m), \\ 2k + \operatorname{dist}(x_1, r_m) + \operatorname{dist}(x_1, x_i) & \text{if } x_1 = x_m, \text{ and } x_i \neq x_1, \\ k + \operatorname{dist}(q_1, r_i) + \delta(\operatorname{dist}(r_m, r_i) - 1) & \text{if } x_1 = x_m = x_i, \text{ and } \delta = \begin{cases} 1 & r_m \in [q_1, r_i] \\ 0 & \text{otherwise} \end{cases} \end{cases}$$

where we set $\operatorname{dist}^*(x_\epsilon, x_i) = 1$ if the core consists of a genus one curve with a rational bridge, and x_i lies between x_1 and x_m on this rational bridge, and $\operatorname{dist}^*(x_\epsilon, x_i) = \operatorname{dist}(x_\epsilon, x_i)$ otherwise.

Proposition 4.4. *Let $(\mathscr{C}, \Sigma_1, \dots, \Sigma_n) \to \Delta$ be a family of pointed semistable curves of arithmetic genus two such that $\mathscr{C}$ has regular total space and smooth generic fiber, and $(\mathscr{C}, \Sigma_1) \to \Delta$ is Weierstrass. Let $(Z, q_1, \dots, q_m)$ be a genus two subcurve of $\mathscr{C}_0$ containing none of the $\Sigma_i(0)$, marked by $Z \cap \overline{\mathscr{C}_0 \setminus Z}$ so that the tail containing Σ_1 is attached to Z at q_1, and satisfying all the shape prescriptions of Proposition 4.2(2). There exists a contraction $\phi \colon \mathscr{C} \to \overline{\mathscr{C}}$ over Δ, with exceptional locus Z, such that $\overline{\mathscr{C}} \to \Delta$ is a family of Gorenstein curves containing a type I_m singularity in the central fiber.*

Proposition 4.5. *Same as above with $(\mathscr{C}, \Sigma_1, \Sigma_2) \to \Delta$ conjugate, $(Z, q_1, \dots, q_m)$ shaped as prescribed by Proposition 4.3(2), and the resulting $\overline{\mathscr{C}} \to \Delta$ containing a type II_m singularity in the central fiber.*

Proof. (of Proposition 4.2) By blowing down all the rational trees on $\overline{\mathscr{C}}_0$, we can assume that the latter does not contain any separating node. Consider then the hyperelliptic cover $\tau \colon \overline{\mathscr{C}} \to \mathbb{P}(\bar{\pi}_* \omega_{\overline{\mathscr{C}}/\Delta})$; restricting to the central fiber, τ contracts all axes, and gives a $2:1$ covering of $\mathbb{P}^1$ by the special branch, ramified at the singularity and at another point; in fact, we can extend the image of this point to a section of $\mathbb{P}(\bar{\pi}_* \omega_{\overline{\mathscr{C}}/\Delta})$ lying inside the branch locus of τ. By pulling this back to $\mathscr{C}$ via $\tau \circ \phi$ we get a horizontal divisor Δ'; clearly, the stable model of $(\mathscr{C}, \Delta')$ is Weierstrass, and its central fiber coincides with the stabilisation of (Z, q_1). This proves the first claim. (The proof of Proposition 4.3(1) is entirely analogous, by noticing that the preimage of a generic hyperplane section of $\mathbb{P}(\bar{\pi}_*\omega_{\overline{\mathscr{C}}/\Delta})$ will mark the two special branches of $\overline{\mathscr{C}}_0$.)

We now come to a proof of the more combinatorial claim (2) of the Proposition. Since $\overline{\mathscr{C}} \to \Delta$ is Gorenstein, and ϕ is assumed to be an isomorphism outside Z, because the dualising sheaf behaves well under restriction to open subschemes, we have an equality of line bundles:

$$\phi^*\omega_{\overline{\mathscr{C}}/\Delta} = \omega_{\mathscr{C}/\Delta}(D),$$

for some effective (Cartier) divisor D on $\mathscr{C}$ supported on Z. The next lemma will help us determine te coefficients of D along the components of Z containing q_i.

Lemma 4.6. *Let $\nu\colon C \to \bar{C}$ be the normalisation of a Gorenstein singularity of genus two, with $\nu^{-1}(q) = \{q_1, \ldots, q_m\}$. Then $\nu^*\omega_{\bar{C}} = \omega_C(2q_1 + \ldots + 2q_{m-1} + 4q_m)$ (type I) or $\nu^*\omega_{\bar{C}} = \omega_C(3q_1 + 2q_2 + \ldots + 2q_{m-1} + 3q_m)$ (type II).*

Proof. The dualising sheaf of a reduced curve admits an explicit description (due to Rosenlicht, see e.g. [AK70, Proposition VIII.1.16]) in terms of residues:

$$\omega_{\bar{C}}(U) = \{\eta \in \Omega_C \otimes K(\nu^{-1}(U)) | \sum_{p_i \in \nu^{-1}(p), p \in U} \mathrm{Res}_{p_i}((\nu^* f)\eta) = 0,\ \forall f \in \mathscr{O}_{\bar{C}}(U)\}.$$

We are going to use the explicit coordinates in (1) and (3). In case I, we know that $\widetilde{\mathfrak{m}}^4 \subseteq R$, therefore we have poles of fourth order at most. It is enough to study the possible polar tails. On the other hand, $t_i^2 \in R$ for all i implies the part of order three is trivial. So let

$$\eta = c_1 \frac{\mathrm{d}\, t_1}{t_1^4} + b_1 \frac{\mathrm{d}\, t_1}{t_1^2} + a_1 \frac{\mathrm{d}\, t_1}{t_1} \oplus \ldots \oplus c_m \frac{\mathrm{d}\, t_m}{t_m^4} + b_m \frac{\mathrm{d}\, t_m}{t_m^2} + a_m \frac{\mathrm{d}\, t_m}{t_m}.$$

From looking at $1 \cdot \eta$ we deduce $\sum_{i=1}^m a_i = 0$; from $x_i \cdot \eta$ we see $b_i + c_m = 0$ for all i, and from $x_i^3 \cdot \eta$ we have $c_i = 0$ for all i. (The statement about third order poles can be evinced from $x_i^2 \cdot \eta$ or from $z \cdot \eta$ indifferently.) Therefore $\omega_C/\nu_*\omega_{\tilde{C}}$ is spanned by

$$\frac{\mathrm{d}\, t_1}{t_1} - \frac{\mathrm{d}\, t_m}{t_m}, \ldots, \frac{\mathrm{d}\, t_{m-1}}{t_{m-1}} - \frac{\mathrm{d}\, t_m}{t_m}, \frac{\mathrm{d}\, t_m}{t_m^2}$$
$$\bar{\eta} = \frac{\mathrm{d}\, t_1}{t_1^2} + \ldots + \frac{\mathrm{d}\, t_{m-1}}{t_{m-1}^2} - \frac{\mathrm{d}\, t_m}{t_m^4}.$$

In particular ω_C is generated by $\bar{\eta}$ as an $\mathscr{O}_C$-module.

In case II, we know that $\widetilde{\mathfrak{m}}^3 \subseteq R$, so we have poles of third order at most. Let

$$\eta = c_1 \frac{\mathrm{d}\, t_1}{t_1^3} + b_1 \frac{\mathrm{d}\, t_1}{t_1^2} + a_1 \frac{\mathrm{d}\, t_1}{t_1} \oplus \ldots \oplus c_m \frac{\mathrm{d}\, t_m}{t_m^3} + b_m \frac{\mathrm{d}\, t_m}{t_m^2} + a_m \frac{\mathrm{d}\, t_m}{t_m}.$$

From looking at $1 \cdot \eta$ we deduce $\sum_{i=1}^m a_i = 0$; from $x_i \cdot \eta$ we see $b_1 + b_m = 0$ (if $i = 1$), and $b_i + c_m = 0$ (if $i = 2, \ldots, m-1$); finally from $x_i^2 \cdot \eta$ we have $c_1 + c_m = 0$

(if $i=1$), and $c_i = 0$ (if $i = 2, \dots, m-1$). Therefore $\omega_C/\nu_*\omega_{\tilde{C}}$ is spanned by

$$\frac{\mathrm{d}\, t_1}{t_1} - \frac{\mathrm{d}\, t_m}{t_m}, \dots, \frac{\mathrm{d}\, t_{m-1}}{t_{m-1}} - \frac{\mathrm{d}\, t_m}{t_m}, \frac{\mathrm{d}\, t_1}{t_1^2} - \frac{\mathrm{d}\, t_m}{t_m^2},$$
$$\bar{\eta} = \frac{\mathrm{d}\, t_1}{t_1^3} + \frac{\mathrm{d}\, t_2}{t_2^2} + \ldots + \frac{\mathrm{d}\, t_{m-1}}{t_{m-1}^2} - \frac{\mathrm{d}\, t_m}{t_m^3}.$$

In particular ω_C is generated by $\bar{\eta}$ as an $\mathscr{O}_C$-module.

□

Corollary 4.7. *The dualising sheaf has multi-degree* $(0, \dots, 0, 2)$ *(case I) and* $(1, 0, \dots, 0, 1)$ *(case II) respectively.*

Remark 4.8. It follows from this computation and Corollary 3.6 that the finiteness condition on automorphism groups, $H^0(\bar{C}, \Omega^\vee_{\bar{C}}(-\sum_{i=1}^n p_i)) = 0$, implies ampleness of $\omega_{\bar{C}}(\sum_{i=1}^n p_i)$.

Let us now go back to the proof of Proposition 4.2. Because $\phi_{|\mathscr{C}_0\setminus Z}$ is the normalisation of $\overline{\mathscr{C}}_0$ at q, and, letting T_i be the tail of $\mathscr{C}_0 \setminus Z$ attached to Z at q_i, we know that $\omega_{\mathscr{C}/\Delta|T_i} = \omega_{T_i}(q_i)$ by adjunction, it follows from Lemma 5.3 that D has multiplicity 3 at the component of Z containing q_m in case I (resp. 2 at the components containing q_1 and q_m in case II), and 1 at all other components containing a q_i. Set $\mathscr{L} = \omega_{\mathscr{C}/\Delta}(D) = \phi^*\omega_{\overline{\mathscr{C}}/\Delta}$; we shall analyse the consequences of $\mathscr{L}_{|Z} = \mathscr{O}_Z$. We think of Z as being the union of a core K and a number ($\leq m$ by semistability) of rational trees.

Let d_A denote the multiplicity of the divisor D along the component A of Z. First, we claim that no component can appear with $d_A = 0$. Assume that this occurred along one of the rational trees. Call S ($S \simeq \mathbb{P}^1$) a component furthest from the core such that $d_S = 0$; R the one that precedes it, and $T_1, \dots, T_h$ the ones that follow it (when sweeping the tree from the core) - so that $h \geq 1$ by the previous paragraph, and $d_{T_i} \geq 1$ by inductive assumption. Then, by adjunction,

$$\deg(\mathscr{L}_{|S}) = -2 + (h+1) + d_R + \sum d_{T_i} = 0,$$

which necessarily implies $h = 1$, and $d_R = d_{T_i} = 0$, contradicting the assumption. The case that S belongs to the core is similar (ω_S might only be more positive).

Let us now consider $d_S = 1$. We stick to the notation above; furthermore, there may be a number k of q_i, $i \in \{(1,)2, \dots, m-1\}$ in case II (resp. I), lying on S. Then, again by adjunction,

$$\deg(\mathscr{L}_{|S}) = -2 + d_R + \sum d_{T_i} = 0.$$

so either $d_R = 2$, $h = 0$ and $k \geq 1$ arbitrary, i.e. S is adjacent to $\overline{C \setminus Z}$; or $d_R = 1$, $h = 1$, and $d_{T_1} = 1$ (with k arbitrary). In the latter case, though, by repeating the same argument on T_1 etc., we would find an infinite chain in Z.

Remark 4.9. More generally, an analogous computation shows that, when balancing a component A of multiplicity d_A, all neighbouring components of multiplicity $d_A - 1$ can be safely ignored (at the same time, the number of such components is bounded only by m, due to the semistability of Z).

We now prove that $d_R > d_S$ holds in general for S on a rational tree. The preceding paragraphs deal with the cases $d_S = 0, 1$; we may therefore assume $d_S > 1$ (which in particular implies $0 \leq k \leq 2$). We have

$$\deg(\mathscr{L}_{|S}) = -2 + d_R - (d_S - 1)(h + k + 1) + \sum d_{T_i} = 0.$$

By proceeding inductively from leaves to root, we can assume that $d_S > d_{T_i}$, $i = 1, \ldots, h$. We may therefore rewrite the previous equality as

$$d_R = (d_S-1)(h+k+1)-\sum d_{T_i}+2 \geq (d_S-1)(k+1)+2 = d_S+1+k(d_S-1) > d_S.$$

In fact, we can prove as on [Smy11a, p.893] that $d_R = d_S + 1$, unless $d_S = 3$ and $q_m \in S$ (type I), or $d_S = 2$ and either q_1 or q_m (or both) are on S (type II). We introduce some terminology to describe the weighted dual graph of D.

Definition 4.10. A g-chain is a weighted graph that is a chain and such that the weight of two adjacent vertices differ by g. We call g the growth rate; the vertex with highest (resp. lowest) weight is called the root (resp. leaf) of the chain. An (a, g)-chain is a g-chain with leaf weight a. The chain C_1 can be attached to the chain C_0 by identifying the root of C_1 with a vertex of C_0 having the same weight. A 1-tree is obtained by attaching a number of $(1, 1)$-chains among themselves.

Let us now look at a component S with $d_S = 2$ and at least one of q_1 and q_m attached to it. The balancing equation is

$$\deg(\mathscr{L}_{|S}) = -2 - (h + k + 1) + d_R + \sum d_{T_i} = 0,$$

with $k \in \{1, 2\}$. The preceding discussion implies that $d_{T_i} = 1$ for all $i = 1, \ldots, h$, so $d_R = 3 + k$. If $k = 2$, i.e. both q_1 and q_m are on S - in which case they are indeed equidistant from the core -, then $d_R = 5$, and it can be shown inductively that the multiplicity of D increases by 3 for every step we make from S towards the core. The same holds in case I, with q_m attached to S and $d_S = 3$.

Finally, say $d_S = 2$ and only $q_1 \in S$. Then $d_R = 4$, and the growth rate along the chain that connects S to the core is 2, unless there is a component S' at which two 2-chains meet.

Definition 4.11. A 2-tree is obtained by attaching a number of $(1, 1)$-chains to a $(2, 2)$-chain. A 3-tree is obtained by attaching a number of $(1, 1)$-chains either to a $(3, 3)$-chain, or to a weighted graph itself obtained by attaching two $(2, 2)$-chains to the leaf of a 3-chain.

From the preceding discussion it is clear that the weighted dual graph of D is obtained by attaching a number of 1-trees, and either (a) one 3-tree or (b) two 2-trees to the dual graph of the core K, weighted in an appropriate fashion.

Finally, let us look at the core K. Consider it as a one-pointed (case (a)), resp. two-pointed (case (b)) curve of genus two, by ignoring all the attachment points of the 1-trees (which works by Remark 5.8), and let $\bar{K} \in \overline{\mathcal{M}}_{2,1}$ (resp. $\overline{\mathcal{M}}_{2,2}$) be its stable model. The following can happen:

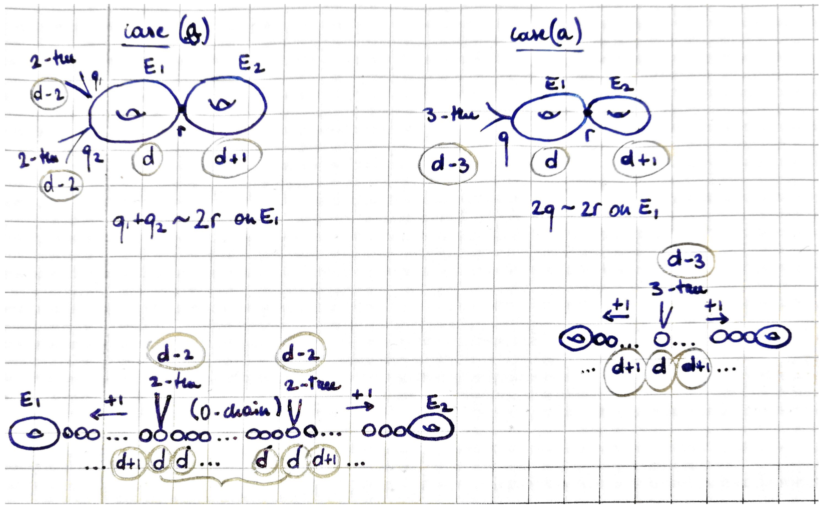
(1) K is a smooth curve of genus two. In case (a), let R be the component adjacent to the core along the 3-tree, and let $x = R \cap K$; then $d_K = d_R + 3$ by balancing R. Now balancing K gives

$$\omega_{\mathscr{C}/\Delta}(d_R R + d_K K)_{|K} = \omega_K(d_R x - (d_R + 2)x) \simeq \mathscr{O}_K,$$

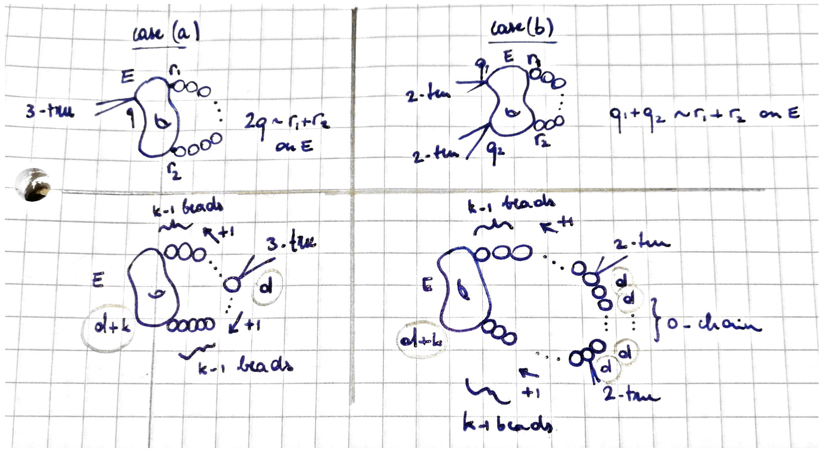
which admits a solution if and only if K is Weierstrass. Similarly case (b) can be balanced if and only if K is conjugate.

(2) K contains two distinct subcurves of genus one E_1 and E_2. We start by solving the balancing equation on one of them, say $E = E_1$. If all but one of the neighbouring components have multiplicity $d_E - 1$, then the last one is forced to have multiplicity $d_E - 1$ as well (by degree reasons). The case that all but two neighbouring components have multiplicity $d_E - 1$ occurs when either one 2-tree or one 3-tree (and exactly one) is attached to E at x; let F be the other component with undetermined multiplicity, which lies between E_1 and E_2 (possibly $F = E_2$), and let $E \cap F = \{y\}$. The case of a 2-tree forces $d_F = d_E$ by degree reason, but then we are left to solve $x \sim y$ in $\mathrm{Pic}(E)$, which is impossible; on the other hand, the case of a 3-tree imposes $d_F = d_E + 1$ and $2x \sim 2y$ in $\mathrm{Pic}(E)$, i.e. K is Weierstrass. By the same token, the two 2-trees have two hit the same genus one curve, say E_1, in nodes x_1, x_2 such that $x_1 + x_2 \sim 2y$ and $d_F = d_E + 1$.

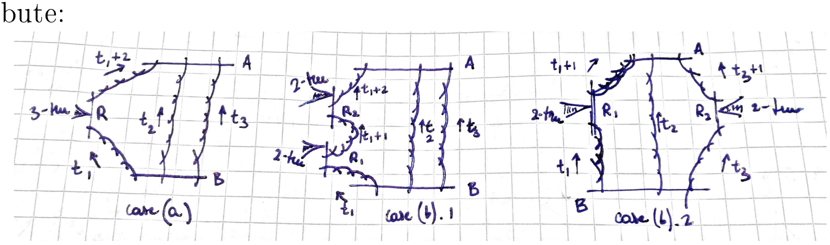
Assume now that there is a chain of rational curves S_i lying between E_1 and E_2 in K, and one of the special trees connects to one of the S_i; in case (b), then, both 2-trees must connect to (possibly different) S_i, by the previous paragraph. Furthermore, the growth rate along the rational chains at E_1 and E_2 has to be 1. This in turn implies that, in case (b), the growth rate along the chain separating the two 2-trees is 0. In particular, the two 2-trees are attached to components with the same multiplicity for D, so q_1 and q_m are equidistant from the core.



(3) $\bar{K} \in \Delta_{irr}$, i.e. K contains an irreducible subcurve of arithmetic genus one E, with two points y_1 and y_2 on E that are joined in K by a (possibly empty) rational chain. We see as above that either a 3-tree is attached to a point $x \in E$ satisfying $2x \sim y_1 + y_2$ in $\mathrm{Pic}(E)$, or two 2-trees are attached to $x_1, x_2 \in E$ satisfying $x_1 + x_2 \sim y_1 + y_2$ in $\mathrm{Pic}(E)$, or the rational chain is not empty and all the distinguished trees are attached to it. In this case, solve the balancing equation on E: let $d = d_E$, d_1 and d_2 be the multiplicities of the rational components attached to y_1 and y_2 respectively; then either $d_1 = d_2 = d - 1$, or $d_1 = d - 1 + k$, $d_2 = d - 1 - k$ and $r_1 - r_2$ is k-torsion in $\mathrm{Pic}(E)$. But, by chasing the balancing equation along the rational necklace, we find that, if $k \geq 1$, then the growth rate increases when passing through a distinguished bead, so that ultimately $d - 1 - k = d_2 > d_1 d - 1 + k$, which is absurd. So the only possibility is to have a rational chain symmetric with respect to the distinguished beads, namely: in case (a) the two pieces of the rational chain standing between the special bead and E have the same length, and in case (b) the distance shortest path between a special bead and E is the same for the two special beads.



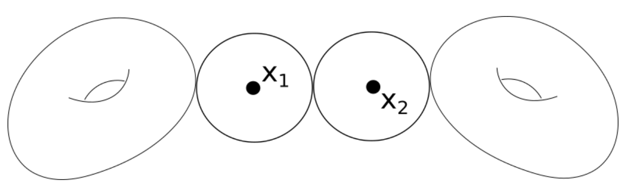
(4) Finally, we consider the case that K has geometric genus 0. There are three possibilities for the dual graph, according to how the distinguished components (denoted by R) and the other stable components (denoted by A and B) distribute:



In case (a) and (b).1, we find that balancing along A or B is equivalent to $\sum_i t_i = 1$. Assume $t_1 \geq 0$; then $d_A > d_B$, therefore $t_2, t_3 > 0$, which contradicts $\sum_i t_i = 1$. Similarly, if $t_1 \leq -2$, then $d_A < d_B$, therefore $t_2, t_3 < 0$, which makes $\sum_i t_i = 1$ again impossible. We find only one solution with $t_1 = -1$ and $t_2 = t_3 = 0$ - notice that it is a degeneration of the case considered in the previous point.

On the other hand, in case (b).2, we find $\sum_i t_i = 1$ when balancing B, and $\sum_i t_i = -1$ when balancing A, which is a contradiction.

Remark 4.12. There is a stable 2-pointed curve that arises as a solution of the balancing equation, yet is not conjugate, namely:



□

Proof. (of Proposition 4.4) By blowing down some rational tails, we can assume that $\mathscr{C}_0 \setminus Z = \sqcup_{i=1}^m T_i$ with each $T_i \simeq \mathbb{P}^1$. □

5. Dualising line bundle and semistable tails - old

This is the most technical and combinatorially delicate section of the paper. We classify the nodal subcurves that can be contracted in a one-parameter smoothing in order to obtain a Gorenstein singularity of genus two. The upshot is that the distinguished (i.e. twin or singular) branches are attached to special (i.e. conjugate or Weierstrass) points of the core (minimal subcurve of genus two). That said, the shape of the contracted curve depends on one parameter only, namely the distance of the distinguished branch(es) from the core - parallel to Smyth's *balanced* condition [Smy11a, Definition 2.11] in genus one, and its generalisation by the strict interior of a circle around the circuit in [RSW17a]. This is going to play a key role in the proof that our moduli spaces are proper.

Lemma 5.1. *Let $\pi\colon \mathscr{C}\to\Delta$ be a family of nodal curves over a DVR, and let $\mathscr{L}$ be a line bundle on $\mathscr{C}$ that is trivial* on a neighbourhood *of a connected subcurve Z of genus two in the central fiber $C=\mathscr{C}_0$, and π-ample everywhere else. Assume that $\mathscr{C}$ is regular along Z. Then $\mathscr{L}$ is π-semiample. Setting*

$$\phi\colon \mathscr{C}\to\mathscr{C}'=\operatorname{Proj}_\Delta\left(\bigoplus_{n\geq 0}\pi_*\mathscr{L}^{\otimes n}\right),$$

$\mathscr{C}'$ is a flat family of reduced curves, and C' has a genus two singularity at $q=\phi(Z)$ with $m=|Z\cap\overline{C\setminus Z}|$ branches, whose normalisation is $\phi_{|\overline{C\setminus Z}}$. Moreover, q is a Gorenstein singularity if and only if $\mathscr{L}$ can be written in the form $\omega_{\mathscr{C}/\Delta}(D+\Sigma)$, where D is supported on Z and Σ is supported away from Z.

Proof. This is a straightforward extension of Smyth's contraction lemma [Smy11a, Lemma 2.13]. Up to substituting $\mathscr{L}$ by a sufficiently high power of itself, we claim that there exists a divisor D' on $\mathscr{C}$, supported on Z, such that $R^1\pi_*\mathscr{L}(-D')=0$. Indeed, if $\operatorname{lev}(Z)=|Z\cap\overline{C\setminus Z}|\geq 3$, it is enough to take $D'=Z$, by cohomology and base-change, and Serre duality. On the other hand, if Z is either a conjugate bridge or a Weierstrass tail (we leave the discussion of the other possible situations, since they will not occur in what follows), we have to go through a case-by-case analysis:

- if Z is irreducible, it is enough to take $D'=3Z$ (in fact, $2Z$ when Z is a bridge);
- if $Z=E_1\sqcup_q E_2$ is the union of two curves of genus one, attached to the rest of C along E_1, then $D'=2E_1+3E_2$ suffices;
- if $Z=E\sqcup_{\{q_1,q_2\}}R$ is the union of a genus one curve with a rational bridge, attached to the rest of C along E, take $D'=2E+3R$; while take $D'=3E+2R$ if the same Z is attached to the rest of C along R;
- if Z is formed of three rational components such that $R_1\cap R_2$ is two nodes, $R_1\cap R_0$ and $R_2\cap R_0$ are each one node, and the rest of C is attached to Z along R_0, then it is enough to choose $D'=3R_0+4R_1+4R_2$.

Similar solutions can be found if Z contains rational trees, or if a rational curve is replaced by a chain in the above. The vanishing of $R^1\pi_*\mathscr{L}(-D')$ implies that

$\pi_*\mathscr{L}$ surjects onto $\pi_*\mathscr{L}_{|D'} = \pi_*\mathscr{O}_{D'}$, by the assumption that $\mathscr{L}$ is trivial on a neighbourhood of Z. On the other hand, $Z = (D')_{\mathrm{red}}$ implies that $H^0(D', \mathscr{O}_{D'}) \twoheadrightarrow H^0(Z, \mathscr{O}_Z) = \mathbf{k}$, hence $\mathscr{L}$ is globally generated along Z. The remaining assertions have been proved by Smyth (*loc. cit.*; see also [RSW17a, Proposition 3.7.3.1]). $\square$

Remark 5.2. The assumption that the total space of $\mathscr{C}$ is regular along Z seems to be necessary for $\mathscr{L}$ to be generated by global sections along Z. Assume $\mathrm{char}(\mathbf{k}) = 0$[2]; using the theory of limit linear series, we show that for certain line bundles $\mathscr{L}$ - trivial on Z and ample elsewhere - on certain one-parameter smoothings of a genus two tail Z, the associated complete linear system has base-points along Z. We do not know how the powers of $\mathscr{L}$ behave, though.

Let X_0 be the nodal curve obtained by attaching $R \simeq \mathbb{P}^1$ to a Weierstrass point q of a smooth genus two curve Z. Choose $d \gg 0$ ($d \geq 5$ is enough), and let us study the moduli space of complete linear systems of degree d on (smoothings of) X_0; with $r = d-2$, the Brill-Noether number is $\rho = 2$ (the dimension of the Jacobian). On the other hand, assume that the R-aspect of the lls has $\mathscr{L}_{Y|Z} \simeq \mathscr{O}_Z$; then the Z-aspect has $\mathscr{L}_{Z|Z} \simeq \mathscr{O}_Z(dq)$, whose vanishing sequence is $\alpha_Z(q) = \{0, 1, \ldots, d-4, d-2, d\}$, from which we deduce for the complementary aspect $\alpha_R(q) \geq \{0, 2, 4, 5, \ldots, d\}$. We want to show that all such aspects are smoothable, by appealing to the Regeneration Theorem [HM98, Theorem 5.41]. Notice that in the case at hand we have a choice of a two-dimensional subspace of $\langle 1, t, t^2, t^3\rangle_{\mathbf{k}} \subseteq H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(d))$ meeting the subspace $\langle t^2, t^3\rangle_{\mathbf{k}}$ non-trivially, i.e. the locus in $\mathrm{Gr}(1, \mathbb{P}^3)$ of lines meeting a fixed line ℓ, which is a Schubert cycle of dimension 3. We therefore need to put X_0 in a family over a base B of dimension 1 at least. We shall do so by considering the family X obtained by attaching R to a moving point of Z, so that X_0 is the fiber of X over $q \in Z$.

Let us start by examining the other possibilities for $\mathscr{G}^{d-2}_d(X_0)$: the R-aspect can in fact restrict to any line bundle of degree 0 on Z, which we are going to write as $\mathscr{O}_Z(p_1 + p_2 - 2q)$ for two moving points p_1, p_2 on Z (think of them as coordinates on $\mathrm{Pic}(Z)$). Then $\mathscr{L}_Z = \mathscr{O}_Z((d-2)q + p_1 + p_2)$.

$\subseteq \mathrm{Pic}(Z)$	dim	$\alpha_{Z;d-3,d-2}$	$\alpha_{R;0,1}$	$\subseteq \mathbb{P}H^0(R, \mathscr{O}_R(d))$	dim
$p_1 + p_2 \sim 2q$	0	$\{d-2, d\}$	$\geq \{0, 2\}$	$\{\ell' \in \mathrm{Gr}(1, \mathbb{P}^3) \mid \ell' \cap \ell \neq\}$	3
$2q \nsim p_1 + p_2 \geq q$	1	$\{d-3, d-1\}$	$\geq \{1, 3\}$	$\mathbb{P}^1$	1
$p_1 + p_2 \ngeq q$	2	$\{d-3, d-2\}$	$\{2, 3\}$	pt	0

If we now let q vary in $B \simeq Z$, we may generically assume that it is not Weierstrass. We then find the following:

$\subseteq \mathrm{Pic}(X)$	dim	$\alpha_{Z;d-3,d-2}$	$\alpha_{R;0,1}$	$\subseteq \mathbb{P}H^0(R, \mathscr{O}_R(d))$	dim
$p_1 + p_2 \sim \omega_Z$	$0+1$	$\{d-2, d-1\}$	$\geq \{1, 2\}$	$(\mathbb{P}^2)^*$	2
$p_1 + p_2 \sim 2q$	$0+1$	$\{d-3, d\}$	$\geq \{0, 3\}$	$\mathbb{P}^2$	2
$\omega_Z, 2q \nsim p_1 + p_2 \geq q$	$1+1$	$\{d-3, d-1\}$	$\geq \{1, 3\}$	$\mathbb{P}^1$	1
$p_1 + p_2 \ngeq q$	$2+1$	$\{d-3, d-2\}$	$\{2, 3\}$	pt	0

[2]This is in fact not an issue in positive characteristic, thanks to a result of S. Keel [Kee99].

We conclude that $\mathscr{G}_d^{d-2}(X/B)$ has pure dimension 3, and we may therefore apply the Regeneration Theorem to deduce that all lls with $\mathscr{L}_{R|Z} \simeq \mathscr{O}_Z$ - in particular those with base-locus Z - are smoothable.[3]

The following lemma is useful in determining the multiplicity of D (as in Lemma 5.1) along components of Z adjacent to $\overline{C \setminus Z}$.

Lemma 5.3. *Let $\nu\colon \tilde{C} \to C$ be the normalisation of a Gorenstein singularity of genus two, with $\nu^{-1}(p) = \{p_1, \ldots, p_m\}$. Then $\nu^*\omega_C = \omega_{\tilde{C}}(3p_1 + 2p_2 + \ldots + 2p_{m-1} + 3p_m)$ (case II) or $\nu^*\omega_C = \omega_{\tilde{C}}(2p_1 + \ldots + 2p_{m-1} + 4p_m)$ (case III).*

Proof. Recall the explicit description of the dualising sheaf for curves:

$$\omega_C(U) = \{\eta \in \Omega_{\tilde{C}} \otimes K(\nu^{-1}(U)) | \sum_{p_i \in \nu^{-1}(p), p \in U} \operatorname{Res}_{p_i}((\nu^* f)\eta) = 0,\ \forall f \in \mathscr{O}_C(U)\}.$$

We are going to use the explicit coordinates in (1) and (3). In case II, we know that $\widetilde{\mathfrak{m}}^3 \subseteq R$, therefore we have poles of third order at most. It is enough to study the possible polar tails. Let

$$\eta = c_1 \frac{\mathrm{d}\, t_1}{t_1^3} + b_1 \frac{\mathrm{d}\, t_1}{t_1^2} + a_1 \frac{\mathrm{d}\, t_1}{t_1} \oplus \ldots \oplus c_m \frac{\mathrm{d}\, t_m}{t_m^3} + b_m \frac{\mathrm{d}\, t_m}{t_m^2} + a_m \frac{\mathrm{d}\, t_m}{t_m}.$$

From looking at $1 \cdot \eta$ we deduce $\sum_{i=1}^m a_i = 0$; from $x_i \cdot \eta$ we see $b_1 + b_m = 0$ (if $i = 1$), and $b_i + c_m = 0$ (if $i = 2, \ldots, m-1$); finally from $x_i^2 \cdot \eta$ we have $c_1 + c_m = 0$ (if $i = 1$), and $c_i = 0$ (if $i = 2, \ldots, m-1$). Therefore $\omega_C/\nu_*\omega_{\tilde{C}}$ is spanned by

$$\frac{\mathrm{d}\, t_1}{t_1} - \frac{\mathrm{d}\, t_m}{t_m}, \ldots, \frac{\mathrm{d}\, t_{m-1}}{t_{m-1}} - \frac{\mathrm{d}\, t_m}{t_m}, \frac{\mathrm{d}\, t_1}{t_1^2} - \frac{\mathrm{d}\, t_m}{t_m^2},$$
$$\bar{\eta} = \frac{\mathrm{d}\, t_1}{t_1^3} + \frac{\mathrm{d}\, t_2}{t_2^2} + \ldots + \frac{\mathrm{d}\, t_{m-1}}{t_{m-1}^2} - \frac{\mathrm{d}\, t_m}{t_m^3}.$$

In particular ω_C is generated by $\bar{\eta}$ as an $\mathscr{O}_C$-module. Hence the first claim.

In case III, we know that $\widetilde{\mathfrak{m}}^4 \subseteq R$, therefore we have poles of fourth order at most. On the other hand $t_i^2 \in R$ for all i implies the part of order three is trivial. So let

$$\eta = c_1 \frac{\mathrm{d}\, t_1}{t_1^4} + b_1 \frac{\mathrm{d}\, t_1}{t_1^2} + a_1 \frac{\mathrm{d}\, t_1}{t_1} \oplus \ldots \oplus c_m \frac{\mathrm{d}\, t_m}{t_m^4} + b_m \frac{\mathrm{d}\, t_m}{t_m^2} + a_m \frac{\mathrm{d}\, t_m}{t_m}.$$

From looking at $1 \cdot \eta$ we deduce $\sum_{i=1}^m a_i = 0$; from $x_i \cdot \eta$ we see $b_i + c_m = 0$ for all i, and from $x_i^3 \cdot \eta$ we have $c_i = 0$ for all i. (The statement about third order poles can be evinced from $x_i^2 \cdot \eta$ or from $z \cdot \eta$ indifferently.) Therefore $\omega_C/\nu_*\omega_{\tilde{C}}$

[3]It has been pointed out to us by F. Carocci that a similar but easier computation can be carried out for a genus one tail as well.

is spanned by

$$\frac{\mathrm{d}\,t_1}{t_1}-\frac{\mathrm{d}\,t_m}{t_m},\ldots,\frac{\mathrm{d}\,t_{m-1}}{t_{m-1}}-\frac{\mathrm{d}\,t_m}{t_m},\frac{\mathrm{d}\,t_m}{t_m^2}$$
$$\bar{\eta}=\frac{\mathrm{d}\,t_1}{t_1^2}+\ldots+\frac{\mathrm{d}\,t_{m-1}}{t_{m-1}^2}-\frac{\mathrm{d}\,t_m}{t_m^4}.$$

In particular ω_C is generated by $\bar{\eta}$ as an $\mathscr{O}_C$-module. Hence the second claim. □

Corollary 5.4. *The dualising sheaf has multi-degree* $(1,0,\ldots,0,1)$ *(case II) and* $(0,\ldots,0,2)$ *(case III) respectively.*

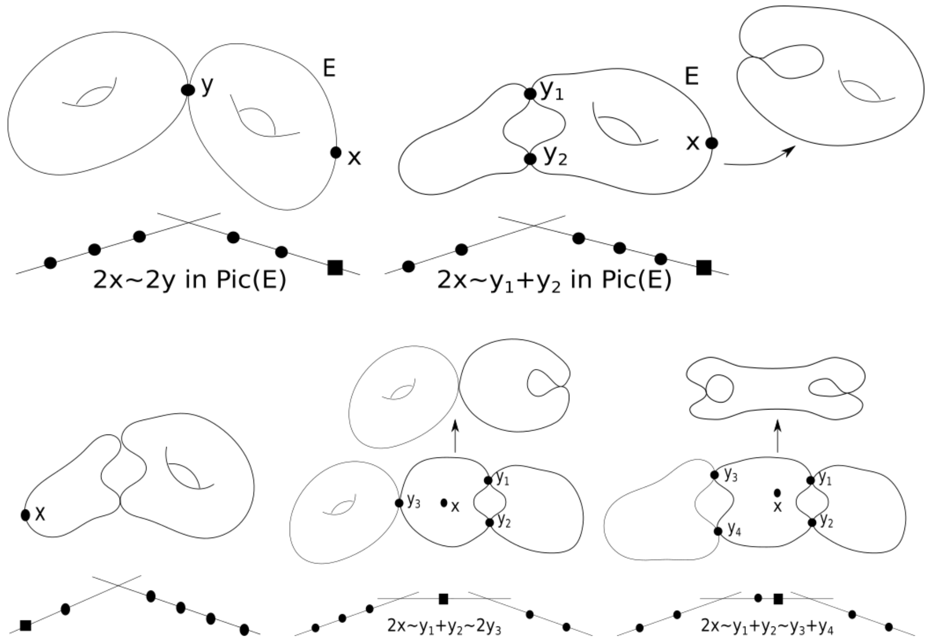
Remark 5.5. $H^0(C,\Omega_C^\vee(-\sum_{i=1}^n p_i))=0$ implies the ampleness of $\omega_C(\sum_{i=1}^n p_i)$.

Remark 5.6. Recall that every smooth curve C of genus two is hyperelliptic; a point $p\in C$ is Weierstrass if it is a ramification point for the unique $\mathfrak{g}^1_2$, while two points $p_1,p_2\in C$ are conjugate if they are swapped under the hyperelliptic involution. The notion can be generalised to nodal curves by declaring (C,p) to be Weierstrass if its stabilisation lies in the closure of

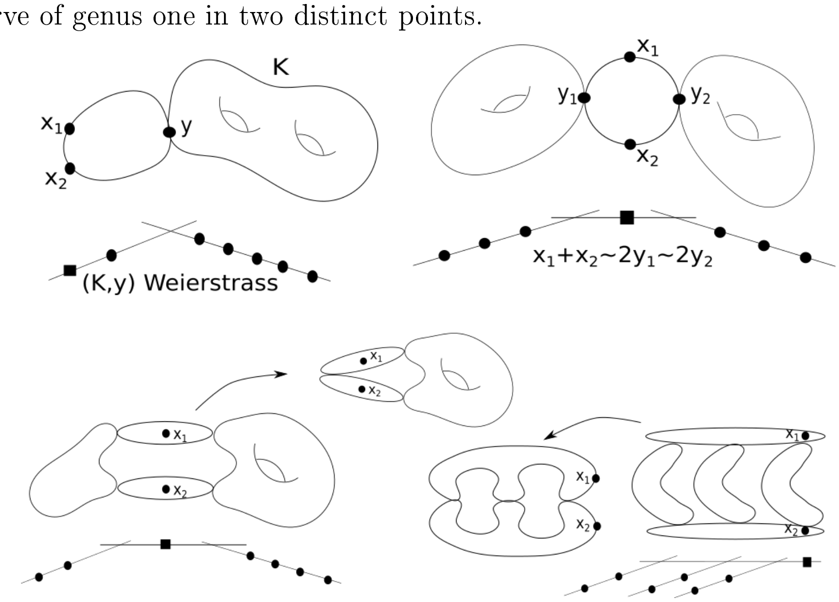
$$\{(C,p)|\ C \text{ smooth and } p \text{ Weierstrass}\}\subseteq\overline{\mathcal{M}}_{2,1},$$

and similarly for conjugate points. We can give a more explicit characterisation of the closure thanks to the theory of admissible covers [HM98, Theorem 5.45]. Another way to see it is that, up to the involution action, the Weierstrass locus is isomorphic to $\overline{\mathcal{M}}_{0,6}/\mathfrak{S}_5$ and the conjugate locus is isomorphic to $\overline{\mathcal{M}}_{0,7}/\mathfrak{S}_6$.

- If p belongs to a component of genus one E, which is attached to another component of genus one at a node q, then p is Weierstrass iff $2p\sim 2q\in\operatorname{Pic}(E)$; if instead E has a self-node that glues q_1 with q_2, then p is Weierstrass iff $2p\sim q_1+q_2\in\operatorname{Pic}(E)$. If instead p is on a rational component R, p is Weierstrass if either R is attached to a genus one curve at two distinct points, or R has a self-node gluing q_1 and q_2 and is attached to a genus one tail at r, in which case we require $\phi(q_1)=\phi(q_2)$ for a double cover $\phi\colon R\to\mathbb{P}^1$ ramified at p and r, or R has two self-nodes gluing q_1 with q_2, and r_1 with r_2, in which case we require p to be a ramification point for a double cover $\phi\colon R\to\mathbb{P}^1$ such that $\phi(q_1)=\phi(q_2)$ and $\phi(r_1)=\phi(r_2)$.



- If p_1 and p_2 are conjugate, they have to map to the same component of the target of the admissible cover. The description of the previous point works by replacing every condition on $2p$ by its analogue for $p_1 + p_2$. There are a few more situations to take into account: p_1 and p_2 could belong to a rational component R bubbling off from a Weierstrass point of a genus two curve, or bridging between two distinct curves of genus one, or p_1 and p_2 could lie on two distinct rational components R_1 and R_2 intersecting at one node and meeting a curve of genus one in two distinct points.



Proposition 5.7 (Semistable tails). *Let (C,q) be a Gorenstein singularity of genus two, with pointed normalisation $\bigsqcup_{i=1}^m(\mathbb{P}^1,q_i)$. Let $\mathscr{C}\to\Delta$ be a one-parameter smoothing of C, and $\phi\colon\mathscr{C}^{ps}\to\mathscr{C}$ a birational contraction from a prestable curve with regular total space. Let $(Z,p_1,\ldots,p_m)$ be $\phi^{-1}(q)$ marked with the intersection points with the rest of $\mathscr{C}_0^{ps}$.*

- *Case II: p_1 and p_m belong to rational tails attached to conjugate points of the core. p_1 and p_m are at the same distance from the core, while all other p_i are further away from it.*
- *Case III: p_m is on a rational tail attached to a Weierstrass point of the core, and all other p_i are further away than p_m from the core.*

Proof. We are going to find conditions on $\mathscr{C}_0^{ps}$ so that a line bundle of the form $\omega_{\mathscr{C}^{ps}/\Delta}(D+\Sigma)$ as in Lemma 5.1 may be found, such that it restricts to the trivial line bundle on a neighbourhood of Z, and it is ample over Δ everywhere else.

By Castelnuovo's criterion, we may assume that Z is semistable, i.e. every rational component contains at least two special points. We may split Z into a core K (minimal subcurve of genus two) and a number (possibly zero) of rational trees. We start by analysing the former.

Suppose we have ϕ as above. $\mathscr{C}$ is a family of hyperelliptic curves, i.e. there is a $2:1$ cover $\mathscr{C}\to\mathbb{P}(\pi_*\omega_{\mathscr{C}/\Delta})$ over Δ. By choosing a hyperplane away from the image of the singular point, we may represent $\omega_{\mathscr{C}/\Delta}$ by a horizontal divisor of degree 2 over Δ that avoids the singularity. Up to a base-change, we may therefore decree $\omega_{\mathscr{C}_\Delta}=\mathscr{O}_{\mathscr{C}}(H_1+H_2)$, where H_1 and H_2 are generically conjugate points, and intersect the central fiber in the special branch(es). By pulling back via ϕ (since the latter is an isomorphism away from Z, H_1 and H_2 define horizontal Cartier divisors on $\mathscr{C}^{ps}$), and taking the pointed stabilisation of $(\mathscr{C}^{ps},H_1,H_2)$, we see that the special branches are attached to conjugate (type II) or Weierstrass (type III) points of K, according to whether $\overline{H_1(0)}\neq\overline{H_2(0)}$ (resp. they coincide).

On the other hand, suppose that K marked by the attaching points of the rational trees containing p_1 and p_m (resp. p_m) is conjugate (resp. Weierstrass). Then, by smoothness of the conjugate (resp. Weierstrass) locus, we may extend these to sections Q_1 and Q_m over Δ. By the regularity assumption, we may obtain $\mathscr{C}_0^{ps}$ by repeatedly blowing up K along (smooth points in the exceptional divisors over) the attaching points of the tails containing $p_1,\ldots,p_m$. By taking the strict transforms of Q_1 and Q_m, and adding a horizontal divisor supported on the rational trees outside of K, we may obtain a line bundle on $\mathscr{C}^{ps}$ of the form prescribed by Lemma 5.1, namely trivial on a neighbourhood of K and generically supercanonical. We come now to a more detailed analysis of the rational chains and trees that may be part of Z.

By Lemma 5.3, in case II both p_1 and p_m are attached to components that appear with multiplicity 2 in D - respectively, in case III, p_m is attached to a component of multiplicity 3 -, while all other markings lie on components along which D has multiplicity 1.

Let d_A denote the multiplicity of the divisor D along the component A of Z. First, we claim that no component can appear with $d_A = 0$. Assume that this occurred along one of the rational trees. Call S ($S \simeq \mathbb{P}^1$) a component furthest from the core such that $d_S = 0$; R the one that precedes it, and $T_1, \dots, T_h$ the ones that follow it (when sweeping the tree from the core) - so that $h \geq 1$ by the previous paragraph, and $d_{T_i} \geq 1$ by assumption. Then, by adjunction,

$$\deg(\mathscr{L}_{|S}) = -2 + (h+1) + d_R + \sum d_{T_i} = 0,$$

which necessarily implies $h = 1$, and $d_R = d_{T_i} = 0$, contradicting the assumption. The case that S belongs to the core is similar (ω_S may only be more positive).

Let us now consider $d_S = 1$. We stick to the notation above; furthermore, there may be a number of p_i, $i \in \{(1,)2, \dots, m-1\}$, lying on S. Then, by adjunction,

$$\deg(\mathscr{L}_{|S}) = -2 + d_R + \sum d_{T_i} = 0.$$

so either $d_R = 2$, $h = 0$ and $k \geq 1$ arbitrary, i.e. S is adjacent to $\overline{C \setminus Z}$; or $d_R = 1$, $h = 1$, and $d_{T_1} = 1$ (with k arbitrary). In the latter case, though, we may repeat the argument on T_1, and we find an infinite chain in Z, which can be excluded.

Remark 5.8. More generally, an analogous computation shows that, when balancing a component A of multiplicity d_A, all neighbouring components of multiplicity $d_A - 1$ can be safely ignored (at the same time, the number of such components is only bounded by m, due to the semistability of Z).

We now prove that $d_R > d_S$ holds in general for S on a rational tree. The preceding paragraphs deal with the cases $d_S = 0, 1$; we may therefore assume $d_S > 1$ (which in particular implies $0 \leq k \leq 2$). We have

$$\deg(\mathscr{L}_{|S}) = -2 + d_R - (d_S - 1)(h + k + 1) + \sum d_{T_i} = 0.$$

By proceeding from leaves to root, we can assume that $d_S > d_{T_i}$, $i = 1, \dots, h$. We may therefore rewrite the previous equality as

$$d_R = (d_S - 1)(h+k+1) - \sum d_{T_i} + 2 \geq (d_S - 1)(k+1) + 2 = d_S + 1 + k(d_S - 1) > d_S.$$

In fact, we can prove as on [Smy11a, p.893] that $d_R = d_S + 1$, unless $d_S = 2$ and either p_1 or p_m (or both) are on S (type II), or $d_S = 3$ and $p_m \in S$ (type III). We introduce some terminology to describe the weighted dual graph of D.

Definition 5.9. A g-chain is a weighted graph that is a chain and such that the weight of two adjacent vertices differ by g. We call g the growth rate; the vertex with highest (resp. lowest) weight is called the root (resp. leaf) of the chain. An (a, g)-chain is a g-chain with leaf weight a. The chain C_1 can be attached to the chain C_0 by identifying the root of C_1 with a vertex of C_0 having the same weight. A 1-tree is obtained by attaching a number of $(1, 1)$-chains among themselves.

Let us now look at a component S with $d_S = 2$ and at least one of p_1 and p_m attached to it. The balancing equation is

$$\deg(\mathscr{L}_{|S}) = -2 - (h + k + 1) + d_R + \sum d_{T_i} = 0,$$

with $k \in \{1, 2\}$. The preceding discussion implies that $d_{T_i} = 1$ for all $i = 1, \ldots, h$, so $d_R = 3 + k$. If $k = 2$, i.e. both p_1 and p_m are on S - in which case they are indeed equidistant from the core -, then $d_R = 5$, and it can be shown inductively that the multiplicity of D increases by 3 for every step we make from S towards the core. The same holds in case III, with p_m attached to S and $d_S = 3$.

Finally, say $d_S = 2$ and only $p_1 \in S$. Then $d_R = 4$, and the growth rate along the chain that connects S to the core is 2, unless there is a component S' at which two 2-chains meet.

Definition 5.10. A 2-tree is obtained by attaching a number of $(1, 1)$-chains to a $(2, 2)$-chain. A 3-tree is obtained by attaching a number of $(1, 1)$-chains either to a $(3, 3)$-chain, or to a weighted graph itself obtained by attaching two $(2, 2)$-chains to the leaf of a 3-chain.

From the preceding discussion it is clear that the weighted dual graph of D is obtained by attaching a number of 1-trees, and either (a) one 3-tree or (b) two 2-trees to the dual graph of the core K, weighted in an appropriate fashion.

Finally, let us look at the core K. Consider it as a one-pointed (case (a)), resp. two-pointed (case (b)) curve of genus two, by ignoring all the attachment points of the 1-trees (which works by Remark 5.8), and let $\bar{K} \in \overline{\mathcal{M}}_{2,1}$ (resp. $\overline{\mathcal{M}}_{2,2}$) be its stable model. The following can happen:

(1) K is a smooth curve of genus two. In case (a), let R be the component adjacent to the core along the 3-tree, and let $q = R \cap K$; then $d_K = d_R + 3$ by balancing R. Now balancing K gives

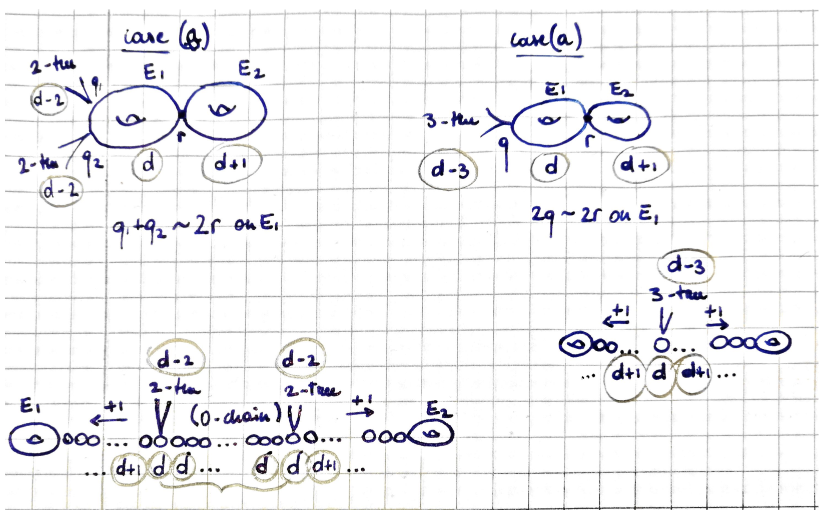
$$\omega_{\mathscr{C}^{ps}/\Delta}(d_R R + d_K K)_{|K} = \omega_K(d_R q - (d_R + 2)q) \simeq \mathscr{O}_K,$$

which admits a solution if and only if K is Weierstrass. Similarly case (b) can be balanced if and only if K is conjugate.

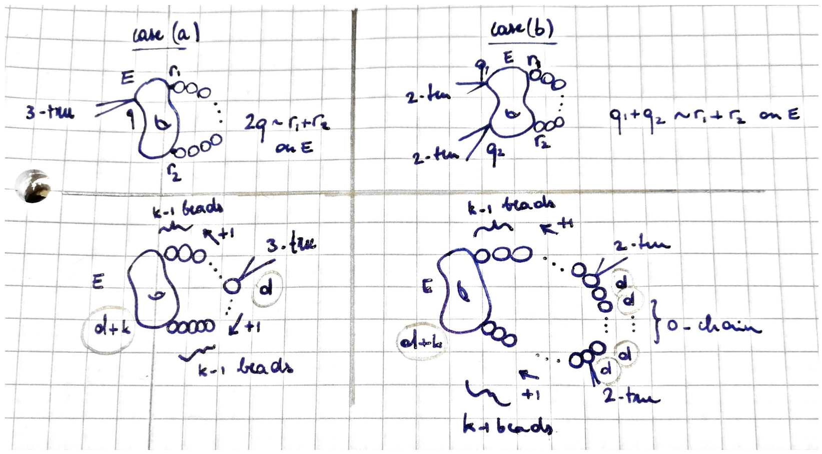
(2) K contains two distinct subcurves of genus one E_1 and E_2. We start by solving the balancing equation on one of them, say $E = E_1$. If all but one of the neighbouring components have multiplicity $d_E - 1$, then the last one is forced to have multiplicity $d_E - 1$ as well (by degree reasons). The case that all but two neighbouring components have multiplicity $d_E - 1$ occurs when either a 2-tree or a 3-tree is attached to E at q; let F be the other component with undetermined multiplicity, which lies between E_1 and E_2 (possibly $F = E_2$), and let $E \cap F = \{r\}$. The case of a 2-tree forces $d_F = d_E$ by degree reason, but then we are left to solve $q \sim r$ in $\mathrm{Pic}(E)$, which is impossible; on the other hand, the case of a 3-tree imposes $d_F = d_E + 1$ and $2q \sim 2r$ in $\mathrm{Pic}(E)$, i.e. K is Weierstrass. By the same token, the two 2-trees have two hit the

same genus one curve, say E_1, in nodes q_1, q_2 such that $q_1 + q_2 \sim 2r$ and $d_F = d_E + 1$.

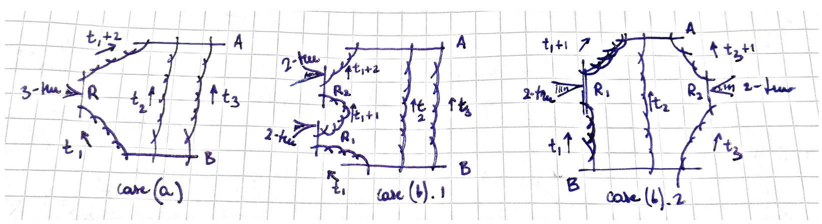
Assume now that there is a chain of rational curves R_i standing between E_1 and E_2 in K, and one of the special trees connects to one of the R_i; in case (b), then, both 2-trees must connect to (possibly different) R_i, by the previous paragraph. Furthermore, the growth rate along the rational chains at E_1 and E_2 has to be 1. This in turn implies that, in case (b), the growth rate along the chain separating the two 2-trees is 0. In particular, the two 2-trees are attached to components with the same multiplicity for D, so p_1 and p_m are equidistant from the core.



(3) $\bar{K} \in \Delta_{irr}$, i.e. K contains an irreducible subcurve of arithmetic genus one E, with two points r_1 and r_2 on E that are joined in K by a (possibly empty) rational chain. We see as above that either a 3-tree is attached to a point $q \in E$ satisfying $2q \sim r_1 + r_2$ in $\mathrm{Pic}(E)$, or two 2-trees are attached to $q_1, q_2 \in E$ satisfying $q_1 + q_2 \sim r_1 + r_2$ in $\mathrm{Pic}(E)$, or the rational chain is not empty and all the distinguished trees are attached to it. In this case, solve the balancing equation on E: let $d = d_E$, d_1 and d_2 be the multiplicities of the rational components attached to r_1 and r_2 respectively; then either $d_1 = d_2 = d - 1$, or $d_1 = d - 1 + k$, $d_2 = d - 1 - k$ and $r_1 - r_2$ is k-torsion in $\mathrm{Pic}(E)$. But, by chasing the balancing equation along the rational necklace, we find that, if $k \geq 1$, then the growth rate increases when passing through a distinguished bead, so that ultimately $d - 1 - k = d_2 > d_1 d - 1 + k$, which is absurd. So the only possibility is to have a rational chain symmetric with respect to the distinguished beads, namely: in case (a) the two pieces of the rational chain standing between the special bead and E have the same length, and in case (b) the distance shortest path between a special bead and E is the same for the two special beads.



(4) Finally, we consider the case that K has geometric genus 0. There are three possibilities for the dual graph, according to how the distinguished components (denoted by R) and the other stable components (denoted by A and B) distribute:

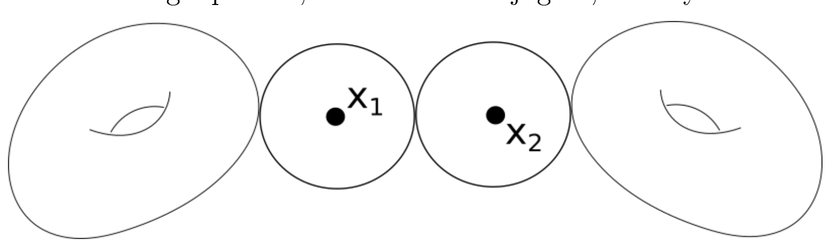


In case (a) and (b).1, we find that balancing along A or B is equivalent to $\sum_i t_i = 1$. Assume $t_1 \geq 0$; then $d_A > d_B$, therefore $t_2, t_3 > 0$, which contradicts $\sum_i t_i = 1$. Similarly, if $t_1 \leq -2$, then $d_A < d_B$, therefore $t_2, t_3 < 0$, which makes $\sum_i t_i = 1$ again impossible. We find only one solution with $t_1 = -1$ and $t_2 = t_3 = 0$ - notice that it is a degeneration of the case considered in the previous point.

On the other hand, in case (b).2, we find $\sum_i t_i = 1$ when balancing B, and $\sum_i t_i = -1$ when balancing A, which is a contradiction.

□

Remark 5.11. There are two cases of stable 2-pointed curves that arise as solutions of the balancing equation, but are not conjugate, namely:



6. The new moduli functors

Definition 6.1. Let $(C, p_1, \ldots, p_n)$ be a reduced curve, marked by smooth points. For a ~~nodally attached~~ subcurve $D \subseteq C$, no matter what the singularities at the intersection with the rest of C, we define the *level* of D to be the number

$$\mathrm{lev}(D) = |D \cap \overline{C \setminus D}| + |\{p_1, \ldots, p_n\} \cap D|.$$

Definition 6.2. Fix positive integers $m < n$. Let $(C, p_1, \ldots, p_n)$ be a connected, reduced, complete curve of arithmetic genus two, marked by smooth points. We say that C is m-stable if:

1. C has only nodes; elliptic l-fold points, $l \leq m+1$; type $II_{\leq m+1}$, and type $III_{\leq m}$ genus two singularities as singular points.
2. If Z is a connected subcurve of arithmetic genus two, then $\mathrm{lev}(Z) > m$.
3. If E is a ~~nodally attached~~ subcurve of arithmetic genus one, then $\mathrm{lev}(E) > m+1$.
4. $H^0(C, \Omega_C^\vee(-\sum_{i=1}^n p_i)) = 0$.
5. If C contains a singularity of genus two, p_1 is connected (through a rational chain) to one of the distinguished branches.

Remark 6.3. The definition is not $\mathfrak{S}_n$-symmetric. In the contraction arguments below, we use the asymmetry to write down the dualising line bundle of a genus two (sub)curve Z as $\omega_Z \simeq \mathscr{O}_Z(q_1 + \bar{q}_1)$, where q_1 is the point on Z which is closest to p_1. Compare this with the genus one situation, where the dualising line bundle of a Gorenstein curve is trivial.

Remark 6.4. If there is a subcurve of genus one, condition (3) and condition (4) jointly imply condition (2). Indeed, $\mathrm{lev}(Z) \geq \mathrm{lev}(E) - 1$, and the only cases in which the level drops by one are: when $Z = (E, p_1, \ldots, p_{l-2}, q_1, q_2) \sqcup_{\{q_1,q_2\}} (\mathbb{P}^1, q_1, q_2, p_{l-1})$; and when $Z = (E, p_1, \ldots, p_{l-1}, q) \sqcup_q (E', q)$, where (E', q) is a one-pointed curve of genus one.

Lemma 6.5 (boundedness)**.** *If $(C, p_1, \ldots, p_n)$ is an m-stable curve of genus two, the N-th power of $A = \omega_C(\sum_{i=1}^n p_i)$ is very ample for every $N > 2 + 8(m+1)$.*

Proof. We need to show that, for every point $p, q \in C$ (possibly equal)

1. *basepoint-freeness*: $H^1(C, A^{\otimes N} \otimes I_p) = 0$;
2. *separating points and tangent vectors*: $H^1(C, A^{\otimes N} \otimes I_p I_q) = 0$.

By Serre duality we may equivalently show that $H^0(C, \omega_C \otimes A^{-N} \otimes (I_p I_q)^\vee) = 0$. Let $\nu\colon \tilde{C} \to C$ be the normalisation, and let $\nu^{-1}(p) = \{p_1, \ldots, p_h\}$, $\nu^{-1}(q) = \{q_1, \ldots, q_k\}$, with $h, k \leq m+1$. It follows from Proposition 2.2 (and the analogous result of Smyth) that $\nu_* \mathscr{O}_{\tilde{C}}(-D) \subseteq I_p I_q$ for $D = 4\sum_{i=1}^h p_i + \sum_{j=1}^k q_j$ (note that $\deg(D) \leq 8(m+1)$); furthermore, the quotient is torsion, therefore, by applying $\mathscr{H}om(-, \mathscr{O}_C)$ and adjunction, we find $(I_p I_q)^\vee \subseteq \nu_* \mathscr{O}_{\tilde{C}}(D)$. It is thus enough to show that $H^0(\tilde{C}, \mathscr{O}_{\tilde{C}}(D) \otimes \nu^*(\omega_C \otimes A^{-N})) = 0$. Finally, $\nu^* \omega_C$ has degree at most two, and ν^*A has degree at least one on any branch of $\tilde{C}$, hence it is enough to take $N > 2 + 8(m+1)$. □

Lemma 6.6 (deformation openness). *Let $(\mathscr{C}, \sigma_1, \ldots, \sigma_n) \to S$ be a family of curves over a Noetherian base scheme with n sections. The locus*

$$\{s \in S | (\mathscr{C}_{\bar{s}}, \sigma_1(\bar{s}), \ldots, \sigma_n(\bar{s})) \text{ is } m\text{-stable}\}$$

is Zariski-open in S.

Proof. Being Gorenstein is an open condition, as much as having connected fibers of arithmetic genus two. This bounds the genus of the singularities that may occur. The case $m = 1$ deserves special attention. In this case, that condition (1) is open follows from acknowledging that $II_2 = A_5$, $III_1 = A_4$, while tacnode, cusp, and node are A_3, A_2, and A_1 respectively, and from a beautiful result of Grothendieck concerning the deformation theory of ADE singularities [Arn72, Dem75]. The case $m \geq 2$ simply follows from upper semicontinuity of embedded dimension and the fact that we have exhausted all possible Gorenstein singularities of genus ≤ 2, and embedding dimension $\leq m + 1$.

Condition (4) translates to: the locus where the automorphism group is unramified is open in the base.

The other conditions are topological, hence constructible. With Noetherian assumptions, it is enough to check their openness over a dvr scheme. Assume that the geometric generic fiber $C_{\bar{\eta}}$ contains two genus one subcurve $E_{1,\bar{\eta}}$ and $E_{2,\bar{\eta}}$; their closures E_1 and E_2 in $\mathscr{C}$ are then flat families of genus one curves over Δ. If $E_{1,\bar{\eta}}$ and $E_{2,\bar{\eta}}$ are disconnected, then so are E_1 and E_2, by local constancy of the number of connected components of fibers of a flat proper morphism with geometrically normal fibers. If $E_{1,\bar{\eta}}$ and $E_{2,\bar{\eta}}$ are joined by a (disconnecting) node $q_{\bar{\eta}}$, then so are $E_{1,0}$ and $E_{2,0}$; indeed, the unique limit of $q_{\bar{\eta}}$ must be a singular point of the projection, but cannot be any worse than a node by local constancy of the arithmetic genus. Finally, if $E_{1,\bar{\eta}}$ and $E_{2,\bar{\eta}}$ share a branch, then so do $E_{1,0}$ and $E_{2,0}$; on the other hand, if $E_{i,\bar{\eta}}$ has more than one branch, then so does E_i. Similarly, if $C_{\bar{\eta}}$ contains only one subcurve of genus one, with two nodes joined by a rational chain, so does C_0. The upshot of this discussion is that

$$|E_{i,\bar{\eta}} \cap \overline{C_{\bar{\eta}} \setminus E_{i,\bar{\eta}}}| = |E_{i,0} \cap \overline{C_0 \setminus E_{i,0}}|.$$

The number of markings on E_i is also constant. Hence we can deduce condition (3) for $C_{\bar{\eta}}$ from the same condition on C_0. Condition (2) follows as in Remark 6.4. Condition (2) can be proved analogously when there is no subcurve of genus one.

Finally, suppose that $C_{\bar{\eta}}$ has a genus two singularity, then so does C_0. The (union of the) distinguished branch(es) $E_{\bar{\eta}}$ of $C_{\bar{\eta}}$ is a genus one singularity, and so is its limit E_0 in C_0. It has to contain the distinguished branch(es) of C_0, because any subcurve contained in the union of the axes of C_0 has genus zero; therefore, by assumption, E_0 contains $p_{1,0}$. Then also $E_{\bar{\eta}}$ contains $p_{1,\bar{\eta}}$. □

Recall the following result of Smyth [Smy11a, Lemma 3.3].

Lemma 6.7. *A Gorenstein curve of arithmetic genus one with no disconnecting nodes Z is either: a smooth elliptic curve; a ring of $r \geq 1$ $\mathbb{P}^1$; or an elliptic m-fold point whose normalisation is the disjoint union of m copies of $\mathbb{P}^1$. In all these cases $\omega_Z \simeq \mathscr{O}_Z$.*

We may provide an analogous description of minimal subcurves of genus two.

Lemma 6.8. *A Gorenstein curve of genus two with no disconnecting nodes Z is either:*

(1) a smooth curve of genus two;
(2) the union of E, a Gorenstein curve of genus one with no disconnecting nodes, and R, a (possibly empty) rational chain, along two distinct nodes;
(3) the union of two copies of $(\mathbb{P}^1, 0, 1, \infty)$ with three (possibly empty) rational chains R_0, R_1, R_∞ joining the homonymous points;
(4) an elliptic m-fold point whose normalisation is the disjoint union of either $m-1$ $\mathbb{P}^1$ (two branches coincide), or $m-1$ $\mathbb{P}^1$ and a Gorenstein curve of genus one with no disconnecting node (i.e. there are two genus one subcurves sharing one branch);
(5) or a singularity of genus two with m-branches, whose normalisation is the disjoint union of m copies of $\mathbb{P}^1$.

In all cases there exists a unique $\mathfrak{g}^1_2$. In cases (4) *and* (5)*, given a smooth point p lying on a special branch, there exists a unique point $\bar{p}$ (possibly equal to p), such that $\omega_Z \simeq \mathscr{O}(p+\bar{p})$.*

Proposition 6.9 (Valuative criterion of properness)**.**

Proof. Existence of limits. We start with a smooth n-pointed curve of genus two over a discrete valuation field. By the semistable reduction theorem [DM69, Corollary 2.7], we may find a finite base-change $\Delta' \to \Delta$ and a semistable curve $\mathscr{C}' \to \Delta'$ with regular total space, such that its generic fiber is isomorphic to the pullback of the curve we started with. By Castelnuovo's criterion, we may further assume that the central fiber contains no rational tails.

Now check whether p_1 afferes to a Weierstrass point or not: in the former case, change base with $\pi'' \mapsto (\pi')^3$, in the latter with $\pi'' \mapsto (\pi')^2$; then resolve. This has the effect of replacing every node with a chain of two (resp. one) -2-curve. It is a technical expedient we find useful in the construction. We drop the primes from notation.

Next we identify a (not necessarily connected) subcurve that needs be contracted in order to find the m-stable limit. The process can be thought of as drawing expanding circles on the dual graph (except, they are not always expanding). We may at any point blow-up the curve at a marking on the central fiber, and consider the strict transform of the corresponding section; thus markings can effectively be considered as legs going to infinity in the dual graph.

We start from the case that the core Z is irreducible. Suppose that the level of Z is $l \leq m$; then we may contract (a subcurve containing) Z as follows. Let q_1 the point of Z closest to p_1.

(1) If (Z, q_1) is Weierstrass, call S_h the -2-curves closest to $\tilde{Z}$, and R_h the second closest. Consider the line bundle
$$\mathscr{L}_{j+1} = \omega_{\mathscr{B}_{j+1}/\Delta}(3\tilde{Z} + 2\sum_{h=1}^{l_j-1} S_h + \sum_{h=1}^{l_j-1} R_h + \tilde{\sigma}_{1,j} + \ldots + \tilde{\sigma}_{n,j}).$$
By the contraction lemma 5.1... By the classification of semistable tails, $\mathscr{C}_{j+1,0}$ acquires a singularity of type III_l (which works out by our initial choice of base-change), and p_1 is connected to the singular branch.

(2) If (Z, q_1) is not Weierstrass, call R_h the -2-curves closest to $\tilde{Z}$. In case there is no rational tail attached to $\bar{q}_1$, blow up the latter point. Consider then the line bundle
$$\mathscr{L}_{j+1} = \omega_{\mathscr{B}_{j+1}/\Delta}(2\tilde{Z} + \sum_{h=1}^{l_j-1} R_h + \tilde{\sigma}_{1,j} + \ldots + \tilde{\sigma}_{n,j});$$
By the contraction lemma 5.1... By the classification of semistable tails, $\mathscr{C}_{j+1,0}$ acquires a singularity of type II_l or II_{l+1} (in which case one of the twin branches is dangling), and p_1 is connected to one of the twin branches.

More generally, in case 1 we may draw circles around Z that at each step expand by 1 along the tail containing p_1 and by 3 along all other tails. Note that at each step the number of branches is the same as the level one step before that thanks to our base-change choice. If l denotes the radius of the circle along T_1, the line bundle
$$\omega_{\mathscr{C}}\left(3lZ + \sum_{R_i \in T_1} 3(l - \operatorname{dist}(R_i, Z))R_i + \sum_{R_i \notin T_1} (l - \operatorname{dist}(R_i, Z))R_i + \sigma_1 + \ldots + \sigma_n\right)$$
performs the desired contraction.

Suppose the minimal subcurve of genus two Z contains two subcurves of genus one; call E_1 and E_2 the minimal such, and assume that p_1 afferes to E_1. Start drawing circles around E_2. If E_2 already has level bigger than $m+1$, stop with the circle of radius 0. Otherwise grow the radius by 1 at a time. The curve to be contracted is the inner disk, so the number of branches is measured by the vertices lying on the circle, and the level by the number of exciding edges. Both are non-decreasing with the radius. We exmine the Weierstrass case; the conjugate is entirely analogous. Note that at this stage we perform one meaningful step every three, due to our choice of base-change.

(1) If level $\geq m+2$ is reached before the circle touches E_1, take the next possible $\equiv 2 \mod 3$ radius, then contract the inner circle by the line

bundle

$$\omega_{\mathscr{C}}((l_2+1)E_2 + \sum_i \max(l_2+1-\mathrm{dist}(E_2,R_i),0)R_i \oplus \sigma_1 \oplus \ldots \oplus \sigma_n)$$

where l_2 is the radius of the circle around E_2. Consider now E_1: if $\mathrm{lev}(E_1) \leq m+1$ start expanding the circle around it. Again, either level $\geq m+2$ can be reached before touching E_2, or, by contracting the maximal balanced subcurve of genus one containing E_1, we produce a curve having two genus one singularities that share a branch. Notice that in this case p_1 afferes to the only genus one subcurve that may have level $\leq m+1$.

(2) Otherwise, one step before reaching E_1, we may contract to produce a genus one singularity with a genus one branch. If the level is $\leq m$ at this point, consider the genus two subcurve Z as a whole. Observe that the line bundle we would like to consider at this point starts with weight 3 instead of 1 along the tail connecting p_1 to E_1. This means that it will be supported two steps further along each rational tail departing from Z except the tail containing p_1. Note also that getting to include E_2 happens at a step $\equiv 0 \mod 3$, therefore including two more components on each rational tail will not make the number of branches grow above m. We may now continue as before, at every step expanding the circle by 1 along T_1 and by 3 along all other rational tails.

In case p_1 is equidistant from E_1 and E_2 (it must then affere to the rational chain joining them), start by expanding a circle around the one with lower level; if they have the same level, expand them simultaneously. If at a later stage p_1 becomes closer to one of the two circles, proceed as above.

Uniqueness of limits. By the theorem of semistable reduction [DM69], we may find a diagram

$$\begin{tikzcd} & \mathscr{C}^{ss} \arrow[dl, "\phi"'] \arrow[dr, "\phi'"] & \\ \mathscr{C} \arrow[dr] & & \mathscr{C}' \arrow[dl] \\ & \Delta & \end{tikzcd}$$

that extends the isomorphism between the generic fibers, with $\mathscr{C}^{ss}$ regular.

Claim: If $\mathscr{C}'_0$ has only singularities of genus $\leq i$ ($i = 0, 1$), then so does $\mathscr{C}_0$.

First assume that $\mathscr{C}'_0$ has only nodes. If $\mathscr{C}_0$ has a singular point x of genus one, $E := \phi^{-1}(x)$ is an unmarked subcurve of arithmetic genus one and level $\leq m+1$ of $\mathscr{C}_0^{ss}$; then so is $\phi'(E)$ (recall that ϕ' is a contraction, therefore it has connected fibers, which excludes the possibility that ϕ' lowers the genus of E by realising a finite cover of a line), contradicting the m-stability of $\mathscr{C}'$. Argue similarly if x is a genus two singularity with $\leq m$ branches. Finally, if x is dangling II_{m+1}, there is a -1-curve R adjacent to $\phi^{-1}(x)$; ϕ' must contract R by DM stability of $\mathscr{C}'$, hence $\phi'(\phi^{-1}(x))$ is a genus two curve of level $\leq m$, which is again absurd.

The case $i=1$ is more delicate; suppose $\mathscr{C}_0$ has a singularity x of genus two; the case of a dangling II_{m+1} can be excluded as above. Then $\mathscr{C}_0^{ss} = Z \cup R_1 \cup \ldots \cup R_l$, with $Z = \phi^{-1}(x)$ and $l \leq m$. If Z has geometric genus two, or is irreducible of geometric genus one, $\phi'(Z)$ violates m-stability of $\mathscr{C}'$. If Z contains a unique subcurve E of genus one, with a rational chain R connecting E to itself, then p_1 must affere to R by the analysis of semistable tails of genus two. Therefore $\operatorname{lev}(E) \leq (l-1)+2 \leq m+1$. Similarly, if Z contains two subcurves of genus one E_1 and E_2, then $(\operatorname{lev}(E_1)-1)+(\operatorname{lev}(E_2)-1) \leq l$, hence at least one of the two has level $\leq m+1$ (the level of the other one has to be positive anyway). In all cases, $\phi'(E)$ contradicts m-stability of $\mathscr{C}'$.

Claim: We may assume that $\mathscr{C}^{ss}$ contains either no -1-curve, or only one, which is contracted by neither ϕ nor ϕ'.

If there is a -1-curve contracted by both, ϕ and ϕ' factor through a smaller regular model. Assume there is a -1-curve not contracted by ϕ. Then, by stability, its image is one of the special branches of a dangling singularity of type II_{l+1}, $l \leq m$; call x the singular point, and $Z = \phi^{-1}(x)$. We may then write $\mathscr{C}_0 = Z \cup R_0 \cup \ldots \cup R_l$, with $R_0 = R$, and R_1 (the tail including) the preimage of the other special branch, which contains p_1. Is it possible that ϕ' contracts a different subcurve than Z? By the previous claim, ϕ' has to contract a subcurve of genus two Z'. If Z' contains R and is of shape II, then it is strictly larger than Z, therefore its contraction will not be dangling and it will have at least $m+1$ branches, by the condition on the level of the minimal genus two subcurve of $\mathscr{C}_0$. Otherwise Z' can be of shape III; this forces R_0 and R_1 to meet on a trunk T, that is attached to a Weierstrass point of the core of $\mathscr{C}_0^{ss}$. If Z' starts from the top of T or further away from the core along R_1, then Z' contains Z strictly, therefore $\mathscr{C}_0'$ will have a singularity of type $III_{\geq m+1}$, by the assumption on the level of the minimal subcurve of genus two of $\mathscr{C}_0$. On the other hand, if Z' starts closer to the core than the top of T, then the level of the minimal subcurve of genus two of $\mathscr{C}'$ will be at most m. In fact this argument proves more, namely that if $\mathscr{C}_0$ has a dangling II singularity, then the exceptional loci of ϕ and ϕ' coincide, therefore $\mathscr{C} \cong \mathscr{C}'$ (see below).

Claim: The exceptional loci of ϕ and ϕ' coincide.

If $\mathscr{C}_0$ has only nodes, then so does $\mathscr{C}_0'$, and we can conclude by the uniqueness part of the stable reduction theorem [DM69]. The meat is in the higher genus components of the exceptional loci.

If $\mathscr{C}_0$ has a genus one singularity x, then it cannot have a genus two singularity, so neither can $\mathscr{C}_0'$. If $\mathscr{C}_0$ has a second genus one singularity y, let $E_1 = \phi^{-1}(x)$ and $E_2 = \phi^{-1}(y)$; they are disjoint balanced subcurves of genus one and level $\leq m+1$ in $\mathscr{C}_0^{ss}$, therefore ϕ' must contract them. Enlarging the contraction radius of any one of them would produce a singularity with at least $m+2$ branches, unless by enlarging we make them touch, in which case we should contract to a genus two singularity, which is also not allowed. The case of a single genus one singularity with a genus one branch, or with a disjoint subcurve of genus one, is entirely similar. In the case of a genus one singularity with two branches joined by a (possibly empty) rational chain R, notice that the argument that expanding the contraction radius increases the number of branches of $\phi'(E')$ above $m+1$ may fail in the case that the contraction circle includes the entire R, but in this case we would find a genus two singularity in $\mathscr{C}'_0$.

Finally, if $\mathscr{C}_0$ has a genus two singularity x - the case of a dangling II_{m+1} was dealt with above -, write $\mathscr{C}_0^{ss} = Z \cup R_1 \cup \ldots \cup R_l$, with $Z = \phi^{-1}(x)$ and $l \leq m$. Now $\phi'(Z)$ must be a point x', by stability considerations. If p_1 afferes to a non-Weierstrass point of Z, x' must be of type II; therefore $Z' = (\phi')^{-1}(x')$ has shape II, which is determined by one parameter. Since $Z \subsetneq Z'$ implies x' has at least $m+1$ branches by level considerations, we conclude $Z = Z'$. If instead p_1 afferes to a Weierstrass point, there are multiple possibilities. If x is type III, and x' as well, the argument is as before; if instead x' is of type II, notice that Z' must start further away from the core than Z (otherwise $Z' \subsetneq Z$, hence the level of $\mathscr{C}'_0$ is not enough), but then $Z \subsetneq Z\prime$, and the singularity of $\mathscr{C}'_0$ is too bad. The other cases are symmetric.

The claim now follows from the observation that the exceptional loci of ϕ and ϕ' are the fibers over higher genus singularity (call them Z) union those rational curves with only two special points that are disjoint from Z.

Claim: The generic isomorphism between $\mathscr{C}$ and $\mathscr{C}'$ extends over Δ.

Follows from [Deb01, Lemma1.13]. □

Definition 6.10. Fix positive integers $m < n$. Let $(C, p_1, \ldots, p_n)$ be a connected, reduced, complete curve of arithmetic genus two, marked by smooth points. We say that C is m-stable if:

(1) C has only nodes; elliptic l-fold points, $l \leq m+1$; type $II_{\leq m}$, dangling II_{m+1}, and type $III_{\leq m}$ genus two singularities as singular points.
(2) If Z is a connected subcurve of arithmetic genus two, then $\mathrm{lev}(Z) > m$.
(3) If E is a nodally attached subcurve of arithmetic genus one, then $\mathrm{lev}(E) > m+1$.
(4) $H^0(C, \Omega_C^\vee(-\sum_{i=1}^n p_i)) = 0$.
(5) If C contains a singularity of genus two, p_1 is connected (through a rational chain) to one of the distinguished branches.
(6) If there is a Gorenstein subcurve of genus one and level less than $m+2$, then it is not nodally attached and p_1 afferes to it.

Remark 6.11. Non-Gorenstein subcurves appear by taking the union of some - but not all - the branches of a Gorenstein singularity of genus one or two.

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