

# Modular compactifications of $\mathcal{M}_{2,n}$ with Gorenstein curves

Luca Battistella

## ABSTRACT

We study the geometry of Gorenstein curve singularities of genus two, and of their stable limits. These singularities come in two families, corresponding to either Weierstrass or conjugate points on a semistable tail. For every  $1 \leq m < n$ , a stability condition - using one of the markings as a reference point, and thus not  $\mathfrak{S}_n$ -symmetric - defines proper Deligne-Mumford stacks  $\overline{\mathcal{M}}_{2,n}^{(m)}$  with a dense open substack representing smooth curves.

## 1. Introduction

We construct alternative compactifications of the moduli stack of smooth  $n$ -pointed curves of genus two. The boundary of the Deligne-Mumford compactification, consisting of stable nodal curves, is gradually replaced by ever more singular curves, complying with more restrictive combinatorial requirements on the dual graph. For  $1 \leq m < n$ , we introduce a notion of  $m$ -stability, that allows Gorenstein singularities of genus one and two while at the same time demanding that higher genus subcurves contain a minimum number of special points. Our main result concerning the stack of  $m$ -stable curves is the following:

**THEOREM.**  $\overline{\mathcal{M}}_{2,n}^{(m)}$  is a proper irreducible Deligne-Mumford stack over  $\mathrm{Spec}(\mathbb{Z}[\frac{1}{30}])$ .

This paper fits into the framework of alternative compactifications and birational geometry of the moduli space of curves, extending work of D.I. Smyth in genus one, but we expect it to find applications to enumerative geometry as well.

We classify Gorenstein singularities of genus two with any number of branches, and their (semi)stable models, highlighting the relation with Brill-Noether theory. The key insight in defining the new stability conditions is that we can avoid non-Gorenstein singularities by modifying the curve at the conjugate point of the special branch; we use one of the markings to identify the latter, and, more generally, to pick the limit in symmetric situations where multiple choices are a priori possible - as a result, our stability conditions are not  $\mathfrak{S}_n$ -symmetric.

We interpret crimping spaces (moduli of curves with a prescribed singularity type) as parameter spaces for the differential geometric data needed in order to construct a higher genus singularity from an ordinary  $m$ -fold point, and verify their impact on the existence of automorphisms, a phenomenon which had not fully emerged in lower genus.

---

2010 *Mathematics Subject Classification* 14H10 (primary), 14H20 (secondary).

*Keywords:* moduli of curves, Gorenstein singularities, genus two, crimping spaces

This project grew out of discussions with Francesca Carocci, whom we thank heartily. We also thank Daniele Agostini, Fabio Bernasconi, Maria Beatrice Pozzetti, and Luca Tasin for helpful conversations. We thank the Max Planck Institute for Mathematics in Bonn for providing financial support and a stimulating research environment.

Though a conspicuous amount of related research has been carried out on the birational geometry of  $\overline{\mathcal{M}}_{2,n}$  for low values of  $n$  [Has05, HL07, Rul01, HL14, FG18, JP18], this appears to be the first proposal of a sequence of modular compactifications for every  $n$ .

### 1.1 From the Deligne-Mumford space to the Hassett-Keel program

One of the most influential results of modern algebraic geometry is the construction of a modular compactification of the stack of smooth pointed curves  $\mathcal{M}_{g,n}$ , due to P. Deligne, D. Mumford, and F. Knudsen, with the introduction of *stable* pointed curves.

DEFINITION 1.1. [DM69] A connected, reduced, complete curve  $C$  over an algebraically closed field  $\mathbf{k}$ , with distinct markings  $(p_1, \dots, p_n)$  lying in the smooth locus of  $C$ , is *stable* if:

- (i)  $C$  admits only nodes (ordinary double points) as singularities;
- (ii) every rational component of  $C$  has at least three special points (markings or nodes), and every elliptic component has at least one.

THEOREM 1.2. [DM69, Knu83] Assume  $2g - 2 + n > 0$ . The moduli stack of stable pointed curves  $\overline{\mathcal{M}}_{g,n}$  is a smooth and proper connected Deligne-Mumford stack over  $\mathrm{Spec}(\mathbb{Z})$ , with projective coarse moduli space  $\overline{\mathbf{M}}_{g,n}$ , and normal crossing boundary representing nodal curves.

On one hand, the Deligne-Mumford compactification has nearly every desirable property one could hope for; on the other, it is certainly not the unique modular compactification of  $\mathcal{M}_{g,n}$ . Classifying all of them is a challenging task, which was set out and partially performed in the inspiring work of Smyth [Smy13]. The motivation comes mostly from birational geometry.

Even though the existence of  $\overline{\mathbf{M}}_{g,n}$  can be deduced from nowadays standard theorems on stacks [KM97], this moduli space was first constructed as a quotient, prompting the development of a powerful technique known as Geometric Invariant Theory [Gie82, MFK94, BS08]. Studying alternative compactifications of  $\mathcal{M}_{g,n}$  sheds some light on the Mori chamber decomposition of  $\overline{\mathbf{M}}_{g,n}$ , and it is not by chance that the first steps in this direction were moved from a GIT perspective - by changing the invariant theory problem or the stability condition under consideration, and analysing the modular properties of the resulting quotients [Sch91, Has05, HH13]. The consequent program that goes under the name of B. Hassett and S. Keel aims to describe all the quotients arising in this way, and to determine whether every step of a log minimal model program for  $\overline{\mathbf{M}}_{g,n}$  enjoys a modular interpretation in terms of curves with worse than nodal singularities [CTV18, CTV19]. Since the early stages of this program, it has developed into a fascinating playground for implementing ideas that originated from (v)GIT into a general structure theory of Artin stacks [AK16, AFSvdW17, AFS17a, AFS17b]. See for instance [Mor11, FS13] for more detailed and comprehensive accounts.

Only few steps of the Hassett-Keel program have been carried out in full generality. Yet, the program has been completed to a larger extent in low genus: with the introduction of Boggi-stable [Bog99] and weighted pointed curves [Has03] in genus zero, and with Smyth's pioneering work in genus one [Smy11a, Smy11b, Smy18], extending earlier work of D. Schubert. In a nutshell, an alternative compactification is defined by allowing a reasonably larger class of curve singularities (*local condition*) while identifying their (semi)stable models, and disallowing the latter by imposing a stronger stability condition (*global condition*, typically combinatorial); the valuative criterion ensures that the resulting moduli problem remains separated and universally closed.

A useful notion in this respect is that of the *genus* of an isolated curve singularity: let  $(C, q)$  be (the germ of) a reduced curve over an algebraically closed field  $\mathbf{k}$  at its unique singular point

$q$ , with normalisation  $\nu: \tilde{C} \rightarrow C$  and  $\mathcal{F} = \nu_* \mathcal{O}_{\tilde{C}} / \mathcal{O}_C$ , a skyscraper sheaf supported at  $q$ .

DEFINITION 1.3. [Smy11a] If  $C$  has  $m$  branches (irreducible components of the normalisation) at  $q$ , and  $\delta$  is the  $\mathbf{k}$ -dimension of  $\mathcal{F}$ , the genus of  $(C, q)$  is defined as:

$$g = \delta - m + 1.$$

The genus can be thought of as the number of conditions that a function must satisfy in order to descend from the seminormalisation (the initial object in the category of universal homeomorphisms  $C' \rightarrow C$ , see [Sta19, Tag 0EUS], or a curve with the same topological space as  $C$  and an ordinary  $m$ -fold point at  $q$ ) to  $C$ . The node, for example, has genus zero (it coincides with its own seminormalisation). That of the genus is a notion that behaves well in families: a complete, reduced curve  $C$  with only one singularity at  $q$ , of  $\delta$ -invariant  $\delta$ , and normalisation a union of  $m$  copies of  $\mathbb{P}^1$ , can only appear in a family of curves of arithmetic genus  $g$ .

Smyth found that, for every fixed number  $m$  of branches, there is a unique germ of Gorenstein singularity of genus one up to isomorphism, namely:

- $m = 1$  the cusp,  $V(y^2 - x^3) \subseteq \mathbb{A}_{x,y}^2$ ;
- $m = 2$  the tacnode,  $V(y^2 - yx^2) \subseteq \mathbb{A}_{x,y}^2$ ;
- $m \geq 3$  the union of  $m$  general lines through the origin of  $\mathbb{A}^{m-1}$ .

Singularities of this kind, with up to  $m$  branches, together with nodes, form a deformation-open class of singularities. Moreover, the elliptic  $m$ -fold point can be obtained by contracting a smooth elliptic curve with  $m$  rational tails in a one-parameter smoothing, and, roughly speaking, all stable models have a shape similar to this.

DEFINITION 1.4. [Smy11a] A connected, reduced, complete curve  $C$  of arithmetic genus one with smooth distinct markings  $(p_1, \dots, p_n)$  is  *$m$ -stable*,  $1 \leq m < n$ , if:

- (i) it admits only nodes and elliptic  $l$ -fold points,  $l \leq m$ , as singularities;
- (ii) for every connected subcurve  $E \subseteq C$  of arithmetic genus one, its *level*:  
 $|E \cap \overline{C \setminus E}| + |\{i: p_i \in E\}|$  is strictly larger than  $m$ ;
- (iii)  $H^0(C, \Omega_C^\vee(-\sum_i p_i)) = 0$  (finiteness of automorphism groups).

The latter can be taken for a decency condition on the moduli stack. The first two, instead, are essential in guaranteeing the uniqueness of  $m$ -stable limits, as per the discussion above. Smyth's main result is the following.

THEOREM 1.5. [Smy11a, Smy11b] *The moduli stack of  $m$ -stable curves  $\overline{\mathcal{M}}_{1,n}(m)$  is a proper irreducible Deligne-Mumford stack over  $\text{Spec } \mathbb{Z}[1/6]$ . It is not smooth for  $m \geq 6$ . The coarse moduli spaces  $\overline{\mathbf{M}}_{1,n}(m)$  arise as birational models of  $\overline{\mathbf{M}}_{1,n}$  for the big line bundles  $D(s) = s\lambda + \psi - \Delta$ , where  $\lambda$  is the Hodge class,  $\psi$  is the sum of the  $\psi$ -classes,  $\Delta$  is a boundary class, and there is an explicit relation between  $s$  and  $m$ .*

Some further information on the geometry and singularities of these spaces (with the restriction  $m = n - 1$ ) has been discovered by Y. Lekili and A. Polishchuk in their study of *strongly non-special* curves [LP17].

## 1.2 Experimenting on a genus two tale

In this subsection, we walk through the motivations and methods at the heart of our construction, exemplifying them in the simplest possible case, that of  $\overline{\mathcal{M}}_{2,2}^{(1)}$ . The facts we mention are either proved or explained in greater detail and generality in the paper. Here is a classical

FACT. *There are two unibranch singularities of genus two, the ramphoid cusp or  $A_4$ -singularity  $V(y^2 - x^5) \subseteq \mathbb{A}_{x,y}^2$ , and the ordinary genus two cusp  $\text{Spec}(\mathbf{k}[t^3, t^4, t^5])$ . The former is Gorenstein, with stable model a Weierstrass tail (a genus two curve attached to a rational one at a Weierstrass point), while the latter is not Gorenstein, with stable model a non-Weierstrass tail of genus two.*

See Lemma 2.1 and Proposition 4.3 below. Recall that every smooth curve of genus two is hyperelliptic, i.e. it can be realised as a two-fold cover of  $\mathbb{P}^1$ , in a unique way up to projectivities. The deck transformation is called the hyperelliptic involution  $\sigma$ ; ramification points (fixed points of  $\sigma$ ) are called Weierstrass, and in general  $\{p, \sigma(p)\}$  are called conjugate points. See Remark 4.1.

Let us try Smyth's approach out on genus two curves, starting with  $\overline{\mathcal{M}}_{2,2}^{(1)}$ . If we are going to require the level of a genus two subcurve to be at least two, it seems that we will need non-Gorenstein singularities in order to keep our moduli space proper. This might lead us into trouble; for example, the (log) dualising line bundle is classically exploited to construct canonical polarisations on stable curves, which in turn are essential in the proof that  $\overline{\mathcal{M}}_{g,n}$  is an algebraic stack (or in the GIT construction of  $\overline{\mathbf{M}}_{g,n}$ ). Yet, there is a way around the singularity  $\mathbf{k}[[t^3, t^4, t^5]]$ .

FACT. *The  $A_5$ -singularity  $V(y^2 - yx^3) \subseteq \mathbb{A}_{x,y}^2$  is a Gorenstein singularity of genus two with two branches. Its stable model is a genus two bridge, with conjugate attaching points. A marked union of two copies of  $\mathbb{P}^1$  along an  $A_5$ -singularity has no non-trivial automorphisms as soon as one of the two branches contains at least two markings.*

See Proposition 2.2 and Corollary 3.6. Going back to  $\overline{\mathcal{M}}_{2,2}$ , suppose  $C$  is the nodal union of a genus two curve  $Z$  with a rational tail  $R$  supporting the two markings, so that  $\text{lev}(Z) = 1$ . If  $R$  is attached to a Weierstrass point of  $Z$ , we may simply contract the latter (in a one-parameter smoothing), thus producing an irreducible ramphoid cusp with two markings. If instead  $R$  is attached to a non-Weierstrass point  $q_1$  of  $Z$ , we may blow-up the one-parameter family at the conjugate point  $\sigma(q_1)$  in the central fibre, and then contract  $Z$  to get a *dangling*  $A_5$ -singularity (meaning that one of the branches is unmarked), which nonetheless has trivial automorphism group. We pursue this strategy, which makes our compactifications not semistable (see [Smy13, Definition 1.2] for the terminology). The necessity to include such curves was prefigured in [AFS16].

To complete the picture, note that, in order to fix a deformation-open class of singularities, we need to allow cusps and tacnodes as well, due to the following

FACT. *The singularities appearing in the miniversal family of an  $A_m$ -singularity are all and only the  $A_l$ -singularities with  $l \leq m$ .*

See Theorem 2.5 for a more general statement - valid for all ADE singularities - due to A. Grothendieck. For the sake of separatedness, we should at the same time require that the level of a genus one subcurve be at least three. Hybrid situations may occur, such as an elliptic curve with a cusp, or an irreducible tacnode; since we need to allow a tacnode and a cusp sharing a branch, we should impose the level condition on genus one subcurves only when they are *nodally attached*. Besides, in the latter example, we need to break the  $\mathfrak{S}_2$ -symmetry (relabelling the markings) in order to have a unique limit: we declare that  $p_1$  must lie on the cuspidal branch. See Figure 1.

We are now in a position to cast a plausible definition of  $\overline{\mathcal{M}}_{2,2}^{(1)}$ .

DEFINITION 1.6. A connected, reduced, complete curve of arithmetic genus two  $C$  over an algebraically closed field  $\mathbf{k}$ , with smooth and disjoint markings  $(p_1, p_2)$ , is *1-stable* if:

- (i)  $C$  has only  $A_1$ -,  $\dots$ ,  $A_4$ - and dangling  $A_5$ - singularities.

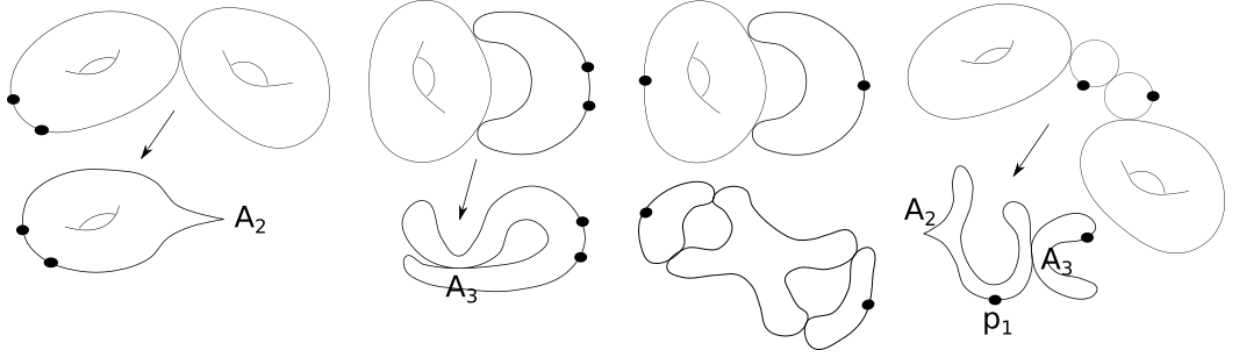


FIGURE 1. Examples of 2-pointed stable curves and their 1-stable counterparts.

- (ii)  $C$  coincides with its minimal subcurve of arithmetic genus two.
- (iii) A subcurve of arithmetic genus one is either nodally attached and of level three, or it is not nodally attached and it contains  $p_1$ .

The main result of the paper is that  $\overline{\mathcal{M}}_{2,2}^{(1)}$  is a proper Deligne-Mumford stack, and the generalisation of this statement to an arbitrary number of markings and a range of stability conditions that we are going to discuss in the next sections.

Let us note in passing that the birational map  $\overline{\mathcal{M}}_{2,2} \dashrightarrow \overline{\mathcal{M}}_{2,2}^{(1)}$  is not defined everywhere. The reason boils down to the following

**FACT.** *There is only one isomorphism class of 2-pointed curves whose normalisation is  $(\mathbb{P}^1, q_1) \sqcup (\mathbb{P}^1, q_2, p_1, p_2)$  and having an  $A_5$ -singularity at  $q_1 = q_2$ . On the other hand, the moduli space of 2-pointed irreducible curves of geometric genus zero with an  $A_4$ -singularity is isomorphic to  $\mathbb{A}^1$ .*

The second statement can be motivated as follows: the pointed normalisation of such a curve is  $(\mathbb{P}^1, q, p_1, p_2)$ , which has neither automorphisms, nor deformations. To produce an  $A_4$ -singularity at  $q$  we may first collapse a non-zero tangent vector at  $q$  (no choice involved), producing a cusp, and then collapse a line in the tangent space at the cusp, avoiding the support of its tangent cone  $\ell$  (therefore, the moduli space is  $\mathbb{P}^1 \setminus \{\ell\} = \mathbb{A}^1$ ). See Lemma 3.2 and the discussion thereafter.

Let  $\Delta = \Delta_{2,\emptyset|0,\{1,2\}} \subseteq \overline{\mathcal{M}}_{2,2}$  be the divisor of rational tails, and  $\mathscr{W} \subseteq \overline{\mathcal{M}}_{2,2}$  the codimension two locus of Weierstrass tails. The 1-stable limit of any point in  $\Delta \setminus \mathscr{W}$  is the dangling  $A_5$ -singularity, while the 1-stable limit of a Weierstrass tail is ill-defined (it depends on the choice of a 1-parameter smoothing); we conjecture that the rational map (identity on the locus of smooth curves) admits a factorisation:

$$\begin{array}{ccc}
 & \text{Bl}_{\mathscr{W}}(\overline{\mathcal{M}}_{2,2}) & \\
 \swarrow & & \searrow \\
 \overline{\mathcal{M}}_{2,2} & \dashrightarrow & \overline{\mathcal{M}}_{2,2}^{(1)}
 \end{array}$$

The blow-up should also encode enough information to contract an unmarked elliptic bridge to a tacnode. We plan to address this point in forthcoming work.

### 1.3 Relation to other work

It would be interesting to compare  $\overline{\mathcal{M}}_{2,2}^{(1)}$  explicitly with Smyth's  $\overline{\mathcal{M}}_{2,2}(\mathcal{Z})$  [Smy13], for the extremal assignment  $\mathcal{Z}$  of unmarked subcurves; here we only note that, while the divisor  $\Delta_{1,\{1\}|1,\{2\}}$  is contracted in  $\overline{\mathcal{M}}_{2,2}^{(1)}$ , the latter contains a copy of  $\overline{\mathcal{M}}_{0,4}$  (see the third column, second row of Figure 1) that is replaced by the class of the rational 4-fold point in  $\overline{\mathcal{M}}_{2,2}(\mathcal{Z})$ .  $\overline{\mathcal{M}}_{2,2}^{(1)}$  seems closely related to the space  $\mathcal{U}_{2,2}^{ns}(ii)$  constructed in [JP18]. More generally, it would be interesting to relate  $\overline{\mathcal{M}}_{2,n}^{(m)}$  (for high values of  $m$ ) to Polishchuk's moduli of curves with nonspecial divisors [Pol15]. Finally, it seems plausible that  $\overline{\mathcal{M}}_{2,n}^{(m)}$  (for low values of  $m$ ) corresponds to a pointed variant of the spaces of admissible hyperelliptic covers with AD singularities constructed in [Fed14].

### 1.4 Outline of results and plan of the paper

In Section 2 we classify all the Gorenstein curve singularities of genus two. They come in two families: the first one (*I*) includes the ramphoid cusp, the  $D_5$ -singularity, and for  $m \geq 3$  the union of a *singular branch* (a cusp) and  $m - 1$  lines living in  $\mathbb{A}^m$ . The second one (*II*) includes the  $A_5$ - and  $D_6$ -singularities, and for  $m \geq 4$  the union of *two tangent branches* (forming a tacnode) with  $m - 2$  lines in  $\mathbb{A}^{m-1}$ . See Proposition 2.2.

In Section 3 we translate the condition that a complete pointed curve of genus two has no infinitesimal automorphisms into a *mostly* combinatorial criterion. For every fixed number of branches  $m$  and genus two singularity type  $\in \{I, II\}$ , there are two isomorphism classes of pointed curves whose normalisation is  $\bigsqcup_{i=1}^m (\mathbb{P}^1, q_i, p_i)$  and having a singularity of the prescribed type at  $q$ ; one of them has  $\text{Aut}(C, p) = \mathbb{G}_m$ , while the other one has trivial automorphism group. This phenomenon is a novelty to genus two. We take a detour into moduli spaces of singularities to justify the claim, and explain how to interpret the *crimping spaces* geometrically in terms of the information we need to construct a genus two singularity from a (non-Gorenstein) singularity of lower genus. This is not strictly necessary in what follows, since the singularity with one-pointed branches never satisfies the level condition we demand from our curves, yet this description is expected to be useful in analysing the indeterminacy of  $\overline{\mathcal{M}}_{2,n}^{(m_1)} \dashrightarrow \overline{\mathcal{M}}_{2,n}^{(m_2)}$ .

In Section 4 we study the (semi)stable limits; starting from a 1-parameter family of semistable curves with smooth generic fibre and regular total space, we show that the shape of a subcurve of the central fibre that can be contracted into a Gorenstein singularity is strongly constrained. Singularities of type *I* arise when the special branch (corresponding to the cusp in the contraction) is attached to a *Weierstrass point* of the minimal subcurve of genus two (the *core*), while singularities of type *II* occur when the special branches (corresponding to the tacnode in the contraction) are attached to *conjugate points*. Furthermore, the size of the curve to be contracted only depends on one number - roughly speaking, the distance of the special branches from the core. The first statement is a consequence of the following simple observation: if  $\phi: \tilde{\mathcal{C}} \rightarrow \mathcal{C}$  is a contraction to a family of Gorenstein curves,  $\phi^* \omega_{\mathcal{C}}$  is trivial on a neighbourhood of the exceptional locus of  $\phi$ , and it coincides with  $\omega_{\tilde{\mathcal{C}}}$  outside it. Now, whereas the dualising line bundle of a Gorenstein curve of genus one with no separating nodes is trivial (see [Smy11a, Lemma 3.3]) and all smooth points display the same behaviour (in the sense that they are non-special), the simplest instance of Brill-Noether theory manifests itself in genus two, with the distinction between Weierstrass and non-Weierstrass points, and the expression  $\omega_{\mathcal{Z}} = \mathcal{O}_{\mathcal{Z}}(q + \sigma(q))$ . The correct extension of these concepts to nodal curves was formulated in the '80s within the theory of admissible covers and limit linear series, and we spend some time to discuss the relevant combinatorics.

In Section 5 we define the notion of *m-stable* *n*-pointed curve of genus two, for every  $1 \leq m < n$ . The basic idea is to trade worse singularities - of both genus one and two, bounded by *m* in the sense of the embedding dimension - with more constraints on the combinatorics of the dual graph - the *level* condition, which bounds below in terms of *m* the number of special points (nodes and markings) that any subcurve of genus one or two has to contain. On the other hand, it is already clear from the discussion above that we need to break the  $\mathfrak{S}_n$ -symmetry, in order to write the dualising line bundle of the minimal subcurve of genus two as  $\mathcal{O}_Z(q_1 + \sigma(q_1))$ , in other words to choose which branches of a semistable model are to be dubbed special. We do so by using the first marking as a reference point, so that  $q_1$  comes to denote the point of  $Z$  closest to  $p_1$ . This shapes our algorithm to construct the *m*-stable limit of a given 1-parameter smoothing. Unavoidably, the formulation of the stability condition is slightly involved, including a prescription of the interplay between  $p_1$  and the singularity. We prove that the moduli stack of *m*-stable curves is algebraic, and it satisfies the valuative criterion of properness.

### 1.5 Future directions of work

Besides regarding this paper as a case-study of the birational geometry of moduli spaces of curves, we are looking forwards to explore its consequences in Gromov-Witten theory. We set up some questions we would like to come back to in future work.

- (i) Resolve the indeterminacy of the rational map  $\overline{\mathcal{M}}_{2,n}^{(m_1)} \dashrightarrow \overline{\mathcal{M}}_{2,n}^{(m_2)}$ , and study the intersection theory of these spaces; we expect the construction to rely on a semistable compactification of the crimping spaces of the genus two singularities, as in [vdW10, §1.10] and [Smy18], as well as on logarithmic geometry, as in [RSW17a]. The study of the Picard group and the projectivity of the coarse moduli space will firmly site this work in the context of the Hassett-Keel program, as in [Smy11b]; we hope techniques similar to [AFSvdW17] and [CTV18] apply. More generally, a question outstanding to our knowledge is whether the whole program fits in the framework of stability developed in [Hal14].
- (ii) Applications to enumerative geometry: the link between reduced Gromov-Witten invariants in genus one (see for example [VZ08, Zin09, LZ09]) and maps from singular curves (see [Vis12]) was partially uncovered in [BCM18], and brought in plain view by [RSW17a, RSW17b]. With F. Carocci we are investigating whether similar techniques may serve to desingularise the main component of the space of genus two maps to projective space. We expect there will be a (n iso)morphism to the modular blow-up constructed in [HLN12]. Maps from singular curves would provide a conceptual definition of reduced invariants for projective complete intersections and beyond, and hopefully make comparison results (standard vs. reduced) accessible. This would lead to a modular interpretation of Gopakumar-Vafa invariants [Pan99].

## 2. Gorenstein curve singularities of genus two

In this and the next sections we work over an algebraically closed field  $\mathbf{k}$  of characteristic different from 2, 3, 5. We produce an algebraic classification of the (complete) local rings of Gorenstein curve singularities of genus two.

Let  $(C, q)$  be the germ of a reduced curve singularity, and let  $(R, \mathfrak{m})$  denote  $(\hat{\mathcal{O}}_{C,q}, \mathfrak{m}_q)$ , with normalisation  $(\tilde{R}, \tilde{\mathfrak{m}}) \simeq (\mathbf{k}[[t_1]] \oplus \dots \oplus \mathbf{k}[[t_m]], \langle t_1, \dots, t_m \rangle)$ . Here *m* is the number of branches of *C* at *q*. Recall the Definition 1.3 of the genus:

$$g = \delta - m + 1;$$

so, for genus two,  $\delta = m + 1$ . Following [Smy11a, Appendix A], we consider  $\tilde{R}/R$  as a  $\mathbb{Z}$ -graded module with:

$$(\tilde{R}/R)_i := \tilde{\mathfrak{m}}^i / (\tilde{\mathfrak{m}}^i \cap R) + \tilde{\mathfrak{m}}^{i+1};$$

furthermore, adapting Smyth's remarks in *loc. cit.* to our situation:

- (i)  $m + 1 = \delta(p) = \sum_{i \geq 0} \dim_{\mathbf{k}}(\tilde{R}/R)_i$ ;
- (ii)  $2 = g = \sum_{i \geq 1} \dim_{\mathbf{k}}(\tilde{R}/R)_i$ ;
- (iii) if  $(\tilde{R}/R)_i = (\tilde{R}/R)_j = 0$  then  $(\tilde{R}/R)_{i+j} = 0$ .

We will also make use of the following observations:

- (iv)  $\sum_{i \geq j} (\tilde{R}/R)_i$  is a grading of  $\tilde{\mathfrak{m}}^j / (\tilde{\mathfrak{m}}^j \cap R)$ ;
- (v) there is an exact sequence of  $R/\mathfrak{m} = \mathbf{k}$ -modules:

$$0 \rightarrow A_i := \frac{\tilde{\mathfrak{m}}^i \cap R}{\tilde{\mathfrak{m}}^{i+1} \cap R} \rightarrow \frac{\tilde{\mathfrak{m}}^i}{\tilde{\mathfrak{m}}^{i+1}} \rightarrow (\tilde{R}/R)_i \rightarrow 0$$

LEMMA 2.1. *There are two unibranch curve singularities of genus two; only one of them is Gorenstein, namely the  $A_4$ -singularity or ramphoid cusp:  $V(y^2 - x^5) \subseteq \mathbb{A}_{x,y}^2$ .*

*Proof.* In the unibranch case  $\dim_{\mathbf{k}}(\tilde{R}/R)_1 \leq 1$ , hence equality holds (by observation (iii) above). We are left with two cases:

- Either  $\dim_{\mathbf{k}}(\tilde{R}/R)_2 = 1$  and  $\dim_{\mathbf{k}}(\tilde{R}/R)_i = 0$  for all  $i \geq 3$ : in this case  $\tilde{\mathfrak{m}}^3 \subseteq \mathfrak{m}$  by observation (iv). From (v) we see that  $\tilde{\mathfrak{m}}^3 = \mathfrak{m}$ , hence  $R \simeq \mathbf{k}[[t^3, t^4, t^5]]$ , a non-Gorenstein singularity sitting in 3-space, which is obtained by collapsing a second-order infinitesimal neighbourhood of the origin in  $\mathbb{A}^1$ .
- Or  $\dim_{\mathbf{k}}(\tilde{R}/R)_3 = 1$  and  $\dim_{\mathbf{k}}(\tilde{R}/R)_i = 0$  for  $i = 2$  and for all  $i \geq 4$ : in this case  $\tilde{\mathfrak{m}}^4 \subseteq \mathfrak{m}$  by observation (iv). On the other hand from  $\dim_{\mathbf{k}}(\tilde{\mathfrak{m}}^2 \cap R / \tilde{\mathfrak{m}}^3 \cap R) = 1$  we deduce that there is a generator of degree 2, and from  $\dim_{\mathbf{k}}(\tilde{\mathfrak{m}}^3 \cap R / \tilde{\mathfrak{m}}^4 \cap R) = 0$  there is none of degree 3. We may write the generator as  $x = t^2 + ct^3$ , and  $\mathfrak{m} = \langle x \rangle + \tilde{\mathfrak{m}}^4$ . Up to a coordinate change (i.e. automorphism of  $\mathbf{k}[[t]]$ ), we may take  $x = t^2$ , and

$$\mathfrak{m}/\mathfrak{m}^2 = \langle t^2, t^5 \rangle,$$

so  $R \simeq \mathbf{k}[[x, y]]/(x^5 - y^2)$ , as anticipated. □

From now on, we only look for Gorenstein singularities. With notation as above, let  $I = (R : \tilde{R}) = \text{Ann}_R(\tilde{R}/R)$  be the *conductor ideal* of the singularity. Recall e.g. [AK70, Proposition VIII.1.16]:  $(C, q)$  is Gorenstein if and only if

$$\dim_{\mathbf{k}}(R/I) = \dim_{\mathbf{k}}(\tilde{R}/R)(= \delta).$$

Recall from [Ste96, Definition 2-1] that a curve singularity  $(C, q)$  is *decomposable* if  $C$  is the union of two curves  $C_1$  and  $C_2$  that lie in distinct smooth spaces intersecting each other transversely in  $q$ . Given a parametrisation  $x_i = x_i(t_1, \dots, t_m)$ ,  $i = 1, \dots, l$ , this means that there is a partition  $S \sqcup S' = \{1, \dots, m\}$  such that  $x_i$  only depends on  $t_s, s \in S$ , or  $s \in S'$ , for all  $i$ . Aside from the node, Gorenstein singularities are never decomposable [AFS16, Proposition 2.1].

PROPOSITION 2.2. *For every fixed integer  $m \geq 2$ , there are exactly two Gorenstein curve singularities of genus two with  $m$  branches.*



*Proof.* We only need to find a basis for  $\mathfrak{m}/\mathfrak{m}^2$ , because a map of complete local rings that is surjective on cotangent spaces is surjective. From observation (iii) again, we find three possibilities for the vector  $(d_1, d_2, d_3)$ ,  $d_i = \dim_{\mathbf{k}}(\tilde{R}/R)_i$ .

**Case (2, 0, 0).** We see that  $\tilde{\mathfrak{m}}^2 \subseteq I$ , so by the Gorenstein assumption and (v):

$$m + 1 = \delta = \dim_{\mathbf{k}}(R/I) \leq \dim_{\mathbf{k}}(R/\tilde{\mathfrak{m}}^2) = \dim_{\mathbf{k}} A_0 + \dim_{\mathbf{k}} A_1 = 1 + (m - 2) = m - 1,$$

a contradiction. Note: the singularity turns out to be decomposable in this case.

**Case (1, 1, 0).** We have  $\tilde{\mathfrak{m}}^3 \subseteq I$ . We are going to write down the  $m - 1$  generators of  $A_1 \pmod{\tilde{\mathfrak{m}}^3}$ <sup>1</sup>. The first generator, call it  $x_1$ , has a non-trivial linear term in at least one of the variables, say  $t_1$ . By scaling  $x_1$  and possibly adding a multiple of  $x_1^2$ , we can make it into the form:  $x_1 = t_1 \oplus p_{1,2}(t_2) \oplus \dots \oplus p_{1,m}(t_m) \pmod{\tilde{\mathfrak{m}}^3}$ . Now we can use  $x_1$  and  $x_1^2$  to make sure the second generator does not involve  $t_1$  at all. It will still have a linear term independent of  $t_1$ , say non-trivial in  $t_2$ . By scaling and adding a multiple of  $x_2^2$ , we can write  $x_2 = 0 \oplus t_2 \oplus \dots \oplus p_{2,m}(t_m) \pmod{\tilde{\mathfrak{m}}^3}$ . By taking a linear combination of  $x_1$  with  $x_2$  and  $x_2^2$ , we may now reduce  $x_1$  to the form  $t_1 \oplus 0 \oplus p_{1,3}(t_3) \oplus \dots \oplus p_{1,m}(t_m) \pmod{\tilde{\mathfrak{m}}^3}$ . Therefore, by Gaussian elimination with the generators and their squares, we may assume that

$$\begin{aligned} x_1 &= t_1 \oplus 0 \oplus \dots \oplus \alpha_{1,m} t_m + \beta_{1,m} t_m^2 \\ x_2 &= 0 \oplus t_2 \oplus \dots \oplus \alpha_{2,m} t_m + \beta_{2,m} t_m^2 \\ &\dots \\ x_{m-1} &= 0 \oplus \dots \oplus t_{m-1} \oplus \alpha_{m-1,m} t_m + \beta_{m-1,m} t_m^2 \pmod{\tilde{\mathfrak{m}}^3}. \end{aligned}$$

We must have  $R/I = \langle 1, x_1, \dots, x_{m-1}, y \rangle$  by the Gorenstein condition (if  $x_i \in I$  for some  $i$ , then  $t_i \in R$ , and the singularity would be decomposable). Hence  $x_i^2 \in I$  for all but at most one  $i$ , say  $i = 1$ . Then  $t_i^2 \in R$  for  $i = 2, \dots, m - 1$ . If  $\alpha_{i,m} \neq 0$  for some  $i$  in this range, then  $t_m^2 \in R$  as well, so  $t_1^2 = x_1^2 - O(t_m^2) \in R$ , contradicting  $\dim_{\mathbf{k}}(\tilde{R}/R)_2 = 1$ . Therefore  $\alpha_{i,m} = 0$  for  $i \in \{2, \dots, m - 1\}$ . If  $\alpha_{1,m} = 0$ , then we need a further generator of  $\mathfrak{m}/\mathfrak{m}^2$ , namely  $z = 0 \oplus \dots \oplus t_m^3$ . In this case, though, both  $x_1^2$  and  $z$  belong to  $I$ , so  $\dim_{\mathbf{k}}(R/I) = m$ , and the singularity cannot be Gorenstein. We are reduced to the following expression:

$$\begin{aligned} x_1 &= t_1 \oplus 0 \oplus \dots \oplus \alpha_{1,m} t_m + \beta_{1,m} t_m^2 \\ x_2 &= 0 \oplus t_2 \oplus \dots \oplus \beta_{2,m} t_m^2 \\ &\dots \\ x_{m-1} &= 0 \oplus \dots \oplus t_{m-1} \oplus \beta_{m-1,m} t_m^2 \pmod{\tilde{\mathfrak{m}}^3}, \end{aligned} \tag{1}$$

with  $\beta_{1,m} \in \mathbf{k}$  and  $\alpha_{1,m}, \beta_{i,m} \in \mathbf{k}^\times$ ,  $i = 2, \dots, m - 1$  (by indecomposability). Finally, we may change coordinates in  $t_m$  and rescale the other  $t_i$  to obtain:

$$\begin{aligned} x_1 &= t_1 \oplus 0 \oplus \dots \oplus t_m \\ x_2 &= 0 \oplus t_2 \oplus \dots \oplus t_m^2 \\ &\dots \\ x_{m-1} &= 0 \oplus \dots \oplus t_{m-1} \oplus t_m^2 \pmod{\tilde{\mathfrak{m}}^3}. \end{aligned} \tag{2}$$

---

<sup>1</sup>To put them into the simplest possible form, we allow ourselves to perform elementary operations from linear algebra at first, while keeping ourselves from changing coordinates until the end - the benefit of this two-step process will become apparent in the next section.

We check that  $R/I = \langle 1, x_1, \dots, x_{m-1}, x_1^2 \rangle$  and  $\tilde{R}/R$  is of type  $(1, 1, 0)$ . In case  $m = 2$ , we need an extra generator  $y = t_2^3$ . Equations are given by:

- $y(y - x_1^3)$  if  $m = 2$  ( $A_5$ -singularity or *oscnode*);
- $x_1 x_2 (x_2 - x_1^2)$  if  $m = 3$  ( $D_6$ -singularity);
- $\langle x_3 (x_1^2 - x_2), x_i (x_j - x_k) \rangle_{1 \leq i < j < k \leq m-1 \text{ or } 1 < j < k < i \leq m-1}$  if  $m \geq 4$ .

**Case  $(1, 0, 1)$ .** We have  $\tilde{\mathfrak{m}}^4 \subseteq I$ . By an argument similar to the above one, we write generators for  $A_1$  as  $x_i = \dots \oplus t_i \oplus \dots \oplus \alpha_{i,m} t_m + \beta_{i,m} t_m^2 + \gamma_{i,m} t_m^3$ , for  $i = 1, \dots, m-1$ . Then  $R/I = \langle 1, x_1, \dots, x_{m-1}, y \rangle$ . For all but at most one  $i$ ,  $x_i^2 \in I$ , but definitely  $x_i^3 \in I$  for all  $i$ . On the other hand  $t_m^3 \notin R$ , because otherwise  $t_i^3 = x_i^3 - \alpha_{i,m}^3 t_m^3 + O(t_m^4)$  would belong to  $R$  as well, contradicting  $\dim_{\mathbf{k}}(\tilde{R}/R)_3 = 1$ . From this we deduce that  $\alpha_{i,m} = 0$  for all  $i = 1, \dots, m-1$ . Since  $\dim_{\mathbf{k}}(\tilde{R}/R)_2 = 0$ , there has to be another generator of  $\mathfrak{m}/\mathfrak{m}^2$  of degree two in  $t_m$ , which we may write as  $x_m = t_m^2 + \gamma_{m,m} t_m^3$ . We can use  $x_m$  to remove all the  $t_m^2$  pieces from  $x_1, \dots, x_{m-1}$ , so we are reduced to

$$\begin{aligned} x_1 &= t_1 \oplus 0 \oplus \dots \oplus \gamma_{1,m} t_m^3 \\ x_2 &= 0 \oplus t_2 \oplus \dots \oplus \gamma_{2,m} t_m^3 \\ &\dots \\ x_{m-1} &= 0 \oplus \dots \oplus t_{m-1} \oplus \gamma_{m-1,m} t_m^3 \\ x_m &= 0 \oplus \dots \oplus t_m^2 + \gamma_{m,m} t_m^3 \pmod{\tilde{\mathfrak{m}}^4}, \end{aligned} \tag{3}$$

with  $\gamma_{m,m} \in \mathbf{k}$  and  $\gamma_{i,m} \in \mathbf{k}^\times$ ,  $i = 1, \dots, m-1$  (by indecomposability). Finally, we may change coordinates in  $t_m$  and rescale the other  $t_i$  to obtain:

$$\begin{aligned} x_1 &= t_1 \oplus 0 \oplus \dots \oplus t_m^3 \\ x_2 &= 0 \oplus t_2 \oplus \dots \oplus t_m^3 \\ &\dots \\ x_{m-1} &= 0 \oplus \dots \oplus t_{m-1} \oplus t_m^3 \\ x_m &= 0 \oplus \dots \oplus t_m^2 \pmod{\tilde{\mathfrak{m}}^4}. \end{aligned} \tag{4}$$

We check that  $R/I = \langle 1, x_1, \dots, x_{m-1}, x_m \rangle$  and  $\tilde{R}/R$  is of type  $(1, 0, 1)$ . It recovers the unique Gorenstein singularity of Lemma 2.1 when  $m = 1$ . Equations are:

- $x^5 - y^2$  if  $m = 1$  ( $A_4$ -singularity or *ramphoid cusp*, with  $x = t^2, y = t^5$ );
- $y(y^3 - x^2)$  if  $m = 2$  ( $D_5$ -singularity, with  $x = x_1, y = x_2$ );
- $\langle x_3 (x_1 - x_2), x_3^3 - x_1 x_2 \rangle$  if  $m = 3$ ;
- $\langle x_i (x_j - x_k), x_m (x_i - x_j), x_m^3 - x_1 x_2 \rangle_{i,j,k \in \{1, \dots, m-1\} \text{ all different}}$  if  $m \geq 4$ .

□

**DEFINITION 2.3.** In case  $(1, 0, 1)$ , we say the singularity is *of type I*, and the branch parameterised by  $t_m$  is called *singular*; in case  $(1, 1, 0)$ , we say the singularity is *of type II*, and the branches parameterised by  $t_1$  and  $t_m$  are called *twin*. We shall refer to the singular or twin branches as *special* or *distinguished*; all other branches are *axes*. *Branch* remains a generic name, indicating any of the previous ones.

**Remark 2.4.** Singularities of type *I* do appear in the miniversal family of singularities of type *II*, and vice versa. For low values of  $m$  - which serves as the playground for our speculations -, this follows from a neat result of Grothendieck ([CML13], see also [Arn72, Dem75]):

**THEOREM 2.5.** *Let  $(C, q)$  be a curve singularity of ADE type. Singularities that appear in the miniversal deformation of  $(C, q)$  are all and only those ADE, whose Dynkin diagram can be obtained as a full subgraph of the diagram of  $(C, q)$ .*

### 3. Tangent sheaf, crimping space, and automorphisms

In this section we analyse the tangent sheaf of a genus two singularity. For a complete pointed Gorenstein curve of genus two, we translate the absence of infinitesimal automorphisms into a (mostly) combinatorial condition. The crimping space naturally makes an appearance.

**LEMMA 3.1.** *Let  $(C, q)$  be a Gorenstein curve singularity of genus two, with pointed normalisation  $\nu: (\tilde{C}, \{q_i\}_{i=1, \dots, m}) \rightarrow (C, q)$ , and assume  $\text{char}(\mathbf{k}) \neq 2, 3, 5$ . There are exact sequences of sheaves*

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \nu_* \Omega_{\tilde{C}}^\vee(-\sum_i 3q_i) & \longrightarrow & \nu_* \Omega_{\tilde{C}}^\vee(-\sum_i q_i) & \longrightarrow & \nu_* \bigoplus_i \Omega_{\tilde{C}}^\vee(-q_i)_{|2q_i} \longrightarrow 0 \\
 & & \parallel & & \uparrow & & \square & \uparrow \phi \\
 0 & \longrightarrow & \nu_* \Omega_{\tilde{C}}^\vee(-\sum_i 3q_i) & \longrightarrow & \Omega_C^\vee & \longrightarrow & \mathbf{k}^{\oplus m} \longrightarrow 0
 \end{array}$$

The rightmost vertical map admits an explicit description in local coordinates.

*Proof.* Let  $K(\tilde{C})$  denote the constant sheaf of rational functions on  $\tilde{C}$ . A section of  $\Omega_{\tilde{C}}^\vee \otimes K(\tilde{C})$  is contained in  $\Omega_{\tilde{C}}^\vee$  if and only if its image under the push-forward

$$\nu_*: \nu_* \mathcal{H}om(\Omega_{\tilde{C}}, K(\tilde{C})) \rightarrow \mathcal{H}om(\Omega_C, K(\tilde{C}))$$

lies in the subspace  $\mathcal{H}om(\Omega_C, \mathcal{O}_C)$ . We may work locally around the singular point in the coordinates we found in Section 2.

(A<sub>4</sub>): In the coordinates  $x = t^2 + ct^3, y = t^4, z = t^5$  (they are redundant, but this will be irrelevant), the section  $f(t) \frac{d}{dt} \in \nu_* \Omega_{\tilde{C}}^\vee \otimes K(\tilde{C})$  pushes forward to

$$\nu_* \left( f(t) \frac{d}{dt} \right) = (2t + 3ct^2) f(t) \frac{d}{dx} + 4t^3 f(t) \frac{d}{dy} + 5t^4 f(t) \frac{d}{dz},$$

from which, writing  $f(t) = f_0 + f_1 t + f_2 t^2 + O(t^3)$ , we see that

$$(2t + 3ct^2) f(t), 4t^3 f(t), 5t^4 f(t) \in \hat{\mathcal{O}}_{C,p} \Leftrightarrow f_0 = 0, cf_1 + 2f_2 = 0.$$

(A<sub>5</sub>): In the coordinates  $x = t_1 \oplus at_2 + bt_2^2, y = t_1^3$  (we have  $a \neq 0$ ), the section  $f_1(t_1) \frac{d}{dt_1} \oplus f_2(t_2) \frac{d}{dt_2}$  pushes forward to

$$\nu_* \left( f_1(t_1) \frac{d}{dt_1} \oplus f_2(t_2) \frac{d}{dt_2} \right) = (f_1(t_1) \oplus (a + 2bt_2) f_2(t_2)) \frac{d}{dx} + 3t_1^2 f_1(t_1) \frac{d}{dy},$$

from which, writing  $f_i(t_i) = f_{i0} + f_{i1} t_i + f_{i2} t_i^2 + O(t_i^3), i = 1, 2$ , we see that

$$f_1(t_1) \oplus (a + 2bt_2) f_2(t_2), 3t_1^2 f_1(t_1) \in \hat{\mathcal{O}}_{C,p} \Leftrightarrow \begin{cases} f_{10} = f_{20} = 0, \\ f_{11} = f_{21}, \\ 2bf_{21} + af_{22} = a^2 f_{12}. \end{cases}$$

$(I_{m \geq 2})$  : In the coordinates of (3),

$$\nu_* \left( \sum_{i=1}^m f_i(t_i) \frac{d}{dt_i} \right) = \sum_{i=1}^{m-1} (f_i(t_i) \oplus 3\gamma_{i,m} t_m^2 f_m(t_m)) \frac{d}{dx_i} + (2t_m + 3\gamma_{m,m} t_m^2) f_m(t_m) \frac{d}{dx_m},$$

hence we deduce that

$$\nu_* \left( \sum_{i=1}^m f_i(t_i) \frac{d}{dt_i} \right) \in \Omega_C^\vee \otimes \hat{\mathcal{O}}_{C,p} \Leftrightarrow \begin{cases} f_{i0} = 0 & i = 1, \dots, m, \\ f_{i1} = 3f_{m1}, & i = 1, \dots, m-1, \\ 3\gamma_{m,m} f_{m1} + 2f_{m2} = 0. \end{cases}$$

$(II_{m \geq 3})$  : In the coordinates of (1),

$$\nu_* \left( \sum_{i=1}^m f_i(t_i) \frac{d}{dt_i} \right) = (f_1(t_1) \oplus (\alpha_{1,m} + 2\beta_{1,m} t_m) f_m(t_m)) \frac{d}{dx_1} + \sum_{i=2}^m (f_i(t_i) \oplus 2\beta_{i,m} t_m f_m(t_m)) \frac{d}{dx_i},$$

hence we deduce that

$$\nu_* \left( \sum_{i=1}^m f_i(t_i) \frac{d}{dt_i} \right) \in \Omega_C^\vee \otimes \hat{\mathcal{O}}_{C,p} \Leftrightarrow \begin{cases} f_{i0} = 0 & i = 1, \dots, m, \\ 2f_{11} = f_{i1} = 2f_{m1}, & i = 2, \dots, m-1, \\ \beta_{1,m} f_{m1} + \alpha_{1,m} f_{m2} = \alpha_{1,m}^2 f_{12}. \end{cases}$$

□

We anticipate that the letters  $\alpha, \beta$  and  $\gamma$  will play a role in determining the automorphism group of a complete curve with markings. We recall some key concepts from F. van der Wyck's thesis. Working over  $\mathbf{k}$ , he considers the stacks:

- $\mathcal{S}$  of reduced one-dimensional (1d)  $\mathbf{k}$ -algebras  $R$ ,
- $\mathcal{T}$  of reduced 1d algebras with resolution  $(R \hookrightarrow (S, J))$ , where  $S$  is a smooth one-dimensional  $\mathbf{k}$ -algebra, and  $J$  the radical of the conductor of  $R \subseteq S$ .

Basically,  $R$  represents the (local) ring of a reduced curve with one singular point,  $S$  is its normalisation, and  $J$  is the ideal of the reduced fibre over the singular point of  $\text{Spec}(R)$ .  $\mathcal{S}$  and  $\mathcal{T}$  are limit-preserving stacks over  $\text{Spec}(\mathbf{k})$  [vdW10, Proposition 1.21]. Furthermore, we may fix a reduced 1d algebra with resolution  $\tau_0 : (R_0 \hookrightarrow (S_0, J_0))$ , and consider the substack  $\mathcal{T}(\tau_0)$  of reduced 1d algebras with singularity type  $\tau_0$  (i.e. isomorphic to  $\tau_0$  locally on both the base and the curve, see [vdW10, Definition 1.64]; that various notions of “locally” coincide is proved in [vdW10, Proposition 1.50]). There is a forgetful morphism  $\mathcal{T} \rightarrow \mathcal{S}$ , and the *crimping space* of  $\tau_0$  is defined to be the fibre over  $R_0$  of the restriction of such morphism to  $\mathcal{T}(\tau_0)$ . The crimping space is a smooth  $\mathbf{k}$ -scheme [vdW10, Theorems 1.70 and 1.73]; indeed, it is isomorphic to the quotient of  $\text{Aut}_{(S_0, J_0)/\mathbf{k}}$  by  $\text{Aut}_{(S_0, J_0)/R_0}$ , the latter consisting of automorphisms of the normalisation that preserve the subalgebra of the singularity; moreover, the quotient can be computed after modding out the lowest power of  $J$  contained in  $R$  [vdW10, Theorem 1.53]. Crimping spaces can be thought of as moduli for the normalisation map.

LEMMA 3.2. *The crimping space of a genus two singularity with  $m$  branches is (a number - depending on the type - of copies of)  $\mathbb{A}^1 \times (\mathbb{A}^1 \setminus \{0\})^{m-1}$ .*

*Proof.* We resume notation from the previous section. We are going to fix the subalgebra  $\tau_0$  given in coordinates by (4) and (2) respectively.

**Type I:** recall that in this case  $\tilde{\mathfrak{m}}^4 \subseteq R$ . For a  $\mathbf{k}$ -algebra  $A$ , let

$$G_i(A) = \{t_i \mapsto g_{i1}t_i + g_{i2}t_i^2 + g_{i3}t_i^3, t_j \mapsto t_j \mid g_{i1} \in A^\times, g_{i2}, g_{i3} \in A\},$$

and notice that

$$\mathrm{Aut}_{(\tilde{R}, \tilde{\mathfrak{m}})}^{\mathrm{mod} \tilde{\mathfrak{m}}^4}(A) = (G_1 \times \dots \times G_m) \rtimes \mathfrak{S}_m(A).$$

Consider now the action of a group element of the form  $(g_1, \dots, g_m; \mathrm{id}_{\mathfrak{S}_m})$  on the given generators of  $R$ :

$$\begin{aligned} x_i &\mapsto \dots \oplus g_{i1}t_i + g_{i2}t_i^2 + g_{i3}t_i^3 \oplus \dots \oplus g_{m1}^3t_m^3, \quad \text{for } i = 1, \dots, m-1; \\ x_m &\mapsto \dots \oplus g_{m1}^2t_m^2 + 2g_{m1}g_{m2}t_m^3 \pmod{\tilde{\mathfrak{m}}^4}. \end{aligned}$$

The former belongs to  $R$  iff  $g_{i1} = g_{m1}^3$ ; the latter does iff  $g_{m2} = 0$ . Thus such elements span a subgroup isomorphic to  $\mathbb{G}_m \times \mathbb{G}_a^{m-1} \times \mathbb{G}_a^m(A)$ . On the other hand, there is a special (singular) branch, parametrised by  $t_m$ . We conclude that

$$\mathrm{Aut}_{\tau_0}^{\mathrm{mod} \tilde{\mathfrak{m}}^3}(A) = (\mathbb{G}_m \times \mathbb{G}_a^{m-1} \times \mathbb{G}_a^m) \rtimes (\mathfrak{S}_{m-1})(A).$$

The quotient is therefore isomorphic to  $m$  copies of  $\mathbb{A}^1 \times (\mathbb{A}^1 \setminus \{0\})^{m-1}$ .

**Type II:** recall that in this case  $\tilde{\mathfrak{m}}^3 \subseteq R$ . For a  $\mathbf{k}$ -algebra  $A$ , let

$$G_i(A) = \{t_i \mapsto g_{i1}t_i + g_{i2}t_i^2, t_j \mapsto t_j \mid g_{i1} \in A^\times, g_{i2} \in A\},$$

and notice that

$$\mathrm{Aut}_{(\tilde{R}, \tilde{\mathfrak{m}})}^{\mathrm{mod} \tilde{\mathfrak{m}}^3}(A) = (G_1 \times \dots \times G_m) \rtimes \mathfrak{S}_m(A).$$

Consider now the action of a group element of the form  $(g_1, \dots, g_m; \mathrm{id}_{\mathfrak{S}_m})$  on the given generators of  $R$ :

$$\begin{aligned} x_i &\mapsto \dots \oplus g_{i1}t_i + g_{i2}t_i^2 \oplus \dots \oplus g_{m1}^2t_m^2, \quad \text{for } i = 2, \dots, m-1; \\ x_1 &\mapsto g_{11}t_1 + g_{12}t_1^2 \oplus \dots \oplus g_{m1}t_m + g_{m2}t_m^2 \pmod{\tilde{\mathfrak{m}}^3}. \end{aligned}$$

The former belongs to  $R$  iff  $g_{11} = g_{m1}$  and  $g_{12} = g_{m2}$ ; the latter does iff  $g_{i1} = g_{m1}^2$ . Thus, such elements span a subgroup isomorphic to  $\mathbb{G}_m \times \mathbb{G}_a^{m-1}(A)$ . On the other hand, all branches are smooth (therefore, isomorphic to each other), but two of them (parametrised by  $t_1$  and  $t_m$  respectively) are tangent, thus forming a distinguished pair. We conclude that

$$\mathrm{Aut}_{\tau_0}^{\mathrm{mod} \tilde{\mathfrak{m}}^3}(A) = (\mathbb{G}_m \times \mathbb{G}_a^{m-1}) \rtimes (\mathfrak{S}_2 \times \mathfrak{S}_{m-2})(A).$$

The quotient is then isomorphic to  $\binom{m}{2}$  copies of  $\mathbb{A}^1 \times (\mathbb{A}^1 \setminus \{0\})^{m-1}$ .  $\square$

The benefit of a two-step classification should now be clear: if we do not allow ourselves to change coordinates (i.e. act by automorphisms of the normalisation) until the end, the crimping space appears in the expressions (1) and (3) for the generators of the singularity subalgebra.

There is a more geometric way to realise the crimping spaces. It is well-known that an ordinary cusp of genus one can be obtained by collapsing (*push-out*) any non-zero tangent vector at  $p \in \mathbb{A}^1$ . More generally, a Gorenstein singularity of genus one and  $m$  branches can be obtained

by collapsing a generic (not contained in any coordinate linear subspace) tangent line at an ordinary  $m$ -fold point (a non-Gorenstein singularity of genus zero) [Smy11a, Lemma 2.2]. Therefore, the crimping space of the elliptic  $m$ -fold point, which is isomorphic to  $(\mathbb{A}^1 \setminus \{0\})^{m-1}$ , can be realised as the complement of the coordinate hyperplanes inside  $\mathbb{P}(T_p R_m) \simeq \mathbb{P}^{m-1}$ , where  $(R_m, p)$  is the rational  $m$ -fold point. Besides, this gives rise to a natural compactification of the crimping space supporting a universal family of curves - in fact, two: either we collapse non-generic tangent vectors, obtaining non-Gorenstein singularities along the boundary (this family  $\mathcal{C}$  admits a common (semi)normalisation by the trivial family  $\tilde{\mathcal{C}} = R_m \times \mathbb{P}(T_p R_m)$ ); or we blow  $\tilde{\mathcal{C}}$  up along the boundary (*sprouting*), so that the non-Gorenstein singularities are replaced by elliptic  $m$ -fold points having strictly semistable branches [Smy11b, §2.2-3].

Similarly, a Gorenstein singularity of genus two can be obtained by collapsing a generic tangent line to a non-Gorenstein singularity of genus one. Indeed,  $\tau_0^I$  admits a partial normalisation by  $\sigma_0^I$ , which is the decomposable union of a cusp (parametrised by  $t_m$ ) together with  $m-1$  axes, adjoining the generator  $t_m^3$ ;  $\tau_0^{II}$  admits a partial normalisation by  $\sigma_0^{II}$ , which is the decomposable union of a tacnode in the  $(t_1, t_m)$ -plane together with  $m-2$  axes, adjoining the generator  $t_m^2$ .

These fit together nicely in a unifying picture: if we restrict  $\mathcal{C}$  from the previous paragraph to the union of the coordinate lines in  $\mathbb{P}(T_p R_m)$ , we obtain  $m$  copies of  $\sigma_0^I$  over the coordinate points, together with  $\binom{m}{2}$  copies of the universal curve of type  $\sigma_0^{II}$  over its crimping space - which is isomorphic to  $\mathbb{A}^1 \setminus \{0\}$  -, identified with the line minus two points. Let  $P = \mathbb{P}(T_p \mathcal{C}_{|\cup \text{lines}})$  be the projectivised tangent space at the singular point of the fibre. For each of the  $\binom{m}{2}$  coordinate lines,  $P$  has one component  $P_i^{II}$  that is a  $\mathbb{P}^{m-1}$ -bundle over the line; besides,  $P$  has  $m$  components  $P_j^I$  isomorphic to  $\mathbb{P}^m$  and supported over the points. The crimping space of the genus two singularities with  $m$  branches (of type  $I$  and  $II$  together) can be realised as an open subscheme of  $P$ , obtained by removing from the  $\mathbb{P}^{m-1}$ -fibres of  $P^{II}$  the  $m-1$  hyperplanes generated by (a) the tangent line to the tacnode and the  $m-2$  axes, and (b) the plane containing the tacnode and all but one of the  $m-2$  axes; and from each  $P_j^I$  the  $m$  planes generated by (a) the tangent cone of the cusp and the  $m-1$  axes, and (b) the plane containing the cusp and all but one of the  $m-1$  axes.

*Remark 3.3.* Crimping spaces are related to the space of arrows  $\phi$  in the diagram of Lemma 3.1.

We interpret  $H^0(\Omega_{\mathbb{P}^1}^\vee(-p)_{|2p})$  as the tangent space to the subgroup of  $PGL_2$  fixing the point  $p \in \mathbb{P}^1$ , thus it inherits a natural Lie algebra structure, isomorphic to the unique non-abelian Lie algebra of dimension two  $V$ . It has a basis:

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{with } [e_1, e_2] = -2e_2.$$

The vector  $(\varphi, \psi)$  is seen to correspond to the infinitesimal automorphism:

$$t \mapsto \frac{1 + \epsilon \varphi t}{1 - \epsilon(\varphi t + \psi)} = t + \epsilon(2\varphi t - \psi t^2).$$

We are interested in arrows  $\phi$  that are embeddings (i.e.  $\in \text{Gr}(m, V^{\oplus m})$ ) of Lie subalgebras, such that the corresponding groups of infinitesimal automorphisms fix a (unique?) subalgebra of a singularity of genus two inside  $\mathbf{k}[[t_1]] \oplus \dots \oplus \mathbf{k}[[t_m]]$ .

We start with some heuristics. Here is the unibranch case: the subalgebra of  $\mathbf{k}[[t]]$  generated by  $x = t^2 + ct^3$  is preserved by  $(\varphi, \psi)$  if and only if

$$(1 + 2\varphi)^2 t^2 - 2\psi(1 + 2\varphi)t^3 + c(1 + 2\varphi)^3 t^3 \text{ is a multiple of } t^2 + ct^3,$$

which reduces to  $\varphi(1 + 2\varphi)c = \psi$ . This further determines  $c$  if and only if  $\varphi \neq 0$ . Note that in

this case (dimension one) the Lie subalgebra condition is automatically satisfied. We have found  $(\varphi, \psi) \in \mathbf{k}^\times \times \mathbf{k}$ .

The case of type  $II_2$ -algebras is more interesting. Let  $x = (t_1, \alpha t_2 + \beta t_2^2)$  be the generator of such an algebra. The image of  $x$  under  $(\varphi_1, \psi_1, \varphi_2, \psi_2)$  is:

$$((1 + 2\varphi_1)t_1 - \psi_1 t_1^2, \alpha(1 + 2\varphi_2)t_2 + (\beta(1 + 4\varphi_2) - \alpha\psi_2)t_2^2),$$

from which we deduce:

$$\varphi_1 = \varphi_2 \quad \text{and} \quad 2\beta\varphi_2 - \alpha\psi_2 = -\alpha^2\psi_1. \quad (5)$$

Now let  $\phi: \mathbf{k}^2 \rightarrow V^{\oplus 2}$  be given by  $\begin{pmatrix} \varphi_{11} & \psi_{11} & \varphi_{12} & \psi_{12} \\ \varphi_{21} & \psi_{21} & \varphi_{22} & \psi_{22} \end{pmatrix}$ , with Plücker coordinates  $w_{ij}$  for the minor of the  $i$ -th and  $j$ -th columns. The first condition in (5) immediately implies

$$w_{13} = 0 \quad \text{and} \quad w_{12} = -w_{23}, w_{14} = w_{34}. \quad (6)$$

The second condition in (5) implies

$$\alpha w_{12} = w_{14} \quad \text{and} \quad 2\beta w_{12} = \alpha w_{24} \quad (7)$$

so that  $(\alpha, \beta) \in \mathbf{k}^\times \times \mathbf{k}$  is determined as soon as  $w_{12}, w_{14} \neq 0$ . Notice that (6) implies in particular the Plücker equation

$$w_{12}w_{34} - w_{13}w_{24} + w_{14}w_{23}.$$

It is easy to see that the condition for  $\phi$  to be a sub-Lie algebra is

$$\text{rk} \begin{pmatrix} \varphi_{11} & \psi_{11} & \varphi_{12} & \psi_{12} \\ \varphi_{21} & \psi_{21} & \varphi_{22} & \psi_{22} \\ 0 & w_{12} & 0 & w_{34} \end{pmatrix} = 2,$$

translating into

$$\begin{aligned} w_{12}w_{13} &= 0 & w_{12}(w_{34} - w_{14}) &= 0 \\ w_{13}w_{34} &= 0 & (w_{12} + w_{23})w_{34} &= 0 \end{aligned}$$

automatically satisfied after (6) too. These equations cut inside  $\mathbb{P}_{[w_{ij}]}^5$  the locus

$$(\mathbb{A}_{w_{14}/w_{12}}^1 \setminus \{0\}) \times \mathbb{A}_{w_{24}/w_{12}}^1.$$

More generally, given a type  $II$  subalgebra  $R_{\alpha,\beta}$  of  $W = \bigoplus_{i=1}^m \mathbf{k}[[t_i]]/(t_i^2)$ , with generators of the form described in (1), the subalgebra of  $V_{(\varphi_i, \psi_i)_{i=1, \dots, m}}^{\oplus m}$  preserving  $R_{\alpha,\beta}$  is isomorphic to  $\mathbf{k}^{\oplus m}$  with equations (see Lemma 3.1):

$$\begin{cases} 2\varphi_1 = \varphi_i = 2\varphi_m, & \text{for } i = 2, \dots, m-1, \\ 2\beta_{1,m}\varphi_m - \alpha_{1,m}\psi_m = -\alpha_{1,m}^2\psi_1; \end{cases}$$

it is easily seen that such a subalgebra of  $V^{\oplus m}$  does not determine  $R_{\alpha,\beta}$ , but it does determine  $(\alpha_{1,m}, \beta_{1,m})$ . The case of type  $I$  is analogous.

We apply the preceding discussion to the study of automorphism groups of complete marked curves with a genus two singularity. The relevant category has been formalised in van der Wyck's thesis, see [vdW10, Proposition 1.102, Theorem 1.105 and Corollary 1.106], where he introduces the concept of type  $T$  reduced complete pointed curves with resolution, and the algebraic stack  $\mathcal{N}_T$  of such objects. The type encodes the number and isomorphism class of the singularities, the distribution of genus and markings among the components of the normalisation, and the adjacency data between components and singular points.

In the case that  $T$  has a unique singularity of genus two, with  $m$  one-marked rational branches, the stack  $\mathcal{N}_T$  is isomorphic to  $[\mathbb{A}^1/\mathbb{G}_m]$  (see [vdW10, Examples 1.111-112]), so it has two points: one with  $\mathbb{G}_m$ , and the other with trivial stabiliser.

DEFINITION 3.4. The *atom* of type  $I_m$  (the name is borrowed from [AFSvdW17]) is obtained by gluing the subalgebra of  $\mathbf{k}[t_1] \oplus \dots \oplus \mathbf{k}[t_m]$  generated by  $x_1, \dots, x_m$  as in (4) with  $m$  copies of  $(\mathbf{k}[s], (s))$  under the identification  $s_i = t_i^{-1}$ .  $\mathbb{G}_m$  acts on the atom by  $\lambda.t_i = \lambda^3 t_i$  for  $i = 1, \dots, m-1$  and  $\lambda.t_i = \lambda t_i$  for  $i = m$ .

Similarly, the atom of type  $II_m$  is obtained by gluing the subalgebra of  $\mathbf{k}[t_1] \oplus \dots \oplus \mathbf{k}[t_m]$  generated by  $x_1, \dots, x_{m-1}$  (and  $y$ ) as in (2) (and following lines) with  $m$  copies of  $(\mathbf{k}[s], (s))$  under the identification  $s_i = t_i^{-1}$ . There is a  $\mathbb{G}_m$ -action on the type  $II$  atom by  $\lambda.t_i = \lambda t_i$  for  $i = 1, m$  and  $\lambda.t_i = \lambda^2 t_i$  for  $i = 2, \dots, m-1$ .

The curve with a genus two singularity and one-marked rational branches that has trivial automorphism group will be called the *non-atom*.

Again, here is a more geometric way to realise the dicotomy. The non-Gorenstein genus one singularity of type  $\sigma_0^{II}$  (resp.  $\sigma_0^I$ ), with one-marked rational branches, has automorphism group  $\mathbb{G}_m^{m-1}$  (resp.  $\mathbb{G}_m^m$ ). This acts on the tangent space at the singular point: of the lines fixed by this action, only one (call it  $\ell'$ ) sits inside the open subset corresponding to the crimping space; all other lines in the crimping space are identified under the group action (call  $\ell$  their equivalence class). Collapsing  $\ell$  yields the non-atom, while collapsing  $\ell'$  yields the atom.

As a third viewpoint, automorphisms can be studied by twisting the exact sequences of Lemma 3.1 by the ideal of the markings, and then taking global sections. The dicotomy arises then from the map  $\phi$ : if the last condition imposed on infinitesimal automorphisms interweaves first and second order non-trivially (i.e. when  $\beta_{1,m}$ , resp.  $\gamma_{m,m}$ , are non-zero) then it is enough that automorphisms are trivial to second order on every branch for them to be trivial for good.

Finally, we shall rephrase the finiteness of automorphism groups explicitly in terms of types. Recall Smyth's description of genus one curves with no infinitesimal automorphisms [Smy11a, Proposition 2.3, Corollary 2.4].

DEFINITION 3.5. Let  $(C, p_1, \dots, p_n)$  be a reduced pointed curve. A connected subcurve  $D \subseteq C$  is said to be *nodally attached* if  $D \cap \overline{C \setminus D}$  consists of nodes only. We say that  $C$  is *residually DM* (rDM) if every nodal and nodally attached subcurve  $D$  of  $C$ , marked by  $\{p_i \in D\} \cup (D \cap \overline{C \setminus D})$ , is Deligne-Mumford stable. As usual, by *special points* we mean markings and nodes.

COROLLARY 3.6. Let  $(C, p_1, \dots, p_n)$  be a Gorenstein pointed curve of arithmetic genus two.  $H^0(C, \Omega_C^\vee(-\sum_{i=1}^n p_i)) = 0$  is equivalent to either of the following:

- (i)  $C$  has a singularity of type  $I_{m \geq 1}$ : either all branches contain exactly one special point and  $C$  is the non-atom; or each of its axes contains at least one special point, and at least one branch has at least two. Furthermore  $C$  is rDM.
- (ii)  $C$  has a singularity of type  $II_{m \geq 2}$ : either all branches contain exactly one special point and  $C$  is the non-atom; or at least one of its twin branches contains a special point, each of its axes contains at least one, and at least one branch has at least two. Furthermore  $C$  is rDM.
- (iii)  $C$  has two elliptic  $m$ -fold points: each of their branches contains at least one special point or is shared, and at least one branch for each singular point contains at least one extra special point. Furthermore  $C$  is rDM.



- (iv)  $C$  has one elliptic  $m$ -fold point: one of its branches is a genus one curve, and every other branch contains at least one special point; otherwise, all branches contain at least one, and either two of its branches coincide, or at least one branch has at least two special points. Furthermore  $C$  is rDM.
- (v)  $C$  contains only nodes and is Deligne-Mumford stable.

#### 4. Dualising line bundle and semistable tails

Given a family of prestable (pointed) curves of genus two over the spectrum of a discrete valuation ring  $\mathcal{C} \rightarrow \Delta$ , with smooth generic fibre  $\mathcal{C}_\eta$  and regular total space, we classify the subcurves of the central fibre  $\mathcal{C}_0$  that can be contracted to yield a Gorenstein singularity of genus two. In the genus one case, Smyth answered the analogous question by identifying the class of *balanced* subcurves [Smy11a, Definition 2.11]: subcurves of arithmetic genus one, such that, when breaking them into a *core* (minimal subcurve of genus one, not containing any separating node) and a number of rational trees (with root corresponding to the component adjacent to the core, and leaves corresponding to the components adjacent to the portion of  $\mathcal{C}_0$  that is not contracted), the distance between any leaf and the root of any such tree is constant, not depending on the tree either. In the case at hand, the answer turns out to be slightly more complicated: first, the special branch(es) of a type *I* (resp. *II*) singularity are connected through rational chains to a Weierstrass (resp. two conjugate) point(s) of the core. Second, the lengths of the rational trees may vary according to where their attaching points lie, but the special chains are always the shortest, and, together with the configuration of the attaching points on the core, they determine the length of any other chain.

*Remark 4.1.* While there are no special points on a smooth curve of genus zero or one, the simplest instance of Brill-Noether theory involves smooth curves of genus two. Every such  $C$  is *hyperelliptic*: it admits a unique (up to reparametrisation) two-fold cover  $\phi: C \rightarrow \mathbb{P}^1$ , induced by the complete canonical linear system, i.e.  $|K_C|$  is the unique  $\mathfrak{g}_2^1$  on  $C$ ; said otherwise, there is a unique element  $\sigma \in \text{Aut}(C)$ , called the *hyperelliptic involution*, such that  $C/\langle \sigma \rangle \simeq \mathbb{P}^1$ . A point  $x \in C$  is called *Weierstrass* if it is a ramification point for  $\phi$  (or, equivalently, a fixed point for  $\sigma$ ); from the Riemann-Hurwitz formula it follows that there are six Weierstrass points on every smooth curve of genus two. Two points  $x_1, x_2$  are said to be conjugate (write  $x_2 = \bar{x}_1$ ) if there exists a point  $z \in \mathbb{P}^1$  such that  $\phi^{-1}(z) = \{x_1, x_2\}$  (or, equivalently,  $\sigma(x_1) = x_2$ ). These notions may be extended to nodal curves by declaring  $(C, x)$  to be Weierstrass if its stabilisation lies in the closure of

$$\mathcal{W} = \{(C, x) \mid C \text{ smooth and } x \text{ Weierstrass}\} \subseteq \overline{\mathcal{M}}_{2,1},$$

and similarly for conjugate points. We then need to study the limiting behaviour of Weierstrass points when a smooth curve degenerates to a nodal one. This is a difficult problem when it comes to higher genus curves; it has received considerable attention since the '70s, in work of E. Arbarello, D. Eisenbud, J. Harris, and many others. In our case it boils down to understanding admissible covers [HM82] of degree two with a branch locus of degree six; said otherwise, up to the involution action, the Weierstrass locus is isomorphic to  $\overline{\mathcal{M}}_{0,6}/\mathfrak{S}_5$ , and the conjugate locus is isomorphic to  $\overline{\mathcal{M}}_{0,7}/\mathfrak{S}_6$ . We remark that  $(C, x)$  being Weierstrass is an intrinsic notion if  $C$  is of compact type (or, more generally, tree-like), but it may depend on the smoothing otherwise (i.e. the fibre of  $\overline{\mathcal{W}} \rightarrow \overline{\mathcal{M}}_2$  may have positive dimension); we have benefited from the exposition in [Dia85, Appendix 2], [Cuk89, Proposition (3.0.6)], and [HM98, Theorem 5.45].

- If  $x$  belongs to a component of genus one  $E$ , which is attached to another component of genus one at a node  $y$ , then  $x$  is Weierstrass iff  $2x \sim 2y \in \text{Pic}(E)$ ; if instead  $E$  has a self-node that glues  $y_1$  with  $y_2$ , then  $x$  is Weierstrass iff  $2x \sim y_1 + y_2 \in \text{Pic}(E)$ .

If  $x$  is on a rational component  $R$ ,  $x$  is Weierstrass if either  $R$  is attached to a genus one curve at two distinct points; or  $R$  has a self-node gluing  $y_1$  and  $y_2$  and is attached to a genus one tail at  $y_3$ , in which case we require  $\phi(y_1) = \phi(y_2)$  for a double cover  $\phi: R \rightarrow \mathbb{P}^1$  ramified at  $x$  and  $y_3$ ; or  $R$  has two self-nodes gluing  $y_1$  with  $y_2$ , and  $y_3$  with  $y_4$ , in which case we require  $x$  to be a ramification point for a double cover  $\phi: R \rightarrow \mathbb{P}^1$  such that  $\phi(y_1) = \phi(y_2)$  and  $\phi(y_3) = \phi(y_4)$  - geometrically, if we embed  $\mathbb{P}^1$  as a conic  $C \subseteq \mathbb{P}^2$ , the line through  $x$  and  $\overline{y_1 y_2} \cap \overline{y_3 y_4}$  should be tangent to  $C$  at  $x$ . See Figure 2.

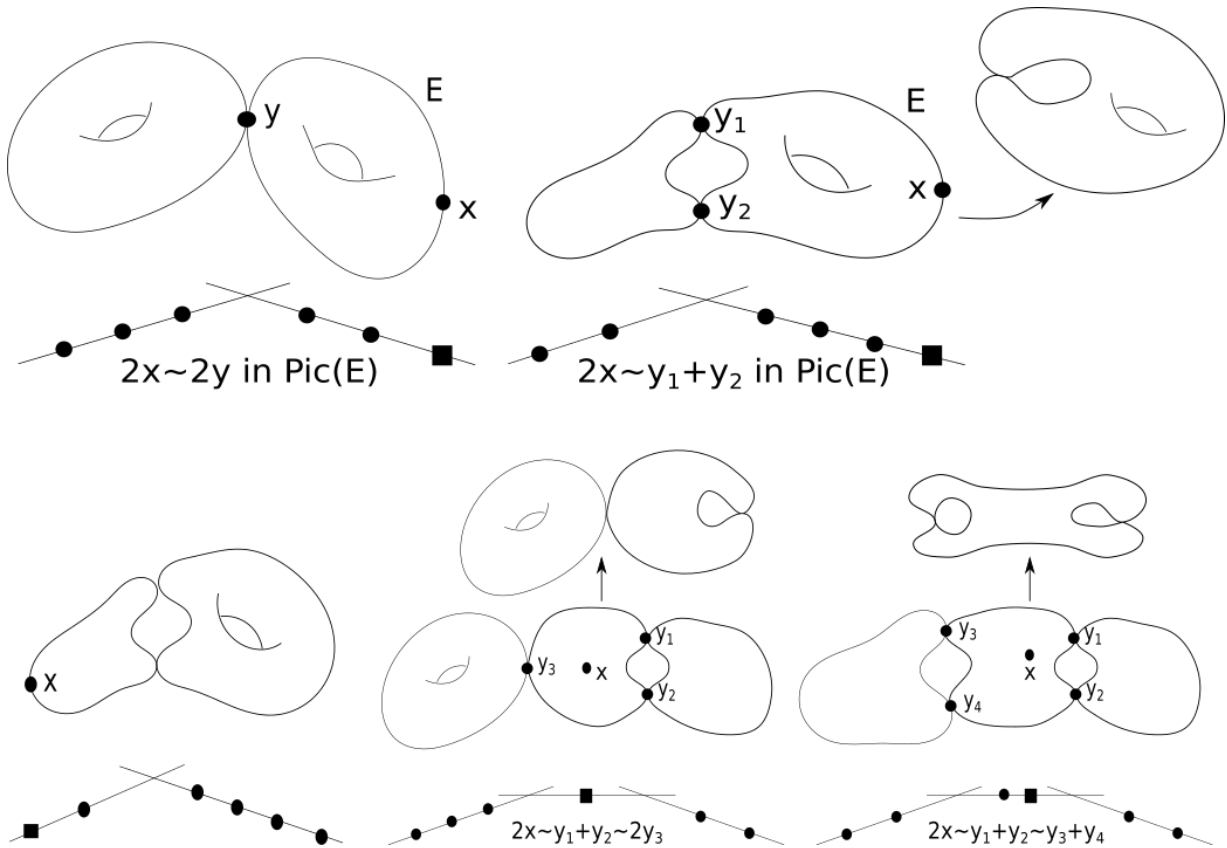


FIGURE 2. Admissible covers and Weierstrass points.

- If  $x_1$  and  $x_2$  are conjugate, they have to map to the same component of the target of the admissible cover. We may adapt the description of the previous point by replacing every condition on  $2x$  by its analogue for  $x_1 + x_2$ . There are a few more situations to take into account:  $x_1$  and  $x_2$  could belong to a rational component  $R$  bubbling off from a Weierstrass point of a genus two curve; or bridging between two distinct curves of genus one; or  $x_1$  and  $x_2$  could lie on two distinct rational components  $R_1$  and  $R_2$  intersecting each other at one node and meeting a curve of genus one in two distinct points ( $\dagger$ ); or  $R_1$  and  $R_2$  intersecting each other in three points. See Figure 3.

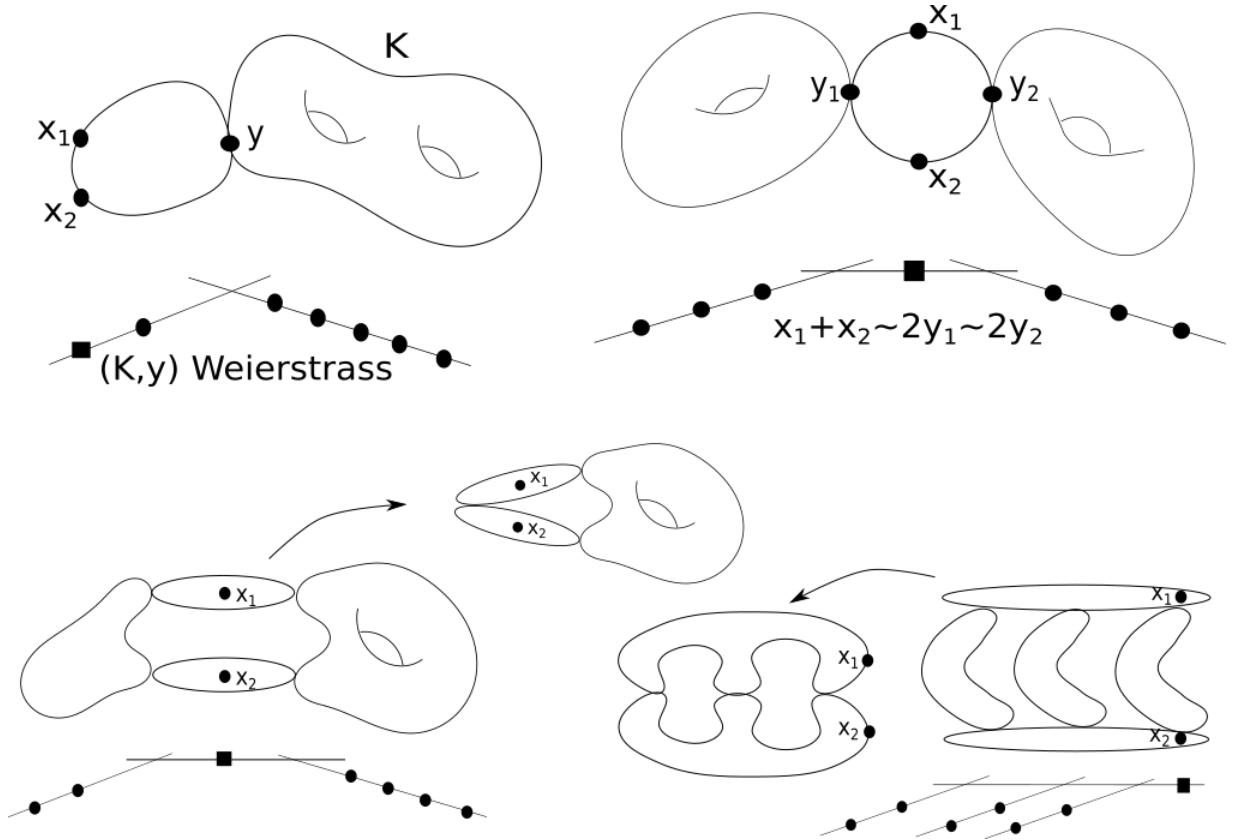


FIGURE 3. Admissible covers and conjugate points.

*Remark 4.2.* In case (†), the singularity of the total space of a smoothing  $\mathcal{C} \rightarrow \Delta$  at the two distinguished nodes (separating the elliptic component from the rational chain) are both  $A_k$  for the same  $k$ , because they map to the same node of the target in the admissible cover. This consideration is stable under base change, and it therefore entails a symmetry of the rational chain in the model with regular total space.

Let us set some notation before providing a description of semistable tails. In what follows, we shall determine where on the core the special chains may *cleave* (be attached), and how their length relates to that of the other chains. The length of a chain is the number of components it is formed of. We shall consider the dual graph of the core (unmarked), dubbing special the components to which a special tail cleaves; we then assign a weight to each edge of the dual graph: the weight is 1 if the edge lies on a path connecting a special and a stable non-special component, passing only through strictly semistable non-special components; the weight is 0 otherwise. For two points  $a$  and  $b$  on the core, their distance is the minimum weight of a path joining them:

$$\text{dist}^*(a, b) = \min_{a \xrightarrow{p} b} \left\{ \sum_{e \in p} w(e) \right\}.$$

For two points on a special chain, their distance is the usual path distance on the dual graph (we could have included special chains besides the core, and marked their edges with weight 1).

**PROPOSITION 4.3.** *Let  $\phi: \mathcal{C} \rightarrow \overline{\mathcal{C}}$  be a birational contraction over the spectrum of a discrete valuation ring  $\Delta$ , where:  $\mathcal{C} \rightarrow \Delta$  is a family of prestable (reduced, nodal) curves of arithmetic*

genus two, with regular total space and smooth generic fibre  $\mathcal{C}_\eta$ ;  $\overline{\mathcal{C}} \rightarrow \Delta$  is a family of Gorenstein curves, and  $\overline{\mathcal{C}}_0$  contains a genus two singularity of type  $I_m$  at  $q$ . Denote by  $(Z; q_1, \dots, q_m)$  the exceptional locus  $\text{Exc}(\phi) = \phi^{-1}(q)$ , marked with  $Z \cap \overline{\mathcal{C}}_0 \setminus Z$ , where  $q_m$  corresponds to the special branch of  $\overline{\mathcal{C}}_0$ . Then:

- (i)  $(Z, q_m)$  is Weierstrass.
- (ii) Let  $x_m$  be the point of the core of  $Z$  closest to  $q_m$ , and let  $k$  be the length of  $R_m$ , the rational chain separating  $q_m$  from  $x_m$ . With similar notation, for every  $i = 1, \dots, m-1$ ,  $R_i$  has length

$$\begin{cases} 3k + 2 + \text{dist}^*(x_m, x_i) & \text{if } x_i \neq x_m, \\ k + 2 + 2 \text{dist}(q_m, r_i) & \text{if } x_i = x_m, \end{cases}$$

where  $r_i$  is the point of  $R_m$  closest to  $q_i$ .

PROPOSITION 4.4. Same as above with  $\overline{\mathcal{C}}_0$  containing a genus two singularity of type  $II_m$ , and  $q_1, q_m$  corresponding to the special branches. Then:

- (i)  $(Z, q_1, q_m)$  is conjugate.
- (ii)  $R_1$  and  $R_m$  have the same length  $k$ , and, for  $i = 2, \dots, m-1$ ,  $R_i$  has length

$$\begin{cases} 2k + 1 + \min_{\epsilon \in \{1, m\}} \text{dist}^*(x_\epsilon, x_i) & \text{if } x_1 \neq x_m, \text{ and } x_i \notin \{x_1, x_m\}, \\ k + 1 + \text{dist}(q_j, r_i) & \text{if } x_1 \neq x_m, \text{ and } x_i = x_j, j \in \{1, m\}, \\ 2k + 1 + \text{dist}(x_1, r_m) + \text{dist}^*(x_1, x_i) & \text{if } x_1 = x_m, \text{ and } x_i \neq x_1, \\ k + 1 + \text{dist}(q_1, r_i) + \delta \text{dist}(r_m, r_i) & \text{if } x_1 = x_m = x_i, \delta = \begin{cases} 1 & r_m \in [q_1, r_i] \\ 0 & \text{otherwise.} \end{cases} \end{cases}$$

PROPOSITION 4.5. Let  $(\mathcal{C}, p_1, \dots, p_n) \rightarrow \Delta$  be a family of pointed semistable curves of arithmetic genus two such that  $\mathcal{C}$  has regular total space and smooth generic fibre, and  $(\mathcal{C}, p_1) \rightarrow \Delta$  is Weierstrass. Let  $(Z, q_1, \dots, q_m)$  be a genus two subcurve of  $\mathcal{C}_0$  containing none of the  $p_i(0)$ , marked by  $Z \cap \overline{\mathcal{C}}_0 \setminus Z$  so that the tail containing  $p_1$  is attached to  $Z$  at  $q_m$ , and satisfying all the shape prescriptions of Proposition 4.3(2). There exists a contraction  $\phi: \mathcal{C} \rightarrow \overline{\mathcal{C}}$  over  $\Delta$ , with exceptional locus  $Z$ , such that  $\overline{\mathcal{C}} \rightarrow \Delta$  is a family of Gorenstein curves containing a type  $I_m$  singularity in the central fibre.

PROPOSITION 4.6. Same as above with  $(\mathcal{C}, p_1, \bar{p}_1) \rightarrow \Delta$  conjugate,  $(Z, q_1, \dots, q_m)$  shaped as prescribed by Proposition 4.4(2),  $q_1, q_m$  corresponding to the branches containing  $p_1, \bar{p}_1$  respectively. The resulting  $\overline{\mathcal{C}} \rightarrow \Delta$  contains a type  $II_m$  singularity in the central fibre.

*Proof.* (of Proposition 4.3) By blowing down all the rational trees on  $\overline{\mathcal{C}}_0$ , we can assume that the latter does not contain any separating node. Consider then the hyperelliptic cover  $\tau: \overline{\mathcal{C}} \rightarrow \mathbb{P}(\pi_* \omega_{\overline{\mathcal{C}}/\Delta})$ ; restricting to the central fibre,  $\tau$  contracts all axes, and induces a two-fold cover of  $\mathbb{P}^1$  by the special branch, ramified at the singularity and at another point; in fact, we can extend the image of this point to a section of  $\mathbb{P}(\pi_* \omega_{\overline{\mathcal{C}}/\Delta})$  lying inside the branch locus of  $\tau$ . By pulling this back to  $\mathcal{C}$  via  $\tau \circ \phi$  we get a horizontal divisor  $\Delta'$ ; clearly, the stable model of  $(\mathcal{C}, \Delta')$  is Weierstrass, and its central fibre coincides with the stabilisation of  $(Z, q_m)$ . This proves the first claim. (The proof of Proposition 4.4(1) is entirely analogous, by noticing that the preimage of a generic hyperplane section of  $\mathbb{P}(\pi_* \omega_{\overline{\mathcal{C}}/\Delta})$  will mark the two special branches of  $\overline{\mathcal{C}}_0$ .)

We now come to a proof of the more combinatorial claim (2) of the Proposition. Since  $\overline{\mathcal{C}} \rightarrow \Delta$  is Gorenstein, and  $\phi$  is assumed to be an isomorphism outside  $Z$ , because the dualising sheaf

behaves well under restriction to open subschemes, we have an equality of line bundles:

$$\phi^* \omega_{\bar{\mathcal{C}}/\Delta} = \omega_{\mathcal{C}/\Delta}(D),$$

for some effective (Cartier) divisor  $D$  on  $\mathcal{C}$  supported on  $Z$ . The next lemma will help us determine the coefficients of  $D$  along the components of  $Z$  containing  $q_i$ .

LEMMA 4.7. *Let  $\nu: C \rightarrow \bar{C}$  be the normalisation of a Gorenstein singularity of genus two, with  $\nu^{-1}(q) = \{q_1, \dots, q_m\}$ . Then  $\nu^* \omega_{\bar{C}} = \omega_C(2q_1 + \dots + 2q_{m-1} + 4q_m)$  (type I) or  $\nu^* \omega_{\bar{C}} = \omega_C(3q_1 + 2q_2 + \dots + 2q_{m-1} + 3q_m)$  (type II).*

*Proof.* The dualising sheaf of a reduced curve admits an explicit description (due to Rosenlicht, see e.g. [AK70, Proposition VIII.1.16]) in terms of residues:

$$\omega_{\bar{C}}(U) = \{\eta \in \Omega_C \otimes K(\nu^{-1}(U)) \mid \sum_{p_i \in \nu^{-1}(p), p \in U} \text{Res}_{p_i}((\nu^* f)\eta) = 0, \forall f \in \mathcal{O}_{\bar{C}}(U)\}.$$

We are going to use the explicit coordinates in (1) and (3). In case I, we know that  $\tilde{\mathfrak{m}}^4 \subseteq R$ , therefore we have poles of fourth order at most. It is enough to study the possible polar tails. On the other hand,  $t_i^2 \in R$  for all  $i$  implies the part of order three is trivial. So let

$$\eta = c_1 \frac{dt_1}{t_1^4} + b_1 \frac{dt_1}{t_1^2} + a_1 \frac{dt_1}{t_1} \oplus \dots \oplus c_m \frac{dt_m}{t_m^4} + b_m \frac{dt_m}{t_m^2} + a_m \frac{dt_m}{t_m}.$$

From looking at  $1 \cdot \eta$  we deduce  $\sum_{i=1}^m a_i = 0$ ; from  $x_i \cdot \eta$  we see  $b_i + c_m = 0$  for all  $i$ , and from  $x_i^3 \cdot \eta$  we have  $c_i = 0$  for all  $i$ . (The statement about third order poles can be evinced from  $x_i^2 \cdot \eta$  or from  $z \cdot \eta$  indifferently.) Therefore  $\omega_C/\nu_* \omega_{\bar{C}}$  is spanned by

$$\begin{aligned} & \frac{dt_1}{t_1} - \frac{dt_m}{t_m}, \dots, \frac{dt_{m-1}}{t_{m-1}} - \frac{dt_m}{t_m}, \frac{dt_m}{t_m^2} \\ & \bar{\eta} = \frac{dt_1}{t_1^2} + \dots + \frac{dt_{m-1}}{t_{m-1}^2} - \frac{dt_m}{t_m^4}. \end{aligned}$$

In particular  $\omega_C$  is generated by  $\bar{\eta}$  as an  $\mathcal{O}_C$ -module.

In case II, we know that  $\tilde{\mathfrak{m}}^3 \subseteq R$ , so we have poles of third order at most. Let

$$\eta = c_1 \frac{dt_1}{t_1^3} + b_1 \frac{dt_1}{t_1^2} + a_1 \frac{dt_1}{t_1} \oplus \dots \oplus c_m \frac{dt_m}{t_m^3} + b_m \frac{dt_m}{t_m^2} + a_m \frac{dt_m}{t_m}.$$

From looking at  $1 \cdot \eta$  we deduce  $\sum_{i=1}^m a_i = 0$ ; from  $x_i \cdot \eta$  we see  $b_1 + b_m = 0$  (if  $i = 1$ ), and  $b_i + c_m = 0$  (if  $i = 2, \dots, m-1$ ); finally from  $x_i^2 \cdot \eta$  we have  $c_1 + c_m = 0$  (if  $i = 1$ ), and  $c_i = 0$  (if  $i = 2, \dots, m-1$ ). Therefore  $\omega_C/\nu_* \omega_{\bar{C}}$  is spanned by

$$\begin{aligned} & \frac{dt_1}{t_1} - \frac{dt_m}{t_m}, \dots, \frac{dt_{m-1}}{t_{m-1}} - \frac{dt_m}{t_m}, \frac{dt_1}{t_1^2} - \frac{dt_m}{t_m^2}, \\ & \bar{\eta} = \frac{dt_1}{t_1^3} + \frac{dt_2}{t_2^2} + \dots + \frac{dt_{m-1}}{t_{m-1}^2} - \frac{dt_m}{t_m^3}. \end{aligned}$$

In particular  $\omega_C$  is generated by  $\bar{\eta}$  as an  $\mathcal{O}_C$ -module. □

COROLLARY 4.8. *The multi-degree of  $\omega_C$  is  $(0, \dots, 0, 2)$  in case I, and  $(1, 0, \dots, 0, 1)$  in case II.*

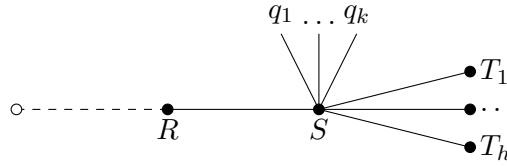
REMARK 4.9. It follows from this and Corollary 3.6 that the finiteness of automorphism groups,  $H^0(\bar{C}, \Omega_{\bar{C}}^\vee(-\sum_{i=1}^n p_i)) = 0$ , implies  $\omega_{\bar{C}}(\sum_{i=1}^n p_i)$  is ample.

Let us go back to the proof of Proposition 4.3. Note that  $\phi|_{\mathcal{C}_0 \setminus Z}$  is the normalisation of  $\overline{\mathcal{C}_0}$  at  $q$ . Letting  $T_i$  be the tail of  $\mathcal{C}_0 \setminus Z$  attached to  $Z$  at  $q_i$ , by the adjunction formula  $\omega_{\mathcal{C}/\Delta|T_i} = \omega_{T_i}(q_i)$ ; it follows from Lemma 4.7 that  $D$  has multiplicity 3 at the component of  $Z$  containing  $q_m$  (case *I*) - resp. 2 at the components containing  $q_1$  and  $q_m$  (case *II*) -, and 1 at any other component containing a  $q_i$ . Set  $\mathcal{L} = \omega_{\mathcal{C}/\Delta}(D) = \phi^* \omega_{\overline{\mathcal{C}}/\Delta}$ ; we draw the consequences of  $\mathcal{L}|_Z = \mathcal{O}_Z$ . We split  $Z$  into a core  $K$  and a number of rational trees. By Castelnuovo criterion, we may assume that  $Z$  is semistable (hence, there are at most  $m$  rational trees).

Let  $d_A$  denote the multiplicity of the divisor  $D$  along the component  $A$  of  $Z$ . First, we claim that no component can appear with  $d_A = 0$ . Assume that this occurred along one of the rational trees. Call  $S$  ( $S \simeq \mathbb{P}^1$ ) a component furthest from the core such that  $d_S = 0$ ;  $R$  the one that precedes it, and  $T_1, \dots, T_h$  the ones that follow it (when sweeping the tree from the core) - so that  $h \geq 1$  by the previous paragraph, and  $d_{T_i} \geq 1$  by inductive assumption. Then, by adjunction,

$$\deg(\mathcal{L}|_S) = -2 + (h+1) + d_R + \sum d_{T_i} = 0,$$

which necessarily implies  $h = 1$ , and  $d_R = d_{T_i} = 0$ , contradicting the assumption. The case that  $S$  belongs to the core is similar ( $\omega_S$  might only be more positive).



Let us now consider  $d_S = 1$ . We stick to the notation above; furthermore, there may be a number  $k$  of  $q_i$ ,  $i \in \{(1), 2, \dots, m-1\}$  in case *II* (resp. *I*), lying on  $S$ . Then, again by adjunction,

$$\deg(\mathcal{L}|_S) = -2 + d_R + \sum d_{T_i} = 0.$$

so either  $d_R = 2$ ,  $h = 0$  and  $k \geq 1$  arbitrary, i.e.  $S$  is adjacent to  $\overline{C \setminus Z}$ ; or  $d_R = 1$ ,  $h = 1$ , and  $d_{T_1} = 1$  (with  $k$  arbitrary). In the latter case, though, by repeating the same argument on  $T_1$  etc., we would find an infinite chain in  $Z$ .

*Remark 4.10.* More generally, an analogous computation shows that, when balancing a component  $A$  of multiplicity  $d_A$ , all neighbouring components of multiplicity  $d_A - 1$  can be safely ignored.

We now prove that  $d_R > d_S$  holds in general for  $S$  on a rational tree. The preceding paragraphs deal with the cases  $d_S = 0, 1$ ; we may therefore assume  $d_S > 1$  (which in particular implies  $0 \leq k \leq 2$ ). We have

$$\deg(\mathcal{L}|_S) = -2 + d_R - (d_S - 1)(h + k + 1) + \sum d_{T_i} = 0.$$

By proceeding inductively from leaves to root, we can assume that  $d_S > d_{T_i}$ ,  $i = 1, \dots, h$ . We may therefore rewrite the previous equality as

$$d_R = (d_S - 1)(h + k + 1) - \sum d_{T_i} + 2 \geq (d_S - 1)(k + 1) + 2 = d_S + 1 + k(d_S - 1) > d_S.$$

In fact, we can prove as on [Smy11a, p.893] that  $d_R = d_S + 1$ , unless  $d_S = 3$  and  $q_m \in S$  (type *I*), or  $d_S = 2$  and either  $q_1$  or  $q_m$  (or both) are on  $S$  (type *II*). We introduce some terminology to describe the weighted dual graph of  $D$ .

DEFINITION 4.11. A  $t$ -chain is a weighted graph that is a chain and such that the weight of two adjacent vertices differ by  $t$ . We call  $t$  the *trend*; the vertex with highest (resp. lowest) weight is called the root (resp. leaf) of the chain. An  $(a, t)$ -chain is a  $t$ -chain with leaf weight  $a$ . The chain  $C_1$  can be attached to the chain  $C_0$  by identifying the root of  $C_1$  with a vertex of  $C_0$  having the same weight. A 1-tree is obtained by attaching a number of  $(1, 1)$ -chains among themselves.

Let us now look at a component  $S$  with  $d_S = 2$  and at least one of  $q_1$  and  $q_m$  attached to it (case *II*). The balancing equation is

$$\deg(\mathcal{L}|_S) = -2 - (h + k + 1) + d_R + \sum d_{T_i} = 0,$$

with  $k \in \{1, 2\}$ . The preceding discussion implies that  $d_{T_i} = 1$  for all  $i = 1, \dots, h$ , so  $d_R = 3 + k$ . If  $k = 2$ , i.e. both  $q_1$  and  $q_m$  are on  $S$  - in which case they are indeed equidistant from the core -, then  $d_R = 5$ , and it can be shown inductively that the trend on the chain connecting  $S$  to the core is 3. The same holds in case *I*, with  $q_m$  attached to  $S$  and  $d_S = 3$ .

Finally, say  $d_S = 2$  and only  $q_1 \in S$ . Then  $d_R = 4$ , and the trend along the chain that connects  $S$  to the core is 2, unless there is a component  $S'$  at which two 2-chains meet, in which case the trend becomes 3 after  $S'$ .

DEFINITION 4.12. A 2-tree is obtained by attaching a number of  $(1, 1)$ -chains to a  $(2, 2)$ -chain. A 3-tree is obtained by attaching a number of  $(1, 1)$ -chains either to a  $(3, 3)$ -chain, or to a weighted graph which is itself obtained by attaching two  $(2, 2)$ -chains to the leaf of a 3-chain.

From the preceding discussion, it is clear that the weighted dual graph of  $D$  is obtained by attaching a number of 1-trees, and either (a) one 3-tree or (b) two 2-trees to the dual graph of the core  $K$ , weighted in an appropriate fashion.

Finally, let us look at the core  $K$ . Consider it as a one-pointed (case (a)), resp. two-pointed (case (b)) curve of genus two, by ignoring all the attachment points of the 1-trees (it does not alter the balancing equation along  $K$  by Remark 4.10), and let  $\bar{K} \in \overline{\mathcal{M}}_{2,1}$  (resp.  $\overline{\mathcal{M}}_{2,2}$ ) be its stable model. The following can happen:

- (i)  $K$  is a smooth curve of genus two. In case (a), let  $R$  be the component adjacent to the core along the 3-tree, and let  $x = R \cap K$ ; then  $d_K = d_R + 3$  by balancing  $R$ . Balancing  $K$  gives

$$\omega_{\mathcal{C}/\Delta}(d_R R + d_K K)|_K = \omega_K(d_R x - (d_R + 2)x) \simeq \mathcal{O}_K,$$

which admits a solution if and only if  $K$  is Weierstrass. Similarly, case (b) can be balanced if and only if  $K$  is conjugate.

- (ii)  $K$  contains two distinct subcurves of genus one  $E_1$  and  $E_2$ . We start by solving the balancing equation on one of them, say  $E = E_1$ . If all but one of the neighbouring components have multiplicity  $d_E - 1$ , then the last one is forced to have multiplicity  $d_E - 1$  as well (by degree reasons). The case that all but two neighbouring components have multiplicity  $d_E - 1$  occurs when either one 2-tree or one 3-tree (and exactly one) is attached to  $E$  at  $x$ ; let  $F$  be the other component with undetermined multiplicity, which lies between  $E_1$  and  $E_2$  (possibly  $F = E_2$ ), and let  $E \cap F = \{y\}$ . The case of a 2-tree forces  $d_F = d_E$  by degree reason, but then we are left to solve  $x \sim y$  in  $\text{Pic}(E)$ , which is impossible; on the other hand, the case of a 3-tree imposes  $d_F = d_E + 1$  and  $2x \sim 2y$  in  $\text{Pic}(E)$ , i.e.  $K$  is Weierstrass. By the same token, the two 2-trees have to hit the same genus one curve, say  $E_1$ , in points  $x_1, x_2$  such that  $x_1 + x_2 \sim 2y \in \text{Pic}(E_1)$  and  $d_F = d_E + 1$ .

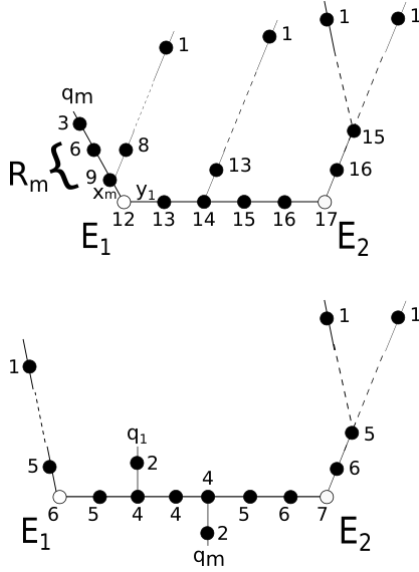


FIGURE 4. Examples of  $Z$  with two genus one components (the white discs). The numbers indicate the multiplicity of  $D$ . Above, one 3-tree attached to  $E_1$ ; below, two 2-trees attached to the rational chain separating  $E_1$  from  $E_2$ .

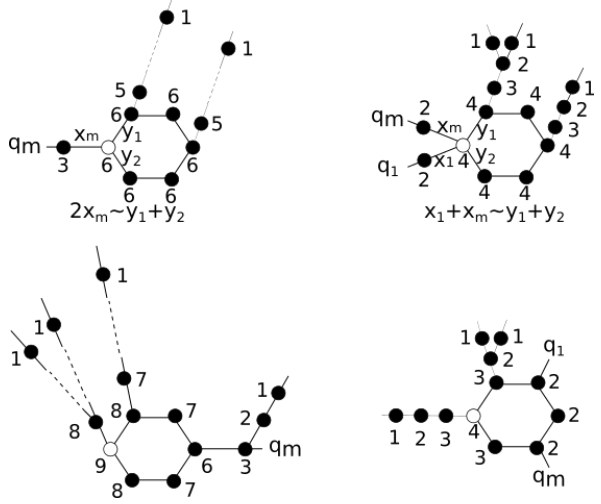


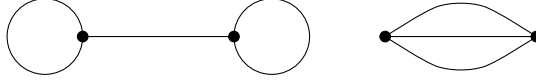
FIGURE 5.  $K$  is a genus one curve  $E$  with a rational bridge  $R$ . Left column, one 3-tree; right column, two 2-trees. Above: the special trees connect to  $E$ ; below: they connect to  $R$  - note that  $R$  is necessarily symmetric in this case.

Assume now there is a chain of rational curves  $S_i$  between  $E_1$  and  $E_2$  in  $K$ , and one of the special trees cleaves to one of the  $S_i$ ; in case (b), then, both 2-trees must connect to (possibly different)  $S_i$ , by the previous paragraph. Moreover, the trend along the rational chains at  $E_1$  and  $E_2$  has to be 1. This in turn implies that the trend along the chain separating the two 2-trees is 0 (case (b)). In particular, the two 2-trees are attached to components with the same multiplicity for  $D$ , so  $q_1$  and  $q_m$  are equidistant from the core. See the examples in Figure 4.

- (iii)  $K$  contains an irreducible subcurve of arithmetic genus one  $E$ , with two points  $y_1$  and  $y_2$  on  $E$  that are joined in  $K$  by a (possibly empty) rational chain. We see as above that either a 3-tree is attached to a point  $x \in E$  satisfying  $2x \sim y_1 + y_2$  in  $\text{Pic}(E)$ , or two 2-trees are attached to  $x_1, x_2 \in E$  satisfying  $x_1 + x_2 \sim y_1 + y_2$  in  $\text{Pic}(E)$ , or the rational chain is not empty and all the distinguished trees are attached to it. In this case, solve the balancing equation on  $E$ : let  $d = d_E$ ,  $d_1$  and  $d_2$  be the multiplicities of the rational components attached to  $y_1$  and  $y_2$  respectively; then either  $d_1 = d_2 = d - 1$ , or  $d_1 = d - 1 + k$ ,  $d_2 = d - 1 - k$  and  $r_1 - r_2$  is  $k$ -torsion in  $\text{Pic}(E)$ . But, by chasing the balancing equation along the rational necklace, we find that, if  $k \geq 1$ , then the trend increases when passing through a distinguished bead, so that ultimately  $d - 1 - k = d_2 > d_1 = d - 1 + k$ , contradiction. So the only possibility is to have a rational chain symmetric with respect to the distinguished beads, namely: in case (a) the two portions of the rational chain lying between the special bead and  $E$  have the same length, and in case (b) the distance of the shortest path between a special bead and  $E$  is the same for the two special beads. Compare with Remark 4.2 and Figure 5.

- (iv) Finally, consider the case of geometric genus 0. There are two trivalent graphs of genus two.





The graph on the left can be dealt with as a degeneration of cases (ii) and (iii) above; note that if only one of the 2-trees cleaves to a rational bridge, the balancing equation admits no solution, and, furthermore, if the special trees cleave to a rational loop, then the latter needs to be symmetric (Remark 4.2).

The graph on the right is the union of two stable components along three semistable chains. It is easy to see that, if a special tree cleaves to a stable component, the only chance that the balancing condition may be satisfied is that there are two 2-trees and they cleave to different stable components; the semistable chains have arbitrary length and  $D$  has the same multiplicity along every component of the core; see the upper left corner of Figure 6. There remain three possibilities for the dual graph, according to how the distinguished components (denoted by  $R$ ) and the other stable components (denoted by  $A$  and  $B$ ) distribute themselves.

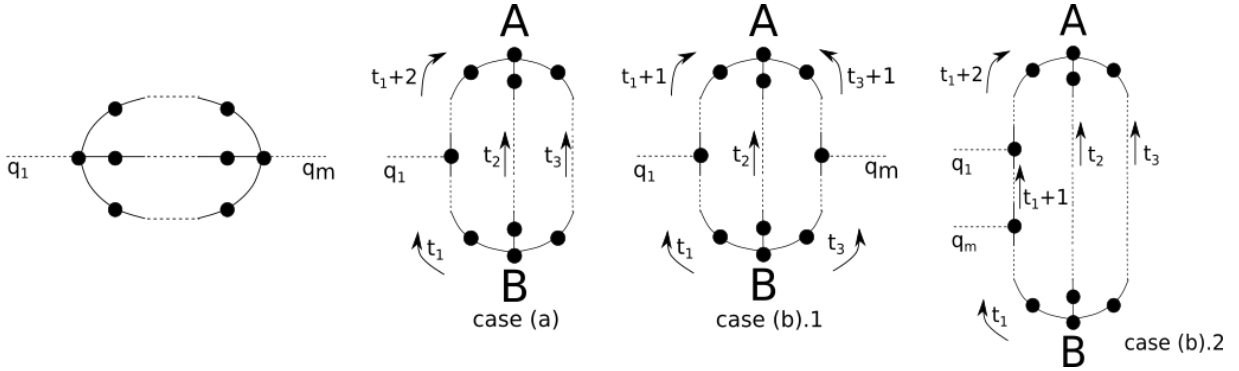


FIGURE 6. When  $K$  consists of rational curves.

Denoting by  $t$  the trend along various rational chains, we find that in case (a) and (b).2 balancing along  $A$  or  $B$  is equivalent to  $\sum_i t_i = 1$ . Assume  $t_1 \geq 0$ ; then  $d_A > d_B$ , therefore  $t_2, t_3 > 0$ , which contradicts  $\sum_i t_i = 1$ . Similarly, if  $t_1 \leq -2$ , then  $d_A < d_B$ , therefore  $t_2, t_3 < 0$ , which makes  $\sum_i t_i = 1$  again impossible. We find only one solution with  $t_1 = -1$  and  $t_2 = t_3 = 0$  - notice that it is a degeneration of case iii above. In case (b).1, we find  $\sum_i t_i = 1$  when balancing  $B$ , and  $\sum_i t_i = -1$  when balancing  $A$ , which is a contradiction.

This concludes the proof of Propositions 4.3(2) and 4.4(2).  $\square$

*Remark 4.13.* When the special trees cleave to a rational component  $R$ , solving the balancing equation imposes no restriction on the relative position of special points on  $R$ . In case they do not satisfy the Weierstrass condition (Remark 4.1), we may still find an effective  $D$  such that  $\omega_{\mathcal{C}/\Delta}(D)$  is trivial on  $Z$ ; we wonder whether this line bundle might not be semiample though.

*Proof.* (of Proposition 4.5) By blowing down some rational tails outside  $Z$ , we can assume that  $\mathcal{C}_0 \setminus Z = \sqcup_{i=1}^m T_i$  with each  $T_i \simeq \mathbb{P}^1$ . The image of  $p_i(0)$  and  $p_j(0)$  might now coincide for  $i \neq j$ . The total space of the curve can still be assumed to be smooth. By abuse of notation, we denote the resulting family of pointed curves by  $(\mathcal{C}, p_1, \dots, p_n)$ . By assumption on the shape of  $Z$ , we can find an effective Cartier  $D$  supported on  $Z$  such that  $\mathcal{L} = \omega_{\mathcal{C}/\Delta}(D + \sum p_i)$  is trivial on  $Z$  and relatively ample elsewhere (both on  $T_i$  and on the generic fibre). Consider a second line bundle  $\mathcal{L}' = \mathcal{O}(2p_1 + \sum p_i)$ . Since we assumed  $p_1$  to be Weierstrass,  $\mathcal{L}_\eta \simeq \mathcal{L}'_\eta$ . On the other hand it is easy to see that the multi-degrees of  $\mathcal{L}_0$  and  $\mathcal{L}'_0$  coincide, as  $Z$  is unmarked and each

rational tail is isomorphic to  $\mathbb{P}^1$ ; it follows from the separatedness of  $\text{Pic}_{\mathcal{C}/\Delta}^0 \rightarrow \Delta$  (see [Del85, p. 136] or [BLR90, §9.4]) that  $\mathcal{L}$  and  $\mathcal{L}'$  are isomorphic line bundles, so that, in particular,  $\mathcal{L}$  is trivial on a neighbourhood of  $Z$ . Observe now that

$$R^1\pi_*\mathcal{L}(-D) = R^1\pi_*\omega_{\mathcal{C}/\Delta}(\sum p_i) = 0$$

by semistability, hence  $\pi_*\mathcal{L} \rightarrow \pi_*(\mathcal{L}|_D) = \pi_*\mathcal{O}_D$ , which contains the constants, showing that  $\mathcal{L}$  is semiample along  $Z$ ; that it is along the  $T_i$  is easier.

We therefore have a well-defined morphism:

$$\mathcal{C} \xrightarrow{\phi} \overline{\mathcal{C}} = \text{Proj}_{\Delta} \left( \bigoplus_{n \geq 0} \pi_*\mathcal{L}^{\otimes n} \right) \rightarrow \Delta$$

associated to  $\mathcal{L}$ . The proof that  $\overline{\mathcal{C}} \rightarrow \Delta$  is a flat family of Gorenstein curves goes along the lines of [Smy11a, Lemma 2.13] or [RSW17a, Proposition 3.7.3.1]. It is clear from the classification that it contains a type  $I_m$ -singularity.

The proof of Proposition 4.6 is entirely analogous.  $\square$

*Remark 4.14.* It follows that genus two Gorenstein singularities are smoothable.

*Caveat.* Given a family of semistable curves  $\pi: \mathcal{C} \rightarrow \Delta$  over a discrete valuation ring, and a line bundle  $\mathcal{L}$  on  $\mathcal{C}$  that is trivial on a higher genus subcurve  $Z$  of  $\mathcal{C}_0$  and  $\pi$ -ample elsewhere, is  $\mathcal{L}$   $\pi$ -semiample? is it relatively generated by global sections? In positive characteristic, a positive answer follows from results of S. Keel [Kee99]. If  $\text{char}(\mathbf{k}) = 0$ , the answer depends on the family; in particular, the assumption that the total space of  $\mathcal{C}$  is regular along  $Z$  seems to be relevant. We construct a counterexample using the theory of limit linear series (l.l.s.): we produce a linear series that can be smoothed while having basepoints along a Weierstrass tail.<sup>2</sup> For any such smoothing, the corresponding line bundle  $\mathcal{L}$  is not globally generated along  $Z$  - though we do not know how powers of  $\mathcal{L}$  behave, so we cannot conclude that  $\mathcal{L}$  is not semiample.

Let  $X_0$  be the nodal curve obtained by attaching  $R \simeq \mathbb{P}^1$  to a Weierstrass point  $q$  of a smooth genus two curve  $Z$ . Choose  $d \gg 0$  ( $d \geq 5$  is enough), and let us study the moduli space of complete linear systems of degree  $d$  on (smoothings of)  $X_0$ ; with  $r = d - 2$ , the Brill-Noether number is  $\rho = 2$  (the dimension of the Jacobian). On the other hand, assume that the  $R$ -aspect of the l.l.s. has  $\mathcal{L}_{Y|Z} \simeq \mathcal{O}_Z$ ; then the  $Z$ -aspect has  $\mathcal{L}_{Z|Z} \simeq \mathcal{O}_Z(dq)$ , whose vanishing sequence is  $\alpha_Z(q) = \{0, 1, \dots, d-4, d-2, d\}$ , from which we deduce for the complementary aspect  $\alpha_R(q) \geq \{0, 2, 4, 5, \dots, d\}$ . We want to show that all such aspects are smoothable, by appealing to the Regeneration Theorem [HM98, Theorem 5.41]. Notice that in the case at hand we have a choice of a two-dimensional subspace of  $\langle 1, t, t^2, t^3 \rangle_{\mathbf{k}} \subseteq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$  meeting the subspace  $\langle t^2, t^3 \rangle_{\mathbf{k}}$  non-trivially, i.e. the locus in  $\text{Gr}(1, \mathbb{P}^3)$  of lines meeting a fixed line  $\ell$ , which is a Schubert cycle of dimension 3. We therefore need to put  $X_0$  in a family over a base  $B$  of dimension 1 at least. We shall do so by considering the family  $X$  obtained by attaching  $R$  to a moving point of  $Z$ , so that  $X_0$  is the fibre of  $X$  over  $q \in Z$ .

Let us start by examining the other possibilities for  $\mathcal{G}_d^{d-2}(X_0)$ : the  $R$ -aspect can in fact restrict to any line bundle of degree 0 on  $Z$ , which we are going to write as  $\mathcal{O}_Z(p_1 + p_2 - 2q)$  for two moving points  $p_1, p_2$  on  $Z$  (think of them as coordinates on  $\text{Pic}(Z)$ ). Then  $\mathcal{L}_Z = \mathcal{O}_Z((d-2)q + p_1 + p_2)$ .

---

<sup>2</sup>It was pointed out by F. Carocci that a similar computation can be carried out for a genus one tail as well. This shows that the regularity assumption of [Smy11a, Lemma 2.13] is necessary.

| $\subseteq \text{Pic}(Z)$     | dim | $\alpha_{Z;d-3,d-2}$ | $\alpha_{R;0,1}$ | $\subseteq \mathbb{P}H^0(R, \mathcal{O}_R(d))$                              | dim |
|-------------------------------|-----|----------------------|------------------|---|-----|
| $p_1 + p_2 \sim 2q$           | 0   | $\{d-2, d\}$         | $\geq \{0, 2\}$  | $\{\ell' \in \text{Gr}(1, \mathbb{P}^3)   \ell' \cap \ell \neq \emptyset\}$ | 3   |
| $2q \approx p_1 + p_2 \geq q$ | 1   | $\{d-3, d-1\}$       | $\geq \{1, 3\}$  | $\mathbb{P}^1$  | 1   |
| $p_1 + p_2 \not\sim q$        | 2   | $\{d-3, d-2\}$       | $\{2, 3\}$       | pt  | 0   |

Letting  $q$  vary in  $B = Z$ , we may generically assume that it is not Weierstrass. We then find:

| $\subseteq \text{Pic}(X)$               | dim     | $\alpha_{Z;d-3,d-2}$ | $\alpha_{R;0,1}$ | $\subseteq \mathbb{P}H^0(R, \mathcal{O}_R(d))$ | dim |
|---|---------|----------------------|------------------|--|-----|
| $p_1 + p_2 \sim \omega_Z$               | $0 + 1$ | $\{d-2, d-1\}$       | $\geq \{1, 2\}$  | $(\mathbb{P}^2)^*$                             | 2   |
| $p_1 + p_2 \sim 2q$                     | $0 + 1$ | $\{d-3, d\}$         | $\geq \{0, 3\}$  | $\mathbb{P}^2$                                 | 2   |
| $\omega_Z, 2q \approx p_1 + p_2 \geq q$ | $1 + 1$ | $\{d-3, d-1\}$       | $\geq \{1, 3\}$  | $\mathbb{P}^1$                                 | 1   |
| $p_1 + p_2 \not\sim q$                  | $2 + 1$ | $\{d-3, d-2\}$       | $\{2, 3\}$       | pt   | 0   |

We conclude that  $\mathcal{G}_d^{d-2}(X/B)$  has pure dimension 3, and we may therefore apply the Regeneration Theorem to deduce that all l.i.s. with  $\mathcal{L}_{R|Z} \simeq \mathcal{O}_Z$  - in particular those containing  $Z$  in the base-locus - are smoothable.

## 5. The new moduli functors

The following is a slight generalisation of [Smy11a, Definition 3.4].

DEFINITION 5.1. Let  $(C, p_1, \dots, p_n)$  be a reduced curve, marked by smooth points. For a nodally attached subcurve  $D \subseteq C$ , we define its *level* as

$$\text{lev}(D) = |D \cap \overline{C \setminus D}| + |\{p_1, \dots, p_n\} \cap D|.$$

In what follows, we impose a level condition only on nodally attached curves, hence we may remain agnostic as to how components attached along worse-than-nodal singularities should be counted towards the level.

We say a Gorenstein curve  $C$  is *minimal* if it contains no node  $x$  such that the normalisation of  $C$  at  $x$  consists of two connected components, one of which has genus zero. When  $C$  has arithmetic genus one, this is the same as saying that  $C$  contains no separating nodes. Recall [Smy11a, Lemma 3.3].

LEMMA 5.2. *A minimal Gorenstein curve  $Z$  of arithmetic genus one can be: a smooth elliptic curve; a ring of  $r \geq 1$  copies of  $\mathbb{P}^1$ ; or an elliptic  $m$ -fold point whose normalisation is the disjoint union of  $m$  copies of  $\mathbb{P}^1$ . In any case  $\omega_Z \simeq \mathcal{O}_Z$ .*

We may similarly describe minimal (sub)curves of genus two.

LEMMA 5.3. *A minimal Gorenstein curve of arithmetic genus two can be either:*

- (i) *a smooth curve of genus two;*
- (ii) *the union of two minimal Gorenstein curves of genus one,  $E_1$  and  $E_2$ , nodally separated by a (possibly empty) rational chain  $R$ ;*
- (iii) *the union of a minimal Gorenstein curve of genus one  $E$ , and a (possibly empty) rational chain  $R$ , along two distinct nodes;*
- (iv) *the union of two copies of  $(\mathbb{P}^1, 0, 1, \infty)$  with three (possibly empty) rational chains  $R_0, R_1, R_\infty$  joining the homonymous points;*

- (v) an elliptic  $m$ -fold point whose pointed normalisation is the disjoint union of either  $m - 2$  copies of  $(\mathbb{P}^1, 0)$  and a semistable rational chain  $(R, 0, \infty)$ , or  $m - 1$  copies of  $(\mathbb{P}^1, 0)$  and a 1-pointed minimal Gorenstein curve of genus one (if the latter is not irreducible and  $m \neq 1$ , there are two genus one subcurves sharing a rational branch);
- (vi) or a singularity of genus two with  $m$ -branches, whose normalisation is the disjoint union of  $m$  copies of  $\mathbb{P}^1$ .

*Remark 5.4.* In both cases (v) and (vi) there are *special* branches supporting the degree of  $\omega_Z$  (compare with Definition 2.3 and Corollary 4.8; recall that the restriction of the dualising sheaf to a component introduces a twist by the conductor ideal, see Noether's formula [Cat82, Proposition 1.2]). Notice that the notion of conjugate points is not always intrinsic to the curve.

Similarly to [Smy11a, Corollary 3.2, Lemma 3.5] we can prove the following.

**LEMMA 5.5.** *Let  $(C, p_1, \dots, p_n)$  be a pointed semistable curve of arithmetic genus two, with minimal genus two subcurve  $Z$ . For every subcurve  $Z' \subseteq C$  of genus two, we have an inclusion  $Z \subseteq Z'$  and  $\text{lev}(Z) \leq \text{lev}(Z')$ .*

**DEFINITION 5.6.** We say that a point *cleaves* to a component of a curve if it is connected to it by a possibly empty chain of rational curves.

We finally come to the definition of  $m$ -stability for curves of genus two.

**DEFINITION 5.7.** Fix positive integers  $1 \leq m < n$ . Let  $(C, p_1, \dots, p_n)$  be a connected, reduced, complete curve of arithmetic genus two, marked by smooth distinct points. We say that  $C$  is  $m$ -stable if:

- (i)  $C$  is Gorenstein with only: nodes; elliptic  $l$ -fold points,  $l \leq m + 1$ ; type  $I_{\leq m}$ , type  $II_{\leq m}$ , and dangling  $II_{m+1}$  singularities of genus two, as singular points.
- (ii) If  $Z$  is a connected subcurve of arithmetic genus two, then  $\text{lev}(Z) > m$ .
- (iii) If  $E$  is a nodally attached subcurve of arithmetic genus one,  $\text{lev}(E) > m + 1$ .
- (iv)  $H^0(C, \Omega_C^\vee(-\sum_{i=1}^n p_i)) = 0$ .
- (v) If  $C$  contains a singularity of genus two, or an elliptic  $l$ -fold with a self-branch or a genus one branch,  $p_1$  cleaves to one of the special branches (see Remark 5.4). If  $C$  contains two genus one subcurves sharing a branch, and  $E_1$  has level less than  $m + 2$ , then  $p_1$  cleaves to  $E_1$ .

*Remark 5.8.* The definition is not  $\mathfrak{S}_n$ -symmetric. In the argument below, we exploit the asymmetry to write the dualising line bundle of a genus two (sub)curve  $Z$  as  $\omega_Z \simeq \mathcal{O}_Z(q_1 + \bar{q}_1)$ , where  $q_1$  is the point of  $Z$  closest to  $p_1$ , and  $\bar{q}_1$  its conjugate, sometimes depending on a one-parameter smoothing. Compare with the situation in genus one, where the dualising line bundle of a minimal Gorenstein curve is trivial (all smooth points are non-special). We also use refer to  $p_1$  when deciding which genus one subcurve to contract first in case there are two of the same level.

*Remark 5.9.* If there is a nodally attached subcurve of genus one, condition (iii) and condition (iv) jointly imply condition (ii). Indeed, from Corollary 3.6 we have  $\text{lev}(Z) \geq \text{lev}(E) - 1$ . The only cases (up to relabelling) in which the level drops by one are: when  $Z = (E, p_1, \dots, p_{l-2}, q_1, q_2) \sqcup_{\{q_1, q_2\}} (\mathbb{P}^1, q_1, q_2, p_{l-1})$ ; and when  $Z = (E_1, p_1, \dots, p_{l-1}, q) \sqcup_q (E_2, q)$ , where all the  $E$  have genus one.

**LEMMA 5.10** boundedness. *If  $(C, p_1, \dots, p_n)$  is an  $m$ -stable curve of genus two, the  $N$ -th power of  $A = \omega_C(\sum_{i=1}^n p_i)$  is very ample for every  $N > 2 + 8(m + 1)$ .*

*Proof.* It is enough to show that, for every pair of points  $p, q \in C$  (possibly equal):

- (i) *basepoint-freeness*:  $H^1(C, A^{\otimes N} \otimes I_p) = 0$ ;
- (ii) *separating points and tangent vectors*:  $H^1(C, A^{\otimes N} \otimes I_p I_q) = 0$ .

By Serre duality we may equivalently show that  $H^0(C, \omega_C \otimes A^{-N} \otimes (I_p I_q)^\vee) = 0$ . Let  $\nu: \tilde{C} \rightarrow C$  be the normalisation, and let  $\nu^{-1}(p) = \{p_1, \dots, p_h\}$ ,  $\nu^{-1}(q) = \{q_1, \dots, q_k\}$ , with  $h, k \leq m+1$ . It follows from Proposition 2.2 (and [Smy11a, Proposition A.3]) that  $\nu_* \mathcal{O}_{\tilde{C}}(-D) \subseteq I_p I_q$  for  $D = 4(\sum_{i=1}^h p_i + \sum_{j=1}^k q_j)$  (note that  $\deg(D) \leq 8(m+1)$ ); furthermore, the quotient is torsion, therefore, by applying  $\mathcal{H}om(-, \mathcal{O}_C)$  and adjunction, we find  $(I_p I_q)^\vee \subseteq \nu_* \mathcal{O}_{\tilde{C}}(D)$ . It is thus enough to show that  $H^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(D) \otimes \nu^*(\omega_C \otimes A^{-N})) = 0$ . Finally,  $\nu^* \omega_C$  has degree at most two, and  $\nu^* A$  has degree at least one on any branch of  $\tilde{C}$ , hence it is enough to take  $N > 2 + 8(m+1)$ .  $\square$

LEMMA 5.11 deformation openness. *Let  $(\mathcal{C}, \sigma_1, \dots, \sigma_n) \rightarrow S$  be a family of curves over a Noetherian base scheme with  $n$  sections. The locus*

$$\{s \in S \mid (\mathcal{C}_{\bar{s}}, \sigma_1(\bar{s}), \dots, \sigma_n(\bar{s})) \text{ is } m\text{-stable}\}$$

*is Zariski-open in  $S$ .*

*Proof.* Having connected fibres which are Gorenstein curves of arithmetic genus two is an open condition (see for example [Sta19, Tag 0E1M]). Only singularities of genus zero (nodes), one (elliptic  $l$ -folds), and two may then occur.

The case  $m = 1$  deserves special attention. In this case, that condition (i) is open follows from acknowledging that  $I_1 = A_4$ ,  $I_2 = A_5$ , while tacnodes, cusps, and nodes are  $A_3$ ,  $A_2$ , and  $A_1$ -singularities respectively, and from Grothendieck's result on the deformation theory of ADE singularities (see Theorem 2.5 above).

The case  $m \geq 2$  simply follows from upper semicontinuity of embedded dimension and the fact that we have exhausted all possible Gorenstein singularities of genus  $\leq 2$ , and embedding dimension  $\leq m+1$ .

Condition (iv) translates to: the locus where the automorphism group is unramified is open in the base. Homogeneity can be used to prove that being unramified, which is open in the source, is also open in the target, for the structural morphism of a group scheme; see the end of the proof of [Smy11a, Lemma 3.10].

The other conditions are topological, hence constructible. With Noetherian assumptions, it is enough to check their openness over the spectrum of a discrete valuation ring. Assume that the geometric generic fibre  $C_{\bar{\eta}}$  contains two genus one subcurve  $E_{1,\bar{\eta}}$  and  $E_{2,\bar{\eta}}$ ; their closures  $E_1$  and  $E_2$  in  $\mathcal{C}$  are then flat families of genus one curves over  $\Delta$ . If  $E_{1,\bar{\eta}}$  and  $E_{2,\bar{\eta}}$  are disconnected, then so are  $E_1$  and  $E_2$ , by local constancy of the number of connected components (from the Zariski decomposition and [Sta19, Tag 0E0D]). If  $E_{1,\bar{\eta}}$  and  $E_{2,\bar{\eta}}$  are joined by a disconnecting node  $q_{\bar{\eta}}$ , then so are  $E_{1,0}$  and  $E_{2,0}$ ; indeed, the unique limit of  $q_{\bar{\eta}}$  must be a singular point of the projection, but cannot be any worse than a node (because we have already used up all of our genus allowance). Finally, if  $E_{1,\bar{\eta}}$  and  $E_{2,\bar{\eta}}$  share a branch, then so do  $E_{1,0}$  and  $E_{2,0}$ ; on the other hand, if  $E_{i,\bar{\eta}}$  has more than one branch, then so does  $E_i$ . Similarly, if  $C_{\bar{\eta}}$  contains only one subcurve of genus one, with two nodes joined by a rational chain, so does  $C_0$ . To summarise,

$$|E_{i,\bar{\eta}} \cap \overline{C_{\bar{\eta}} \setminus E_{i,\bar{\eta}}}| = |E_{i,0} \cap \overline{C_0 \setminus E_{i,0}}|.$$

The number of markings on  $E_i$  is also constant. Hence we can deduce condition (iii) for  $C_{\bar{\eta}}$  from the same condition on  $C_0$ . Condition (ii) follows in this case from Remark 5.9; it can be proven analogously when there is no subcurve of genus one.

Finally, suppose that  $C_{\bar{\eta}}$  has a genus two singularity, then so does  $C_0$ . The (union of the) distinguished branch(es)  $E_{\bar{\eta}}$  of  $C_{\bar{\eta}}$  is a genus one singularity, and so is its limit  $E_0$  in  $C_0$ . It has to contain the distinguished branch(es) of  $C_0$ , because any subcurve not containing them has genus zero; therefore, by assumption,  $E_0$  contains  $p_{1,0}$ . Then also  $E_{\bar{\eta}}$  contains  $p_{1,\bar{\eta}}$ , because the markings are contained in the non-singular locus of the curve. Similarly, if  $C_{\bar{\eta}}$  has a genus one singularity with a self-branch, the limit of such a branch is a genus one subcurve  $E_0$  of  $C_0$ ; the latter may very well acquire a genus two singularity, but  $E_0$  will contain the special branches of it, so it will be connected to  $p_1$ . We conclude as above. The case that  $C_{\bar{\eta}}$  contains a genus one subcurve of low level is analogous. We have thus proved that condition (v) is open.  $\square$

DEFINITION 5.12. We shall denote by  $\overline{\mathcal{M}}_{2,n}^{(m)}$  the moduli stack of  $n$ -pointed  $m$ -stable curves of genus two.

It follows from the previous lemmas and standard arguments that  $\overline{\mathcal{M}}_{2,n}^{(m)}$  is represented by a Deligne-Mumford stack of finite type.

PROPOSITION 5.13 Valuative criterion of properness for  $\overline{\mathcal{M}}_{2,n}^{(m)}$ . *Given a smooth  $n$ -pointed curve of genus two  $C_{\eta}$  over a discrete valuation field  $\eta = \text{Spec}(K) \hookrightarrow \Delta$ , there exists a finite base-change  $\Delta' \rightarrow \Delta$  after which  $C_{\eta}$  can be completed to an  $m$ -stable curve over  $\Delta'$ . Two such models are always dominated by a third one.*

*Existence of limits.* By the semistable reduction theorem [DM69, Corollary 2.7], we may find a finite base-change  $\Delta' \rightarrow \Delta$  and a semistable curve  $\mathcal{C}' \rightarrow \Delta'$  with regular total space, such that its generic fibre is isomorphic to the pullback of  $C_{\eta}$ . By Castelnuovo's criterion, we may further assume that the central fibre contains no rational tails.

We check whether  $p_1$  is Weierstrass or not: in the former case, change base with  $\pi'' \mapsto (\pi')^3$  and resolve; in the latter, mark  $\mathcal{C}'$  with an extra section  $\bar{p}_1$  given by the closure of the conjugate point  $\bar{p}_1(\eta)$  (it might coincide with one of  $p_1, \dots, p_n$ ; if it coincides with  $p_1$ , we have a Weierstrass point indeed), then change base with  $\pi'' \mapsto (\pi')^2$  and resolve. We drop the primes from notation.  $\mathcal{C}_0$  is now marked with a(n extra) smooth point  $\bar{p}_1$ . The base-change is a technical expedient we find useful in the forthcoming construction.

We claim there is a unique genus two subcurve  $Z$  of  $\mathcal{C}_0$  that satisfies the shape requirements of Proposition 4.3 (resp. 4.4) - or consists of two disjoint balanced subcurves of genus one - such that the curve we obtain by contracting  $Z$  has bounded-above singularities and bounded-below level as in (i),(ii), and (iii) of Definition 5.7.

We think of this process as drawing a family of expanding circles on the dual graph (except, they are not exactly circles), as we have learned from [RSW17a]. Note that we may at any point blow the curve up at a marking on the central fibre, and consider the strict transform of the corresponding section; thus markings can effectively be considered as infinite legs in the dual graph - this is important for all the valence considerations below. We are going to contract the strict interior of the circle; note that the number of branches of the resulting singularity is determined by the inner valence of the circle, and the level by the outer valence.

For simplicity, we start by examining the case that the core of  $\mathcal{C}_0$  is irreducible, and  $p_1$  is Weierstrass. Step 0: if the core  $K$  has level  $\geq m+1$ , then the curve is already  $m$ -stable. Otherwise, draw a first circle comprising  $K$ , and reaching every second closest rational component along any rational tree attached to  $K$ , except for the tree containing  $p_1$ . Note that the inner valence is exactly  $\text{lev}(K)$ , thanks to the base-change we have performed earlier, and in particular it is no

larger than  $m$ ; on the other hand, semistability implies that the outer valence is non-decreasing: if it is  $\geq m+1$  we stop, otherwise we repeat the process. Calling  $K_1$  the union of the components strictly inside the first circle,  $\mathcal{C}_0$  is the union of  $K_1$  and a number of rational trees; if the outer valence at the first step is still  $\leq m$ , we enlarge the circle by reaching one step further along the rational tree containing  $p_1$ , and three steps further along every other. Because a circle of very large radius has both inner and outer valence equal to  $n > m$ , by increasing the radius step by step we will eventually reach level  $m+1$  or higher. If we stop at the  $l$ -th step, the line bundle we will use to perform the contraction is

$$\begin{aligned} \mathcal{L} = \omega_{\mathcal{C}/\Delta}(3lK + \sum_{R \in [p_1, K]} [3l - 3 \operatorname{dist}(R, K)]_+ R + \sum_{R \notin T_1} [3l - \operatorname{dist}(R, K)]_+ R + \\ \sum_{R \in T_1, R \notin [p_1, K]} [3l - 3 \operatorname{dist}(T_1 \wedge T_R, K) - \operatorname{dist}(R, T_1 \wedge T_R)]_+ R + \sum_{i=1}^n p_i) \end{aligned}$$

where  $T_1$  is the rational tree (connected component of  $\mathcal{C}_0 \setminus K$ ) containing  $p_1$ ,  $T_R$  the one containing  $R$ ,  $T_1 \wedge T_R$  their common component furthest from the core,  $\operatorname{dist}$  is the distance on the dual graph, and  $[k]_+ = \max\{0, k\}$  for any integer  $k$ . If we write  $\mathcal{L} = \omega_{\mathcal{C}/\Delta}^{\log}(D)$  for an effective vertical divisor  $D$  whose support is  $Z$ , the shape prescription being satisfied by construction, it follows from Proposition 4.5 that  $\mathcal{L}$  is  $\pi$ -semiample, and the associated contraction yields a singularity of type  $I$  with  $p_1$  cleaving to the special branch. Note that  $\mathcal{L}$  contracts as well the semistable rational components that are disjoint from  $Z$ , hence the resulting curve has no (infinitesimal) automorphisms. The level condition is satisfied by construction, therefore  $\overline{\mathcal{C}}_0$  is  $m$ -stable.

The case that the core is irreducible and  $p_1$  is not Weierstrass is dealt with in a similar fashion. Remember that in this case we have constructed a conjugate section  $\bar{p}_1$ ; this is an auxiliary marking that will be forgotten in the end, *and should not be taken into account when computing the level*. At every step we draw a larger circle by including one more component along the trees containing  $p_1$  and  $\bar{p}_1$ , and two more along every other tree; the inner valence of the new circle is the same as the outer valence of the old one, thanks to the base-change we performed at the beginning. At the  $l$ -th step we are going to use the line bundle

$$\begin{aligned} \mathcal{L} = \omega_{\mathcal{C}/\Delta}(2lK + \sum_{\substack{R \in [p_1, K] \\ \text{or } [\bar{p}_1, K]}} [2l - 2 \operatorname{dist}(R, K)]_+ R + \sum_{R \notin T_1, \bar{T}_1} [2l - \operatorname{dist}(R, K)]_+ R + \\ \sum_{\substack{R \in T_1, R \notin [p_1, K] \text{ or} \\ R \in \bar{T}_1, R \notin [\bar{p}_1, K]}} [2l - 2 \operatorname{dist}(T_1 \wedge T_R, K) - \operatorname{dist}(R, T_1 \wedge T_R)]_+ R + \sum_{i=1}^n p_i + \bar{p}_1) \end{aligned}$$

to contract the strict interior of the circle. It follows from Proposition 4.6 that  $\mathcal{L}$  is  $\pi$ -semiample, and the associated contraction contains a singularity of type  $II$  with  $p_1$  cleaving to one of the twin branches. Note that a  $II_{m+1}$  singularity will occur only if the level is exactly  $m$  and  $\bar{p}_1$  does not coincide with any other special point, so that one of the twin branches remains dangling after forgetting  $\bar{p}_1$ . Again, the stability condition is satisfied by construction.

Suppose next that the minimal subcurve of genus two  $Z$  contains two subcurves of genus one; call  $E_1$  and  $E_2$  the minimal such, and assume that  $p_1$  cleaves to  $E_1$  in a point that is 2-torsion with respect to the node separating  $E_1$  from  $E_2$ . We start by drawing expanding circles around  $E_2$  until the level condition for it is satisfied, and then we do the same for  $E_1$ , so that (v) holds; observe, though, that as soon as the two circles touch, the contraction will not have

two distinguished genus one subcurves anymore, therefore we need to check (ii) rather than (iii). Here is a more detailed description of the process.

- If  $\text{lev}(E_2) \geq m + 2$  is attained before the circle around  $E_2$  gets to touch  $E_1$ , take the next  $l_2 \equiv 2 \pmod{3}$  (thus “undoing” the  $3 : 1$  base-change, which was unnecessary in this case), then contract the inner disc by the line bundle

$$\mathcal{L}_2 = \omega_{\mathcal{C}/\Delta}((l_2 + 1)E_2 + \sum [l_2 + 1 - \text{dist}(E_2, R)]_+ R + \sum_{i=1}^n p_i).$$

Smyth’s contraction lemma [Smy11a, Lemma 2.13] applies;  $E_2$  is contracted to an elliptic  $l$ -fold point  $q_2$  ( $l \leq m + 1$ ). Consider now  $E_1$ . If  $\text{lev}(E_1) \leq m + 1$ , start drawing expanding circles around it. Either level  $\geq m + 2$  can be reached before touching the singularity at  $q_2$ , or contracting the maximal balanced subcurve of genus one containing  $E_1$  and not  $q_2$  yields a curve with two genus one singularities sharing a branch. Note that  $p_1$  cleaves to the only genus one subcurve that may have level  $\leq m + 1$ .

- Otherwise, one step before including  $E_1$ , we may contract the disc around  $E_2$  to yield a genus one singularity with a genus one branch. If  $\text{lev}_2 \leq m$  at this point, we need to contract a genus two subcurve. At this *critical step* the multiplicity of  $D$  along  $E_1$  grows from 0 to 3, hence  $D$  will be supported three steps further along each rational tail departing from  $E_2$ . Since the critical step occurs at  $\equiv 1 \pmod{3}$ , the length of the rational tails is and remains  $\equiv 2 \pmod{3}$ , hence we perform only one “meaningful” step forward, thanks to the preliminary base-change. See Figure 7. In particular, the inner valence of the disc will be  $\leq m$ . We proceed now as in the irreducible case, expanding the circle by 1 along  $T_1$  and by 3 along all other rational tails.

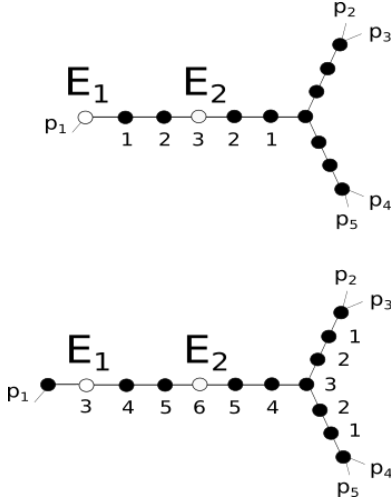


FIGURE 7. An instance of the critical step, from contracting the two genus one subcurves separately, to contracting the genus two subcurve as a whole. Above, the contraction yields a tacnode with a genus one branch, having  $\text{lev}_2 = 3$ . Below, the contraction yields a singularity of type  $I_3$ . Note that we had to blow-up at  $p_1$ .

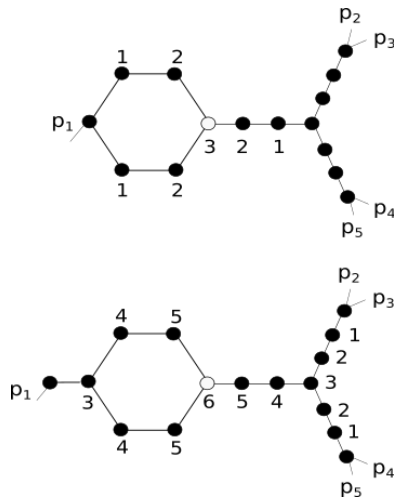


FIGURE 8. Another instance of the critical step, from contracting a genus one subcurve, to contracting a genus two subcurve. Above, the contraction yields an elliptic 3-fold point with two coinciding branches, having  $\text{lev}_2 = 3$ . Below, the contraction yields a singularity of type  $I_3$ . Note that in the meantime we have performed a blow-up at  $p_1$ .



The case that the central fibre contains two subcurves of genus one, and  $p_1$  cleaves to a non-Weierstrass point, is analogous: it is enough to replace the number 3 by the number 2, and 2 by 1, in the previous argument. The only novelty is, it can happen that  $p_1$  is equidistant from  $E_1$  and  $E_2$ , cleaving to the rational chain joining them. In this case we start by expanding a circle around the one with the lowest level; if they have the same level, expand them simultaneously. If at a later stage  $p_1$  becomes closer to one of the two circles, we proceed as above, namely by enlarging the circle furthest from  $p_1$ .

When the core consists of an elliptic curve  $E$  with a rational bridge  $R$ , and  $p_1$  cleaves to a Weierstrass point - either on  $E$  or on  $R$  -, we have noticed above (see the end of Remark 4.1), that  $R$  comprises an odd number  $k = 2h + 1$  of rational components. We start as above by enlarging a balanced circle around  $E$  in order to establish the level condition (iii); there is a critical step (after which we contract a genus two subcurve, and have to satisfy (ii) instead) when the circle touches itself along  $R$ . This happens after the  $(3h + 2)$ -th step, therefore extending the circle by 1 along the tail containing  $p_1$ , and by 3 on all the other ones, only performs one meaningful step. See Figure 8. The case of a non-Weierstrass point on the elliptic bridge is analogous.

Finally, the case that the central fibre has geometric genus zero can be seen as a degeneration of the previous ones, since we only contract a genus zero subcurve when it is a semistable chain.  $\square$

*Uniqueness of limits.* Up to a further base-change, there is a diagram

$$\begin{array}{ccc}
 & \mathcal{C}^{ss} & \\
 \phi \swarrow & & \searrow \phi' \\
 \mathcal{C} & & \mathcal{C}' \\
 & \searrow & \swarrow \\
 & \Delta &
 \end{array}$$

extending the isomorphism between the generic fibres, where  $\mathcal{C}^{ss}$  has semistable central fibre and regular total space, by the semistable reduction theorem.

*Claim 1.* If  $\mathcal{C}'_0$  has only singularities of genus  $\leq i$  ( $i = 0, 1$ ), then so does  $\mathcal{C}_0$ .

First, assume that  $\mathcal{C}'_0$  has only nodes. If  $\mathcal{C}_0$  has a singular point  $x$  of genus one,  $E := \phi^{-1}(x)$  is an *unmarked* subcurve of arithmetic genus one and level  $\leq m + 1$  of  $\mathcal{C}_0^{ss}$ . Then so is  $\phi'(E)$ : indeed,  $\phi'$  being a contraction, it has connected fibres, which bars  $\phi'|_E$  from being a finite cover of a  $\mathbb{P}^1$ . This contradicts the  $m$ -stability of  $\mathcal{C}'$ . We may argue similarly if  $x$  is a genus two singularity with  $\leq m$  branches. On the other hand, if  $x$  is dangling  $I_{m+1}$ , there is a  $-1$ -curve  $R$  adjacent to  $\phi^{-1}(x)$ ;  $\phi'$  must contract  $R$  by DM stability of  $\mathcal{C}'$ , hence  $\phi'(\phi^{-1}(x))$  is again a genus two curve of level  $\leq m$ .

Assume now that  $\mathcal{C}'_0$  has at worst singularities of genus one, while  $\mathcal{C}_0$  has a singularity  $x$  of genus two; the case of a dangling  $I_{m+1}$  can be excluded as above. Then  $\mathcal{C}_0^{ss} = Z \cup R_1 \cup \dots \cup R_l$ , with  $Z = \phi^{-1}(x)$  and  $l \leq m$ . If  $Z$  has geometric genus two, or is irreducible of geometric genus one,  $\phi'(Z)$  violates the  $m$ -stability of  $\mathcal{C}'$ . If  $Z$  contains a unique subcurve  $E$  of genus one, with a rational bridge  $R$ , then at least one of  $R_1, \dots, R_l$  must be connected to  $R$ , otherwise  $\mathcal{C}'_0$  - which is obtained by contracting a balanced subcurve around  $E$ , not including the entire  $R$  - would have a positive dimensional automorphism group (scaling a semistable component of  $R$ ). Therefore  $\text{lev}(E) \leq (l - 1) + 2 \leq m + 1$ . Similarly, if  $Z$  contains two subcurves of genus one  $E_1$  and  $E_2$ ,

then  $(\text{lev}(E_1) - 1) + (\text{lev}(E_2) - 1) \leq l$ , hence at least one of the two has level  $\leq m + 1$ . In any case,  $\phi'(E)$  contradicts the  $m$ -stability of  $\mathcal{C}'$ .

*Claim 2.* We may assume that  $\mathcal{C}^{ss}$  contains either no  $-1$ -curve, or only one, which is contracted by neither  $\phi$  nor  $\phi'$ .

If there is a  $-1$ -curve contracted by both,  $\phi$  and  $\phi'$  factor through a smaller regular model. Assume there is a  $-1$ -curve not contracted by  $\phi$ . Then, by condition (iv), its image has to be one of the special branches of a genus two singularity; on the other hand, by condition (v), the only special branch of a singularity of type  $I$  must contain some special points; therefore we conclude that the singular point  $x$  of  $\mathcal{C}_0$  is dangling of type  $II_{l+1}$ ,  $l \leq m$ . If we let  $Z = \phi^{-1}(x)$ , we may write  $\mathcal{C}_0 = Z \cup R_0 \cup \dots \cup R_l$ , with  $R_0 = R$ , and  $R_1$  the tail containing  $p_1$ . By Claim 1,  $\phi'$  has to contract a genus two subcurve  $Z'$  as well. If  $Z'$  is of the shape described in Proposition 4.6 and it contains  $R$ , then  $Z \subsetneq Z'$  is easily seen, which implies  $\mathcal{C}'_0$  has a singularity of type  $II$  with more than  $m + 1$  branches, by the level condition (ii) on  $\mathcal{C}_0$ ; on the other hand, if  $Z' \subsetneq Z$  were disjoint from  $R$ , then  $\mathcal{C}'_0$  would not satisfy condition (ii). Similarly, if  $Z'$  were to be of the shape described in Proposition 4.5, then  $R_0$  and  $R_1$  would have to meet in a “trunk”  $T$  attached to a Weierstrass point of the core of  $\mathcal{C}_0^{ss}$ ; if  $Z'$  started down along the trunk closer to the core, then  $Z' \subsetneq Z$  (so  $\mathcal{C}'_0$  would not satisfy (ii)), while if  $Z'$  started at the top of the trunk or further away from the core, then  $Z \subsetneq Z'$  (so  $\mathcal{C}'_0$  would not satisfy (i)). We conclude that, if  $\mathcal{C}_0$  has a dangling  $II_{l+1}$  singularity, then so does  $\mathcal{C}'_0$ .

*Claim 3.* The exceptional loci of  $\phi$  and  $\phi'$  coincide.

If  $\mathcal{C}_0$  has only nodes, then so does  $\mathcal{C}'_0$  by Claim 1, and we can conclude by the uniqueness part of the stable reduction theorem.

If  $\mathcal{C}_0$  has a genus one singularity  $x$ , it cannot have a genus two singularity as well, so neither can  $\mathcal{C}'_0$  by Claim 1. If  $\mathcal{C}_0$  has a second genus one singularity  $y$ , let  $E_1 = \phi^{-1}(x)$  and  $E_2 = \phi^{-1}(y)$ ; they are disjoint balanced subcurves of genus one and level  $\leq m + 1$  in  $\mathcal{C}_0^{ss}$ , therefore  $\phi'$  must contract them. Enlarging the contraction radius of any one of them would yield a singularity with at least  $m + 2$  branches (by condition (iii) on  $\mathcal{C}_0$ ), unless by enlarging we make them touch, in which case we would contract to a genus two singularity; but this is not possible, by Claim 1. The case of a single genus one singularity with a genus one branch, or with a disjoint subcurve of genus one, or with two branches joined by a (possibly empty) rational chain, is similar.

Finally, the case that  $\mathcal{C}_0$  has a genus two singularity, has already been discussed at the end of Claim 2. To summarise, writing  $\mathcal{C}_0^{ss} = Z \cup R_1 \cup \dots \cup R_l$ , with  $Z = \phi^{-1}(x)$  and  $l \leq m$  - the case of a dangling  $II_{m+1}$  was dealt with before -,  $\phi'(Z)$  must be a point  $x'$ , by stability considerations. Call  $Z' = (\phi')^{-1}(x')$  and note that  $Z \subseteq Z'$ . If  $x$  and  $x'$  are singularities of the same type,  $Z = Z'$  is easily deduced by level/singularity (i.e. outer/inner valence) considerations, the key point being that the shape of the curve has only one parameter (the “radius” of the circle), which is determined by  $m$ -stability. On the other hand, if  $x$  were of type  $II$  and  $x'$  of type  $I$ , the two special trees determined by  $x$  would have to share a trunk attached to a Weierstrass point of the core, and  $Z \subseteq Z'$  would imply  $Z \subsetneq Z'$ , which together with condition (ii) for  $\mathcal{C}'$  would make  $x'$  into a singularity with too many branches.

The claim follows from observing that the exceptional locus of  $\phi$  (resp.  $\phi'$ ) is the union of the fibres  $F$  over higher genus singularities of  $\mathcal{C}_0$  (resp.  $\mathcal{C}'_0$ ), and the rational components with only two special points that are disjoint from  $F$ .

*Claim 4.* The generic isomorphism between  $\mathcal{C}$  and  $\mathcal{C}'$  extends over  $\Delta$ .

Follows from [Deb01, Lemma 1.13]. □

Summing up, we have proved the following:

**THEOREM 5.14.** *For  $1 \leq m < n$ ,  $m$ -stability defines a proper Deligne-Mumford stack  $\overline{\mathcal{M}}_{2,n}^{(m)}$  over  $\mathrm{Spec}(\mathbb{Z}[\frac{1}{30}])$ , containing  $\mathcal{M}_{2,n}$  as a dense open substack.*

The restriction on coefficients avoids ramification of the automorphism groups.

## REFERENCES

- AFS16 Jarod Alper, Maksym Fedorchuk, and David Ishii Smyth. Singularities with  $\mathbb{G}_m$ -action and the log minimal model program for  $\overline{\mathcal{M}}_g$ . *J. Reine Angew. Math.*, 721:1–41, 2016.
- AFS17a Jarod Alper, Maksym Fedorchuk, and David Ishii Smyth. Second flip in the Hassett-Keel program: existence of good moduli spaces. *Compos. Math.*, 153(8):1584–1609, 2017.
- AFS17b Jarod Alper, Maksym Fedorchuk, and David Ishii Smyth. Second flip in the Hassett-Keel program: projectivity. *Int. Math. Res. Not. IMRN*, (24):7375–7419, 2017.
- AFSvdW17 Jarod Alper, Maksym Fedorchuk, David Ishii Smyth, and Frederick van der Wyck. Second flip in the Hassett-Keel program: a local description. *Compos. Math.*, 153(8):1547–1583, 2017.
- AK70 Allen Altman and Steven Kleiman. *Introduction to Grothendieck duality theory*. Lecture Notes in Mathematics, Vol. 146. Springer-Verlag, Berlin-New York, 1970.
- AK16 Jarod Alper and Andrew Kresch. Equivariant versal deformations of semistable curves. *Michigan Math. J.*, 65(2):227–250, 2016.
- Arn72 Vladimir Igorevič Arnol’d. Normal forms of functions near degenerate critical points, the Weyl groups  $A_k, D_k, E_k$  and Lagrangian singularities. *Funkcional. Anal. i Priložen.*, 6(4):3–25, 1972.
- BCM18 Luca Battistella, Francesca Carocci, and Cristina Manolache. Reduced invariants from cuspidal maps. *arXiv e-prints*, page arXiv:1801.07739, Jan 2018.
- BLR90 Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. *Néron models*, volume 21 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1990.
- Bog99 Marco Boggi. Compactifications of configurations of points on  $\mathbb{P}^1$  and quadratic transformations of projective space. *Indag. Math. (N.S.)*, 10(2):191–202, 1999.
- BS08 Elizabeth Baldwin and David Swinarski. A geometric invariant theory construction of moduli spaces of stable maps. *Int. Math. Res. Pap. IMRP*, (1):Art. ID rp. 004, 104, 2008.
- Cat82 Fabrizio Catanese. Pluricanonical-Gorenstein-curves. In *Enumerative geometry and classical algebraic geometry (Nice, 1981)*, volume 24 of *Progr. Math.*, pages 51–95. Birkhäuser Boston, Boston, MA, 1982.
- CML13 Sebastian Casalaina-Martin and Radu Laza. Simultaneous semi-stable reduction for curves with ADE singularities. *Trans. Amer. Math. Soc.*, 365(5):2271–2295, 2013.
- CTV18 Giulio Codogni, Luca Tasin, and Filippo Viviani. On the first steps of the minimal model program for the moduli space of stable pointed curves. *arXiv e-prints*, page arXiv:1808.00231, Aug 2018.
- CTV19 Giulio Codogni, Luca Tasin, and Filippo Viviani. On some modular contractions of the moduli space of stable pointed curves. *arXiv e-prints*, page arXiv:1904.13212, Apr 2019.
- Cuk89 Fernando Cukierman. Families of Weierstrass points. *Duke Math. J.*, 58(2):317–346, 1989.
- Deb01 Olivier Debarre. *Higher-dimensional algebraic geometry*. Universitext. Springer-Verlag, New York, 2001.

- Del85 Pierre Deligne. Le lemme de Gabber. *Astérisque*, (127):131–150, 1985. Seminar on arithmetic bundles: the Mordell conjecture (Paris, 1983/84).
- Dem75 Michel Demazure. Classification des germes à point critique isolé et à nombres de modules 0 ou 1 (d’après V. I. Arnol’d). pages 124–142. *Lecture Notes in Math.*, Vol. 431, 1975.
- Dia85 Steven Diaz. Exceptional Weierstrass points and the divisor on moduli space that they define. *Mem. Amer. Math. Soc.*, 56(327):iv+69, 1985.
- DM69 Pierre Deligne and David Mumford. The irreducibility of the space of curves of given genus. *Inst. Hautes Études Sci. Publ. Math.*, (36):75–109, 1969.
- Fed14 Maksym Fedorchuk. Moduli spaces of hyperelliptic curves with A and D singularities. *Math. Z.*, 276(1-2):299–328, 2014.
- FG18 Maksym Fedorchuk and Matthew Grimes. VGIT presentation of the second flip of  $\overline{M}_{2,1}$ . *arXiv e-prints*, page arXiv:1805.08581, May 2018.
- FS13 Maksym Fedorchuk and David Ishii Smyth. Alternate compactifications of moduli spaces of curves. In *Handbook of moduli. Vol. I*, volume 24 of *Adv. Lect. Math. (ALM)*, pages 331–413. Int. Press, Somerville, MA, 2013.
- Gie82 David Gieseker. *Lectures on moduli of curves*, volume 69 of *Tata Institute of Fundamental Research Lectures on Mathematics and Physics*. Published for the Tata Institute of Fundamental Research, Bombay; Springer-Verlag, Berlin-New York, 1982.
- Hal14 Daniel Halpern-Leistner. On the structure of instability in moduli theory. *arXiv e-prints*, page arXiv:1411.0627, Nov 2014.
- Has03 Brendan Hassett. Moduli spaces of weighted pointed stable curves. *Adv. Math.*, 173(2):316–352, 2003.
- Has05 Brendan Hassett. Classical and minimal models of the moduli space of curves of genus two. In *Geometric methods in algebra and number theory*, volume 235 of *Progr. Math.*, pages 169–192. Birkhäuser Boston, Boston, MA, 2005.
- HH13 Brendan Hassett and Donghoon Hyeon. Log minimal model program for the moduli space of stable curves: the first flip. *Ann. of Math. (2)*, 177(3):911–968, 2013.
- HL07 Donghoon Hyeon and Yongnam Lee. Stability of tri-canonical curves of genus two. *Math. Ann.*, 337(2):479–488, 2007.
- HL14 Donghoon Hyeon and Yongnam Lee. A birational contraction of genus 2 tails in the moduli space of genus 4 curves I. *Int. Math. Res. Not. IMRN*, (13):3735–3757, 2014.
- HLN12 Yi Hu, Jun Li, and Jingchen Niu. Genus Two Stable Maps, Local Equations and Modular Resolutions. *arXiv e-prints*, page arXiv:1201.2427, Jan 2012.
- HM82 Joe Harris and David Mumford. On the Kodaira dimension of the moduli space of curves. *Invent. Math.*, 67(1):23–88, 1982. With an appendix by William Fulton.
- HM98 Joe Harris and Ian Morrison. *Moduli of curves*, volume 187 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998.
- JP18 Drew Johnson and Alexander Polishchuk. Birational models of  $\overline{M}_{2,2}$  arising as moduli of curves with nonspecial divisors. *arXiv e-prints*, page arXiv:1807.09746, Jul 2018.
- Kee99 Seán Keel. Basepoint freeness for nef and big line bundles in positive characteristic. *Ann. of Math. (2)*, 149(1):253–286, 1999.
- KM97 Seán Keel and Shigefumi Mori. Quotients by groupoids. *Ann. of Math. (2)*, 145(1):193–213, 1997.
- Knu83 Finn F. Knudsen. The projectivity of the moduli space of stable curves. II. The stacks  $M_{g,n}$ . *Math. Scand.*, 52(2):161–199, 1983.
- LP17 Yankı Lekili and Alexander Polishchuk. A modular compactification of  $\mathcal{M}_{1,n}$  from  $\mathcal{A}_\infty$  structures. *J. Reine Angew. Math.*, 2017.

- LZ09 Jun Li and Aleksey Zinger. On the genus-one Gromov-Witten invariants of complete intersections. *J. Differential Geom.*, 82(3):641–690, 2009.
- MFK94 David Mumford, John Fogarty, and Frances Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]*. Springer-Verlag, Berlin, third edition, 1994.
- Mor11 Ian Morrison. Mori theory of moduli spaces of stable curves. <http://www.projectivepress.com/moduli/moristablecurves.pdf>, 2011.
- Pan99 Rahul Pandharipande. Hodge integrals and degenerate contributions. *Comm. Math. Phys.*, 208(2):489–506, 1999.
- Pol15 Alexander Polishchuk. Moduli of curves with nonspecial divisors and relative moduli of  $\mathcal{A}_\infty$ -structures. *arXiv e-prints*, page arXiv:1511.03797, Nov 2015.
- RSW17a Dhruv Ranganathan, Keli Santos-Parker, and Jonathan Wise. Moduli of stable maps in genus one and logarithmic geometry I. *arXiv e-prints*, page arXiv:1708.02359, August 2017.
- RSW17b Dhruv Ranganathan, Keli Santos-Parker, and Jonathan Wise. Moduli of stable maps in genus one and logarithmic geometry II. *arXiv e-prints*, page arXiv:1709.00490, September 2017.
- Rul01 William F. Rulla. *The birational geometry of  $\overline{M}_{2,1}$  and  $\overline{M}_3$* . PhD thesis, University of Texas at Austin, 2001.
- Sch91 David Schubert. A new compactification of the moduli space of curves. *Compositio Math.*, 78(3):297–313, 1991.
- Smy11a David Ishii Smyth. Modular compactifications of the space of pointed elliptic curves I. *Compos. Math.*, 147(3):877–913, 2011.
- Smy11b David Ishii Smyth. Modular compactifications of the space of pointed elliptic curves II. *Compos. Math.*, 147(6):1843–1884, 2011.
- Smy13 David Ishii Smyth. Towards a classification of modular compactifications of  $\mathcal{M}_{g,n}$ . *Invent. Math.*, 192(2):459–503, 2013.
- Smy18 David Ishii Smyth. Intersections of psi-classes on moduli spaces of m-stable curves. *arXiv e-prints*, page arXiv:1808.03214, Aug 2018.
- Sta19 The Stacks Project Authors. *Stacks Project*. <https://stacks.math.columbia.edu>, 2019.
- Ste96 Jan Stevens. On the classification of reducible curve singularities. In *Algebraic geometry and singularities (La Rábida, 1991)*, volume 134 of *Progr. Math.*, pages 383–407. Birkhäuser, Basel, 1996.
- vdW10 Frederick van der Wyck. *Moduli of singular curves and crimping*. PhD thesis, Harvard University, 2010.
- Vis12 Michael Viscardi. Alternate compactifications of the moduli space of genus one maps. *Manuscripta Math.*, 139(1-2):201–236, 2012.
- VZ08 Ravi Vakil and Aleksey Zinger. A desingularization of the main component of the moduli space of genus-one stable maps into  $\mathbb{P}^n$ . *Geom. Topol.*, 12(1):1–95, 2008.
- Zin09 Aleksey Zinger. Reduced genus-one Gromov-Witten invariants. *J. Differential Geom.*, 83(2):407–460, 2009.

Luca Battistella [battistella@mpim-bonn.mpg.de](mailto:battistella@mpim-bonn.mpg.de)

Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany