MODULAR COMPACTIFICATIONS OF $\mathcal{M}_{2,n}$ I

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ABSTRACT. We study the geometry of Gorenstein curve singularities of genus two, and of their stable limits. There are two families of such singularities, corresponding to either Weierstrass or conjugate points on a semistable tail. For every $1 \leq m < n$, a stability condition - using one of the markings as a reference point, and therefore not $\mathfrak{S}_n$-symmetric - defines proper Deligne-Mumford stacks $\overline{\mathcal{M}}_{2,n}^{(m)}$ containing the locus of smooth curves as a dense open.

CONTENTS

1. INTRODUCTION

We construct alternative compactifications of the moduli stack of smooth n-pointed curves of genus two. The boundary of the Deligne-Mumford compactification, consisting of stable nodal curve, is gradually replaced by ever more singular curves, complying with more restrictive combinatorial requisites on the dual graph. For $1 \leq m < n$, the notion of m-stability allows genus one and two singularities, while demanding that every higher genus subcurve contain at least a number of special points. Our main result on the stack of m-stable curves is:

Theorem. $\overline{\mathcal{M}}_{2,n}^{(m)}$ *is a* proper *irreducible Deligne-Mumfod stack over* $\mathrm{Spec}(\mathbb{Z}[\frac{1}{30}])$.

This paper fits into the context of birational geometry of moduli spaces, extending work of D.I. Smyth in genus one, but we also expect it to find applications in enumerative geometry. We classify singularities of genus two and their (semi)stable models, highlighting the relation with Brill-Noether theory. We interpret crimping spaces as moduli spaces for the differential geometric data needed in order to construct a higher genus singularity from an ordinary m-fold point, and verify their impact on the existence of automorphisms, a phenomenon that had not fully emerged in lower genus.

1.1. **From the Deligne-Mumford space to the Hassett-Keel program.**
One of the most fascinating and influential results of modern algebraic geometry is the construction of a modular compactification of the stack of smooth pointed curves $\mathcal{M}_{g,n}$, due to P. Deligne, D. Mumford, and F. Knudsen, via the introduction of *stable* pointed curves.

Definition 1.1. [DM69] A connected, reduced, complete curve C, with distinct markings $(p_1, \ldots, p_n)$ lying in the smooth locus of C, is *stable* if:
(1) C admits only nodes (ordinary double points) as singularities;
(2) every rational component of C has at least three special points (markings or nodes), and every elliptic component has at least one.

Theorem 1.2. [DM69, Knu83] *Assume $2g - 2 + n > 0$. The moduli stack of stable pointed curves $\overline{\mathcal{M}}_{g,n}$ is a smooth and proper connected Deligne-Mumford stack over $\mathrm{Spec}(\mathbb{Z})$, with projective coarse moduli space $\overline{\mathbf{M}}_{g,n}$. The boundary $\overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$, representing nodal curves, is a normal crossing divisor.*

On one hand, the Deligne-Mumford compactification has nearly every desirable property one could hope for; on the other, it is certainly not the unique modular compactification of $\mathcal{M}_{g,n}$. Classifying all of them is a challenging task, which has so far found only a partial solution in the beautiful work of Smyth [Smy13].

Even though the existence of $\overline{\mathbf{M}}_{g,n}$ can be deduced from nowadays standard general theory of stacks [KM97], this moduli space was first constructed as a quotient, prompting the development of the powerful techniques of Geometric Invariant Theory [Gie82, MFK94, BS08]. The study of alternative compactifications of $\mathcal{M}_{g,n}$ is motivated as well by an interest in the birational geometry of $\overline{\mathbf{M}}_{g,n}$, and it is not by chance that the first steps in this direction were moved from a GIT perspective - by changing the invariant theory problem or the stability condition under consideration, and analysing the modular properties of the resulting quotient [Sch91, Has05]. The consequent program that goes under the name of B. Hassett and S. Keel aims to describe all the quotients arising in this way, and to determine whether every step of a minimal model program for $\overline{\mathbf{M}}_{g,n}$ enjoys a modular interpretation in terms of curves with worse than nodal singularities [CTV18, CTV19]. Since the early stages of this program, it has developed into a fascinating playground for implementing ideas that originated from (v)GIT into a general structure theory of Artin stacks [AK16, AFSvdW17, AFS17a, AFS17b]. See for instance [Mor11, FS13] for more comprehensive and instructive accounts.

Only few steps of the Hassett-Keel program have been carried out in full generality. On the other hand, the program has been completed to a larger extent in low genus: with the introduction of weighted pointed curves [Has03] in genus zero, and with Smyth's pioneering work in genus one [Smy11a, Smy11b, Smy18], extending earlier work of D. Schubert in a direction. The philosophy is that an alternative compactification is defined by allowing a reasonably larger class of curve singularities (*local condition*), while identifying their (semi)stable models, and disallowing them by imposing a stronger stability condition (*global condition*,

usually combinatorial); this ensures that the resulting moduli problem be again separated and universally closed, by the valuative criterion.

A useful notion in this respect is that of the *genus* of an isolated curve singularity: let (C, q) be (the germ of) a reduced curve over an algebraically closed field $\mathbf{k}$ at its unique singular point q, with normalisation $\nu \colon \widetilde{C} \to C$.

Definition 1.3. [Smy11a] If C has m branches at q, and δ is the $\mathbf{k}$-dimension of $\nu_* \mathscr{O}_{\widetilde{C}} / \mathscr{O}_C$, a skyscraper sheaf supported at q, the genus of (C, q) is defined as:

$$g = \delta - m + 1.$$

The genus can be thought of as the number of conditions that a function must satisfy in order to descend from the seminormalisation (the initial object in the category of universal homeomorphism $C' \to C$, see [Sta19, Tag 0EUS], or a curve with the same topological space as C and an ordinary m-fold point at q) to C. The node, for example, has genus zero (it coincides with its own seminormalisation). It is a notion that behaves well in families: a complete, reduced curve C with only one singularity at q, of δ-invariant δ, and normalisation a union of m copies of $\mathbb{P}^1$, can only appear in a family of curves of arithmetic genus g.

Smyth found that, for every fixed number of branches m, there is a unique germ of Gorenstein singularity of genus one up to isomorphism, namely:

$m = 1$: the cusp, $V(y^2 - x^3) \subseteq \mathbb{A}^2_{x,y}$;

$m = 2$: the tacnode, $V(y^2 - yx^2) \subseteq \mathbb{A}^2_{x,y}$;

$m \geq 3$: the union of m general lines through the origin of $\mathbb{A}^{m-1}$.

Singularities of this kind, with up to m branches, together with nodes, form a deformation-open class of singularities. Furthermore, the elliptic m-fold point can be obtained by contracting a smooth elliptic curve with m rational tails in a one-parameter smoothing, and, roughly speaking, all stable models look like this.

Definition 1.4. [Smy11a] A connected, reduced, complete curve C of arithmetic genus one with smooth distinct markings $(p_1, \dots, p_n)$ is *m-stable*, $1 \leq m < n$, if:

(1) it admits only nodes and elliptic l-fold points, $l \leq m$, as singularities;
(2) for every connected subcurve $E \subseteq C$ of arithmetic genus one, its *level*:
   $|E \cap \overline{C \setminus E}| + |\{i \colon p_i \in E\}|$ is strictly larger than m;
(3) $H^0(C, \Omega_C^{\vee}(-\sum_i p_i)) = 0$ (i.e. $\mathrm{Aut}(C, \underline{p})$ is a finite group).

The latter can be taken for a decency condition on the moduli stack. The second one, instead, is essential in guaranteeing the uniqueness of m-stable limits, as per the discussion above. Smyth's main result is the following.

Theorem 1.5. [Smy11a, Smy11b] *The moduli stack of m-stable curves $\overline{\mathcal{M}}_{1,n}(m)$ is a proper irreducible Deligne-Mumford stack over $\mathrm{Spec}\,\mathbb{Z}[1/6]$. It is* not *smooth for $m \geq 6$. The coarse moduli spaces $\overline{\mathbf{M}}_{1,n}(m)$ arise as birational models of $\overline{\mathbf{M}}_{1,n}$ for the big line bundles $D(s) = s\lambda + \psi - \Delta$, where ψ is the sum of the ψ-classes, Δ is a boundary class, and there is an explicit relation between s and m.*

1.2. **Experimenting on a genus two tale.** In this subsection, we walk through the motivations and methods at the heart of our construction, exemplifying them in the simplest possible case, that of $\overline{\mathcal{M}}_{2,2}^{(1)}$. The facts we mention are either proved or explained in greater detail and generality in the paper. Here is a classical

Fact. *There are two unibranch singularities of genus two, namely the* ramphoid *cusp or A_4-singularity $V(y^2 - x^5) \subseteq \mathbb{A}^2_{x,y}$, and the singularity $\mathrm{Spec}(\mathbf{k}[t^3, t^4, t^5])$ in 3-space. The former is Gorenstein, and its stable model is a Weierstrass tail (a genus two curve attached to a rational one at a Weierstrass point), while the latter is not Gorenstein, and its stable model is a non-Weierstrass genus two tail.*

See Lemma 2.1 and Proposition 4.2 below. Recall that every smooth curve of genus two is hyperelliptic, i.e. it can be realised as a two-fold cover of $\mathbb{P}^1$, in a unique way up to projectivities. The deck transformation is called the hyperelliptic involution σ; ramification points (fixed points of σ) are called Weierstrass, while in general $\{p, \sigma(p)\}$ are called conjugate points. See Remark 4.1 below.

Let us try out Smyth's approach on genus two curves, starting from $\overline{\mathcal{M}}_{2,2}^{(1)}$. If we are going to require the level of a genus two subcurve to be at least 2, it seems we need to include non-Gorenstein singularities in order to keep our moduli space proper. This might lead us into trouble; for example, the (log) dualising line bundle is classically used to construct canonical polarisations on stable curves, which in turn are essential in the proof that $\overline{\mathcal{M}}_{g,n}$ is an algebraic stack (or in the construction of $\overline{\mathbf{M}}_{g,n}$). Yet, there is a way around the awful singularity $\mathbf{k}[t^3, t^4, t^5]$.

Fact. *The A_5-singularity $V(y^2 - yx^3) \subseteq \mathbb{A}^2_{x,y}$ is a Gorenstein singularity of genus two with two branches. Its stable model is a genus two bridge, with the two nodes being conjugate points. A marked union of two copies of $\mathbb{P}^1$ along an A_5-singularity has no non-trivial automorphisms as soon as one of the two branches contains at least two markings.*

See Proposition 2.2 and Corollary 3.6 below. Let us go back to $\overline{\mathcal{M}}_{2,2}$. Suppose C is the nodal union of a genus two curve Z with a rational tail R supporting the two markings, so that $\mathrm{lev}(Z) = 1$. If R is attached to a Weierstrass point of Z, we may simply contract the latter (in a 1-parameter smoothing), thus producing an irreducible ramphoid cusp with two markings. If instead R is attached to a non-Weierstrass point q_1 of Z, we may blow up the one-parameter family at the conjugate point $\sigma(q_1)$ in the central fiber, and then contract Z to get a *dangling* A_5-singularity (meaning that one of the branches is unmarked), which nonetheless has trivial automorphism group. We pursue this strategy, which makes our compactifications not semistable (see [Smy13, Definition 1.2] for the terminology). The necessity to include such curves was prefigured in [AFS16].

To complete the picture, note that, in order to fix a deformation-open class of singularities, we need to allow cusps and tacnodes as well.

Fact. *The singularities appearing in the miniversal family of an A_m-singularity are all and only the A_l-singularities with $l \leq m$.*

See Theorem 2.5 below for a more general statement - valid for all ADE singularities - due to A. Grothendieck. For the sake of separatedness, we should at the same time require that the level of a genus one subcurve be at least 3. Note that hybrid situations may emerge: e.g. an elliptic curve with a cusp, or an irreducible tacnode; it is worth pointing out that, as we need to allow a tacnode and a cusp sharing a branch, we should impose the level condition on genus one subcurves only when they are *nodally attached* - in this case, any nodally attached subcurve of genus one will violate 1-stability. Besides, in the last example we need to break the $\mathfrak{S}_2$-symmetry (relabelling the markings) in order to have a unique limit: we do so by declaring that p_1 lie on the cuspidal branch. See Figure 1.
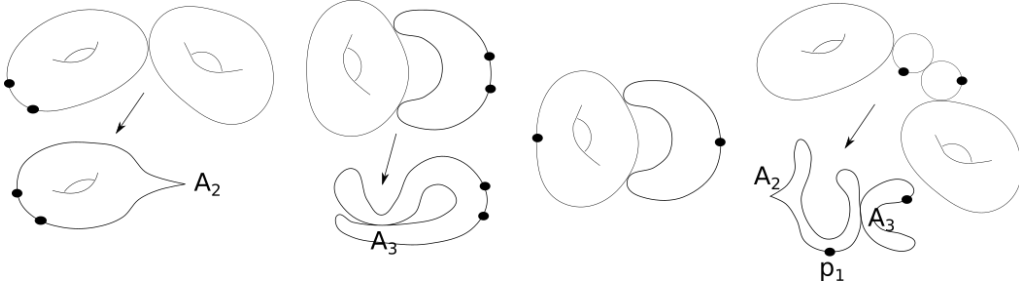

FIGURE 1. Examples of 2-pointed stable curves and their 1-stable limits.

We are now in a position to cast a plausible definition of $\overline{\mathcal{M}}_{2,2}^{(1)}$.

Definition 1.6. A connected, reduced, complete curve of arithmetic genus two C over $\mathbf{k} = \bar{\mathbf{k}}$, with smooth and disjoint markings p_1, p_2, is 1-*stable* if:

(1) C has only $A_1-, \ldots, A_4-$ and dangling A_5- singularities.
(2) There is no proper subcurve of arithmetic genus two.
(3) A subcurve of arithmetic genus one is either nodally attached and of level 3, or it is not nodally attached and it contains p_1.

The main result of the paper is that $\overline{\mathcal{M}}_{2,2}^{(1)}$ is a proper Deligne-Mumford stack, and the generalisation of this statement to an arbitrary number of markings and a range of stability conditions that we are going to discuss in the next subsection.

Let us note in passing that the birational map $\overline{\mathcal{M}}_{2,2} \dashrightarrow \overline{\mathcal{M}}_{2,2}^{(1)}$ is not defined everywhere. The reason boils down to the following

Fact. *There is only one isomorphism class of 2-pointed curves whose normalisation is $(\mathbb{P}^1, q_1) \sqcup (\mathbb{P}^1, q_2, p_1, p_2)$ and having an A_5-singularity at $q_1 = q_2$. On the other hand, the moduli space of 2-pointed irreducible curves of geometric genus zero and having an A_4-singularity is isomorphic to $\mathbb{A}^1$.*

The second statement can be motivated as follows: the pointed normalisation of such a curve is $(\mathbb{P}^1, q, p_1, p_2)$, which has neither automorphisms, nor deformations. To produce an A_4-singularity at q we may first collapse a non-zero tangent vector

at q (all the choices are equivalent), producing a cusp, and then collapse a line in the tangent space to the cusp, avoiding the tangent cone ℓ of the cusp (the moduli space is therefore $\mathbb{P}^1 \setminus \{\ell\} = \mathbb{A}^1$). See Lemma 3.2 and the discussion thereafter.

Let $\Delta = \Delta_{1,\emptyset|0,\{1,2\}} \subseteq \overline{\mathcal{M}}_{2,2}$ be the divisor of rational tails, and $\mathscr{W} \subseteq \overline{\mathcal{M}}_{2,2}$ the codimension 2 locus of Weierstrass tails. The 1-stable limit of any point in $\Delta \setminus \mathscr{W}$ is the dangling A_5-singularity, while the 1-stable limit of a Weierstrass tail is ill-defined (it depends on the choice of a 1-parameter smoothing); we conjecture that the rational map (identity on the locus of smooth curves) admits a factorisation:

$$\mathrm{Bl}_{\mathscr{W}}(\overline{\mathcal{M}}_{2,2})$$

$$\overline{\mathcal{M}}_{2,2} \dashrightarrow \overline{\mathcal{M}}_{2,2}^{(1)}$$

We plan to address this point in forthcoming work.

1.3. **Outline of results and plan of the paper.** In Section 2 we classify all the Gorenstein curve singularities of genus two. They come in two families: the first (I) one includes the ramphoid cusp, the D_5-singularity, and for $m \geq 3$ the union of *a singular branch* (a cusp) and $m-1$ lines living in $\mathbb{A}^m$. The second (II) one includes the A_5- nd D_6-singularities, and for $m \geq 4$ the union of *two tangent branches* (forming a tacnode) with $m-2$ lines in $\mathbb{A}^{m-1}$. See Proposition 2.2.

In Section 3 we translate the condition that a complete pointed curve of genus two have no infinitesimal automorphisms into a *mostly* combinatorial criterion. For every fixed number of branches m and genus two singularity type $\in \{I, II\}$, there are two isomorphism classes of pointed curves whose normalisation is $\bigsqcup_{i=1}^{m}(\mathbb{P}^1, q_i, p_i)$ and having a singularity of the prescribed type at q; one of them has $\mathrm{Aut}(C, p) = \mathbb{G}_{\mathrm{m}}$, while the other one has trivial automorphism group. This is a novelty phenomenon to genus two. We take a detour into moduli spaces of singularities to justify the claim, and explain how to interpret the *crimping spaces* geometrically in terms of the information we need to construct a genus two singularity from a (non-Gorenstein) singularity of lower genus. This is not strictly necessary in what follows, since the singularity with one-pointed branches does not satisfy the level condition we demand from our curves, yet this description is expected to be useful in analysing the indeterminacy of $\overline{\mathcal{M}}_{2,n}^{(m_1)} \dashrightarrow \overline{\mathcal{M}}_{2,n}^{(m_2)}$.

In Section 4 we study the (semi)stable limits; starting from a 1-parameter family of semistable curves with smooth generic fiber and regular total space, we show that the shape of a subcurve of the central fiber that can be contracted into a Gorenstein singularity is strongly constrained. Singularities of type I arise when the special branch (corresponding to the cusp in the contraction) is attached to a *Weierstrass point* of the minimal subcurve of genus two (the *core*), while singularities of type II occur when the special branches (corresponding to the tacnode in the contraction) are attached to *conjugate points*. Furthermore, the size of the curve to be contracted only depends on one number - roughly speaking,

the distance of the special branches from the core. The first statement is a consequence of the following simple observation: if $\phi \colon \widetilde{\mathscr{C}} \to \mathscr{C}$ is a contraction to a family of Gorenstein curves, $\phi^* \omega_{\mathscr{C}}$ is trivial on a neighbourhood of the exceptional locus of ϕ, and it coincides with $\omega_{\widetilde{\mathscr{C}}}$ outside it. Now, whereas the dualising line bundle of a Gorenstein curve of genus one with no separating nodes is trivial (see [Smy11a, Lemma 3.3]) and all smooth points look the same (i.e. they are not special), the simplest instance of Brill-Noether theory manifests itself in genus two, with the distinction between Weierstrass and non-Weierstrass points, and the expression $\omega_Z = \mathscr{O}_Z(q + \sigma(q))$. The correct extension of these concepts to nodal curves was formulated in the '80s within the theory of admissible covers and limit linear series, and we spend some time to discuss the relevant combinatorics.

In Section 5 we define the notion of m-*stable* n-pointed curve of genus two, for every $1 \le m < n$. The basic idea is to trade worse singularities - of both genus one and two, bounded by m in the sense of the embedding dimension - with more constraints on the combinatorics of the dual graph - the *level* condition, which bounds below in terms of m the number of special points (nodes and markings) that any subcurve of genus one or two has to contain. On the other hand, it is already clear from the discussion above that we need to break the $\mathfrak{S}_n$-symmetry, in order to write the dualising line bundle of the minimal subcurve of genus two as $\mathscr{O}_Z(q_1 + \sigma(q_1))$, in other words to choose which branches of a semistable model are to be dubbed special. We do so by using the first marking as a reference point, so that q_1 comes to denote the point of Z closest to p_1. This shapes our algorithm to construct the m-stable limit of a given 1-parameter smoothing. Unavoidably, the formulation of the stability condition is slightly involved, including a prescription of the interplay between p_1 and the singularity. We prove that the moduli stack of m-stable curves is algebraic, and it satisfies the valuative criterion of properness.

1.4. **Future directions of work.** Besides regarding this as a case-study of the birational geometry of moduli spaces, we are looking forwards to applications in enumerative geometry. We set up some questions we would like to come back to.

(1) Resolve the indeterminacy of the rational map $\overline{\mathcal{M}}_{2,n}^{(m_1)} \dashrightarrow \overline{\mathcal{M}}_{2,n}^{(m_2)}$, and study the intersection theory of these spaces; we expect the construction to rely on a semistable compactification of the crimping spaces of the genus two singularities, as in [vdW10, §1.10] and [Smy18]. It would be interesting to put this work in the context of the Hassett-Keel program, as in [Smy11b].

More generally, a question outstanding to our knowledge is whether the whole program fits in the theoretical framework developed in [Hal14].

(2) We expect this work to find applications in Gromov-Witten theory. In genus one, the link between reduced Gromov-Witten invariants (see for example [VZ08, Zin09, LZ09]) and maps from singular curves (see [Vis12]) was partially uncovered in [BCM18], and brought in plain view by [RSW17a, RSW17b]. With F. Carocci we are investigating whether similar techniques may serve to desingularise the main component of the space of genus two

maps to projective space. We expect there will be a(n iso)morphism to the modular blow-up constructed in [HLN12]. Maps from singular curves would provide a conceptual definition of reduced invariants for projective complete intersections and beyond, and hopefully make comparison results (standard vs. reduced) accessible. It would be interesting to relate them to Gopakumar-Vafa formulae [Pan99].

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2. GORENSTEIN CURVE SINGULARITIES OF GENUS TWO

In this and the next sections we work over an algebraically closed field $\mathbf{k}$ of characteristic different from $2, 3, 5$. We produce an algebraic classification of the (complete) local rings of Gorenstein curve singularities of genus two.

Let (C, q) be the germ of a reduced curve singularity, and let $(R, \mathfrak{m})$ denote $(\widehat{\mathscr{O}}_{C,q}, \mathfrak{m}_q)$, with normalisation $(\widetilde{R}, \widetilde{\mathfrak{m}}) \simeq (\mathbf{k}\llbracket t_1 \rrbracket \oplus \ldots \oplus \mathbf{k}\llbracket t_m \rrbracket, \langle t_1, \ldots, t_m \rangle)$. Here m is the number of branches of C at q. Recall the Definition 1.3 of the genus:

$$g = \delta - m + 1;$$

so, for genus two, $\delta = m + 1$. Following [Smy11a, Appendix A], we consider $\widetilde{R}/R$ as a $\mathbb{Z}$-graded module with:

$$(\widetilde{R}/R)_i := \widetilde{\mathfrak{m}}^i / (\widetilde{\mathfrak{m}}^i \cap R) + \widetilde{\mathfrak{m}}^{i+1};$$

furthermore, adapting Smyth's remarks in *loc. cit.* to our situation:

(1) $m + 1 = \delta(p) = \sum_{i \geq 0} \dim_{\mathbf{k}}(\widetilde{R}/R)_i$;
(2) $2 = g = \sum_{i \geq 1} \dim_{\mathbf{k}}(\widetilde{R}/R)_i$;
(3) if $(\widetilde{R}/R)_i = (\widetilde{R}/R)_j = 0$ then $(\widetilde{R}/R)_{i+j} = 0$.

We will also make use of the following observations:

(4) $\sum_{i \geq k} (\widetilde{R}/R)_i$ is a grading of $\widetilde{\mathfrak{m}}^k / (\widetilde{\mathfrak{m}}^k \cap R)$;
(5) there is an exact sequence of $R/\mathfrak{m} = \mathbf{k}$-modules:

$$0 \to A_i := \frac{\widetilde{\mathfrak{m}}^i \cap R}{\widetilde{\mathfrak{m}}^{i+1} \cap R} \to \frac{\widetilde{\mathfrak{m}}^i}{\widetilde{\mathfrak{m}}^{i+1}} \to \left(\widetilde{R}/R \right)_i \to 0$$

Lemma 2.1. *There are two unibranch curve singularities of genus two; only one of them is Gorenstein, the A_4-singularity or ramphoid cusp: $V(y^2 - x^5) \subseteq \mathbb{A}^2_{x,y}$.*

Proof. In the unibranch case $\dim_{\mathbf{k}}(\widetilde{R}/R)_1 \leq 1$, hence equality holds (by observation (3) above). We are left with two cases:

- Either $\dim_{\mathbf{k}}(\widetilde{R}/R)_2 = 1$ and $\dim_{\mathbf{k}}(\widetilde{R}/R)_i = 0$ for all $i \geq 3$: in this case $\widetilde{\mathfrak{m}}^3 \subseteq \mathfrak{m}$ by observation (4). From (5) we see that $\widetilde{\mathfrak{m}}^3 = \mathfrak{m}$, hence $R \simeq \mathbf{k}[\![t^3, t^4, t^5]\!]$, a spatial non-Gorenstein singularity.
- Or $\dim_{\mathbf{k}}(\widetilde{R}/R)_3 = 1$ and $\dim_{\mathbf{k}}(\widetilde{R}/R)_i = 0$ for $i = 2$ and for all $i \geq 4$: in this case $\widetilde{\mathfrak{m}}^4 \subseteq \mathfrak{m}$ by observation (4). On the other hand from $\dim_{\mathbf{k}}(\widetilde{\mathfrak{m}}^2 \cap R/\widetilde{\mathfrak{m}}^3 \cap R) = 1$ we deduce that there is a generator of degree 2, and from $\dim_{\mathbf{k}}(\widetilde{\mathfrak{m}}^3 \cap R/\widetilde{\mathfrak{m}}^4 \cap R) = 0$ there is none of degree 3. We may write the generator as $x = t^2 + ct^3$, and $\mathfrak{m} = \langle x \rangle + \widetilde{\mathfrak{m}}^4$. Up to a coordinate change (i.e. automorphism of $\mathbf{k}[\![t]\!]$), we may take $x = t^2$, and

$$\mathfrak{m}/\mathfrak{m}^2 = \langle t^2, t^5 \rangle,$$

so $R \simeq \mathbf{k}[\![x, y]\!]/(x^5 - y^2)$, as anticipated.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

From now on, we only look for Gorenstein singularities. With notation as above, let $I = (R : \tilde{R}) = \mathrm{Ann}_R(\tilde{R}/R)$ be the *conductor ideal* of the singularity. Recall e.g. [AK70, Proposition VIII.1.16]: (C, q) is Gorenstein if and only if

$$\dim_{\mathbf{k}}(R/I) = \dim_{\mathbf{k}}(\widetilde{R}/R)(= \delta).$$

Recall from [Ste96, Definition 2-1] that a curve singularity (C, q) is *decomposable* if C is the union of two curves C_1 and C_2 that lie in distinct smooth spaces intersecting each other transversely in q. Given a parametrisation $x_i = x_i(t_1, \ldots, t_m)$, $i = 1, \ldots, l$, this means that there is a partition $S \sqcup S' = \{1, \ldots, m\}$ such that x_i only depends on $t_s, s \in S$, or $s \in S'$, for all i. Aside from the node, Gorenstein singularities are never decomposable [AFS16, Proposition 2.1].

Proposition 2.2. *For every fixed integer $m \geq 2$, there are exactly two Gorenstein curve singularities of genus two with m branches.*

Proof. We only need to find a basis for $\mathfrak{m}/\mathfrak{m}^2$, because a map of complete local rings that is surjective on cotangent spaces is surjective. From observation (3) again, we find three possibilities for the vector (d_1, d_2, d_3), $d_k = \dim_{\mathbf{k}}(\widetilde{R}/R)_k$.

Case $(2, 0, 0)$. We see that $\widetilde{\mathfrak{m}}^2 \subseteq I$, so by the Gorenstein assumption we have:

$$m + 1 = \delta = \dim_{\mathbf{k}}(R/I) \leq \dim_{\mathbf{k}}(R/\widetilde{\mathfrak{m}}^2) = \dim_{\mathbf{k}} A_0 + \dim_{\mathbf{k}} A_1 = m - 1,$$

a contradiction. Note: the singularity turns out to be decomposable in this case.

Case $(1, 1, 0)$. We have $\widetilde{\mathfrak{m}}^3 \subseteq I$. We are going to write down the $m - 1$ generators of $A_1 \pmod{\widetilde{\mathfrak{m}}^3}$[1]. The first generator, call it x_1, has a non-trivial linear term in at least one of the variables, say t_1. By scaling x_1 and possibly adding a multiple of x_1^2, we can make it into the form: $x_1 = t_1 \oplus p_{1,2}(t_2) \oplus \ldots \oplus p_{1,m}(t_m)$ $\pmod{\widetilde{\mathfrak{m}}^3}$. Now we can use x_1 and x_1^2 to make sure the second generator does not involve t_1 at all. It will still have a linear term independent of t_1, say non-trivial in

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[1]To put them into the simplest possible form, we allow ourselves to perform elementary operations from linear algebra at first, while keeping ourselves from changing coordinates until the end - the benefit of this two-step process will become apparent in the next section.

t_2. By scaling and adding a multiple of x_2^2, we can write $x_2 = 0 \oplus t_2 \oplus \ldots \oplus p_{2,m}(t_m)$ (mod $\widetilde{\mathfrak{m}}^3$). By taking a linear combination of x_1 with x_2 and x_2^2, we may now reduce x_1 to the form $t_1 \oplus 0 \oplus p_{1,3}(t_3) \oplus \ldots \oplus p_{1,m}(t_m)$ (mod $\widetilde{\mathfrak{m}}^3$). Therefore, by Gaussian elimination with the generators and their squares, we may assume that

$$
\begin{aligned}
x_1 =& t_1 \oplus 0 \oplus \ldots \oplus \alpha_{1,m} t_m + \beta_{1,m} t_m^2 \\
x_2 =& 0 \oplus t_2 \oplus \ldots \oplus \alpha_{2,m} t_m + \beta_{2,m} t_m^2 \\
&\ldots \\
x_{m-1} =& 0 \oplus \ldots \oplus t_{m-1} \oplus \alpha_{m-1,m} t_m + \beta_{m-1,m} t_m^2 \quad (\text{mod } \widetilde{\mathfrak{m}}^3).
\end{aligned}
$$

We must have $R/I = \langle 1, x_1, \ldots, x_{m-1}, y \rangle$ by the Gorenstein condition (if $x_i \in I$ for some i, then $t_i \in R$, and the singularity would be decomposable). Hence $x_i^2 \in I$ for all but at most one i, say $i = 1$. Then $t_i^2 \in R$ for $i = 2, \ldots, m-1$. If $\alpha_{i,m} \neq 0$ for some i in this range, then $t_m^2 \in R$ as well, so $t_1^2 = x_1^2 - O(t_m^2) \in R$, contradicting $\dim_{\mathbf{k}}(\widetilde{R}/R)_2 = 1$. Therefore $\alpha_{i,m} = 0$ for $i \in \{2, \ldots, m-1\}$. If $\alpha_{1,m} = 0$, then we need a further generator of $\mathfrak{m}/\mathfrak{m}^2$, namely $z = 0 \oplus \ldots \oplus t_m^3$. In this case, though, both x_1^2 and z belong to I, so $\dim_k(R/I) = m$, and the singularity cannot be Gorenstein. We are reduced to the following expression:

$$
\begin{aligned}
x_1 =& t_1 \oplus 0 \oplus \ldots \oplus \alpha_{1,m} t_m + \beta_{1,m} t_m^2 \\
x_2 =& 0 \oplus t_2 \oplus \ldots \oplus \beta_{2,m} t_m^2 \\
&\ldots \\
x_{m-1} =& 0 \oplus \ldots \oplus t_{m-1} \oplus \beta_{m-1,m} t_m^2 \quad (\text{mod } \widetilde{\mathfrak{m}}^3),
\end{aligned}
\tag{1}
$$

with $\beta_{1,m} \in \mathbf{k}$ and $\alpha_{1,m}, \beta_{i,m} \in \mathbf{k}^\times$, $i = 2, \ldots, m-1$ (by indecomposability). Finally, we may change coordinates in t_m and scale the other t_i to obtain:

$$
\begin{aligned}
x_1 =& t_1 \oplus 0 \oplus \ldots \oplus t_m \\
x_2 =& 0 \oplus t_2 \oplus \ldots \oplus t_m^2 \\
&\ldots \\
x_{m-1} =& 0 \oplus \ldots \oplus t_{m-1} \oplus t_m^2 \quad (\text{mod } \widetilde{\mathfrak{m}}^3).
\end{aligned}
\tag{2}
$$

We check that $R/I = \langle 1, x_1, \ldots, x_{m-1}, x_1^2 \rangle$ and $\widetilde{R}/R$ is of type $(1,1,0)$. In case $m = 2$, we need an extra generator $y = t_2^3$. Equations are given by:

- $y(y - x_1^3)$ if $m = 2$ (A_5-singularity);
- $x_1 x_2 (x_2 - x_1^2)$ if $m = 3$ (D_6-singularity);
- $\langle x_3(x_1^2 - x_2), x_i(x_j - x_k) \rangle_{1 \leq i < j < k \leq m-1 \text{ or } 1 < j < k < i \leq m-1}$ if $m \geq 4$.

Case $(1,0,1)$. We have $\widetilde{\mathfrak{m}}^4 \subseteq I$. By an argument similar to the above one, we write generators for A_1 as $x_i = \ldots \oplus t_i \oplus \ldots \oplus \alpha_{i,m} t_m + \beta_{i,m} t_m^2 + \gamma_{i,m} t_m^3$, for $i = 1, \ldots, m-1$. Then $R/I = \langle 1, x_1, \ldots, x_{m-1}, y \rangle$. For all but at most one i, $x_i^2 \in I$, but definitely $x_i^3 \in I$ for all i. On the other hand $t_m^3 \notin R$, because otherwise $t_i^3 = x_i^3 - \alpha_{i,m}^3 t_m^3 + O(t_m^4)$ would belong to R as well, contradicting $\dim_{\mathbf{k}}(\widetilde{R}/R)_3 = 1$. From this we deduce that $\alpha_{i,m} = 0$ for all $i = 1, \ldots, m-1$.

Since $\dim_{\mathbf{k}}(\widetilde{R}/R)_2 = 0$, there has to be another generator of $\mathfrak{m}/\mathfrak{m}^2$ of degree two in t_m, which we may write as $x_m = t_m^2 + \gamma_{m,m}t_m^3$. We can use x_m to remove all the t_m^2 pieces from $x_1, \ldots, x_{m-1}$, so we are reduced to

$$
\begin{aligned}
x_1 &= t_1 \oplus 0 \oplus \ldots \oplus \gamma_{1,m}t_m^3 \\
x_2 &= 0 \oplus t_2 \oplus \ldots \oplus \gamma_{2,m}t_m^3 \\
&\quad \cdots \\
x_{m-1} &= 0 \oplus \ldots \oplus t_{m-1} \oplus \gamma_{m-1,m}t_m^3 \\
x_m &= 0 \oplus \ldots \oplus t_m^2 + \gamma_{m,m}t_m^3 \pmod{\widetilde{\mathfrak{m}}^4},
\end{aligned}
$$

(3)

with $\gamma_{m,m} \in \mathbf{k}$ and $\gamma_{i,m} \in \mathbf{k}^{\times}$, $i = 1, \ldots, m-1$ (by indecomposability). Finally, we may change coordinates in t_m and scale the other t_i to obtain:

$$
\begin{aligned}
x_1 &= t_1 \oplus 0 \oplus \ldots \oplus t_m^3 \\
x_2 &= 0 \oplus t_2 \oplus \ldots \oplus t_m^3 \\
&\quad \cdots \\
x_{m-1} &= 0 \oplus \ldots \oplus t_{m-1} \oplus t_m^3 \\
x_m &= 0 \oplus \ldots \oplus t_m^2 \pmod{\widetilde{\mathfrak{m}}^4}.
\end{aligned}
$$

(4)

We check that $R/I = \langle 1, x_1, \ldots, x_{m-1}, x_m \rangle$ and $\widetilde{R}/R$ is of type $(1, 0, 1)$. It recovers the unique Gorenstein singularity of Lemma 2.1 when $m = 1$. Equations are:

- $x^5 - y^2$ if $m = 1$ (A_4-singularity or *ramphoid cusp*, with $x = t^2, y = t^5$);
- $y(y^3 - x^2)$ if $m = 2$ (D_5-singularity, with $x = x_1, y = x_2$);
- $\langle x_3(x_1 - x_2), x_3^3 - x_1 x_2 \rangle$ if $m = 3$;
- $\langle x_i(x_j - x_k), x_m(x_i - x_j), x_m^3 - x_1 x_2 \rangle_{i,j,k \in \{1,\ldots,m-1\}}$ all different if $m \geq 4$.

□

Definition 2.3. In case $(1, 0, 1)$, we say the singularity is *of type I*, and the branch parametrised by t_m is called *singular*; in case $(1, 1, 0)$, we say the singularity is *of type II*, and the branches parametrised by t_1 and t_m are called *twin*. We shall refer to the singular or twin branches as *special* or *distinguished*; all other branches are *axes*. *Branch* remains a generic name, indicating any of the previous ones.

Remark 2.4. Singularities of type I do appear in the miniversal family of singularities of type II, and viceversa. For low values of m - which serves as the playground for our speculations -, this follows from a neat result of Grothendieck that we have learnt from [CML13] (see also [Arn72, Dem75]):

Theorem 2.5. *Let (C, q) be a curve singularity of ADE type. Singularities that appear in the miniversal deformation of (C, q) are all and only those ADE, whose Dynkin diagram can be obtained as a full subgraph of the diagram of (C, q).*

3. Tangent sheaf, crimping space, and automorphisms

In this section we analyse the tangent sheaf of a genus two singularity. For a complete Gorenstein curve of genus two with markings, we translate the absence of infinitesimal automorphisms into a (mostly) combinatorial condition. The crimping space naturally makes an appearance in the process.

Lemma 3.1. *Let (C, q) be a Gorenstein curve singularity of genus two, with pointed normalisation $\nu\colon (\tilde{C}, \{q_i\}_{i=1,\dots,m}) \to (C, q)$, and assume $\mathrm{char}(\mathbf{k}) \neq 2, 3, 5$. There is a diagram of exact sequences of sheaves*

$$0 \longrightarrow \nu_*\Omega_{\tilde{C}}^{\vee}(-\textstyle\sum_i 3q_i) \longrightarrow \nu_*\Omega_{\tilde{C}}^{\vee}(-\textstyle\sum_i q_i) \longrightarrow \nu_*\bigoplus_i \Omega_{\tilde{C}}^{\vee}(-q_i)_{|2q_i} \longrightarrow 0$$

$$0 \longrightarrow \nu_*\Omega_{\tilde{C}}^{\vee}(-\textstyle\sum_i 3q_i) \longrightarrow \Omega_C^{\vee} \longrightarrow \mathbf{k}^{\oplus m} \longrightarrow 0$$

The rightmost vertical map admits an explicit description in local coordinates.

Proof. Let $K(\tilde{C})$ denote the constant sheaf of rational functions on $\tilde{C}$. A section of $\Omega_{\tilde{C}}^{\vee} \otimes K(\tilde{C})$ is contained in $\Omega_C^{\vee}$ if and only if its image under push-forward

$$\nu_*\colon \nu_*\mathscr{H}om(\Omega_{\tilde{C}}, K(\tilde{C})) \to \mathscr{H}om(\Omega_C, K(\tilde{C}))$$

lies in the subspace $\mathscr{H}om(\Omega_C, \mathscr{O}_C)$. We may work locally around the singular point in the coordinates we found in Section 2.

A_4: In the coordinates $x = t^2 + ct^3, y = t^4, z = t^5$ (they are redundant, but this will not cause any problem), the section $f(t)\frac{d}{dt} \in \nu_*\Omega_{\tilde{C}}^{\vee} \otimes K(\tilde{C})$ pushes forward to

$$\nu_*\left(f(t)\frac{d}{dt}\right) = (2t + 3ct^2)f(t)\frac{d}{dx} + 4t^3 f(t)\frac{d}{dy} + 5t^4 f(t)\frac{d}{dz},$$

from which, writing $f(t) = f_0 + f_1 t + f_2 t^2 + O(t^3)$, we see that

$$(2t + 3ct^2)f(t), 4t^3 f(t), 5t^4 f(t) \in \hat{\mathscr{O}}_{C,p} \Leftrightarrow f_0 = 0, cf_1 + 2f_2 = 0.$$

A_5: In the coordinates $x = t_1 \oplus at_2 + bt_2^2, y = t_1^3$, the section $f_1(t_1)\frac{d}{dt_1} \oplus f_2(t_2)\frac{d}{dt_2}$ pushes forward to

$$\nu_*\left(f_1(t_1)\frac{d}{dt_1} \oplus f_2(t_2)\frac{d}{dt_2}\right) = (f_1(t_1) \oplus (a + 2bt_2)f_2(t_2))\frac{d}{dx} + 3t_1^2 f_1(t_1)\frac{d}{dy},$$

from which, writing $f_i(t_i) = f_{i0} + f_{i1}t_i + f_{i2}t_i^2 + O(t_i^3), i = 1, 2$, we see that

$$f_1(t_1) \oplus (a + 2bt_2)f_2(t_2), 3t_1^2 f_1(t_1) \in \hat{\mathscr{O}}_{C,p} \Leftrightarrow \begin{cases} f_{10} = f_{20} = 0, \\ f_{11} = f_{21}, \\ 2bf_{21} + af_{22} = a^2 f_{12}. \end{cases}$$

$I_{m\geq 2}$: In the coordinates of (3),

$$\nu_*\left(\sum_{i=1}^m f_i(t_i)\frac{d}{dt_i}\right) = \sum_{i=1}^{m-1}\left(f_i(t_i)\oplus 3\gamma_{i,m}t_m^2 f_m(t_m)\right)\frac{d}{dx_i}+$$

$$(2t_m+3\gamma_{m,m}t_m^2)f_m(t_m)\frac{d}{dx_m},$$

hence we deduce that

$$\nu_*\left(\sum_{i=1}^m f_i(t_i)\frac{d}{dt_i}\right)\in\Omega_C^\vee\otimes\hat{\mathscr{O}}_{C,p}\Leftrightarrow\begin{cases}f_{i0}=0 & \text{for } i=1,\ldots,m,\\ f_{i1}=3f_{m1}, & \text{for } i=1,\ldots,m-1,\\ 3\gamma_{m,m}f_{m1}+2f_{m2}=0.\end{cases}$$

$II_{m\geq 3}$: In the coordinates of (1),

$$\nu_*\left(\sum_{i=1}^m f_i(t_i)\frac{d}{dt_i}\right) = (f_1(t_1)\oplus(\alpha_{1,m}+2\beta_{1,m}t_m)f_m(t_m))\frac{d}{dx_1}+$$

$$\sum_{i=2}^m (f_i(t_i)\oplus 2\beta_{i,m}t_m f_m(t_m))\frac{d}{dx_i},$$

hence we deduce that

$$\nu_*\left(\sum_{i=1}^m f_i(t_i)\frac{d}{dt_i}\right)\in\Omega_C^\vee\otimes\hat{\mathscr{O}}_{C,p}\Leftrightarrow\begin{cases}f_{i0}=0 & \text{for } i=1,\ldots,m,\\ 2f_{11}=f_{i1}=2f_{m1}, & \text{for } i=2,\ldots,m-1,\\ \beta_{1,m}f_{m1}+\alpha_{1,m}f_{m2}=\alpha_{1,m}^2 f_{12}.\end{cases}$$

$\square$

We anticipate that the letters α,β and γ will play a role in determining the automorphism group of a complete curve with markings. We recall some key concepts from F. van der Wyck's thesis. Working over $\mathbf{k}$, he considers the stacks:

- $\mathscr{S}$ of reduced one-dimensional (1d) $\mathbf{k}$-algebras R,
- $\mathscr{T}$ of reduced 1d algebras with resolution $(R\hookrightarrow(S,J))$, where S is a smooth one-dimensional $\mathbf{k}$-algebra, and J the radical of the conductor of $R\subseteq S$.

Basically, R represents the (local) ring of a reduced curve with one singular point, S is its normalisation, and J is the ideal of the reduced fiber over the singular point of $\mathrm{Spec}(R)$. $\mathscr{S}$ and $\mathscr{T}$ are limit-preserving stacks over $\mathrm{Spec}(\mathbf{k})$ [vdW10, Proposition 1.21]. Furthermore, we may fix a reduced 1d algebra with resolution $\tau_0:(R_0\hookrightarrow(S_0,J_0))$, and consider the substack $\mathscr{T}(\tau_0)$ of reduced 1d algebras with singularity type τ_0 (i.e. isomorphic to τ_0 locally on both the base and the curve, see [vdW10, Definition 1.64]; that various notions of "locally" coincide is proved in [vdW10, Proposition 1.50]). There is a forgetful morphism $\mathscr{T}\to\mathscr{S}$, and the *crimping space* of τ_0 is defined to be the fiber over R_0 of the restriction of such morphism to $\mathscr{T}(\tau_0)$. The crimping space is a smooth $\mathbf{k}$-scheme [vdW10, Theorems 1.70 and 1.73]; indeed, it is isomorphic to the quotient of $\mathrm{Aut}_{(S_0,J_0)/\mathbf{k}}$

by $\mathrm{Aut}_{(S_0,J_0)/R_0}$, the latter consisting of automorphisms of the normalisation that preserve the subalgebra of the singularity; moreover, the quotient can be computed after modding out the lowest power of J contained in R [vdW10, Theorem 1.53]. Crimping spaces can be thought of as moduli for the normalisation map.

Lemma 3.2. *The crimping space of a genus two singularity with m branches is (a number - depending on the type - of copies of)* $\mathbb{A}^1 \times (\mathbb{A}^1 \setminus \{0\})^{m-1}$.

Proof. We resume notation from the previous section. We are going to fix the subalgebra τ_0 given in coordinates by (4) and (2) above respectively.

Type *I*: recall that in this case $\widetilde{\mathfrak{m}}^4 \subseteq R$. For a **k**-algebra A, let

$$G_i(A) = \{t_i \mapsto g_{i1}t_i + g_{i2}t_i^2 + g_{i3}t_i^3, t_j \mapsto t_j \mid g_{i1} \in A^*, g_{i2}, g_{i3} \in A\},$$

and notice that

$$\mathrm{Aut}_{(\widetilde{R},\widetilde{\mathfrak{m}})}^{\mathrm{mod}\ \widetilde{\mathfrak{m}}^4}(A) = (G_1 \times \ldots \times G_m) \rtimes \mathfrak{S}_m(A).$$

Consider now the action of a group element of the form $(g_1, \ldots, g_m; \mathrm{id}_{\mathfrak{S}_m})$ on the given generators of R:

$$x_i \mapsto \ldots \oplus g_{i1}t_i + g_{i2}t_i^2 + g_{i3}t_i^3 \oplus \ldots \oplus g_{m1}^3 t_m^3, \quad \text{for } i = 1, \ldots, m-1;$$
$$x_m \mapsto \ldots \oplus g_{m1}^2 t_m^2 + 2g_{m1}g_{m2}t_m^3.$$

The former belongs to R iff $g_{i1} = g_{m1}^3$; the latter does iff $g_{m2} = 0$. Thus such elements span a subgroup isomorphic to $\mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{a}}^{m-1} \times \mathbb{G}_{\mathrm{a}}^m(A)$. On the other hand, there is a special (singular) branch, parametrised by t_m. We conclude that

$$\mathrm{Aut}_{\tau_0}^{\mathrm{mod}\ \widetilde{\mathfrak{m}}^3}(A) = (\mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{a}}^{m-1} \times \mathbb{G}_{\mathrm{a}}^m) \rtimes (\mathfrak{S}_{m-1})(A).$$

The quotient is therefore isomorphic to m copies of $\mathbb{A}^1 \times (\mathbb{A}^1 \setminus \{0\})^{m-1}$.

Type *II*: recall that in this case $\widetilde{\mathfrak{m}}^3 \subseteq R$. For a **k**-algebra A, let

$$G_i(A) = \{t_i \mapsto g_{i1}t_i + g_{i2}t_i^2, t_j \mapsto t_j \mid g_{i1} \in A^\times, g_{i2} \in A\},$$

and notice that

$$\mathrm{Aut}_{(\widetilde{R},\widetilde{\mathfrak{m}})}^{\mathrm{mod}\ \widetilde{\mathfrak{m}}^3}(A) = (G_1 \times \ldots \times G_m) \rtimes \mathfrak{S}_m(A).$$

Consider now the action of a group element of the form $(g_1, \ldots, g_m; \mathrm{id}_{\mathfrak{S}_m})$ on the given generators of R:

$$x_1 \mapsto g_{11}t_1 + g_{12}t_1^2 \oplus \ldots \oplus g_{m1}t_m + g_{m2}t_m^2;$$
$$x_i \mapsto \ldots \oplus g_{i1}t_i + g_{i2}t_i^2 \oplus \ldots \oplus g_{m1}^2 t_m^2, \quad \text{for } i = 2, \ldots, m-1.$$

The former belongs to R iff $g_{11} = g_{m1}$ and $g_{12} = g_{m2}$; the latter does iff $g_{i1} = g_{m1}^2$. Thus, such elements span a subgroup isomorphic to $\mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{a}}^{m-1}(A)$. On the other hand, all branches are isomorphic to one another, but there is a pair of distinguished ones (parametrised by t_1 and t_m respectively). We conclude that

$$\mathrm{Aut}_{\tau_0}^{\mathrm{mod}\ \widetilde{\mathfrak{m}}^3}(A) = (\mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{a}}^{m-1}) \rtimes (\mathfrak{S}_2 \times \mathfrak{S}_{m-2})(A).$$

The quotient is then isomorphic to $\binom{m}{2}$ copies of $\mathbb{A}^1 \times (\mathbb{A}^1 \setminus \{0\})^{m-1}$. $\qquad\square$

The benefit of a two-step classification should now be clear: if we do not allow ourselves to change coordinates (i.e. act by automorphisms of the normalisation) until the end, the crimping space appears already from the expressions (1) and (3) for the generators of the subalgebra of the singularity.

There is a more geometric way to realise the crimping spaces. It is well-known that a cusp can be obtained by collapsing (*push-out*) a generic (i.e. non-zero) tangent vector at $p \in \mathbb{P}^1$. More generally, a Gorenstein singularity of genus one and m branches can be obtained by collapsing a generic (not contained in any coordinate linear subspace) tangent line at an ordinary m-fold point (which is a non-Gorenstein singularity of genus zero) [Smy11a, Lemma 2.2]. Therefore, the crimping space of the elliptic m-fold point, isomorphic to $(\mathbb{A}^1 \setminus \{0\})^{m-1}$, can be realised as the complement of the coordinate hyperplanes inside $\mathbb{P}(T_p R_m) \simeq \mathbb{P}^{m-1}$, where (R_m, p) is the rational m-fold point. Besides, this gives rise to a natural compactification of the crimping space supporting a universal family of curves - in fact, two: either we collapse non-generic tangent vectors, obtaining non-Gorenstein singularities along the boundary (this family $\mathscr{C}$ has a common normalisation, that is the trivial family $\widetilde{\mathscr{C}} = R_m \times \mathbb{P}(T_p R_m)$); or we blow $\widetilde{\mathscr{C}}$ up along the boundary (*sprouting*), and we replace the non-Gorenstein singularities with elliptic m-fold points having strictly semistable branches [Smy11b, §2.2-3].

Similarly, a Gorenstein singularity of genus two can be obtained by collapsing a generic tangent line to a non-Gorenstein singularity of genus one. Indeed, type τ_0^I admits a partial normalisation by σ_0^I, which is the decomposable union of a cusp (parametrised by t_m) together with $m-1$ axes, by adjoining the generator t_m^3; while type τ_0^{II} admits a partial normalisation by σ_0^{II}, which is the decomposable union of a tacnode in the (t_1, t_m)-plane together with $m-2$ axes, by adjoining the generator t_m^2.

These fit together nicely in a unifying picture: if we restrict $\mathscr{C}$ from the previous paragraph to the union of the coordinate lines in $\mathbb{P}(T_p R_m)$, we obtain m copies of σ_0^I over the points, together with $\binom{m}{2}$ copies of the universal curve of type σ_0^{II} over its crimping space - which is isomorphic to $\mathbb{A}^1 \setminus \{0\}$ -, identified with the line minus two points. Let $P = \mathbb{P}(T_p \mathscr{C}_{|\cup \text{lines}})$ be the projectivised tangent space at the singular point in the fiber of this family of genus one singularities. For each of the $\binom{m}{2}$ coordinate lines, P has one component P_i^{II} that is a $\mathbb{P}^{m-1}$-bundle over such a line; besides, P has m components P_j^I isomorphic to $\mathbb{P}^m$ and supported over the points. The crimping space of the genus two singularities with m branches (of type I and II together) can be realised as an open subscheme of P: it is obtained by removing from the $\mathbb{P}^{m-1}$-fibers of P^{II} the $m-1$ hyperplanes generated by (a) the tangent line to the tacnode and the $m-2$ axes, and (b) the plane containing the tacnode and all but one of the $m-2$ axes; and from each component of P^I the m planes generated by (a) the tangent cone of the cusp and the $m-1$ axes, and (b) the plane containing the cusp and all but one of the $m-1$ axes.

Remark 3.3. The crimping space is related to the moduli space of arrows ϕ in the diagram of Lemma 3.1.

We interpret $H^0(\Omega^\vee_{\mathbb{P}^1}(-p)_{|2p})$ as the tangent space to the subgroup of PGL_2 fixing the point $p \in \mathbb{P}^1$, thus it inherits a natural Lie algebra structure, isomorphic to the only non-abelian Lie algebra of dimension two V. It has a basis:

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{with } [e_1, e_2] = -2e_2.$$

The vector (φ, ψ) is seen to correspond to the infinitesimal automorphism:

$$t \mapsto \frac{1 + \epsilon\varphi t}{1 - \epsilon(\varphi t + \psi)} = t + \epsilon(2\varphi t - \psi t^2).$$

We are interested in arrows ϕ that are embeddings (i.e. $\in \mathrm{Gr}(m, V^{\oplus m})$) of Lie subalgebras, such that the corresponding groups of infinitesimal automorphisms fix a (unique?) subalgebra of a singularity of genus two inside $\mathbf{k}[\![t_1]\!] \oplus \ldots \oplus \mathbf{k}[\![t_m]\!]$.

We start with some heuristics. Here is the unibranch case: the subalgebra of $\mathbf{k}[\![t]\!]$ generated by $x = t^2 + ct^3$ is preserved by (φ, ψ) if and only if

$$(1 + 2\varphi)^2 t^2 - 2\psi(1 + 2\varphi)t^3 + c(1 + 2\varphi)^3 t^3 \text{ is a multiple of } t^2 + ct^3,$$

which reduces to $\varphi(1 + 2\varphi)c = \psi$. This further determines c if and only if $\varphi \neq 0$. Note that in this case (dimension one) the Lie subalgebra condition is automatically satisfied. We have found $(\varphi, \psi) \in \mathbf{k}^\times \times \mathbf{k}$.

The case of type II_2-algebras is more interesting. Let $x = (t_1, \alpha t_2 + \beta t_2^2)$ be the generator of such an algebra. The image of x under $(\varphi_1, \psi_1, \varphi_2, \psi_2)$ is:

$$\left((1 + 2\varphi_1)t_1 - \psi_1 t_1^2, \alpha(1 + 2\varphi_2)t_2 + (\beta(1 + 4\varphi_2) - \alpha\psi_2)t_2^2\right),$$

from which we deduce:

(5) $$\varphi_1 = \varphi_2 \quad \text{and} \quad 2\beta\varphi_2 - \alpha\psi_2 = -\alpha^2\psi_1.$$

Now let $\phi \colon \mathbf{k}^2 \to V^{\oplus 2}$ be given by $\begin{pmatrix} \varphi_{11} & \psi_{11} & \varphi_{12} & \psi_{12} \\ \varphi_{21} & \psi_{21} & \varphi_{22} & \psi_{22} \end{pmatrix}$, with Plücker coordinates w_{ij} for the minor of the i-th and j-th columns. The first condition in (5) immediately implies

(6) $$w_{13} = 0 \quad \text{and} \quad w_{12} = -w_{23}, w_{14} = w_{34}.$$

The second condition in (5) implies

(7) $$\alpha w_{12} = w_{14} \quad \text{and} \quad 2\beta w_{12} = \alpha w_{24}$$

so that $(\alpha, \beta) \in \mathbf{k}^\times \times \mathbf{k}$ is determined as soon as $w_{12}, w_{14} \neq 0$. Notice that (6) implies in particular the Plücker equation

$$w_{12}w_{34} - w_{13}w_{24} + w_{14}w_{23}.$$

It is easy to see that the condition for ϕ to be a sub-Lie algebra is

$$\mathrm{rk} \begin{pmatrix} \varphi_{11} & \psi_{11} & \varphi_{12} & \psi_{12} \\ \varphi_{21} & \psi_{21} & \varphi_{22} & \psi_{22} \\ 0 & w_{12} & 0 & w_{34} \end{pmatrix} = 2,$$

translating into

$$w_{12}w_{13} = 0 \qquad\qquad\qquad w_{12}(w_{34} - w_{14}) = 0$$
$$w_{13}w_{34} = 0 \qquad\qquad\qquad (w_{12} + w_{23})w_{34} = 0$$

automatically satisfied after (6) too. These equations cut inside $\mathbb{P}^5_{[w_{ij}]}$ the locus

$$(\mathbb{A}^1_{w_{14}/w_{12}} \setminus \{0\}) \times \mathbb{A}^1_{w_{24}/w_{12}}.$$

More generally, given a type II subalgebra $R_{\alpha,\beta}$ of $W = \bigoplus_{i=1}^m \mathbf{k}[\![t_i]\!]/(t_i^2)$, with generators of the form described in (1), the subalgebra of $V^{\oplus m}_{(\varphi_i,\psi_i)_{i=1,\dots,m}}$ preserving $R_{\alpha,\beta}$ is isomorphic to $\mathbf{k}^{\oplus m}$ with equations (see Lemma 3.1):

$$\begin{cases} 2\varphi_1 = \varphi_i = 2\varphi_m, & \text{for } i = 2, \dots, m-1, \\ 2\beta_{1,m}\varphi_m - \alpha_{1,m}\psi_m = -\alpha_{1,m}^2\psi_1; \end{cases}$$

it is easily seen that such a subalgebra of $V^{\oplus m}$ does not determine $R_{\alpha,\beta}$, but it does determine $(\alpha_{1,m}, \beta_{1,m})$. The case of type I is analogous.

We apply the preceding discussion to the study of automorphism groups of complete marked curves with a genus two singularity. The relevant category has been formalised in van der Wyck's thesis, see [vdW10, Proposition 1.102, Theorem 1.105 and Corollary 1.106], where he introduces the concept of type T reduced pointed curve with resolution, and the algebraic stack $\mathcal{N}_T$ of such objects. The type encodes the number and isomorphism class of the singularities, the distribution of genus and markings among the components of the normalisation, and the adjacency data between components and singular points. In the case that T has a unique singularity of genus two, with m one-marked rational branches, the stack $\mathcal{N}_T$ is isomorphic to $[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$ (see [vdW10, Examples 1.111-112]), so it has two points: one with $\mathbb{G}_{\mathrm{m}}$, and the other with trivial stabiliser.

Definition 3.4. The *atom* of type I_m (the name is borrowed from [AFSvdW17]) is obtained by gluing the subalgebra of $\mathbf{k}[t_1] \oplus \dots \oplus \mathbf{k}[t_m]$ generated by $x_1, \dots, x_m$ as in (4) with m copies of $(\mathbf{k}[s], (s))$ under the identification $s_i = t_i^{-1}$. $\mathbb{G}_{\mathrm{m}}$ acts on the atom by $\lambda.t_i = \lambda^3 t_i$ for $i = 1, \dots, m-1$ and $\lambda.t_i = \lambda t_i$ for $i = m$.

Similarly, the atom of type II_m is obtained by gluing the subalgebra of $\mathbf{k}[t_1] \oplus \dots \oplus \mathbf{k}[t_m]$ generated by $x_1, \dots, x_{m-1}$ (and y) as in (2) (and following lines) with m copies of $(\mathbf{k}[s], (s))$ under the identification $s_i = t_i^{-1}$. There is a $\mathbb{G}_{\mathrm{m}}$-action on the type II atom by $\lambda.t_i = \lambda t_i$ for $i = 1, m$ and $\lambda.t_i = \lambda^2 t_i$ for $i = 2, \dots, m-1$.

The curve with a genus two singularity and one-marked rational branches that has trivial automorphism group will be called the *non-atom*.

There is a more geometric way to realise this dicotomy. The non-Gorenstein genus one singularity of type σ_0^{II} (resp. σ_0^I), with one-marked rational branches, has automorphism group $\mathbb{G}_m^{m-1}$ (resp. $\mathbb{G}_m^m$). This acts on the tangent space at the singular point: of the lines fixed by this action, only one (call it ℓ') sits inside the open subset corresponding to the crimping space; all other lines in the crimping space are identified under the group action (call ℓ their equivalence class). Collapsing ℓ yields the non-atom, while collapsing ℓ' yields the atom.

As a third viewpoint, automorphisms can be studied by twisting the exact sequences of Lemma 3.1 by the ideal of the markings, and then taking global sections. The dicotomy arises then from the map ϕ: if the last condition imposed on infinitesimal automorphisms interweaves first and second order non-trivially (i.e. when $\beta_{1,m}$, resp. $\gamma_{m,m}$, are non-zero) then it is enough that automorphisms be trivial to second order on every branch for them to be trivial for good.

Finally, we shall turn the condition that the automorphism group be finite into an explicit one in terms of types. Recall Smyth's description of Gorenstein curves of genus one with no automorphisms [Smy11a, Proposition 2.3, Corollary 2.4].

Definition 3.5. Let $(C, p_1, \ldots, p_n)$ be a reduced pointed curve. A connected subcurve $D \subseteq C$ is said to be *nodally attached* if $D \cap \overline{C \setminus D}$ consists of nodes only. We say that C is *residually DM* (rDM) if every nodal and nodally attached subcurve D of C, marked by $\{p_i \in D\} \cup (D \cap \overline{C \setminus D})$, is Deligne-Mumford stable. As usual, by *special points* we mean markings and nodes.

Corollary 3.6. *Let* $(C, p_1 \ldots, p_n)$ *be a Gorenstein pointed curve of arithmetic genus two.* $H^0(C, \Omega_C^\vee(-\sum_{i=1}^n p_i)) = 0$ *is equivalent to either of the following:*

(1) *C has a singularity of type $I_{m \geq 1}$: either all branches contain exactly one special point and C is the non-atom; or each of its axes contains at least one special point, and at least one branch has at least two. Furthermore C is rDM.*

(2) *C has a singularity of type $II_{m \geq 2}$: either all branches contain exactly one special point and C is the non-atom; or at least one of its twin branches contains a special point, each of its axes contains at least one special point, and at least one branch has at least two. Furthermore C is rDM.*

(3) *C has two elliptic m-fold points: each of their branches contains at least one special point, and either they share a branch, or at least one branch of each singular point contains at least two special points. Furthermore C is rDM.*

(4) *C has one elliptic m-fold point: one of its branches is a genus one curve, and every other branch contains at least one special point; otherwise, all branches contain at least one, and either two of its branches coincide, or at least one branch has at least two special points. Furthermore C is rDM.*

(5) *C contains only nodes and is Deligne-Mumford stable.*

4. Dualising line bundle and semistable tails

Given a family of prestable (pointed) curves of genus two over the spectrum of a discrete valuation ring $\mathscr{C} \to \Delta$, with smooth generic fiber $\mathscr{C}_\eta$ and regular

total space, we classify the subcurves of the central fiber $\mathscr{C}_0$ that can be con-
tracted to yield a Gorenstein singularity of genus two. In the genus one case,
Smyth answered the analogous question by identifying the class of *balanced* sub-
curves: subcurves of arithmetic genus one, such that, when breaking them into
a *core* (minimal subcurve of genus one, not containing any separating node) and
a number of rational trees (with root corresponding to the component adjacent
to the core, and leaves corresponding to the components adjacent to the portion
of $\mathscr{C}_0$ that is not contracted), the distance between any leaf and the root for any
such tree is constant. In the case at hand, the answer turns out to be slightly
more complicated: first, the special branch(es) of a type *I* (resp. *II*) singular-
ity are connected through rational chains to a Weierstrass (resp. two conjugate)
point(s) of the core. Second, the lengths of the rational trees may vary according
to where their attaching points lie on the core (if the latter is reducible), but the
special chains are always the shortest, and together with the configuration of the
attaching points on the core they determine the length of any other tree.

Remark 4.1. While there are no special points on a smooth curve of genus zero or
one, the simplest instance of Brill-Noether theory involves smooth curves of genus
two. Every such C is *hyperelliptic*: it admits a unique (up to reparametrisation)
$2:1$ cover $\phi\colon C \to \mathbb{P}^1$, induced by the complete canonical linear system, i.e. $|K_C|$
is the unique $\mathfrak{g}_2^1$ on C; said otherwise, there is a unique element $\sigma \in \mathrm{Aut}(C)$,
called the *hyperelliptic involution*, such that $C/\langle\sigma\rangle \simeq \mathbb{P}^1$. A point $x \in C$ is called
Weierstrass if it is a ramification point for ϕ (or, equivalently, a fixed point for σ);
from the Riemann-Hurwitz formula it follows that there are six Weierstrass points
on every smooth curve of genus two. Two points x_1, x_2 are said to be conjugate
(write $x_2 = \bar{x}_1$) if there exists a point $z \in \mathbb{P}^1$ such that $\phi^{-1}(z) = \{x_1, x_2\}$ (or,
equivalently, $\sigma(x_1) = x_2$). These notions may be extended to nodal curves by
declaring (C, x) to be Weierstrass if its stabilisation lies in the closure of

$$\mathscr{W} = \{(C, x)|\ C \text{ smooth and } x \text{ Weierstrass}\} \subseteq \overline{\mathcal{M}}_{2,1},$$

and similarly for conjugate points. We then need to study the limiting behaviour
of Weierstrass points when a smooth curve degenerates to a nodal one. This is a
difficult problem when it comes to higher genus curves; it has received considerable
attention since the '70s, in work of Arbarello, Eisenbud-Harris, and many others.
In our case it boils down to understanding admissible covers [HM82] of degree two
with a branch locus of degree six; said otherwise, up to the involution action, the
Weierstrass locus is isomorphic to $\overline{\mathcal{M}}_{0,6}/\mathfrak{S}_5$, and the conjugate locus is isomorphic
to $\overline{\mathcal{M}}_{0,7}/\mathfrak{S}_6$. We remark that (C, x) being Weierstrass is an intrinsic notion if
C is of compact type (or, more generally, tree-like), but it may depend on the
smoothing otherwise (i.e. the fiber of $\overline{\mathscr{W}} \to \overline{\mathcal{M}}_2$ may have positive dimension); we
have benefited from the exposition in [Dia85, Appendix 2], [Cuk89, Proposition
(3.0.6)], and [HM98, Theorem 5.45].

- If x belongs to a component of genus one E, which is attached to another
  component of genus one at a node y, then x is Weierstrass iff $2x \sim 2y \in \mathrm{Pic}(E)$;

if instead E has a self-node that glues y_1 with y_2, then x is Weierstrass iff $2x \sim y_1 + y_2 \in \text{Pic}(E)$.

If x is on a rational component R, x is Weierstrass if either R is attached to a genus one curve at two distinct points; or R has a self-node gluing y_1 and y_2 and is attached to a genus one tail at y_3, in which case we require $\phi(y_1) = \phi(y_2)$ for a double cover $\phi \colon R \to \mathbb{P}^1$ ramified at x and y_3; or R has two self-nodes gluing y_1 with y_2, and y_3 with y_4, in which case we require x to be a ramification point for a double cover $\phi \colon R \to \mathbb{P}^1$ such that $\phi(y_1) = \phi(y_2)$ and $\phi(y_3) = \phi(y_4)$. See Figure 2.
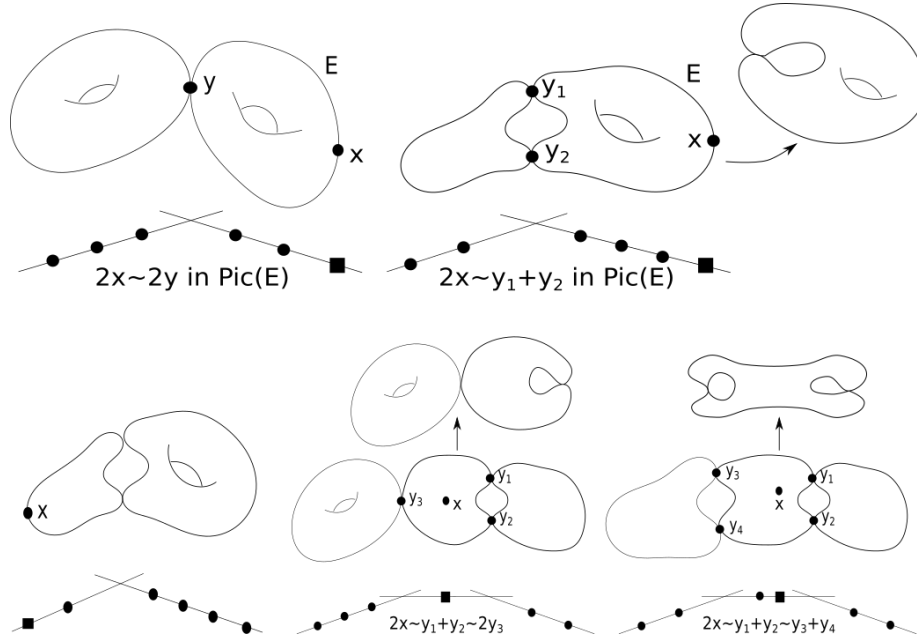

FIGURE 2. Admissible covers and Weierstrass points.

- If x_1 and x_2 are conjugate, they have to map to the same component of the target of the admissible cover. We may readapt the description of the previous point by replacing every condition on $2x$ by its analogue for $x_1 + x_2$. There are a few more situations to take into account: x_1 and x_2 could belong to a rational component R bubbling off from a Weierstrass point of a genus two curve; or bridging between two distinct curves of genus one; or x_1 and x_2 could lie on two distinct rational components R_1 and R_2 intersecting at one node and meeting a curve of genus one in two distinct points (†); or R_1 and R_2 intersecting each other in three points. See Figure 3.

We observe that in case (†), the singularity of the total space of a smoothing $\mathscr{C} \to \Delta$ at the two distinguished nodes (separating the elliptic component from the rational chain) are both A_k for the same k, because they map to the same
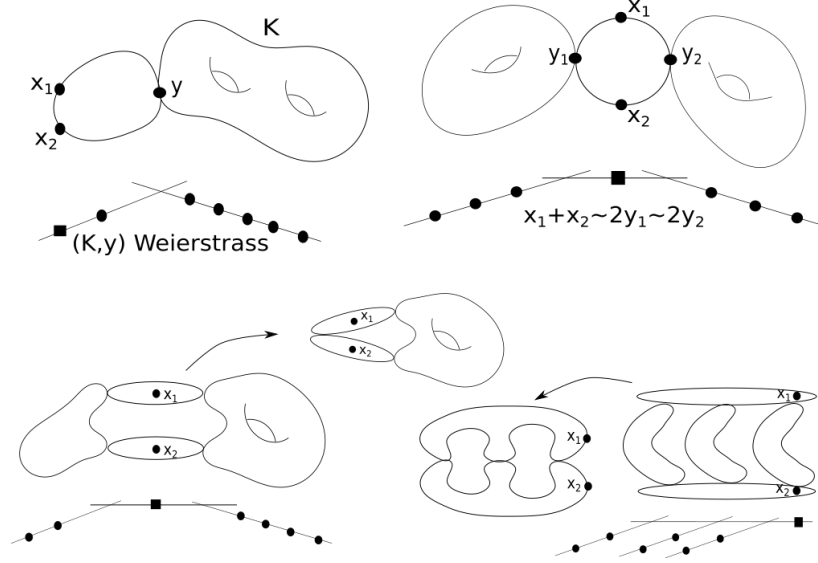
FIGURE 3. Admissible covers and conjugate points.

node of the target in the admissible cover. This is stable under base change, so it entails a symmetry of the rational chain in the model with regular total space.

The length of a chain is the number of its beads. For two points a, b on a curve whose dual graph is a chain, we denote by $\operatorname{dist}(a, b)$ the length of the chain (so for example it is 1 if a and b lie on the same irreducible component but $a \neq b$).

Proposition 4.2. *Let* $\phi\colon \mathcal{C} \to \overline{\mathcal{C}}$ *be a birational contraction over the spectrum of a discrete valuation ring* Δ, *where:* $\mathcal{C} \to \Delta$ *is a family of prestable (reduced, nodal) curves of arithmetic genus two, with regular total space and smooth generic fiber* $\mathcal{C}_\eta$; $\overline{\mathcal{C}} \to \Delta$ *is a family of Gorenstein curves of arithmetic genus two, and* $\overline{\mathcal{C}}_0$ *containing a genus two singularity of type* I_m *at* q. *Denote by* $(Z; q_1, \ldots, q_m)$ *the exceptional locus* $\operatorname{Exc}(\phi) = \phi^{-1}(q)$, *marked with* $Z \cap \overline{\mathcal{C}_0 \setminus Z}$, *where* q_m *corresponds to the special branch of* $\overline{\mathcal{C}}_0$. *Then:*

(1) (Z, q_m) *is Weierstrass.*

(2) Let x_m *be the point of the core of* Z *closest to* q_m, *and let* k *be the length of* R_m, *the rational chain separating* q_m *from* x_m. *With similar notation, for every* $i = 1, \ldots, m-1$, R_i *has length*

$$
\begin{cases}
3k + 1 + \operatorname{dist}(x_m, x_i) & \text{if } x_i \neq x_m, \\
k + 2\operatorname{dist}(q_m, r_i) & \text{if } x_i = x_m,
\end{cases}
$$

where r_i *is the point of* R_m *closest to* q_i.

Proposition 4.3. *Same as above with* $\overline{\mathcal{C}}_0$ *containing a genus two singularity of type* II_m, *and* q_1, q_m *corresponding to the special branches. Then:*

(1) (Z, q_1, q_m) *is conjugate.*

(2) R_1 *and* R_m *have the same length* k, *and, for* $i = 2, \ldots, m-1$, R_i *has length*

$$
\begin{cases}
2k + \min_{\epsilon \in \{1, m\}} \operatorname{dist}^*(x_\epsilon, x_i) & \text{if } x_1 \neq x_m, \text{ and } x_i \notin \{x_1, x_m\}, \\
k + \operatorname{dist}(q_1, r_i) & \text{if } x_1 \neq x_m, \text{ and } x_i = x_1, \\
k + \operatorname{dist}(q_m, r_i) & \text{if } x_1 \neq x_m, \text{ and } x_i = x_m, \\
2k + \operatorname{dist}(x_1, r_m) + \operatorname{dist}(x_1, x_i) & \text{if } x_1 = x_m, \text{ and } x_i \neq x_1, \\
k + \operatorname{dist}(q_1, r_i) + \delta(\operatorname{dist}(r_m, r_i) - 1) & \text{if } x_1 = x_m = x_i, \delta = \begin{cases} 1 & r_m \in [q_1, r_i] \\ 0 & otherwise \end{cases}
\end{cases}
$$

Here we have set $\operatorname{dist}^*(x_\epsilon, x_i) = 1$ if the core consists of a genus one curve with a rational bridge, and x_i lies between x_1 and x_m on this rational bridge, while $\operatorname{dist}^*(x_\epsilon, x_i) = \operatorname{dist}(x_\epsilon, x_i)$ otherwise.

Proposition 4.4. *Let* $(\mathscr{C}, p_1, \ldots, p_n) \to \Delta$ *be a family of pointed semistable curves of arithmetic genus two such that* $\mathscr{C}$ *has regular total space and smooth generic fiber, and* $(\mathscr{C}, p_1) \to \Delta$ *is Weierstrass. Let* $(Z, q_1, \ldots, q_m)$ *be a genus two subcurve of* $\mathscr{C}_0$ *containing none of the* $p_i(0)$, *marked by* $Z \cap \overline{\mathscr{C}_0 \setminus Z}$ *so that the tail containing* p_1 *is attached to* Z *at* q_m, *and satisfying all the shape prescriptions of Proposition 4.2(2). There exists a contraction* $\phi \colon \mathscr{C} \to \overline{\mathscr{C}}$ *over* Δ, *with exceptional locus* Z, *such that* $\overline{\mathscr{C}} \to \Delta$ *is a family of Gorenstein curves containing a type* I_m *singularity in the central fiber.*

Proposition 4.5. *Same as above with* $(\mathscr{C}, p_1, \bar{p}_1) \to \Delta$ *conjugate,* $(Z, q_1, \ldots, q_m)$ *shaped as prescribed by Proposition 4.3(2), with* q_1, q_m *corresponding to the banches containing* p_1 *and* $\bar{p}_1$. *The resulting* $\overline{\mathscr{C}} \to \Delta$ *contains a type* II_m *singularity in the central fiber.*

Proof. (of Proposition 4.2) By blowing down all the rational trees on $\overline{\mathscr{C}}_0$, we can assume that the latter does not contain any separating node. Consider then the hyperelliptic cover $\tau \colon \overline{\mathscr{C}} \to \mathbb{P}(\bar{\pi}_* \omega_{\overline{\mathscr{C}}/\Delta})$; restricting to the central fiber, τ contracts all axes, and induces a two-fold cover of $\mathbb{P}^1$ by the special branch, ramified at the singularity and at another point; in fact, we can extend the image of this point to a section of $\mathbb{P}(\bar{\pi}_* \omega_{\overline{\mathscr{C}}/\Delta})$ lying inside the branch locus of τ. By pulling this back to $\mathscr{C}$ via $\tau \circ \phi$ we get a horizontal divisor Δ'; clearly, the stable model of $(\mathscr{C}, \Delta')$ is Weierstrass, and its central fiber coincides with the stabilisation of (Z, q_1). This proves the first claim. (The proof of Proposition 4.3(1) is entirely analogous, by noticing that the preimage of a generic hyperplane section of $\mathbb{P}(\bar{\pi}_* \omega_{\overline{\mathscr{C}}/\Delta})$ will mark the two special branches of $\overline{\mathscr{C}}_0$.)

We now come to a proof of the more combinatorial claim (2) of the Proposition. Since $\overline{\mathscr{C}} \to \Delta$ is Gorenstein, and ϕ is assumed to be an isomorphism outside Z, because the dualising sheaf behaves well under restriction to open subschemes, we have an equality of line bundles:

$$\phi^* \omega_{\overline{\mathscr{C}}/\Delta} = \omega_{\mathscr{C}/\Delta}(D),$$

for some effective (Cartier) divisor D on $\mathscr{C}$ supported on Z. The next lemma will help us determine the coefficients of D along the components of Z containing q_i.

Lemma 4.6. *Let $\nu\colon C \to \bar{C}$ be the normalisation of a Gorenstein singularity of genus two, with $\nu^{-1}(q) = \{q_1, \ldots, q_m\}$. Then $\nu^*\omega_{\bar{C}} = \omega_C(2q_1 + \ldots + 2q_{m-1} + 4q_m)$ (type I) or $\nu^*\omega_{\bar{C}} = \omega_C(3q_1 + 2q_2 + \ldots + 2q_{m-1} + 3q_m)$ (type II).*

Proof. The dualising sheaf of a reduced curve admits an explicit description (due to Rosenlicht, see e.g. [AK70, Proposition VIII.1.16]) in terms of residues:

$$\omega_{\bar{C}}(U) = \{\eta \in \Omega_C \otimes K(\nu^{-1}(U)) | \sum_{p_i \in \nu^{-1}(p), p \in U} \mathrm{Res}_{p_i}((\nu^*f)\eta) = 0, \ \forall f \in \mathscr{O}_{\bar{C}}(U)\}.$$

We are going to use the explicit coordinates in (1) and (3). In case I, we know that $\widetilde{\mathfrak{m}}^4 \subseteq R$, therefore we have poles of fourth order at most. It is enough to study the possible polar tails. On the other hand, $t_i^2 \in R$ for all i implies the part of order three is trivial. So let

$$\eta = c_1 \frac{\mathrm{d}\,t_1}{t_1^4} + b_1 \frac{\mathrm{d}\,t_1}{t_1^2} + a_1 \frac{\mathrm{d}\,t_1}{t_1} \oplus \ldots \oplus c_m \frac{\mathrm{d}\,t_m}{t_m^4} + b_m \frac{\mathrm{d}\,t_m}{t_m^2} + a_m \frac{\mathrm{d}\,t_m}{t_m}.$$

From looking at $1 \cdot \eta$ we deduce $\sum_{i=1}^m a_i = 0$; from $x_i \cdot \eta$ we see $b_i + c_m = 0$ for all i, and from $x_i^3 \cdot \eta$ we have $c_i = 0$ for all i. (The statement about third order poles can be evinced from $x_i^2 \cdot \eta$ or from $z \cdot \eta$ indifferently.) Therefore $\omega_C/\nu_*\omega_{\bar{C}}$ is spanned by

$$\frac{\mathrm{d}\,t_1}{t_1} - \frac{\mathrm{d}\,t_m}{t_m}, \ldots, \frac{\mathrm{d}\,t_{m-1}}{t_{m-1}} - \frac{\mathrm{d}\,t_m}{t_m}, \frac{\mathrm{d}\,t_m}{t_m^2}$$

$$\bar{\eta} = \frac{\mathrm{d}\,t_1}{t_1^2} + \ldots + \frac{\mathrm{d}\,t_{m-1}}{t_{m-1}^2} - \frac{\mathrm{d}\,t_m}{t_m^4}.$$

In particular ω_C is generated by $\bar{\eta}$ as an $\mathscr{O}_C$-module.

In case II, we know that $\widetilde{\mathfrak{m}}^3 \subseteq R$, so we have poles of third order at most. Let

$$\eta = c_1 \frac{\mathrm{d}\,t_1}{t_1^3} + b_1 \frac{\mathrm{d}\,t_1}{t_1^2} + a_1 \frac{\mathrm{d}\,t_1}{t_1} \oplus \ldots \oplus c_m \frac{\mathrm{d}\,t_m}{t_m^3} + b_m \frac{\mathrm{d}\,t_m}{t_m^2} + a_m \frac{\mathrm{d}\,t_m}{t_m}.$$

From looking at $1 \cdot \eta$ we deduce $\sum_{i=1}^m a_i = 0$; from $x_i \cdot \eta$ we see $b_1 + b_m = 0$ (if $i = 1$), and $b_i + c_m = 0$ (if $i = 2, \ldots, m-1$); finally from $x_i^2 \cdot \eta$ we have $c_1 + c_m = 0$ (if $i = 1$), and $c_i = 0$ (if $i = 2, \ldots, m-1$). Therefore $\omega_C/\nu_*\omega_{\bar{C}}$ is spanned by

$$\frac{\mathrm{d}\,t_1}{t_1} - \frac{\mathrm{d}\,t_m}{t_m}, \ldots, \frac{\mathrm{d}\,t_{m-1}}{t_{m-1}} - \frac{\mathrm{d}\,t_m}{t_m}, \frac{\mathrm{d}\,t_1}{t_1^2} - \frac{\mathrm{d}\,t_m}{t_m^2},$$

$$\bar{\eta} = \frac{\mathrm{d}\,t_1}{t_1^3} + \frac{\mathrm{d}\,t_2}{t_2^2} + \ldots + \frac{\mathrm{d}\,t_{m-1}}{t_{m-1}^2} - \frac{\mathrm{d}\,t_m}{t_m^3}.$$

In particular ω_C is generated by $\bar{\eta}$ as an $\mathscr{O}_C$-module. $\qquad\square$

Corollary 4.7. *The dualising sheaf has multi-degree $(0, \ldots, 0, 2)$ (case I) and $(1, 0, \ldots, 0, 1)$ (case II) respectively.*

Remark 4.8. It follows from this and Corollary 3.6 that the finiteness of auto-morphism groups, $H^0(\bar{C}, \Omega_{\bar{C}}^\vee(-\sum_{i=1}^n p_i)) = 0$, implies $\omega_{\bar{C}}(\sum_{i=1}^n p_i)$ is ample.

Let us go back to the proof of Proposition 4.2. Note that $\phi_{|\mathscr{C}_0 \setminus Z}$ is the nor-malisation of $\overline{\mathscr{C}}_0$ at q. Letting T_i be the tail of $\mathscr{C}_0 \setminus Z$ attached to Z at q_i, by the adjunction formula $\omega_{\mathscr{C}/\Delta|T_i} = \omega_{T_i}(q_i)$; it follows from Lemma 4.6 that D has multiplicity 3 at the component of Z containing q_m (case I) - resp. 2 at the com-ponents containing q_1 and q_m (case II) -, and 1 at any component containing the other q_i. Set $\mathscr{L} = \omega_{\mathscr{C}/\Delta}(D) = \phi^* \omega_{\overline{\mathscr{C}}/\Delta}$; we draw the consequences of $\mathscr{L}_{|Z} = \mathscr{O}_Z$. We split Z into a core K and a number ($\leq m$ by semistability) of rational trees.

Let d_A denote the multiplicity of the divisor D along the component A of Z. First, we claim that no component can appear with $d_A = 0$. Assume that this occurred along one of the rational trees. Call S ($S \simeq \mathbb{P}^1$) a component furthest from the core such that $d_S = 0$; R the one that precedes it, and $T_1, \ldots, T_h$ the ones that follow it (when sweeping the tree from the core) - so that $h \geq 1$ by the previous paragraph, and $d_{T_i} \geq 1$ by inductive assumption. Then, by adjunction,

$$\deg(\mathscr{L}_{|S}) = -2 + (h+1) + d_R + \sum d_{T_i} = 0,$$

which necessarily implies $h = 1$, and $d_R = d_{T_i} = 0$, contradicting the assumption. The case that S belongs to the core is similar (ω_S might only be more positive).

Let us now consider $d_S = 1$. We stick to the notation above; furthermore, there may be a number k of q_i, $i \in \{(1,)2, \ldots, m-1\}$ in case II (resp. I), lying on S. Then, again by adjunction,

$$\deg(\mathscr{L}_{|S}) = -2 + d_R + \sum d_{T_i} = 0.$$

so either $d_R = 2$, $h = 0$ and $k \geq 1$ arbitrary, i.e. S is adjacent to $\overline{C \setminus Z}$; or $d_R = 1$, $h = 1$, and $d_{T_1} = 1$ (with k arbitrary). In the latter case, though, by repeating the same argument on T_1 etc., we would find an infinite chain in Z.

Remark 4.9. More generally, an analogous computation shows that, when bal-ancing a component A of multiplicity d_A, all neighbouring components of mul-tiplicity $d_A - 1$ can be safely ignored (at the same time, the number of such components is bounded only by m, due to the semistability of Z).

We now prove that $d_R > d_S$ holds in general for S on a rational tree. The preceding paragraphs deal with the cases $d_S = 0, 1$; we may therefore assume $d_S > 1$ (which in particular implies $0 \leq k \leq 2$). We have

$$\deg(\mathscr{L}_{|S}) = -2 + d_R - (d_S - 1)(h + k + 1) + \sum d_{T_i} = 0.$$

By proceeding inductively from leaves to root, we can assume that $d_S > d_{T_i}$, $i = 1, \ldots, h$. We may therefore rewrite the previous equality as

$$d_R = (d_S - 1)(h + k + 1) - \sum d_{T_i} + 2 \geq (d_S - 1)(k + 1) + 2 = d_S + 1 + k(d_S - 1) > d_S.$$

In fact, we can prove as on [Smy11a, p.893] that $d_R = d_S + 1$, unless $d_S = 3$ and $q_m \in S$ (type I), or $d_S = 2$ and either q_1 or q_m (or both) are on S (type II). We introduce some terminology to describe the weighted dual graph of D.

Definition 4.10. A t-chain is a weighted graph that is a chain and such that the weight of two adjacent vertices differ by t. We call t the *trend*; the vertex with highest (resp. lowest) weight is called the root (resp. leaf) of the chain. An (a, t)-chain is a t-chain with leaf weight a. The chain C_1 can be attached to the chain C_0 by identifying the root of C_1 with a vertex of C_0 having the same weight. A 1-tree is obtained by attaching a number of $(1, 1)$-chains among themselves.

Let us now look at a component S with $d_S = 2$ and at least one of q_1 and q_m attached to it (case II). The balancing equation is

$$\deg(\mathscr{L}_{|S}) = -2 - (h + k + 1) + d_R + \sum d_{T_i} = 0,$$

with $k \in \{1, 2\}$. The preceding discussion implies that $d_{T_i} = 1$ for all $i = 1, \ldots, h$, so $d_R = 3 + k$. If $k = 2$, i.e. both q_1 and q_m are on S - in which case they are indeed equidistant from the core -, then $d_R = 5$, and it can be shown inductively that the multiplicity of D increases by 3 for every step we make from S towards the core. The same holds in case I, with q_m attached to S and $d_S = 3$.

Finally, say $d_S = 2$ and only $q_1 \in S$. Then $d_R = 4$, and the trend along the chain that connects S to the core is 2, unless there is a component S' at which two 2-chains meet, in which case the trend becomes 3 after S'.

Definition 4.11. A 2-tree is obtained by attaching a number of $(1, 1)$-chains to a $(2, 2)$-chain. A 3-tree is obtained by attaching a number of $(1, 1)$-chains either to a $(3, 3)$-chain, or to a weighted graph itself obtained by attaching two $(2, 2)$-chains to the leaf of a 3-chain.

From the preceding discussion it is clear that the weighted dual graph of D is obtained by attaching a number of 1-trees, and either (a) one 3-tree or (b) two 2-trees to the dual graph of the core K, weighted in an appropriate fashion.

Finally, let us look at the core K. Consider it as a one-pointed (case (a)), resp. two-pointed (case (b)) curve of genus two, by ignoring all the attachment points of the 1-trees (it does not alter the balancing equation along K by Remark 4.9), and let $\bar{K} \in \overline{\mathcal{M}}_{2,1}$ (resp. $\overline{\mathcal{M}}_{2,2}$) be its stable model. The following can happen:

(1) K is a smooth curve of genus two. In case (a), let R be the component adjacent to the core along the 3-tree, and let $x = R \cap K$; then $d_K = d_R + 3$ by balancing R. Now balancing K gives

$$\omega_{\mathscr{C}/\Delta}(d_R R + d_K K)_{|K} = \omega_K(d_R x - (d_R + 2)x) \simeq \mathscr{O}_K,$$

which admits a solution if and only if K is Weierstrass. Similarly case (b) can be balanced if and only if K is conjugate.

(2) K contains two distinct subcurves of genus one E_1 and E_2. We start by solving the balancing equation on one of them, say $E = E_1$. If all but one

of the neighbouring components have multiplicity $d_E - 1$, then the last one is forced to have multiplicity $d_E - 1$ as well (by degree reasons). The case that all but two neighbouring components have multiplicity $d_E - 1$ occurs when either one 2-tree or one 3-tree (and exactly one) is attached to E at x; let F be the other component with undetermined multiplicity, which lies between E_1 and E_2 (possibly $F = E_2$), and let $E \cap F = \{y\}$. The case of a 2-tree forces $d_F = d_E$ by degree reason, but then we are left to solve $x \sim y$ in $\mathrm{Pic}(E)$, which is impossible; on the other hand, the case of a 3-tree imposes $d_F = d_E + 1$ and $2x \sim 2y$ in $\mathrm{Pic}(E)$, i.e. K is Weierstrass. By the same token, the two 2-trees have two hit the same genus one curve, say E_1, in points x_1, x_2 such that $x_1 + x_2 \sim 2y \in \mathrm{Pic}(E_1)$ and $d_F = d_E + 1$.

Assume now that there is a chain of rational curves S_i lying between E_1 and E_2 in K, and one of the special trees connects to one of the S_i; in case (b), then, both 2-trees must connect to (possibly different) S_i, by the previous paragraph. Furthermore, the trend along the rational chains at E_1 and E_2 has to be 1. This in turn implies that, in case (b), the trend along the chain separating the two 2-trees is 0. In particular, the two 2-trees are attached to components with the same multiplicity for D, so q_1 and q_m are equidistant from the core. See Figure 4 for some examples.
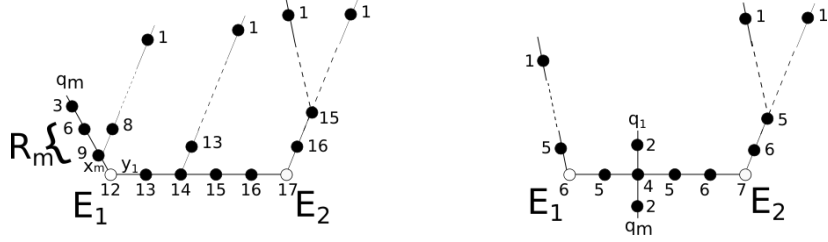

FIGURE 4. Examples of Z with two genus one components (the white dots). The numbers indicate the multiplicity of D. On the left, one 3-tree attached to E_1; on the right, two 2-trees attached to the same bead of the rational chain separating E_1 from E_2.

(3) $\bar{K} \in \Delta_{irr}$, i.e. K contains an irreducible subcurve of arithmetic genus one E, with two points y_1 and y_2 on E that are joined in K by a (possibly empty) rational chain. We see as above that either a 3-tree is attached to a point $x \in E$ satisfying $2x \sim y_1 + y_2$ in $\mathrm{Pic}(E)$, or two 2-trees are attached to $x_1, x_2 \in E$ satisfying $x_1 + x_2 \sim y_1 + y_2$ in $\mathrm{Pic}(E)$, or the rational chain is not empty and all the distinguished trees are attached to it. In this case, solve the balancing equation on E: let $d = d_E$, d_1 and d_2 be the multiplicities of the rational components attached to y_1 and y_2 respectively; then either $d_1 = d_2 = d - 1$, or $d_1 = d - 1 + k$, $d_2 = d - 1 - k$ and $r_1 - r_2$ is k-torsion in $\mathrm{Pic}(E)$. But, by chasing the balancing equation along the rational necklace, we find that, if $k \geq 1$, then the trend increases when passing through a distinguished bead,

so that ultimately $d - 1 - k = d_2 > d_1 = d - 1 + k$, contradiction. So the only possibility is to have a rational chain symmetric with respect to the distinguished beads, namely: in case (a) the two pieces of the rational chain standing between the special bead and E have the same length, and in case (b) the distance of the shortest path between a special bead and E is the same for the two special beads. See Figure 5.
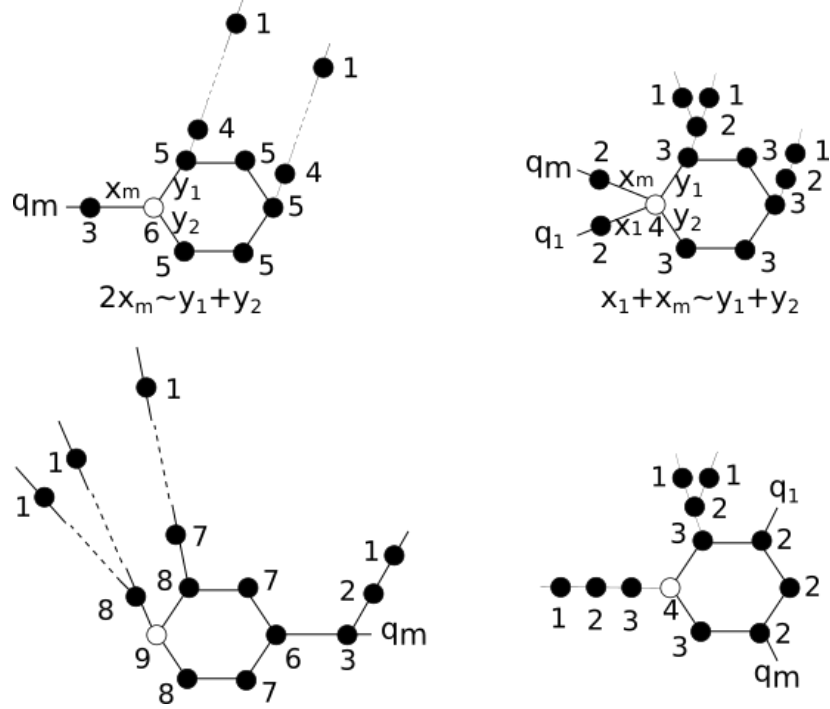

FIGURE 5. K is a genus one curve E with a rational bridge R. Left column, one 3-tree; right column, two 2-trees. Above: the special trees connect to E; below: they connect to R - note that R is necessarily symmetric in this case.

(4) Finally, we consider the case that K has geometric genus 0, namely it is the union of two stable components along three semistable chains. It is easy to see that, if a special tree cleaves to a stable component, the only chance that the balancing condition may be satisfied is that there are two 2-trees and they cleave to different stable components; the semistable chains have arbitrary length and D has the same multiplicity along every component of the core; see the upper left corner of Figure 6. There remain three possibilities for the dual graph, according to how the distinguished components (denoted by R) and the other stable components (denoted by A and B) distribute themselves. Denoting by t the trend along various rational chains, we find that in case (a) and (b).2 balancing along A or B is equivalent to $\sum_i t_i = 1$. Assume $t_1 \geq 0$;
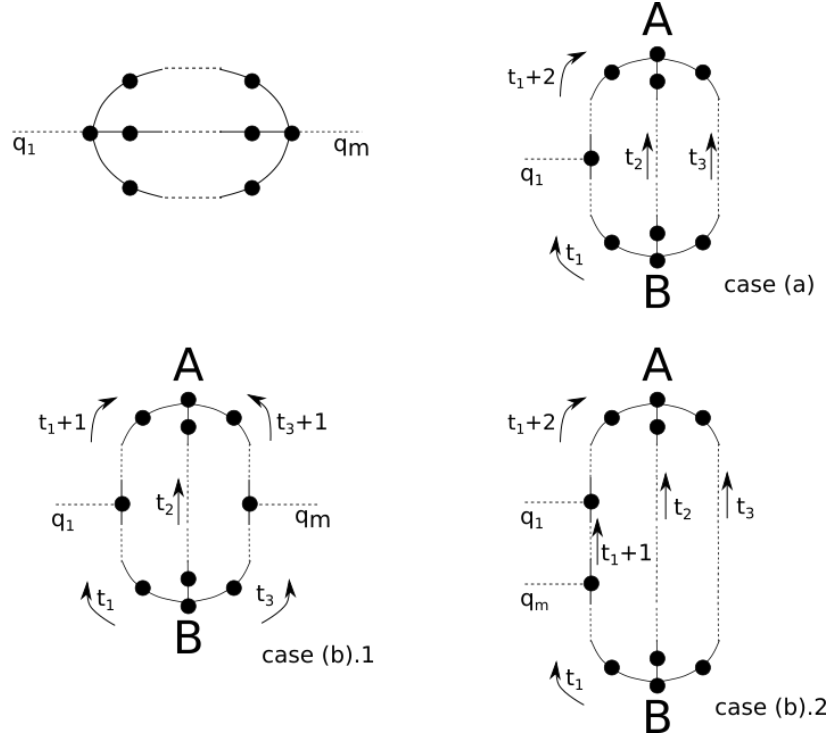
FIGURE 6. When K consists of rational curves.

then $d_A > d_B$, therefore $t_2, t_3 > 0$, which contradicts $\sum_i t_i = 1$. Similarly, if $t_1 \leq -2$, then $d_A < d_B$, therefore $t_2, t_3 < 0$, which makes $\sum_i t_i = 1$ again impossible. We find only one solution with $t_1 = -1$ and $t_2 = t_3 = 0$ - notice that it is a degeneration of case 3 above. In case (b).1, we find $\sum_i t_i = 1$ when balancing B, and $\sum_i t_i = -1$ when balancing A, which is a contradiction.

This concludes the proof of Propositions 4.2(2) and 4.3(2). □

Remark 4.12. There is a stable 2-pointed curve that arises as a solution of the balancing equation, yet is not conjugate, namely:
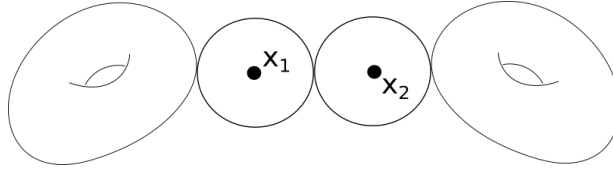
In this case, the line bundle $\omega_{\mathscr{C}/\Delta}(D)$ trivial along Z might not be semiample.

Proof. (of Proposition 4.4) By blowing down some rational tails outside Z, we can assume that $\mathscr{C}_0 \setminus Z = \sqcup_{i=1}^m T_i$ with each $T_i \simeq \mathbb{P}^1$. The image of $p_i(0)$ and $p_j(0)$ may now coincide for $i \neq j$. The total space of the curve can still be assumed to be smooth. By abuse of notation, we denote the resulting family of pointed

curves by $(\mathscr{C}, p_1, \ldots, p_n)$. By assumption on the shape of Z, we can find an effective Cartier D supported on Z such that $\mathscr{L} = \omega_{\mathscr{C}/\Delta}(D + \sum p_i)$ is trivial on Z and relatively ample elsewhere (both on T_i and on the generic fiber). Consider a second line bundle $\mathscr{L}' = \mathcal{O}(2p_1 + \sum p_i)$. Since we assumed p_1 to be Weierstrass, $\mathscr{L}_\eta \simeq \mathscr{L}'_\eta$. On the other hand it is easy to see that the multi-degrees of $\mathscr{L}_0$ and $\mathscr{L}'_0$ coincide; it follows from the separatedness of $\mathrm{Pic}^0_{\mathscr{C}/\Delta} \to \Delta$ (see [Del85, p. 136] or [BLR90, §9.4]) that $\mathscr{L}$ and $\mathscr{L}'$ are isomorphic line bundles, so that in particular $\mathscr{L}$ is trivial *on a neighbourhood of* Z. Observe now that

$$R^1 \pi_* \mathscr{L}(-D) = R^1 \pi_* \omega_{\mathscr{C}/\Delta}(\sum p_i) = 0$$

by semistability, hence $\pi_* \mathscr{L} \twoheadrightarrow \pi_*(\mathscr{L}_{|D}) = \pi_* \mathcal{O}_D$, which contains the constants, showing that $\mathscr{L}$ is semiample along Z (that it is along the T_i is easier). We therefore have a well-defined morphism:

$$\mathscr{C} \xrightarrow{\phi} \overline{\mathscr{C}} = \underline{\mathrm{Proj}}_\Delta \left(\bigoplus_{n \geq 0} \pi_* \mathscr{L}^{\otimes n} \right) \to \Delta$$

associated to $\mathscr{L}$. The proof that $\overline{\mathscr{C}} \to \Delta$ is a flat family of Gorenstein curves goes along the lines of [Smy11a, Lemma 2.13] or [RSW17a, Proposition 3.7.3.1]. It is the clear from the classification that it contains a type I_m-singularity in the central fiber. The proof of Proposition 4.5 is entirely analogous. $\square$

Caveat. Here is a natural question: given a family of semistable curves $\pi \colon \mathscr{C} \to \Delta$ over a discrete valuation ring, and a line bundle $\mathscr{L}$ on $\mathscr{C}$ that is trivial on a higher genus subcurve Z of $\mathscr{C}_0$ and π-ample elsewhere, is $\mathscr{L}$ π-semiample? is it relatively generated by global sections? A positive answer follows from a result of S. Keel in positive characteristic [Kee99]. If $\mathrm{char}(\mathbf{k}) = 0$, the answer seems to depend on the family; in particular, the assumption that the total space of $\mathscr{C}$ is regular along Z seems to be relevant. We construct a counterexample using the theory of limit linear series (l.l.s.): we produce a linear series that can be smoothed while having basepoints along a Weierstrass tail.[2] For any such smoothing, the corresponding line bundle $\mathscr{L}$ is not globally generated along Z - though we do not know how the powers of $\mathscr{L}$ behave.

Let X_0 be the nodal curve obtained by attaching $R \simeq \mathbb{P}^1$ to a Weierstrass point q of a smooth genus two curve Z. Choose $d \gg 0$ ($d \geq 5$ is enough), and let us study the moduli space of complete linear systems of degree d on (smoothings of) X_0; with $r = d - 2$, the Brill-Noether number is $\rho = 2$ (the dimension of the Jacobian). On the other hand, assume that the R-aspect of the l.l.s. has $\mathscr{L}_{Y|Z} \simeq \mathcal{O}_Z$; then the Z-aspect has $\mathscr{L}_{Z|Z} \simeq \mathcal{O}_Z(dq)$, whose vanishing sequence is $\alpha_Z(q) = \{0, 1, \ldots, d-4, d-2, d\}$, from which we deduce for the complementary aspect $\alpha_R(q) \geq \{0, 2, 4, 5, \ldots, d\}$. We want to show that all such aspects are

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[2]It was pointed out by F. Carocci that a similar computation can be carried out for a genus one tail as well. This shows that the regularity assumption of [Smy11a, Lemma 2.13] is necessary.

smoothable, by appealing to the Regeneration Theorem [HM98, Theorem 5.41]. Notice that in the case at hand we have a choice of a two-dimensional subspace of $\langle 1, t, t^2, t^3 \rangle_{\mathbf{k}} \subseteq H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(d))$ meeting the subspace $\langle t^2, t^3 \rangle_{\mathbf{k}}$ non-trivially, i.e. the locus in $\mathrm{Gr}(1, \mathbb{P}^3)$ of lines meeting a fixed line ℓ, which is a Schubert cycle of dimension 3. We therefore need to put X_0 in a family over a base B of dimension 1 at least. We shall do so by considering the family X obtained by attaching R to a moving point of Z, so that X_0 is the fiber of X over $q \in Z$.

Let us start by examining the other possibilities for $\mathscr{G}_d^{d-2}(X_0)$: the R-aspect can in fact restrict to any line bundle of degree 0 on Z, which we are going to write as $\mathscr{O}_Z(p_1 + p_2 - 2q)$ for two moving points p_1, p_2 on Z (think of them as coordinates on $\mathrm{Pic}(Z)$). Then $\mathscr{L}_Z = \mathscr{O}_Z((d-2)q + p_1 + p_2)$.

$\subseteq \mathrm{Pic}(Z)$	dim	$\alpha_{Z;d-3,d-2}$	$\alpha_{R;0,1}$	$\subseteq \mathbb{P}H^0(R, \mathscr{O}_R(d))$	dim
$p_1 + p_2 \sim 2q$	0	$\{d-2, d\}$	$\geq \{0,2\}$	$\{\ell' \in \mathrm{Gr}(1,\mathbb{P}^3) \mid \ell' \cap \ell \neq\}$	3
$2q \nsim p_1 + p_2 \geq q$	1	$\{d-3, d-1\}$	$\geq \{1,3\}$	$\mathbb{P}^1$	1
$p_1 + p_2 \ngeq q$	2	$\{d-3, d-2\}$	$\{2,3\}$	pt	0

If we now let q vary in $B \simeq Z$, we may generically assume that it is not Weierstrass. We then find the following:

$\subseteq \mathrm{Pic}(X)$	dim	$\alpha_{Z;d-3,d-2}$	$\alpha_{R;0,1}$	$\subseteq \mathbb{P}H^0(R, \mathscr{O}_R(d))$	dim
$p_1 + p_2 \sim \omega_Z$	$0+1$	$\{d-2, d-1\}$	$\geq \{1,2\}$	$(\mathbb{P}^2)^*$	2
$p_1 + p_2 \sim 2q$	$0+1$	$\{d-3, d\}$	$\geq \{0,3\}$	$\mathbb{P}^2$	2
$\omega_Z, 2q \nsim p_1 + p_2 \geq q$	$1+1$	$\{d-3, d-1\}$	$\geq \{1,3\}$	$\mathbb{P}^1$	1
$p_1 + p_2 \ngeq q$	$2+1$	$\{d-3, d-2\}$	$\{2,3\}$	pt	0

We conclude that $\mathscr{G}_d^{d-2}(X/B)$ has pure dimension 3, and we may therefore apply the Regeneration Theorem to deduce that all l.l.s. with $\mathscr{L}_{R|Z} \simeq \mathscr{O}_Z$ - in particular those with base-locus Z - are smoothable.

5. The new moduli functors

The following generalises [Smy11a, Definition 3.4].

Definition 5.1. Let $(C, p_1, \ldots, p_n)$ be a reduced curve, marked by smooth points. For a nodally attached subcurve $D \subseteq C$, we define its *level* as

$$\mathrm{lev}(D) = |D \cap \overline{C \setminus D}| + |\{p_1, \ldots, p_n\} \cap D|.$$

We say a Gorenstein curve C is *minimal* if it contains no node x such that the normalisation of C at x consists of two connected components, one of which has genus zero. When C has arithmetic genus one, this is the same as saying that C contains no separating nodes. Recall [Smy11a, Lemma 3.3].

Lemma 5.2. *A minimal Gorenstein curve Z of arithmetic genus one can be: a smooth elliptic curve; a ring of $r \geq 1$ copies of $\mathbb{P}^1$; or an elliptic m-fold point whose normalisation is the disjoint union of m copies of $\mathbb{P}^1$. In any case $\omega_Z \simeq \mathscr{O}_Z$.*

We may similarly describe minimal (sub)curves of genus two.

Lemma 5.3. *A minimal Gorenstein curve Z of arithmetic genus two can be:*

(1) a smooth curve of genus two;
(2) the union of two minimal Gorenstein curves of genus one, E_1 and E_2, nodally separated by a (possibly empty) rational chain R;
(3) the union of a minimal Gorenstein curve of genus one E, and a (possibly empty) rational chain R, along two distinct nodes;
(4) the union of two copies of $(\mathbb{P}^1, 0, 1, \infty)$ with three (possibly empty) rational chains R_0, R_1, R_∞ joining the homonymous points;
(5) an elliptic m-fold point x whose normalisation is the disjoint union of either $m-1$ copies of $\mathbb{P}^1$ (i.e. two branches of x coincide), or $m-1$ copies of $\mathbb{P}^1$ and a minimal Gorenstein curve of genus one (i.e. Z contains two genus one subcurves sharing a branch);
(6) or a singularity of genus two with m-branches, whose normalisation is the disjoint union of m copies of $\mathbb{P}^1$.

Remark 5.4. In both cases (5) and (6) there are special branches supporting the degree of ω_Z (compare with Definition 2.3 and Corollary 4.7; recall that the restriction of the dualising sheaf to a component introduces a twist by the conductor ideal, see Noether's formula [Cat82, Proposition 1.2]). Notice that the notion of conjugate points is not always intrinsic to the curve.

Similarly to [Smy11a, Corollary 3.2, Lemma 3.5] we can prove the following.

Lemma 5.5. Let $(C, p_1, \ldots, p_n)$ be a pointed semistable curve of arithmetic genus two, with minimal genus two subcurve Z. For every subcurve $Z' \subseteq C$ of genus two, we have an inclusion $Z \subseteq Z'$ and $\mathrm{lev}(Z) \leq \mathrm{lev}(Z')$.

Definition 5.6. We say that a point *cleaves* to a component of a curve if it is connected to it by a possibly empty chain of rational curves.

We finally come to the definition of m-stability for curves of genus two.

Definition 5.7. Fix positive integers $1 \leq m < n$. Let $(C, p_1, \ldots, p_n)$ be a connected, reduced, complete curve of arithmetic genus two, marked by smooth distinct points. We say that C is m-stable if:
(1) C is Gorenstein with only: nodes; elliptic l-fold points, $l \leq m+1$; type $I_{\leq m}$, type $II_{\leq m}$, and dangling II_{m+1} singularities of genus two, as singular points.
(2) If Z is a connected subcurve of arithmetic genus two, then $\mathrm{lev}(Z) > m$.
(3) If E is a nodally attached subcurve of arithmetic genus one, $\mathrm{lev}(E) > m+1$.
(4) $H^0(C, \Omega_C^\vee(-\sum_{i=1}^n p_i)) = 0$.
(5) If C contains a singularity of genus two, or an elliptic l-fold with a self-branch or a genus one branch, p_1 cleaves to one of the special branches (see Remark 5.4). If C contains two genus one subcurves sharing a branch, and E_1 has level less than $m+2$, then p_1 cleaves to E_1.

Remark 5.8. The definition is not $\mathfrak{S}_n$-symmetric. In the argument below, we exploit the asymmetry to write the dualising line bundle of a genus two (sub)curve Z as $\omega_Z \simeq \mathscr{O}_Z(q_1 + \bar{q}_1)$, where q_1 is the point of Z closest to p_1, and $\bar{q}_1$ its conjugate,

possibly depending on a one-parameter smoothing. Compare with the situation in genus one, where the dualising line bundle of a minimal Gorenstein curve is trivial (all smooth points are non-special). We also use it to decide which genus one subcurve to contract first in case there are two of the same level.

Remark 5.9. If there is a subcurve of genus one, condition (3) and condition (4) jointly imply condition (2). Indeed, from Corollary 3.6 we see that $\mathrm{lev}(Z) \geq \mathrm{lev}(E) - 1$. The only cases (up to relabelling) in which the level drops by one are: when $Z = (E, p_1, \ldots, p_{l-2}, q_1, q_2) \sqcup_{\{q_1, q_2\}} (\mathbb{P}^1, q_1, q_2, p_{l-1})$; and when $Z = (E_1, p_1, \ldots, p_{l-1}, q) \sqcup_q (E_2, q)$, where all the E have genus one.

Lemma 5.10 (boundedness). *If $(C, p_1, \ldots, p_n)$ is an m-stable curve of genus two, the N-th power of $A = \omega_C(\sum_{i=1}^n p_i)$ is very ample for every $N > 2 + 8(m+1)$.*

Proof. It is enough to show that, for every pair of points $p, q \in C$ (possibly equal):

(1) *basepoint-freeness*: $H^1(C, A^{\otimes N} \otimes I_p) = 0$;
(2) *separating points and tangent vectors*: $H^1(C, A^{\otimes N} \otimes I_p I_q) = 0$.

By Serre duality we may equivalently show that $H^0(C, \omega_C \otimes A^{-N} \otimes (I_p I_q)^\vee) = 0$. Let $\nu \colon \tilde{C} \to C$ be the normalisation, and let $\nu^{-1}(p) = \{p_1, \ldots, p_h\}$, $\nu^{-1}(q) = \{q_1, \ldots, q_k\}$, with $h, k \leq m + 1$. It follows from Proposition 2.2 (and [Smy11a, Proposition A.3]) that $\nu_* \mathcal{O}_{\tilde{C}}(-D) \subseteq I_p I_q$ for $D = 4(\sum_{i=1}^h p_i + \sum_{j=1}^k q_j)$ (note that $\deg(D) \leq 8(m+1)$); furthermore, the quotient is torsion, therefore, by applying $\mathcal{H}om(-, \mathcal{O}_C)$ and adjunction, we find $(I_p I_q)^\vee \subseteq \nu_* \mathcal{O}_{\tilde{C}}(D)$. It is thus enough to show that $H^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(D) \otimes \nu^*(\omega_C \otimes A^{-N})) = 0$. Finally, $\nu^* \omega_C$ has degree at most two, and $\nu^* A$ has degree at least one on any branch of $\tilde{C}$, hence it is enough to take $N > 2 + 8(m+1)$. $\qquad\square$

Lemma 5.11 (deformation openness). *Let $(\mathcal{C}, \sigma_1, \ldots, \sigma_n) \to S$ be a family of curves over a Noetherian base scheme with n sections. The locus*

$$\{s \in S | (\mathcal{C}_{\bar{s}}, \sigma_1(\bar{s}), \ldots, \sigma_n(\bar{s})) \text{ is } m\text{-stable}\}$$

is Zariski-open in S.

Proof. Having connected fibers which are Gorenstein curves of arithmetic genus two is an open condition (see for example [Sta19, Tag 0E1M]). Only singularities of genus zero (nodes), one (elliptic l-folds), and two may then occur.

The case $m = 1$ deserves special attention. In this case, that condition (1) is open follows from acknowledging that $I_1 = A_4$, $II_2 = A_5$, while tacnodes, cusps, and nodes are A_3, A_2, and A_1-singularities respectively, and from Grothendieck's results on the deformation theory of ADE singularities (see Theorem 2.5 above).

The case $m \geq 2$ simply follows from upper semicontinuity of embedded dimension and the fact that we have exhausted all possible Gorenstein singularities of genus ≤ 2, and embedding dimension $\leq m + 1$.

Condition (4) translates to: the locus where the automorphism group is unramified is open in the base. Homogeneity can be used to prove that being unramified,

which is open in the source, is also open in the target, for the structural morphism of a group scheme; see the end of the proof of [Smy11a, Lemma 3.10].

The other conditions are topological, hence constructible. With Noetherian assumptions, it is enough to check their openness over the spectrum of a discrete valuation ring. Assume that the geometric generic fiber $C_{\bar{\eta}}$ contains two genus one subcurve $E_{1,\bar{\eta}}$ and $E_{2,\bar{\eta}}$; their closures E_1 and E_2 in $\mathscr{C}$ are then flat families of genus one curves over Δ. If $E_{1,\bar{\eta}}$ and $E_{2,\bar{\eta}}$ are disconnected, then so are E_1 and E_2, by local constancy of the number of connected components (from the Zariski decomposition and [Sta19, Tag 0E0D]). If $E_{1,\bar{\eta}}$ and $E_{2,\bar{\eta}}$ are joined by a disconnecting node $q_{\bar{\eta}}$, then so are $E_{1,0}$ and $E_{2,0}$; indeed, the unique limit of $q_{\bar{\eta}}$ must be a singular point of the projection, but cannot be any worse than a node (because we have already exhausted all the available genus). Finally, if $E_{1,\bar{\eta}}$ and $E_{2,\bar{\eta}}$ share a branch, then so do $E_{1,0}$ and $E_{2,0}$; on the other hand, if $E_{i,\bar{\eta}}$ has more than one branch, then so does E_i. Similarly, if $C_{\bar{\eta}}$ contains only one subcurve of genus one, with two nodes joined by a rational chain, so does C_0. The upshot of this discussion is that

$$|E_{i,\bar{\eta}} \cap \overline{C_{\bar{\eta}} \setminus E_{i,\bar{\eta}}}| = |E_{i,0} \cap \overline{C_0 \setminus E_{i,0}}|.$$

The number of markings on E_i is also constant. Hence we can deduce condition (3) for $C_{\bar{\eta}}$ from the same condition on C_0. Condition (2) follows in this case from Remark 5.9; it can be proved analogously when there is no subcurve of genus one.

Finally, suppose that $C_{\bar{\eta}}$ has a genus two singularity, then so does C_0. The (union of the) distinguished branch(es) $E_{\bar{\eta}}$ of $C_{\bar{\eta}}$ is a genus one singularity, and so is its limit E_0 in C_0. It has to contain the distinguished branch(es) of C_0, because any subcurve not containing them has genus zero; therefore, by assumption, E_0 contains $p_{1,0}$. Then also $E_{\bar{\eta}}$ contains $p_{1,\bar{\eta}}$, because the markings are contained in the non-singular locus of the curve. Similarly, $C_{\bar{\eta}}$ has a genus one singularity with a self-branch, the limit of such a branch is a genus one subcurve E_0 of C_0; the latter may very well acquire a genus two singularity, but E_0 will contain the special branches of it, so it will be connected to p_1. We conclude as above. The case that $C_{\bar{\eta}}$ contains a genus one subcurve of low level is analogous. We have thus proved that condition 5 is open. $\qquad\square$

Definition 5.12. We shall denote by $\overline{\mathcal{M}}_{2,n}^{(m)}$ the moduli stack of n-pointed m-stable curves of genus two.

It follows from the previous lemmas and standard arguments that $\overline{\mathcal{M}}_{2,n}^{(m)}$ is represented by a Deligne-Mumford stack of finite type.

Proposition 5.13 (Valuative criterion of properness for $\overline{\mathcal{M}}_{2,n}^{(m)}$). *Given a smooth n-pointed curve of genus two C_η over a discrete valuation field $\eta = \operatorname{Spec}(K) \hookrightarrow \Delta$, there exists a finite base-change $\Delta' \to \Delta$ after which C_η can be completed to an m-stable curve over Δ'. Two such models are always dominated by a third one.*

Proof. Existence of limits. We start with a smooth n-pointed curve of genus two over a discrete valuation field. By the semistable reduction theorem [DM69, Corollary 2.7], we may find a finite base-change $\Delta' \to \Delta$ and a semistable curve $\mathscr{C}' \to \Delta'$ with regular total space, such that its generic fiber is isomorphic to the pullback of the curve we started with. By Castelnuovo's criterion, we may further assume that the central fiber contains no rational tails.

We check whether p_1 is Weierstrass or not: in the former case, change base with $\pi'' \mapsto (\pi')^3$ and resolve; in the latter, mark $\mathscr{C}'$ with an extra section $\bar{p}_1$ given by the closure of the conjugate point $\bar{p}_1(\eta)$ (unless it coincides with one of $p_1, \ldots, p_n$; if it coincides with p_1, we have a Weierstrass point indeed), then change base with $\pi'' \mapsto (\pi')^2$ and resolve. We drop the primes from notation. $\mathscr{C}_0$ is now marked with a(n extra) smooth point $\bar{p}_1$. The base-change is a technical expedient we find useful in the forthcoming construction.

We claim there is a unique genus two subcurve Z of $\mathscr{C}_0$ that satisfies the shape requirements of Proposition 4.2 (resp. 4.3) - or consists of two disjoint balanced subcurves of genus one - guaranteeing the curve we obtain by contracting Z has bounded above singularities and bounded below level as in conditions (1),(2), and (3) of Definition 5.7.

We think of this process as drawing a family of expanding circles on the dual graph (except, they are neither exactly circles, nor always expanding). Note that we may at any point blow-up the curve at a marking on the central fiber, and consider the strict transform of the corresponding section; thus markings can effectively be considered as infinite legs in the dual graph. We are going to contract the strict interior of the circle; note that the number of branches of the resulting singularity is determined by the inner valence of the circle, and the level by the outer valence.

For simplicity, we start by examining the case that the core of $\mathscr{C}_0$ is irreducible, and p_1 is Weierstrass. Step 0: if the core K has level $\geq m+1$, then the curve is already m-stable. Otherwise, draw a first circle comprising K, and reaching every second closest rational component along any rational tree attached to K, except for the tree containing p_1. Note that the inner valence is exactly $\mathrm{lev}(K)$, thanks to the base-change we have performed earlier, and in particular it is no larger than m; on the other hand, semistability implies that the outer valence is non-decreasing - it is important to consider markings as infinite legs for this -: if it is $\geq m+1$ we stop, otherwise we repeat the process. Calling K_1 the union of the components strictly inside the first circle, $\mathscr{C}_0$ is the union of K_1 and a number of rational trees; if the outer valence at the first step is still $\leq m$, we enlarge the circle by reaching one step further along the rational tree containing p_1, and three steps along all the other ones. Because a circle of very large radius has both inner and outer valence equal to $n > m$, by increasing the radius step by step we will eventually reach level $m+1$ or higher. If we stop at the l-th step, the line bundle

we will use to perform the contraction is

$$\mathscr{L} = \omega_{\mathscr{C}/\Delta}(3lK + \sum_{R\in[p_1,K]} [3l - 3\operatorname{dist}(R,K)]_+ R + \sum_{R\notin T_1} [3l - \operatorname{dist}(R,K)]_+ R +$$

$$\sum_{R\in T_1, R\notin[p_1,K]} [3l - 3\operatorname{dist}(T_1 \wedge T_R, K) - \operatorname{dist}(R, T_1 \wedge T_R)]_+ R + \sum_{i=1}^{n} p_i)$$

where T_1 is the rational tree (connected component of $\mathscr{C}_0 \setminus K$) containing p_1, T_R the one containing R, $T_1 \wedge T_R$ their common component furthest from the core, dist is the distance on the dual graph, and $[k]_+ = \max\{0, k\}$ for any integer k. If $\mathscr{L}$ differs from $\omega_{\mathscr{C}/\Delta}^{\log}$ by a vertical divisor whose support is Z, the shape prescription being satisfied by construction, it follows from Proposition 4.4 that $\mathscr{L}$ is π-semiample, and the contraction it yields contains a singularity of type I with p_1 cleaving to the special branch. Note that $\mathscr{L}$ contracts as well the semistable rational components that are disjoint from Z, hence the resulting curve has no (infinitesimal) automorphisms. The level condition is satisfied by construction, therefore $\overline{\mathscr{C}}_0$ is m-stable.

The case that the core is irreducible and p_1 is not Weierstrass is dealt with in a similar fashion. Remember that in this case we have constructed a conjugate section $\bar{p}_1$; this is an auxiliary marking that will be forgotten in the end, and should not be taken into account when computing the level. At every step we draw a larger circle by including one more component along the trees containing p_1 and $\bar{p}_1$, and two more along every other tree; the inner valence of the new circle is the same as the outer valence of the old one, thanks to the $2:1$ base-change. At the l-th step we are going to use the line bundle

$$\mathscr{L} = \omega_{\mathscr{C}/\Delta}(2lK + \sum_{\substack{R\in[p_1,K]\\ \text{or } [\bar{p}_1,K]}} [2l - 2\operatorname{dist}(R,K)]_+ R + \sum_{R\notin T_1, \bar{T}_1} [2l - \operatorname{dist}(R,K)]_+ R +$$

$$\sum_{\substack{R\in T_1, R\notin[p_1,K] \text{ or}\\ R\in \bar{T}_1, R\notin[\bar{p}_1,K]}} [2l - 2\operatorname{dist}(T_1 \wedge T_R, K) - \operatorname{dist}(R, T_1 \wedge T_R)]_+ R + \sum_{i=1}^{n} p_i + \bar{p}_1)$$

to contract the strict interior of the circle. It follows from Proposition 4.5 that $\mathscr{L}$ is π-semiample, and the associated contraction contains a singularity of type II with p_1 cleaving to one of the twin branches. Note that a II_{m+1} singularity will occur only if the level is m and $\bar{p}_1$ does not coincide with any other special point, so that one of the twin branches remains dangling after forgetting $\bar{p}_1$. Again, the stability condition is satisfied by construction.

Suppose next that the minimal subcurve of genus two Z contains two subcurves of genus one; call E_1 and E_2 the minimal such, and assume that p_1 cleaves to E_1 in a point that is 2-torsion with respect to the node separating E_1 from E_2. We start by drawing expanding circles around E_2 until the level condition for it is
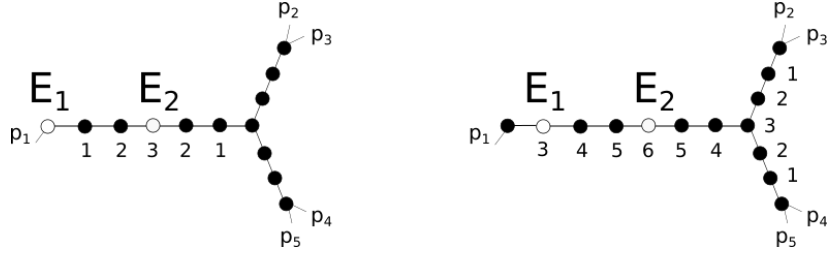
FIGURE 7. An instance of the critical step, when we pass from
contracting the two genus one subcurves separately, to contracting
the genus two subcurve as a whole. On the left, the contraction
produces a tacnode with a genus one branch, having $\mathrm{lev}_2 = 3$. On
the right, the contraction produces a singularity of type I_3. Note
that in the meantime we performed a blow-up at p_1.

satisfied, and then we do the same for E_1, so that (5) holds; observe, though, that
as soon as the two circles touch, the contraction will not have two distinguished
genus one subcurves anymore, therefore we need to check (2) rather than (3).

- If $\mathrm{lev}(E_2) \geq m + 2$ is attained before the circle around E_2 gets to touch E_1,
  take the next $l_2 \equiv 2 \pmod 3$ (thus "undoing" the $3 : 1$ base-change, which was
  unnecessary in this case), then contract the inner disc by the line bundle

$$\mathscr{L}_2 = \omega_{\mathscr{C}/\Delta}((l_2 + 1)E_2 + \sum [l_2 + 1 - \mathrm{dist}(E_2, R)]_+ R + \sum_{i=1}^{n} p_i).$$

  Smyth's contraction lemma [Smy11a, Lemma 2.13] applies, so that E_2 is con-
  tracted to an elliptic l-fold point q_2 ($l \leq m + 1$). Consider now E_1. If
  $\mathrm{lev}(E_1) \leq m+1$, start drawing expanding circles around it. Either level $\geq m+2$
  can be reached before touching the singularity at q_2, or, by contracting the max-
  imal balanced subcurve of genus one containing E_1 and not q_2, we produce a
  curve having two genus one singularities that share a branch. Note that p_1
  cleaves to the only genus one subcurve that may have level $\leq m + 1$.
- Otherwise, one step before including E_1, we may contract the disc around E_2
  to yield a genus one singularity with a genus one branch. If $\mathrm{lev}_2 \leq m$ at this
  point, we need to contract the whole genus two subcurve. What happens at
  this critical step is that the multiplicity of D along E_1 grows from 0 to 3, hence
  D will be supported three steps further along each rational tail departing from
  E_2. Because the critical step is $\equiv 1 \pmod 3$, the length of the rational tails
  is and remains $\equiv 2 \pmod 3$, hence we perform only one "meaningful" step
  forward (i.e. on the original central fiber, before changing base), thanks to the
  preliminary base-change. See Figure 7. In particular, the inner valence of the
  disc will be $\leq m$. We may now proceed as in the irreducible case, expanding
  the circle at every step by 1 along T_1 and by 3 along all other rational tails.

The case that the central fiber contains two subcurves of genus one, and p_1 cleaves to a non-Weierstrass point, is analogous: it is enough to replace the number 3 by the number 2, and 2 by 1, in the previous argument. The only novelty is, it can happen that p_1 is equidistant from E_1 and E_2, cleaving to the rational chain joining them. In this case we start by expanding a circle around the one with the lowest level; if they have the same level, expand them simultaneously. If at a later stage p_1 becomes closer to one of the two circles, proceed as above.

When the core consists of an elliptic curve E with a rational bridge R, and p_1 cleaves to a Weierstrass point - either on E or on R -, we have noticed above (see the end of Remark 4.1), that R comprises an odd number $k = 2h + 1$ of rational components. We start as above by enlarging a balanced circle around E in order to establish the level condition (3); once again there is a critical step (after which we contract a genus two subcurve, and check (2) instead) when the circle touches itself along R, and we observe that this happens after the $(3h+2)$-th step, therefore extending the circle by 1 along the tail containing p_1, and by 3 on all the other ones, only makes one meaningful step. The case of a non-Weierstrass point on the elliptic bridge is analogous.

Finally, the case that the central fiber has geometric genus zero is dealt with as if the core was irreducible, since we do not introduce any genus zero singularity other than the node (the remaining ones are not Gorenstein).

Uniqueness of limits. By the semistable reduction theorem, there is a diagram

$$
\begin{array}{ccc}
 & \mathcal{C}^{ss} & \\
 \phi \swarrow & & \searrow \phi' \\
 \mathcal{C} & & \mathcal{C}' \\
 \searrow & & \swarrow \\
 & \Delta &
\end{array}
$$

extending the isomorphism between the generic fibers, where $\mathcal{C}^{ss}$ has semistable central fiber and regular total space.

Claim 1: If $\mathcal{C}'_0$ has only singularities of genus $\leq i$ $(i = 0, 1)$, then so does $\mathcal{C}_0$. First, assume that $\mathcal{C}'_0$ has only nodes. If $\mathcal{C}_0$ has a singular point x of genus one, $E := \phi^{-1}(x)$ is an *unmarked* subcurve of arithmetic genus one and level $\leq m + 1$ of $\mathcal{C}^{ss}_0$. Then so is $\phi'(E)$: indeed, ϕ' being a contraction, it has connected fibers, which excludes the possibility that ϕ' lowers the genus of E by realising a finite cover of a line. This contradicts the m-stability of $\mathcal{C}'$. We may argue similarly if x is a genus two singularity with $\leq m$ branches. On the other hand, if x is dangling II_{m+1}, there is a -1-curve R adjacent to $\phi^{-1}(x)$; ϕ' must contract R by DM stability of $\mathcal{C}'$, hence $\phi'(\phi^{-1}(x))$ is again a genus two curve of level $\leq m$.

Assume now that $\mathcal{C}'_0$ has at worst singularities of genus one, while $\mathcal{C}_0$ has a singularity x of genus two; the case of a dangling II_{m+1} can be excluded as above. Then $\mathcal{C}^{ss}_0 = Z \cup R_1 \cup \ldots \cup R_l$, with $Z = \phi^{-1}(x)$ and $l \leq m$. If Z has geometric

genus two, or is irreducible of geometric genus one, $\phi'(Z)$ violates the m-stability of $\mathscr{C}'$. If Z contains a unique subcurve E of genus one, with a rational bridge R, then at least one of $R_1, \ldots, R_l$ must be connected to R, for otherwise $\mathscr{C}'_0$ - which is obtained by contracting a balanced subcurve around E, not including the entire R - would have a positive dimensional automorphism group (scaling a semistable component of R). Therefore $\mathrm{lev}(E) \leq (l-1)+2 \leq m+1$. Similarly, if Z contains two subcurves of genus one E_1 and E_2, then $(\mathrm{lev}(E_1) - 1) + (\mathrm{lev}(E_2) - 1) \leq l$, hence at least one of the two has level $\leq m + 1$. In any case, $\phi'(E)$ contradicts the m-stability of $\mathscr{C}'$.

Claim 2: We may assume that $\mathscr{C}^{ss}$ contains either no -1-curve, or only one, which is contracted by neither ϕ nor ϕ'.

If there is a -1-curve contracted by both, ϕ and ϕ' factor through a smaller regular model. Assume there is a -1-curve not contracted by ϕ. Then, by condition (4), its image has to be one of the special branches of a genus two singularity; on the other hand, by condition (5), the only special branch of a singularity of type I must contain some special points; therefore we conclude that the singular point x of $\mathscr{C}_0$ is dangling of type II_{l+1}, $l \leq m$. If we let $Z = \phi^{-1}(x)$, we may write $\mathscr{C}_0 = Z \cup R_0 \cup \ldots \cup R_l$, with $R_0 = R$, and R_1 the tail containing p_1. By Claim 1, ϕ' has to contract a genus two subcurve Z' as well. If Z' is of the shape described in Proposition 4.5 and it contains R, then $Z \subsetneq Z'$ is easily seen, which implies $\mathscr{C}'_0$ has a singularity of type II with more than $m+1$ branches, by the level condition (2) on $\mathscr{C}_0$; on the other hand, if $Z' \subsetneq Z$ were disjoint from R, then $\mathscr{C}'_0$ would not satisfy condition (2). Similarly, if Z' is of the shape described in Proposition 4.4, then R_0 and R_1 must meet in a "trunk" T attached to a Weierstrass point of the core of $\mathscr{C}_0^{ss}$, and it can be argued as above that $Z' = Z$ must hold.

Claim 3: The exceptional loci of ϕ and ϕ' coincide.

If $\mathscr{C}_0$ has only nodes, then so does $\mathscr{C}'_0$ by Claim 1, and we can conclude by the uniqueness part of the stable reduction theorem.

If $\mathscr{C}_0$ has a genus one singularity x, it cannot have a genus two singularity as well, so neither can $\mathscr{C}'_0$ by Claim 1. If $\mathscr{C}_0$ has a second genus one singularity y, let $E_1 = \phi^{-1}(x)$ and $E_2 = \phi^{-1}(y)$; they are disjoint balanced subcurves of genus one and level $\leq m + 1$ in $\mathscr{C}_0^{ss}$, therefore ϕ' must contract them. Enlarging the contraction radius of any one of them would yield a singularity with at least $m+2$ branches (by condition (3) on $\mathscr{C}_0$), unless by enlarging we make them touch, in which case we would contract to a genus two singularity; but this is not possible, by Claim 1. The case of a single genus one singularity with a genus one branch, or with a disjoint subcurve of genus one, or with two branches joined by a (possibly empty) rational chain, is similar.

Finally, the case that $\mathscr{C}_0$ has a genus two singularity, has already been discussed at the end of Claim 2. To summarise, writing $\mathscr{C}_0^{ss} = Z \cup R_1 \cup \ldots \cup R_l$, with $Z = \phi^{-1}(x)$ and $l \leq m$ - the case of a dangling II_{m+1} was dealt with before -, $\phi'(Z)$ must be a point x', by stability considerations. Call $Z' = (\phi')^{-1}(x')$ and note that $Z \subseteq Z'$. If x and x' are singularities of the same type, $Z = Z'$ is

easily deduced by level/singularity (i.e. outer/inner valence) considerations, the key point being that the shape of the curve has only one parameter (the "radius" of the circle), which is determined by m-stability. On the other hand, if x were of type II and x' of type I, the two special trees determined by x would have to share a trunk attached to a Weierstrass point of the core, and $Z \subseteq Z'$ would imply $Z \subsetneq Z'$, which together with condition (2) for $\mathscr{C}'$ would make x' into a singularity with too many branches.

The claim follows from observing that the exceptional locus of ϕ (resp. ϕ') is the union of the fibers F over higher genus singularities of $\mathscr{C}_0$ (resp. $\mathscr{C}_0'$), and the rational components with only two special points that are disjoint from F.

Claim: The generic isomorphism between $\mathscr{C}$ and $\mathscr{C}'$ extends over Δ.

Follows from [Deb01, Lemma 1.13]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Summing up, we have proved the following:

Theorem 5.14. *For $1 \le m < n$, m-stability defines a proper Deligne-Mumford stack $\overline{\mathcal{M}}_{2,n}^{(m)}$ over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{30}])$, containing $\mathcal{M}_{2,n}$ as a dense open substack.*

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