

Modular compactifications of $\mathcal{M}_{2,n}$ with Gorenstein curves

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ABSTRACT

We study the geometry of Gorenstein curve singularities of genus two, and of their stable limits. These singularities come in two families, corresponding to either Weierstrass or conjugate points on a semistable tail. For every $1 \leq m < n$, a stability condition - using one of the markings as a reference point, and thus not \mathfrak{S}_n -symmetric - defines proper Deligne-Mumford stacks $\overline{\mathcal{M}}_{2,n}^{(m)}$ with a dense open substack representing smooth curves.

1. Introduction

We construct alternative compactifications of the moduli stack of smooth n -pointed curves of genus two. The boundary of the Deligne-Mumford compactification, consisting of stable nodal curves, is gradually replaced by ever more singular curves, complying with more restrictive combinatorial requirements on the dual graph. For $1 \leq m < n$, we introduce a notion of m -stability, that allows Gorenstein singularities of genus one and two while at the same time demanding that higher genus subcurves contain a minimum number of special points. Our main result concerning the stack of m -stable curves is the following:

THEOREM. $\overline{\mathcal{M}}_{2,n}^{(m)}$ is a proper irreducible Deligne-Mumford stack over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{6}])$.

This paper fits into the framework of alternative compactifications and birational geometry of the moduli space of curves, extending work of D.I. Smyth in genus one, but we expect it to find applications to enumerative geometry as well.

We classify Gorenstein singularities of genus two with any number of branches, and their (semi)stable models, highlighting the relation with Brill-Noether theory, and adopting the language of piecewise-linear functions on tropical curves. The key insight in defining the new stability conditions is that we can avoid non-Gorenstein singularities by modifying the curve at the conjugate point of the special branch; we use one of the markings to select the latter, and, more generally, to identify the m -stable limit in some very symmetric situations in which multiple choices are possible, a priori - as a result, our stability conditions are not \mathfrak{S}_n -symmetric.

2010 Mathematics Subject Classification 14H10 (primary), 14H20 (secondary).

Keywords: moduli of curves, Gorenstein singularities, genus two, crimping spaces

This project grew out of discussions with Francesca Carocci, whom I thank heartily. I am grateful to Daniele Agostini, Fabio Bernasconi, Sebastian Bozlee, Maria Beatrice Pozzetti, Dhruv Ranganathan, Luca Tasin, and Jonathan Wise for helpful conversations. I thank the anonymous referee, whose detailed comments were of great help in improving the exposition of this material. I thank the Max Planck Institute for Mathematics in Bonn for providing financial support and a stimulating research environment. During the revision of this paper, I was supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy EXC-2181/1 - 390900948 (the Heidelberg STRUCTURES Cluster of Excellence).

We interpret crimping spaces (moduli of curves with a prescribed singularity type) as parameter spaces for the differential geometric data needed in order to construct a higher genus singularity from an ordinary m -fold point, and establish a connection with the existence of infinitesimal automorphisms, a phenomenon which had not fully emerged in lower genus.

Though a conspicuous amount of related research has been carried out on the birational geometry of $\overline{\mathcal{M}}_{2,n}$ for low values of n [Has05, HL07, Rul01, HL14, FG18, JP18], this appears to be the first proposal of a sequence of modular compactifications for every n .

1.1 From the Deligne-Mumford space to the Hassett-Keel program

One of the most influential results of modern algebraic geometry is the construction of a modular compactification of the stack of smooth pointed curves $\mathcal{M}_{g,n}$, due to P. Deligne, D. Mumford, and F. Knudsen, with the introduction of *stable* pointed curves.

DEFINITION 1.1. [DM69] A connected, reduced, complete curve C over an algebraically closed field \mathbf{k} , with distinct markings (p_1, \dots, p_n) lying in the smooth locus of C , is *stable* if:

- (i) C admits only nodes (ordinary double points) as singularities;
- (ii) every rational component of C has at least three special points (markings or nodes), and every elliptic component has at least one.

THEOREM 1.2. [DM69, Knu83] Assume $2g - 2 + n > 0$. The moduli stack of stable pointed curves $\overline{\mathcal{M}}_{g,n}$ is a smooth and proper connected Deligne-Mumford stack over $\mathrm{Spec}(\mathbb{Z})$, with projective coarse moduli space $\overline{\mathbf{M}}_{g,n}$, and normal crossing boundary representing nodal curves.

On one hand, the Deligne-Mumford compactification has nearly every desirable property one could hope for; on the other, it is certainly not the unique modular compactification of $\mathcal{M}_{g,n}$. Classifying all of them is a challenging task, which was set out and partially performed in the inspiring work of Smyth [Smy13] (see also [Boz20] for more recent efforts, bringing logarithmic geometry into the picture). The motivation comes mostly from birational geometry.

Even though the existence of $\overline{\mathbf{M}}_{g,n}$ can be deduced from nowadays standard theorems on stacks [KM97], this moduli space was first constructed as a quotient, prompting the development of a powerful technique known as Geometric Invariant Theory [Gie82, MFK94, BS08]. Studying alternative compactifications of $\mathcal{M}_{g,n}$ sheds some light on the Mori chamber decomposition of $\overline{\mathbf{M}}_{g,n}$, and it is not by chance that the first steps in this direction were moved from a GIT perspective - by changing the invariant theory problem or the stability condition under consideration, and analysing the modular properties of the resulting quotients [Sch91, Has05, HH13]. This program, initiated by B. Hassett and S. Keel, aims to describe all the quotients arising in this way, and to determine whether every step of a log minimal model program for $\overline{\mathbf{M}}_{g,n}$ enjoys a modular interpretation in terms of curves with worse than nodal singularities [CTV18, CTV19]. Since the early stages of this program, it has developed into a fascinating playground for implementing ideas that originated from (v)GIT into a general structure theory of Artin stacks [AK16, AFSvdW17, AFS17a, AFS17b]. See for instance [Mor11, FS13] for more detailed and comprehensive accounts.

Only few steps of the Hassett-Keel program have been carried out in full generality. Yet, the program has been completed to a larger extent in low genus: with the introduction of Boggi-stable [Bog99] and weighted pointed curves [Has03] in genus zero, and with Smyth's pioneering work in genus one [Smy11a, Smy11b, Smy18], extending earlier work of D. Schubert. In a nutshell, an

alternative compactification is defined by allowing a reasonably larger class of curve singularities (*local condition*) while identifying their (semi)stable models, and disallowing the latter by imposing a stronger stability condition (*global condition*, typically combinatorial); the valuative criterion ensures that the resulting moduli problem remains separated and universally closed.

A useful notion in this respect is that of the *genus* of an isolated curve singularity: let (C, q) be (the germ of) a reduced curve over an algebraically closed field \mathbf{k} at its unique singular point q , with normalisation $\nu: \tilde{C} \rightarrow C$ and $\mathcal{F} = \nu_* \mathcal{O}_{\tilde{C}} / \mathcal{O}_C$, a skyscraper sheaf supported at q .

DEFINITION 1.3. [Smy11a] If C has m branches (irreducible components of the normalisation) at q , and δ is the \mathbf{k} -dimension of \mathcal{F} , the genus of (C, q) is defined as:

$$g = \delta - m + 1.$$

The genus can be thought of as the number of conditions that a function must satisfy in order to descend from the seminormalisation (the initial object in the category of universal homeomorphisms $C' \rightarrow C$, see [Sta20, Tag 0EUS], or a curve with the same topological space as C and an ordinary m -fold point at q) to C . The node, for example, has genus zero (it coincides with its own seminormalisation). The genus of a singular point represents its non-topological contribution to the arithmetic genus of the curve containing it.

Smyth found that, for every fixed number m of branches, there is a unique germ of Gorenstein singularity of genus one up to isomorphism, namely:

- $m = 1$ the cusp, $V(y^2 - x^3) \subseteq \mathbb{A}_{x,y}^2$;
- $m = 2$ the tacnode, $V(y^2 - yx^2) \subseteq \mathbb{A}_{x,y}^2$;
- $m \geq 3$ the union of m general lines through the origin of \mathbb{A}^{m-1} .

Singularities of this kind, with up to m branches, together with nodes, form a deformation-open class of singularities. Moreover, the elliptic m -fold point can be obtained by contracting a smooth elliptic curve with m rational tails in a one-parameter smoothing, and, roughly speaking, all stable models have a shape similar to this one.

DEFINITION 1.4. [Smy11a] A connected, reduced, complete curve C of arithmetic genus one with smooth distinct markings (p_1, \dots, p_n) is *m -stable*, $1 \leq m < n$, if:

- (i) it admits only nodes and elliptic l -fold points, $l \leq m$, as singularities;
- (ii) for every connected subcurve $E \subseteq C$ of arithmetic genus one, its *level*: $|E \cap \overline{C} \setminus E| + |\{i: p_i \in E\}|$ is strictly larger than m ;
- (iii) $H^0(C, \Omega_C^\vee(-\sum_i p_i)) = 0$ (finiteness of automorphism groups).

The latter can be taken for a decency condition on the moduli stack. The first two, instead, are essential in guaranteeing the uniqueness of m -stable limits, as per the discussion above. Smyth's main result is the following.

THEOREM 1.5. [Smy11a, Smy11b] *The moduli stack of m -stable curves $\overline{\mathcal{M}}_{1,n}(m)$ is a proper irreducible Deligne-Mumford stack over $\mathrm{Spec} \mathbb{Z}[1/6]$. It is not smooth for $m \geq 6$. The coarse moduli spaces $\overline{\mathbf{M}}_{1,n}(m)$ arise as birational models of $\overline{\mathbf{M}}_{1,n}$ for the big line bundles $D(s) = s\lambda + \psi - \Delta$, where λ is the Hodge class, ψ is the sum of the ψ -classes, Δ is a boundary class, and there is an explicit relation between s and m .*

Some further information on the geometry and singularities of these spaces (with the restriction $m = n - 1$) has been discovered by Y. Lekili and A. Polishchuk in their study of *strongly non-special* curves [LP17].

1.2 Experimenting on a genus two tale

In this subsection, we walk through the motivations and methods at the heart of our construction, exemplifying them in the simplest possible case, that of $\overline{\mathcal{M}}_{2,2}^{(1)}$. The facts we mention are either proved or explained in greater detail and generality in the paper. Here is a classical

FACT. *There are two unibranch singularities of genus two, the ramphoid cusp or A_4 -singularity $V(y^2 - x^5) \subseteq \mathbb{A}_{x,y}^2$, and the ordinary genus two cusp $\mathrm{Spec}(\mathbf{k}[t^3, t^4, t^5])$. The former is Gorenstein, with stable model a Weierstrass tail (a genus two curve attached to a rational one at a Weierstrass point), while the latter is not Gorenstein, with stable model a non-Weierstrass tail of genus two.*

See Lemma 2.1 and Proposition 4.8 below. Recall that every smooth curve of genus two is hyperelliptic, i.e. it can be realised as a two-fold cover of \mathbb{P}^1 , in a unique way up to projectivities. The cover automorphism is called the hyperelliptic involution σ ; ramification points (fixed points of σ) are called Weierstrass, and in general $\{p, \sigma(p)\}$ are called conjugate points. See Section 4.1.

Let us try Smyth's approach out on genus two curves, starting with $\overline{\mathcal{M}}_{2,2}^{(1)}$. If we are going to require the level of a genus two subcurve to be at least two, it seems that we will need non-Gorenstein singularities in order to keep our moduli space proper. This might lead us into trouble; for example, the (log) dualising line bundle is classically exploited to construct canonical polarisations on stable curves, which in turn are essential in the proof that $\overline{\mathcal{M}}_{g,n}$ is an algebraic stack (or in the GIT construction of $\overline{\mathbf{M}}_{g,n}$). Yet, there is a way around the singularity $\mathbf{k}[[t^3, t^4, t^5]]$.

FACT. *The A_5 -singularity $V(y^2 - yx^3) \subseteq \mathbb{A}_{x,y}^2$ is a Gorenstein singularity of genus two with two branches. Its stable model is a genus two bridge, with conjugate attaching points. A marked union of two copies of \mathbb{P}^1 along an A_5 -singularity has no non-trivial automorphisms as soon as one of the two branches contains at least two markings.*

See Proposition 2.2 and Corollary 3.7. Going back to $\overline{\mathcal{M}}_{2,2}$, suppose C is the nodal union of a genus two curve Z with a rational tail R supporting the two markings, so that $\mathrm{lev}(Z) = 1$. If R is attached to a Weierstrass point of Z , we may simply contract the latter (in a one-parameter smoothing), thus producing an irreducible ramphoid cusp with two markings. If instead R is attached to a non-Weierstrass point q_1 of Z , we may blow-up the one-parameter family at the conjugate point $\sigma(q_1)$ in the central fibre, and then contract Z to get a *dangling* A_5 -singularity (meaning that one of the branches is unmarked), which nonetheless has trivial automorphism group. We pursue this strategy, which makes our compactifications not semistable (see [Smy13, Definition 1.2] for the terminology). The necessity to include such curves was prefigured in [AFS16].

To complete the picture, note that, in order to fix a deformation-open class of singularities, we need to allow cusps and tacnodes as well, due to the following

FACT. *The singularities appearing in the miniversal family of an A_m -singularity are all and only the A_l -singularities with $l \leq m$.*

See Theorem 5.5 for a more general statement - valid for all ADE singularities - due to A. Grothendieck. Since the semistable tail of a cusp (resp. tacnode) is an elliptic tail (resp. bridge), if we want our moduli space to remain separated, we should require that the level of a genus one subcurve be at least three at the same time as we introduce cusps and tacnodes. Hybrid situations may occur, such as an elliptic curve with a cusp, or an irreducible tacnode; since we need to allow a tacnode and a cusp sharing a branch, we should impose the level condition on genus one subcurves only when they are *nodally attached*. Besides, in the latter example, we need

to break the \mathfrak{S}_2 -symmetry (relabelling the markings) in order to have a unique limit: we declare that p_1 must lie on the cuspidal branch. See Figure 1.

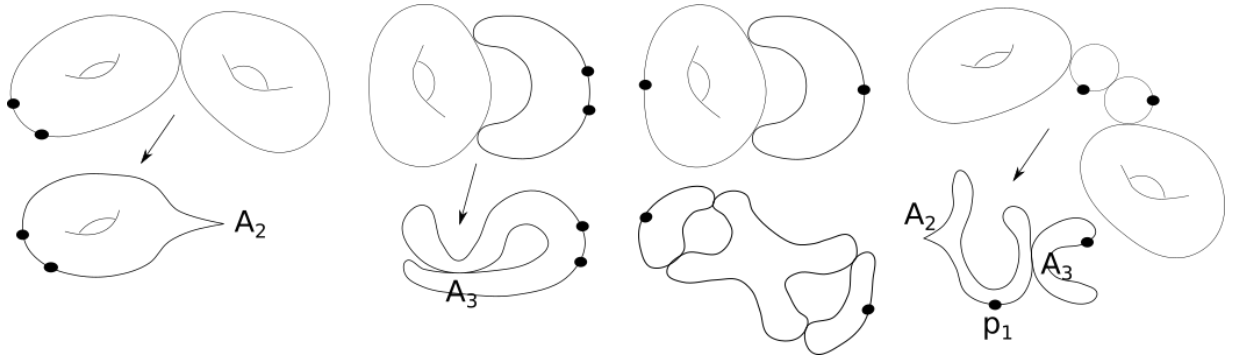


FIGURE 1. Examples of 2-pointed stable curves and their 1-stable counterparts.

We are now in a position to cast a plausible definition of $\overline{\mathcal{M}}_{2,2}^{(1)}$.

DEFINITION 1.6. A connected, reduced, complete curve of arithmetic genus two C over an algebraically closed field \mathbf{k} , with smooth and disjoint markings (p_1, p_2) , is 1-stable if:

- (i) C has only A_1 –, ..., A_4 – and dangling A_5 – singularities.
- (ii) C coincides with its minimal subcurve of arithmetic genus two.
- (iii) A subcurve of arithmetic genus one is either nodally attached and of level three, or it is not nodally attached and it contains p_1 .

The main result of the paper is that $\overline{\mathcal{M}}_{2,2}^{(1)}$ is a proper Deligne-Mumford stack, and the generalisation of this statement to an arbitrary number of markings and a range of stability conditions that we are going to discuss in the next sections.

Let us note in passing that the birational map $\overline{\mathcal{M}}_{2,2} \dashrightarrow \overline{\mathcal{M}}_{2,2}^{(1)}$ is not defined everywhere. The reason boils down to the following

FACT. *There is only one isomorphism class of 2-pointed curves whose normalisation is $(\mathbb{P}^1, q_1) \sqcup (\mathbb{P}^1, q_2, p_1, p_2)$ and having an A_5 -singularity at $q_1 = q_2$. On the other hand, the moduli space of 2-pointed irreducible curves of geometric genus zero with an A_4 -singularity is isomorphic to \mathbb{A}^1 .*

The second statement can be motivated as follows: the pointed normalisation of such a curve is $(\mathbb{P}^1, q, p_1, p_2)$, which has neither automorphisms, nor deformations. To produce an A_4 -singularity at q we may first collapse a non-zero tangent vector at q (no choice involved), producing a cusp, and then collapse a line in the tangent space at the cusp, avoiding the support of its tangent cone ℓ (therefore, the moduli space is $\mathbb{P}^1 \setminus \{\ell\} = \mathbb{A}^1$). See Lemma 3.2 and the discussion thereafter.

Let $\Delta = \Delta_{2,0|0,\{1,2\}} \subseteq \overline{\mathcal{M}}_{2,2}$ be the divisor of rational tails, and $\mathscr{W} \subseteq \overline{\mathcal{M}}_{2,2}$ the codimension two locus of Weierstrass tails. The 1-stable limit of any point in $\Delta \setminus \mathscr{W}$ is the dangling A_5 -singularity, while the 1-stable limit of a Weierstrass tail is ill-defined (it depends on the choice of a 1-parameter smoothing); we conjecture that the rational map (identity on the locus of smooth curves) admits a factorisation:

$$\begin{array}{ccc}
 & \text{Bl}_{\mathcal{W}}(\overline{\mathcal{M}}_{2,2}) & \\
 \swarrow & & \searrow \\
 \overline{\mathcal{M}}_{2,2} & \xrightarrow{\quad\quad\quad} & \overline{\mathcal{M}}_{2,2}^{(1)}
 \end{array}$$

The blow-up should also encode enough information to contract an unmarked elliptic bridge to a tacnode. As it turns out, a modular desingularisation can be obtained by starting from the moduli space of pointed admissible covers, and performing a logarithmic modification based on some piecewise linear function on the tropicalization of the source curve. These methods have been developed in [BC20], and we do not address the desingularisation here.

1.3 Relation to other work

It would be interesting to compare $\overline{\mathcal{M}}_{2,2}^{(1)}$ explicitly with Smyth's $\overline{\mathcal{M}}_{2,2}(\mathcal{Z})$ [Smy13], for the extremal assignment \mathcal{Z} of unmarked subcurves; here we only note that, while the divisor $\Delta_{1,\{1\}|1,\{2\}}$ is contracted in $\overline{\mathcal{M}}_{2,2}^{(1)}$, the latter contains a copy of $\overline{\mathcal{M}}_{0,4}$ (see the third column, second row of Figure 1) that is replaced by the class of the rational 4-fold point in $\overline{\mathcal{M}}_{2,2}(\mathcal{Z})$. $\overline{\mathcal{M}}_{2,2}^{(1)}$ seems closely related to the space $\mathcal{U}_{2,2}^{ns}(ii)$ constructed in [JP18]. More generally, it would be interesting to relate $\overline{\mathcal{M}}_{2,n}^{(m)}$ (for high values of m) to Polishchuk's moduli of curves with nonspecial divisors [Pol15]. Finally, it seems plausible that $\overline{\mathcal{M}}_{2,n}^{(m)}$ (for low values of m) corresponds to a pointed variant of the spaces of admissible hyperelliptic covers with AD singularities constructed in [Fed14].

1.4 Outline of results and plan of the paper

In Section 2 we classify all the Gorenstein curve singularities of genus two. They come in two families: the first one (*I*) includes the ramphoid cusp, the D_5 -singularity, and for $m \geq 3$ the union of a *singular branch* (a cusp) and $m - 1$ lines living in \mathbb{A}^m . The second one (*II*) includes the A_5 - and D_6 -singularities, and for $m \geq 4$ the union of *two tangent branches* (forming a tacnode) with $m - 2$ lines in \mathbb{A}^{m-1} . See Proposition 2.2.

In Section 3 we translate the condition that a complete pointed curve of genus two has no infinitesimal automorphisms into a *mostly* combinatorial criterion. For every fixed number of branches m and genus two singularity type $\in \{I, II\}$, there are two isomorphism classes of pointed curves whose normalisation is $\bigsqcup_{i=1}^m (\mathbb{P}^1, q_i, p_i)$ and having a singularity of the prescribed type at q ; one of them has $\text{Aut}(C, p) = \mathbb{G}_m$, while the other one has trivial automorphism group. This phenomenon is a novelty to genus two. We take a detour into moduli spaces of singularities to justify the claim, and explain how to interpret the *crimping spaces* geometrically in terms of the information we need to construct a genus two singularity from a (non-Gorenstein) singularity of lower genus. This is not strictly necessary in what follows, since the singularity with one-pointed branches never satisfies the level condition we demand from our curves, yet this description is useful in analysing the indeterminacy of $\overline{\mathcal{M}}_{2,n}^{(m_1)} \dashrightarrow \overline{\mathcal{M}}_{2,n}^{(m_2)}$.

In Section 4 we study the (semi)stable limits; starting from a 1-parameter family of semistable curves with smooth generic fibre and regular total space, we show that the shape of a subcurve of the central fibre that can be contracted into a Gorenstein singularity is strongly constrained. Singularities of type *I* arise when the special branch (corresponding to the cusp in the contraction) is attached to a *Weierstrass point* of the minimal subcurve of genus two (the *core*), while singularities of type *II* occur when the special branches (corresponding to the tacnode in the contraction)

are attached to *conjugate points*. Furthermore, the size of the curve to be contracted only depends on one number - roughly speaking, the distance of the special branches from the core. The first statement is a consequence of the following simple observation: if $\phi: \tilde{\mathcal{C}} \rightarrow \mathcal{C}$ is a contraction to a family of Gorenstein curves, $\phi^*\omega_{\mathcal{C}}$ is trivial on a neighbourhood of the exceptional locus of ϕ , and it coincides with $\omega_{\tilde{\mathcal{C}}}$ outside it. Now, whereas the dualising line bundle of a Gorenstein curve of genus one with no separating nodes is trivial (see [Smy11a, Lemma 3.3]) and all smooth points display the same behaviour (in the sense that they are non-special), the simplest instance of Brill-Noether theory manifests itself in genus two, with the distinction between Weierstrass and non-Weierstrass points, and the expression $\omega_Z = \mathcal{O}_Z(q + \sigma(q))$. The correct extension of these concepts to nodal curves was formulated in the '80s within the theory of admissible covers and limit linear series. We phrase the shape restrictions in terms of the existence of a certain piecewise-linear function on the dual graph of the central fibre.

In Section 5 we define the notion of *m-stable n-pointed curve* of genus two, for every $1 \leq m < n$. The basic idea is to trade worse singularities - of both genus one and two, bounded by m in the sense of the embedding dimension - with more constraints on the combinatorics of the dual graph - the *level* condition, which bounds below in terms of m the number of special points (nodes and markings) that any subcurve of genus one or two has to contain. On the other hand, it is already clear from the discussion above that we need to break the \mathfrak{S}_n -symmetry, in order to write the dualising line bundle of the minimal subcurve of genus two as $\mathcal{O}_Z(q_1 + \sigma(q_1))$, in other words to choose which branches of a semistable model are to be dubbed special. We do so by using the first marking as a reference point, so that q_1 comes to denote the point of Z closest to p_1 . This shapes our algorithm to construct the *m-stable* limit of a given 1-parameter smoothing. Unavoidably, the formulation of the stability condition is slightly involved, including a prescription of the interplay between p_1 and the singularity. We prove that the moduli stack of *m-stable* curves is algebraic, and it satisfies the valuative criterion of properness.

1.5 Future directions of work

Besides regarding this paper as a case-study of the birational geometry of moduli spaces of curves, it also has some nontrivial applications to Gromov-Witten theory. We set up some questions we would like to come back to in future work.

- (i) The indeterminacy of the rational map $\overline{\mathcal{M}}_{2,n}^{(m_1)} \dashrightarrow \overline{\mathcal{M}}_{2,n}^{(m_2)}$ can be resolved modularly: a space dominating all the $\overline{\mathcal{M}}_{2,n}^{(m)}$ can be obtained as a logarithmic modification (as in [RSPW19a]) of the space of *admissible covers* (of degree two, with rational target and six ramification points). We shall describe this construction in more details in a forthcoming paper. We wonder whether the models constructed here correspond to the trace of the minimal model program on a two-dimensional slice of the cone of pseudo-effective divisors, as in [Smy11b]. More generally, a question outstanding to our knowledge is whether the whole program fits in the framework of stability developed in [Hal14].
- (ii) Enumerative geometry: the link between reduced Gromov-Witten invariants in genus one (see for example [VZ08, Zin09, LZ09]) and maps from singular curves (see [Vis12]) was partially uncovered in [BCM18], and brought in plain view by [RSPW19a, RSPW19b]. In joint work with F. Carocci [BC20], we exploit similar techniques to desingularise the main component of the space of genus two maps to projective space. We enrich the logarithmic structure by including a compatible admissible cover. A universal morphism to a Gorenstein curve is constructed on a logarithmically étale model of the base, encoding the choice of a *tropical*

canonical divisor. We stress the fact that non-reduced fibres (*singular ribbons*) arise naturally in that context (this in turn suggests the question, whether more birational models of $\mathcal{M}_{2,n}$ can be defined by including non-reduced Gorenstein curves as well). The main component is recovered as those maps that factor through the Gorenstein contraction. Our desingularisation is less efficient than [HLN18], but maps from singular curves provide a conceptual definition of reduced invariants for projective complete intersections and beyond. We hope that they will make comparison results (standard vs. reduced) easier to prove. This would lead to a modular interpretation of Gopakumar-Vafa invariants [Pan99].

2. Gorenstein curve singularities of genus two and their dualising line bundle

We produce an algebraic classification of the (complete) local rings of Gorenstein curve singularities of genus two over an algebraically closed field \mathbf{k} . The proof involves a technical calculation with the conductor ideal. Alternatively, one can look for a local generator of the dualising line bundle at the singularity; we remark on this below.

Let (C, q) be the germ of a reduced curve singularity, and let (R, \mathfrak{m}) denote $(\hat{\mathcal{O}}_{C,q}, \mathfrak{m}_q)$, with normalisation $(\tilde{R}, \tilde{\mathfrak{m}}) \simeq (\mathbf{k}[[t_1]] \oplus \dots \oplus \mathbf{k}[[t_m]], \langle t_1, \dots, t_m \rangle)$. Here m is the number of branches of C at q . Recall the Definition 1.3 of the genus:

$$g = \delta - m + 1;$$

so, for genus two, $\delta = m + 1$. Following [Smy11a, Appendix A], we consider \tilde{R}/R as a \mathbb{Z} -graded module with:

$$(\tilde{R}/R)_i := \tilde{\mathfrak{m}}^i / (\tilde{\mathfrak{m}}^i \cap R) + \tilde{\mathfrak{m}}^{i+1};$$

furthermore, adapting Smyth's remarks in *loc. cit.* to our situation:

- (i) $m + 1 = \delta(p) = \sum_{i \geq 0} \dim_{\mathbf{k}}(\tilde{R}/R)_i$;
- (ii) $2 = g = \sum_{i \geq 1} \dim_{\mathbf{k}}(\tilde{R}/R)_i$;
- (iii) if $(\tilde{R}/R)_i = (\tilde{R}/R)_j = 0$ then $(\tilde{R}/R)_{i+j} = 0$.

We will also make use of the following observations:

- (iv) $\sum_{i \geq j} (\tilde{R}/R)_i$ is a grading of $\tilde{\mathfrak{m}}^j / (\tilde{\mathfrak{m}}^j \cap R)$;
- (v) there is an exact sequence of $R/\mathfrak{m} = \mathbf{k}$ -modules:

$$0 \rightarrow A_i := \frac{\tilde{\mathfrak{m}}^i \cap R}{\tilde{\mathfrak{m}}^{i+1} \cap R} \rightarrow \frac{\tilde{\mathfrak{m}}^i}{\tilde{\mathfrak{m}}^{i+1}} \rightarrow (\tilde{R}/R)_i \rightarrow 0$$

LEMMA 2.1. *There are two unibranch curve singularities of genus two; only one of them is Gorenstein, namely the A_4 -singularity or ramphoid cusp: $V(y^2 - x^5) \subseteq \mathbb{A}_{x,y}^2$.*

Proof. In the unibranch case $\dim_{\mathbf{k}}(\tilde{R}/R)_1 \leq 1$, hence equality holds (by observation (iii) above). We are left with two cases:

- Either $\dim_{\mathbf{k}}(\tilde{R}/R)_2 = 1$ and $\dim_{\mathbf{k}}(\tilde{R}/R)_i = 0$ for all $i \geq 3$: in this case $\tilde{\mathfrak{m}}^3 \subseteq \mathfrak{m}$ by observation (iv). From (v) we see that $\tilde{\mathfrak{m}}^3 = \mathfrak{m}$, hence $R \simeq \mathbf{k}[[t^3, t^4, t^5]]$, a non-Gorenstein singularity sitting in 3-space, which is obtained by collapsing a second-order infinitesimal neighbourhood of the origin in \mathbb{A}^1 (we shall call it an ordinary cusp of genus two).
- Or $\dim_{\mathbf{k}}(\tilde{R}/R)_3 = 1$ and $\dim_{\mathbf{k}}(\tilde{R}/R)_i = 0$ for $i = 2$ and for all $i \geq 4$: in this case $\tilde{\mathfrak{m}}^4 \subseteq \mathfrak{m}$ by observation (iv). On the other hand from $\dim_{\mathbf{k}}(\tilde{\mathfrak{m}}^2 \cap R / \tilde{\mathfrak{m}}^3 \cap R) = 1$ we deduce that there is

a generator of degree 2, and from $\dim_{\mathbf{k}}(\tilde{\mathfrak{m}}^3 \cap R/\tilde{\mathfrak{m}}^4 \cap R) = 0$ there is none of degree 3. We may write the generator as $x = t^2 + ct^3$, and $\mathfrak{m} = \langle x \rangle + \tilde{\mathfrak{m}}^4$. Up to a coordinate change (i.e. automorphism of $\mathbf{k}[[t]]$), we may take $x = t^2$, and

$$\mathfrak{m}/\mathfrak{m}^2 = \langle t^2, t^5 \rangle,$$

so $R \simeq \mathbf{k}[[x, y]]/(x^5 - y^2)$, as anticipated. □

From now on, we only look for Gorenstein singularities. With notation as above, let $I = (R : \tilde{R}) = \text{Ann}_R(\tilde{R}/R)$ be the *conductor ideal* of the singularity. Recall e.g. [AK70, Proposition VIII.1.16]: (C, q) is Gorenstein if and only if

$$\dim_{\mathbf{k}}(R/I) = \dim_{\mathbf{k}}(\tilde{R}/R)(= \delta).$$

Recall from [Ste96, Definition 2-1] that a curve singularity (C, q) is *decomposable* if C is the union of two curves C_1 and C_2 that lie in distinct smooth spaces intersecting each other transversely in q . Given a parametrisation $x_i = x_i(t_1, \dots, t_m)$, $i = 1, \dots, l$, this means that there is a partition $S \sqcup S' = \{1, \dots, m\}$ such that x_i only depends on $t_s, s \in S$, or $s \in S'$, for all i . Aside from the node, Gorenstein singularities are never decomposable [AFS16, Proposition 2.1].

PROPOSITION 2.2. *For every fixed integer $m \geq 2$, there are exactly two Gorenstein curve singularities of genus two with m branches.*

Proof. We only need to find a basis for $\mathfrak{m}/\mathfrak{m}^2$, because a map of complete local rings that is surjective on cotangent spaces is surjective. From observation (iii) again, we find three possibilities for the vector (d_1, d_2, d_3) , $d_i = \dim_{\mathbf{k}}(\tilde{R}/R)_i$; $d_{\geq 4} = 0$ in any case.

Case (2, 0, 0). We see that $\tilde{\mathfrak{m}}^2 \subseteq I$, so, if (C, q) were Gorenstein, (v) would imply:

$$m + 1 = \delta = \dim_{\mathbf{k}}(R/I) \leq \dim_{\mathbf{k}}(R/\tilde{\mathfrak{m}}^2) = \dim_{\mathbf{k}} A_0 + \dim_{\mathbf{k}} A_1 = 1 + (m - 2) = m - 1,$$

a contradiction. Note: the singularity turns out to be decomposable in this case.

Case (1, 1, 0). We have $\tilde{\mathfrak{m}}^3 \subseteq I$. We are going to write down the $m - 1$ generators of $A_1 \pmod{\tilde{\mathfrak{m}}^3}$ ¹. The first generator, call it x_1 , has a non-trivial linear term in at least one of the variables, say t_1 . By scaling x_1 and possibly adding a multiple of x_1^2 , we can make it into the form: $x_1 = t_1 \oplus p_{1,2}(t_2) \oplus \dots \oplus p_{1,m}(t_m) \pmod{\tilde{\mathfrak{m}}^3}$. Now we can use x_1 and x_1^2 to make sure the second generator does not involve t_1 at all. It will still have a linear term independent of t_1 , say non-trivial in t_2 . By scaling and adding a multiple of x_2^2 , we can write $x_2 = 0 \oplus t_2 \oplus \dots \oplus p_{2,m}(t_m) \pmod{\tilde{\mathfrak{m}}^3}$. By taking a linear combination of x_1 with x_2 and x_2^2 , we may now reduce x_1 to the form $t_1 \oplus 0 \oplus p_{1,3}(t_3) \oplus \dots \oplus p_{1,m}(t_m) \pmod{\tilde{\mathfrak{m}}^3}$. Continuing this way, by Gaussian elimination with the generators and their squares, we may write them as:

$$\begin{aligned} x_1 &= t_1 \oplus 0 \oplus \dots \oplus \alpha_{1,m} t_m + \beta_{1,m} t_m^2 \\ x_2 &= 0 \oplus t_2 \oplus \dots \oplus \alpha_{2,m} t_m + \beta_{2,m} t_m^2 \\ &\dots \\ x_{m-1} &= 0 \oplus \dots \oplus t_{m-1} \oplus \alpha_{m-1,m} t_m + \beta_{m-1,m} t_m^2 \pmod{\tilde{\mathfrak{m}}^3}. \end{aligned}$$

¹To express them in the simplest possible form, we perform at first only polynomial manipulations of the generators, while changing coordinates on the normalisation only at the end - the benefit of this two-step process will become apparent in the next section.

If $x_i \in I$ for some i , then $t_i \in R$, and the singularity would be decomposable. So, by the Gorenstein condition, R/I is generated by $1, x_1, \dots, x_{m-1}$, and an extra element y . Hence $x_i^2 \in I$ for all but at most one i , say $i = 1$. Then $t_i^2 \in R$ for $i = 2, \dots, m-1$. If $\alpha_{i,m} \neq 0$ for some i in this range, then $t_m^2 \in R$ as well, so $t_1^2 = x_1^2 - O(t_m^2) \in R$, contradicting $\dim_{\mathbf{k}}(\tilde{R}/R)_2 = 1$. Therefore $\alpha_{i,m} = 0$ for $i \in \{2, \dots, m-1\}$. If $\alpha_{1,m} = 0$, there would be another generator of $\mathfrak{m}/\mathfrak{m}^2$, that could be taken to be $z = 0 \oplus \dots \oplus t_m^3$. In this case, though, both x_1^2 and z would belong to I , so $\dim_{\mathbf{k}}(R/I) = m$, and the singularity could not be Gorenstein. We are reduced to the following expression:

$$\begin{aligned} x_1 &= t_1 \oplus 0 \oplus \dots \oplus \alpha_{1,m} t_m + \beta_{1,m} t_m^2 \\ x_2 &= 0 \oplus t_2 \oplus \dots \oplus \beta_{2,m} t_m^2 \\ &\dots \\ x_{m-1} &= 0 \oplus \dots \oplus t_{m-1} \oplus \beta_{m-1,m} t_m^2 \pmod{\tilde{\mathfrak{m}}^3}, \end{aligned} \tag{1}$$

with $\beta_{1,m} \in \mathbf{k}$ and $\alpha_{1,m}, \beta_{i,m} \in \mathbf{k}^\times$, $i = 2, \dots, m-1$ (by indecomposability). Finally, we change coordinates in t_m (abusing notation, $t_m := \alpha_{1,m} t_m + \beta_{1,m} t_m^2$) and rescale the other t_i to obtain:

$$\begin{aligned} x_1 &= t_1 \oplus 0 \oplus \dots \oplus t_m \\ x_2 &= 0 \oplus t_2 \oplus \dots \oplus t_m^2 \\ &\dots \\ x_{m-1} &= 0 \oplus \dots \oplus t_{m-1} \oplus t_m^2 \pmod{\tilde{\mathfrak{m}}^3}. \end{aligned} \tag{2}$$

We check that $R/I = \langle 1, x_1, \dots, x_{m-1}, x_1^2 \rangle$ and \tilde{R}/R is of type $(1, 1, 0)$. In case $m = 2$, we need an extra generator $y = t_2^3$. Equations are given by:

- $y(y - x_1^3)$ if $m = 2$ (A_5 -singularity or *oscnode*);
- $x_1 x_2 (x_2 - x_1^2)$ if $m = 3$ (D_6 -singularity);
- $\langle x_3 (x_1^2 - x_2), x_i (x_j - x_k) \rangle_{1 \leq i < j < k \leq m-1 \text{ or } 1 < j < k < i \leq m-1}$ if $m \geq 4$.

Case $(1, 0, 1)$. We have $\tilde{\mathfrak{m}}^4 \subseteq I$. By an argument similar to the above one, we write generators for A_1 as $x_i = \dots \oplus t_i \oplus \dots \oplus \alpha_{i,m} t_m + \beta_{i,m} t_m^2 + \gamma_{i,m} t_m^3$, for $i = 1, \dots, m-1$. Then $R/I = \langle 1, x_1, \dots, x_{m-1}, y \rangle$. For all but at most one i , $x_i^2 \in I$, but definitely $x_i^3 \in I$ for all i . On the other hand $t_m^3 \notin R$, because otherwise $t_i^3 = x_i^3 - \alpha_{i,m}^3 t_m^3 + O(t_m^4)$ would belong to R as well, contradicting $\dim_{\mathbf{k}}(\tilde{R}/R)_3 = 1$. From this we deduce that $\alpha_{i,m} = 0$ for all $i = 1, \dots, m-1$. Since $\dim_{\mathbf{k}}(\tilde{R}/R)_2 = 0$, there has to be another generator of $\mathfrak{m}/\mathfrak{m}^2$ of degree two in t_m , which we may write as $x_m = t_m^2 + \gamma_{m,m} t_m^3$. We can use x_m to remove all the t_m^2 pieces from x_1, \dots, x_{m-1} , so we are reduced to the following expression:

$$\begin{aligned} x_1 &= t_1 \oplus 0 \oplus \dots \oplus \gamma_{1,m} t_m^3 \\ x_2 &= 0 \oplus t_2 \oplus \dots \oplus \gamma_{2,m} t_m^3 \\ &\dots \\ x_{m-1} &= 0 \oplus \dots \oplus t_{m-1} \oplus \gamma_{m-1,m} t_m^3 \\ x_m &= 0 \oplus \dots \oplus t_m^2 + \gamma_{m,m} t_m^3 \pmod{\tilde{\mathfrak{m}}^4}, \end{aligned} \tag{3}$$

with $\gamma_{m,m} \in \mathbf{k}$ and $\gamma_{i,m} \in \mathbf{k}^\times$, $i = 1, \dots, m-1$ (by indecomposability). Finally, we change

coordinates in t_m (abusing notation $t_m := t_m \sqrt{1 + \gamma_{m,m} t_m}$)² and rescale the other t_i to obtain:

$$\begin{aligned} x_1 &= t_1 \oplus 0 \oplus \dots \oplus t_m^3 \\ x_2 &= 0 \oplus t_2 \oplus \dots \oplus t_m^3 \\ &\dots \\ x_{m-1} &= 0 \oplus \dots \oplus t_{m-1} \oplus t_m^3 \\ x_m &= 0 \oplus \dots \oplus t_m^2 \pmod{\tilde{\mathfrak{m}}^4}. \end{aligned} \tag{4}$$

We check that $R/I = \langle 1, x_1, \dots, x_{m-1}, x_m \rangle$ and \tilde{R}/R is of type $(1, 0, 1)$. Incidentally, when $m = 1$, we recover the unique Gorenstein singularity of Lemma 2.1. Equations are given by:

- $x^5 - y^2$ if $m = 1$ (A_4 -singularity or *ramploid cusp*, with $x = t^2, y = t^5$);
- $y(y^3 - x^2)$ if $m = 2$ (D_5 -singularity, with $x = x_1, y = x_2$);
- $\langle x_3(x_1 - x_2), x_3^3 - x_1 x_2 \rangle$ if $m = 3$;
- $\langle x_i(x_j - x_k), x_m(x_i - x_j), x_m^3 - x_1 x_2 \rangle_{i,j,k \in \{1, \dots, m-1\}}$ all different if $m \geq 4$.

□

Remark 2.3. We sketch an alternative proof of the above proposition based on meromorphic differentials, see also Corollary 2.5 below. We address the case $(1, 0, 1)$ and leave the (easier) case $(1, 1, 0)$ to the interested reader. The setup is as in [RSPW19b, §2.1]: let C be a projective Gorenstein curve with a unique singularity of genus two at the point q . Let:

$$\tilde{C} \xrightarrow{\nu} \hat{C} \xrightarrow{\mu} C$$

be respectively the normalisation and semi-normalisation of C . We have inclusions:

$$\begin{array}{cccc} \mathcal{O}_C & \subseteq \mu_* \mathcal{O}_{\hat{C}} & \subseteq \mu_* \nu_* \mathcal{O}_{\tilde{C}} & \subseteq K \\ J \supseteq \omega_C & \supseteq \mu_* \omega_{\hat{C}} & \supseteq \mu_* \nu_* \omega_{\tilde{C}} & \end{array}$$

where K is the sheaf of rational functions, and J the sheaf of meromorphic differentials. The rows are dual to each other with respect to the residue pairing $J \otimes K \rightarrow \mathbf{k}$ [AK70, Proposition 1.16(ii)]. The skyscraper sheaf $\omega_{\hat{C}}/\nu_* \omega_{\tilde{C}}$ is generated by the logarithmic differentials:

$$\frac{dt_i}{t_i} - \frac{dt_j}{t_j}, \quad i, j \in \{1, \dots, m\}. \tag{5}$$

The skyscraper sheaves $\mu_* \mathcal{O}_{\hat{C}}/\mathcal{O}_C$ and $\omega_C/\mu_* \omega_{\hat{C}}$ have length two. Let η_1 be a generator of the latter; since C is Gorenstein, we may assume that η_1 is a local generator of ω_C . Since $\tilde{\mathfrak{m}}^4 \subseteq R$ and by Equation (5), we may assume that η_1 takes the following form:

$$\eta_1 = \zeta \frac{dt_1}{t_1} + \sum_{i=1}^m \alpha_i \frac{dt_i}{t_i^2} + \sum_{i=1}^m \beta_i \frac{dt_i}{t_i^3} + \sum_{i=1}^m \gamma_i \frac{dt_i}{t_i^4}. \tag{6}$$

Since the constant functions descend to C , the residue condition implies that $\zeta = 0$. Now, $\mu_* \omega_{\hat{C}}/\omega_C$ has a second generator η_2 of the form $\eta_2 = f \eta_1$ for some $f \in R$. Expanding $f = f_0 + \sum_{i,j} f_{i,j} t_i^j$ we may write:

$$\eta_2 = \sum_{i=1}^m (\beta_i f_{i,1} + \gamma_i f_{i,2}) \frac{dt_i}{t_i^2} + \sum_{i=1}^m \gamma_i f_{i,1} \frac{dt_i}{t_i^3}. \tag{7}$$

²For this to be possible, we have to assume that \mathbf{k} has characteristic different from 2.

Not all the γ_i can be zero, otherwise we would have $\tilde{\mathfrak{m}}^3 \subseteq R$, so say $\gamma_m \neq 0$; this implies that $t_m^3 \notin R$ (and in particular $t_m \notin R$). Up to scaling we have $\gamma_m = 1$. Since A_3 has dimension one, for every $i = 1, \dots, m-1$ there is a linear combination:

$$g_i = t_i^3 + x_i t_m^3 \in R.$$

Pairing with η_1 we find $x_i = -\gamma_i$.

Since $A_2 = 0$, for every $i = 1, \dots, m$ there is a linear combination:

$$h_i = t_i^2 + y_i t_m^3 \in R.$$

Pairing with η_1 we find $y_i = -\beta_i$. Pairing with η_2 we find $\gamma_i f_{i,1} = 0$ (so $f_{m,1} = 0$). Equation (7) becomes:

$$\eta_2 = \left(\sum_{i:\gamma_i=0, f_{i,1} \neq 0} \beta_i f_{i,1} + \sum_{i:\gamma_i \neq 0, f_{i,1}=0} \gamma_i f_{i,2} \right) \frac{dt_i}{t_i^2}.$$

Since A_1 has dimension one and $t_m \notin R$, for every $i = 1, \dots, m-1$ there is a linear combination:

$$l_i = t_i + z_i t_m^3 \in R.$$

Pairing with η_1 we find $z_i = -\alpha_i$. So $\alpha_i \neq 0$ for $i = 1, \dots, m-1$, otherwise C would be decomposable. Pairing with η_2 we find

$$\beta_i f_{i,1} + \gamma_i f_{i,2} = 0, \quad i = 1, \dots, m-1. \quad (8)$$

Taking l_i^2 , we have $t_i^2 \in R, i = 1, \dots, m-1$, as well. Pairing with η_1 , we find $\beta_i = 0$.

Taking l_i^3 , we have $t_i^3 \in R, i = 1, \dots, m-1$, as well. Pairing with η_1 , we find $\gamma_i = 0$. So, ultimately, we find:

$$\eta_2 = \frac{dt_m}{t_m^2}.$$

By taking a linear combination with η_2 , Equation (6) becomes:

$$\eta_1 = \sum_{i=1}^{m-1} \alpha_i \frac{dt_i}{t_i^2} + \beta_m \frac{dt_m}{t_m^3} + \frac{dt_m}{t_m^4}, \text{ with } \alpha_i \in \mathbf{k}^\times, \beta_m \in \mathbf{k}.$$

DEFINITION 2.4. In case $(1, 0, 1)$, we say the singularity is *of type I*, and the branch parameterised by t_m is called *singular*; in case $(1, 1, 0)$, we say the singularity is *of type II*, and the branches parameterised by t_1 and t_m are called *twin*. We shall refer to the singular or twin branches as *special* or *distinguished*; all other branches are *axes*. *Branch* remains a generic name, indicating any of the previous ones.

We gather a description of the dualising line bundle in the following:

COROLLARY 2.5. Let $\nu: C \rightarrow \bar{C}$ be the normalisation of a Gorenstein singularity of genus two, with $\nu^{-1}(q) = \{q_1, \dots, q_m\}$.

(I) With local parametrisation as in (4), $\omega_{\bar{C}}$ is generated by:

$$\frac{dt_1}{t_1^2} + \dots + \frac{dt_{m-1}}{t_{m-1}^2} - \frac{dt_m}{t_m^4},$$

$$\text{and } \nu^* \omega_{\bar{C}} = \omega_C(2q_1 + \dots + 2q_{m-1} + 4q_m).$$

(II) With local parametrisation as in (2), $\omega_{\tilde{C}}$ is generated by:

$$\frac{dt_1}{t_1^3} + \frac{dt_2}{t_2^2} + \dots + \frac{dt_{m-1}}{t_{m-1}^2} - \frac{dt_m}{t_m^3},$$

and $\nu^*\omega_{\tilde{C}} = \omega_C(3q_1 + 2q_2 + \dots + 2q_{m-1} + 3q_m)$.

3. Tangent sheaf, crimping space, and automorphisms

In this section we analyse the tangent sheaf of a genus two singularity. For a complete pointed Gorenstein curve of genus two, we translate the absence of infinitesimal automorphisms into a (mostly) combinatorial criterion. We will use this in Section 5, when we define stability conditions on the stack of pointed Gorenstein curves of genus two, to make sure that the resulting substacks are Deligne-Mumford. The crimping space naturally makes an appearance. Although we do not use it explicitly in this paper, it parametrises singularities of a given type and pointed normalisation. It helps understanding the birational map between two compactifications of $\mathcal{M}_{2,n}$.

In the next lemma, we find explicit conditions for a vector field on the normalisation, vanishing at the preimage of the singular point, to descend to the singular curve.

LEMMA 3.1. *Let (C, q) be a Gorenstein curve singularity of genus two, with pointed normalisation $\nu: (\tilde{C}, \{q_i\}_{i=1,\dots,m}) \rightarrow (C, q)$, and assume $\text{char}(\mathbf{k}) \neq 2, 3$. There are exact sequences of sheaves*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \nu_*\Omega_{\tilde{C}}^\vee(-\sum_i 3q_i) & \longrightarrow & \nu_*\Omega_{\tilde{C}}^\vee(-\sum_i q_i) & \longrightarrow & \nu_*\bigoplus_i \Omega_{\tilde{C}}^\vee(-q_i)_{|2q_i} \longrightarrow 0 \\ & & \parallel & & \uparrow & & \square & \uparrow \phi \\ 0 & \longrightarrow & \nu_*\Omega_{\tilde{C}}^\vee(-\sum_i 3q_i) & \longrightarrow & \Omega_C^\vee & \longrightarrow & \mathbf{k}^{\oplus m} \longrightarrow 0 \end{array}$$

The image of the rightmost vertical map admits an explicit description in local coordinates.

Proof. Let $K(\tilde{C})$ denote the locally constant sheaf of rational functions on \tilde{C} . A section of $\Omega_{\tilde{C}}^\vee \otimes K(\tilde{C})$ is contained in $\Omega_{\tilde{C}}^\vee$ if and only if its image under the push-forward map:

$$\nu_*: \nu_*\mathcal{H}om(\Omega_{\tilde{C}}, K(\tilde{C})) \rightarrow \mathcal{H}om(\Omega_C, K(\tilde{C}))$$

lies in the subspace $\mathcal{H}om(\Omega_C, \mathcal{O}_C)$. Since in any case $\tilde{\mathbf{m}}^4 \subseteq \mathbf{m}$ (see the proof of Proposition 2.2), vector fields vanishing up to order three certainly descend. In order to find the remaining conditions, we may work locally around the singular point in the coordinates of Section 2.

(A₄): In the coordinates $x = t^2 + ct^3, y = t^4, z = t^5$ (they are redundant, but this will be irrelevant), the section $f(t)\frac{d}{dt} \in \nu_*\Omega_{\tilde{C}}^\vee \otimes K(\tilde{C})$ pushes forward to

$$\nu_*\left(f(t)\frac{d}{dt}\right) = (2t + 3ct^2)f(t)\frac{d}{dx} + 4t^3f(t)\frac{d}{dy} + 5t^4f(t)\frac{d}{dz},$$

from which, writing $f(t) = f_0 + f_1t + f_2t^2 + O(t^3)$, we see that

$$(2t + 3ct^2)f(t), 4t^3f(t), 5t^4f(t) \in \hat{\mathcal{O}}_{C,p} \Leftrightarrow f_0 = 0, cf_1 + 2f_2 = 0.$$

(A₅): In the coordinates $x = t_1 \oplus at_2 + bt_2^2, y = t_1^3$ (we have $a \neq 0$), the section $f_1(t_1)\frac{d}{dt_1} \oplus f_2(t_2)\frac{d}{dt_2}$ pushes forward to

$$\nu_*\left(f_1(t_1)\frac{d}{dt_1} \oplus f_2(t_2)\frac{d}{dt_2}\right) = (f_1(t_1) \oplus (a + 2bt_2)f_2(t_2))\frac{d}{dx} + 3t_1^2f_1(t_1)\frac{d}{dy},$$

from which, writing $f_i(t_i) = f_{i0} + f_{i1}t_i + f_{i2}t_i^2 + O(t_i^3)$, $i = 1, 2$, we see that

$$f_1(t_1) \oplus (a + 2bt_2)f_2(t_2), 3t_1^2f_1(t_1) \in \hat{\mathcal{O}}_{C,p} \Leftrightarrow \begin{cases} f_{10} = f_{20} = 0, \\ f_{11} = f_{21}, \\ 2bf_{21} + af_{22} = a^2f_{12}. \end{cases}$$

$(I_{m \geq 2})$: In the coordinates of (3),

$$\nu_* \left(\sum_{i=1}^m f_i(t_i) \frac{d}{dt_i} \right) = \sum_{i=1}^{m-1} (f_i(t_i) \oplus 3\gamma_{i,m} t_m^2 f_m(t_m)) \frac{d}{dx_i} + (2t_m + 3\gamma_{m,m} t_m^2) f_m(t_m) \frac{d}{dx_m},$$

hence we deduce that

$$\nu_* \left(\sum_{i=1}^m f_i(t_i) \frac{d}{dt_i} \right) \in \Omega_C^\vee \otimes \hat{\mathcal{O}}_{C,p} \Leftrightarrow \begin{cases} f_{i0} = 0 & i = 1, \dots, m, \\ f_{i1} = 3f_{m1}, & i = 1, \dots, m-1, \\ 3\gamma_{m,m} f_{m1} + 2f_{m2} = 0. \end{cases}$$

$(II_{m \geq 3})$: In the coordinates of (1),

$$\nu_* \left(\sum_{i=1}^m f_i(t_i) \frac{d}{dt_i} \right) = (f_1(t_1) \oplus (\alpha_{1,m} + 2\beta_{1,m} t_m) f_m(t_m)) \frac{d}{dx_1} + \sum_{i=2}^m (f_i(t_i) \oplus 2\beta_{i,m} t_m f_m(t_m)) \frac{d}{dx_i},$$

hence we deduce that

$$\nu_* \left(\sum_{i=1}^m f_i(t_i) \frac{d}{dt_i} \right) \in \Omega_C^\vee \otimes \hat{\mathcal{O}}_{C,p} \Leftrightarrow \begin{cases} f_{i0} = 0 & i = 1, \dots, m, \\ 2f_{11} = f_{i1} = 2f_{m1}, & i = 2, \dots, m-1, \\ \beta_{1,m} f_{m1} + \alpha_{1,m} f_{m2} = \alpha_{1,m}^2 f_{12}. \end{cases}$$

□

We anticipate that the letters α, β and γ will play a role in determining the automorphism group of a complete curve with markings. We recall some key concepts from F. van der Wyck's thesis. Working over \mathbf{k} , he considers the stacks:

- \mathcal{S} of reduced one-dimensional (1d) \mathbf{k} -algebras R ,
- \mathcal{T} of reduced 1d algebras with resolution $(R \hookrightarrow (S, J))$, where S is a smooth one-dimensional \mathbf{k} -algebra, and J the radical of the conductor of $R \subseteq S$.

Basically, R represents the (local) ring of a reduced curve with one singular point, S is its normalisation, and J is the ideal of the reduced fibre over the singular point of $\text{Spec}(R)$. \mathcal{S} and \mathcal{T} are limit-preserving stacks over $\text{Spec}(\mathbf{k})$ [vdW10, Proposition 1.21]. Furthermore, we may fix a reduced 1d algebra with resolution $\tau_0 : (R_0 \hookrightarrow (S_0, J_0))$, and consider the substack $\mathcal{T}(\tau_0)$ of reduced 1d algebras with singularity type τ_0 (i.e. isomorphic to τ_0 locally on both the base and the curve, see [vdW10, Definition 1.64]; that various notions of “locally” coincide is proved in [vdW10, Proposition 1.50]). There is a forgetful morphism $\mathcal{T} \rightarrow \mathcal{S}$, and the *crimping space* of τ_0 is defined to be the fibre over R_0 of the restriction of such morphism to $\mathcal{T}(\tau_0)$. The crimping space is a smooth \mathbf{k} -scheme [vdW10, Theorems 1.70 and 1.73]; indeed, it is isomorphic to the quotient of $\text{Aut}_{(S_0, J_0)/\mathbf{k}}$ by $\text{Aut}_{(S_0, J_0)/R_0}$, the latter consisting of automorphisms of the normalisation that preserve the subalgebra of the singularity; moreover, by [vdW10, Theorem 1.53] the quotient can

be computed after modding out the lowest power of J contained in R , denoted by $\text{Aut}_{(S,J)}^{\text{mod } J^k}$ respectively $\text{Aut}_{(S,J)/R}^{\text{mod } J^k}$. Crimping spaces can be thought of as moduli for the normalisation map.

LEMMA 3.2. *If $\text{char}(\mathbf{k}) \neq 2, 3$, the crimping space of a genus two singularity of type I (resp. II) with m branches is the disjoint union of m (resp. $\binom{m}{2}$) copies of $\mathbb{A}^1 \times (\mathbb{A}^1 \setminus \{0\})^{m-1}$.*

Proof. We resume notation from the previous section. We are going to fix the subalgebra τ_0 given in coordinates by (4) and (2) respectively.

Type I: recall that in this case $\tilde{\mathfrak{m}}^4 \subseteq R$. For a \mathbf{k} -algebra A , let

$$G_i(A) = \{t_i \mapsto g_{i1}t_i + g_{i2}t_i^2 + g_{i3}t_i^3, t_j \mapsto t_j \mid g_{i1} \in A^\times, g_{i2}, g_{i3} \in A\},$$

and notice that

$$\text{Aut}_{(\tilde{R}, \tilde{\mathfrak{m}})}^{\text{mod } \tilde{\mathfrak{m}}^4}(A) = (G_1 \times \dots \times G_m) \rtimes \mathfrak{S}_m(A).$$

Consider now the action of a group element of the form $(g_1, \dots, g_m; \text{id}_{\mathfrak{S}_m})$ on the given generators of R :

$$\begin{aligned} x_i &\mapsto \dots \oplus g_{i1}t_i + g_{i2}t_i^2 + g_{i3}t_i^3 \oplus \dots \oplus g_{m1}^3t_m^3, \quad \text{for } i = 1, \dots, m-1; \\ x_m &\mapsto \dots \oplus g_{m1}^2t_m^2 + 2g_{m1}g_{m2}t_m^3 \pmod{\tilde{\mathfrak{m}}^4}. \end{aligned}$$

The former belongs to R iff $g_{i1} = g_{m1}^3$; the latter does iff $g_{m2} = 0$. Thus such elements span a subgroup isomorphic to $\mathbb{G}_m \times \mathbb{G}_a^{m-1} \times \mathbb{G}_a^m(A)$. On the other hand, there is a special (singular) branch, parametrised by t_m . We conclude that

$$\text{Aut}_{\tau_0}^{\text{mod } \tilde{\mathfrak{m}}^4}(A) = (\mathbb{G}_m \times \mathbb{G}_a^{m-1} \times \mathbb{G}_a^m) \rtimes (\mathfrak{S}_{m-1})(A).$$

The quotient is therefore isomorphic to m copies of $\mathbb{A}^1 \times (\mathbb{A}^1 \setminus \{0\})^{m-1}$.

Type II: recall that in this case $\tilde{\mathfrak{m}}^3 \subseteq R$. For a \mathbf{k} -algebra A , let

$$G_i(A) = \{t_i \mapsto g_{i1}t_i + g_{i2}t_i^2, t_j \mapsto t_j \mid g_{i1} \in A^\times, g_{i2} \in A\},$$

and notice that

$$\text{Aut}_{(\tilde{R}, \tilde{\mathfrak{m}})}^{\text{mod } \tilde{\mathfrak{m}}^3}(A) = (G_1 \times \dots \times G_m) \rtimes \mathfrak{S}_m(A).$$

Consider now the action of a group element of the form $(g_1, \dots, g_m; \text{id}_{\mathfrak{S}_m})$ on the given generators of R :

$$\begin{aligned} x_i &\mapsto \dots \oplus g_{i1}t_i + g_{i2}t_i^2 \oplus \dots \oplus g_{m1}^2t_m^2, \quad \text{for } i = 2, \dots, m-1; \\ x_1 &\mapsto g_{11}t_1 + g_{12}t_1^2 \oplus \dots \oplus g_{m1}t_m + g_{m2}t_m^2 \pmod{\tilde{\mathfrak{m}}^3}. \end{aligned}$$

The former belongs to R iff $g_{i1} = g_{m1}$ and $g_{i2} = g_{m2}$; the latter does iff $g_{i1} = g_{m1}^2$. Thus, such elements span a subgroup isomorphic to $\mathbb{G}_m \times \mathbb{G}_a^{m-1}(A)$. On the other hand, all branches are smooth (therefore, isomorphic to each other), but two of them (parametrised by t_1 and t_m respectively) are tangent, thus forming a distinguished pair. We conclude that

$$\text{Aut}_{\tau_0}^{\text{mod } \tilde{\mathfrak{m}}^3}(A) = (\mathbb{G}_m \times \mathbb{G}_a^{m-1}) \rtimes (\mathfrak{S}_2 \times \mathfrak{S}_{m-2})(A).$$

The quotient is then isomorphic to $\binom{m}{2}$ copies of $\mathbb{A}^1 \times (\mathbb{A}^1 \setminus \{0\})^{m-1}$. \square

Remark 3.3. The restrictions on the characteristic of the base field in Lemmas 3.1 and 3.2 rule out the sporadic occurrence of infinite families of automorphisms, and its effect on the crimping spaces. For example, when $\text{char}(\mathbf{k}) = 2$, the singularities $\mathbf{k}[[t^2, t^5]]$ and $\mathbf{k}[[t^2 + t^3, t^4, t^5]]$ are not isomorphic, the group of infinitesimal automorphisms has positive dimension, and the crimping space consists of an isolated point [vdW10, Examples 1.79-80].

The benefit of a two-step classification should now be clear: if we do not allow ourselves to change coordinates (i.e. act by automorphisms of the normalisation) until the end, the crimping space appears in the expressions (1) and (3) for the generators of the singularity subalgebra.

There is a more geometric way to realise the crimping spaces. It is well-known that an ordinary cusp of genus one can be obtained by collapsing (*push-out*) any non-zero tangent vector at $p \in \mathbb{A}^1$. More generally, a Gorenstein singularity of genus one and m branches can be obtained by collapsing a generic (not contained in any coordinate linear subspace) tangent line at an ordinary m -fold point (a non-Gorenstein singularity of genus zero) [Smy11a, Lemma 2.2]. Therefore, the crimping space of the elliptic m -fold point, which is isomorphic to $(\mathbb{A}^1 \setminus \{0\})^{m-1}$, can be realised as the complement of the coordinate hyperplanes inside $\mathbb{P}(T_p R_m) \simeq \mathbb{P}^{m-1}$, where (R_m, p) is the rational m -fold point. Besides, this gives rise to a natural compactification of the crimping space supporting a universal family of curves - in fact, two: either we collapse non-generic tangent vectors, obtaining non-Gorenstein singularities along the boundary (this family \mathcal{C} admits a common (semi)normalisation by the trivial family $\tilde{\mathcal{C}} = R_m \times \mathbb{P}(T_p R_m)$); or we blow $\tilde{\mathcal{C}}$ up along the boundary (*sprouting*), so that the non-Gorenstein singularities are replaced by elliptic m -fold points having strictly semistable branches [Smy11b, §2.2-3].

Similarly, a Gorenstein singularity of genus two can be obtained by collapsing a generic line in the tangent space of a non-Gorenstein singularity of genus one. Indeed, τ_0^I admits a partial normalisation by σ_0^I , which is the decomposable union of a cusp (parametrised by t_m) together with $m-1$ axes; the local ring of σ_0^I is obtained from that of τ_0^I by adjoining the generator t_m^3 . τ_0^{II} admits a partial normalisation by σ_0^{II} , which is the decomposable union of a tacnode in the (t_1, t_m) -plane together with $m-2$ axes, adjoining the generator t_m^2 .

These fit together nicely in a unifying picture: if we restrict \mathcal{C} from the previous paragraph to the union of the coordinate lines in $\mathbb{P}(T_p R_m)$, we obtain m copies of σ_0^I over the coordinate points, together with $\binom{m}{2}$ copies of the universal curve of type σ_0^{II} over its crimping space - which is isomorphic to $\mathbb{A}^1 \setminus \{0\}$ - identified with the line minus two points. Let $P = \mathbb{P}(T_{\mathcal{C}/\mathbb{P}, p} \cup \text{lines})$ be the projectivised tangent space of the fibre at the singular point. For each of the $\binom{m}{2}$ coordinate lines, P has one component P_i^{II} that is a \mathbb{P}^{m-1} -bundle over the line; besides, P has m components P_j^I isomorphic to \mathbb{P}^m and supported over the points. The crimping space of the genus two singularities with m branches (of type I and II together) can be realised as an open subscheme of P , obtained by removing from the \mathbb{P}^{m-1} -fibres of P^{II} the $m-1$ hyperplanes generated by (a) the tangent line to the tacnode and the $m-2$ axes, and (b) the plane containing the tacnode and all but one of the $m-2$ axes; and from each P_j^I the m planes generated by (a) the tangent cone of the cusp and the $m-1$ axes, and (b) the plane containing the cusp and all but one of the $m-1$ axes.

Remark 3.4. Crimping spaces are related to the space of arrows ϕ in the diagram of Lemma 3.1.

We interpret $H^0(\Omega_{\mathbb{P}^1}^\vee(-p)_{|2p})$ as the tangent space to the subgroup of PGL_2 fixing the point $p \in \mathbb{P}^1$, thus it inherits a natural Lie algebra structure, isomorphic to the unique non-abelian Lie algebra of dimension two V . It has a basis:

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{with } [e_1, e_2] = -2e_2.$$

The vector (φ, ψ) is seen to correspond to the infinitesimal automorphism:

$$t \mapsto \frac{1 + \epsilon \varphi t}{1 - \epsilon(\varphi t + \psi)} = t + \epsilon(2\varphi t - \psi t^2).$$

We are interested in arrows ϕ that are embeddings (i.e. $\in \text{Gr}(m, V^{\oplus m})$) of Lie subalgebras, such

that the corresponding groups of infinitesimal automorphisms fix the subalgebra of a singularity of genus two inside $\mathbf{k}[[t_1]] \oplus \dots \oplus \mathbf{k}[[t_m]]$.

We start with some heuristics. Here is the unibranch case: the subalgebra of $\mathbf{k}[[t]]$ generated by $x = t^2 + ct^3$ is preserved by (φ, ψ) if and only if

$$(1 + 2\varphi)^2 t^2 - 2\psi(1 + 2\varphi)t^3 + c(1 + 2\varphi)^3 t^3 \text{ is a multiple of } t^2 + ct^3,$$

which reduces to $\varphi(1 + 2\varphi)c = \psi$. This further determines c if and only if $\varphi \neq 0$. Note that in this case (dimension one) the Lie subalgebra condition is automatically satisfied. We have found $(\varphi, \psi) \in \mathbf{k}^\times \times \mathbf{k}$.

The case of type II_2 -algebras is more interesting. Let $x = (t_1, \alpha t_2 + \beta t_2^2)$ be the generator of such an algebra. The image of x under $(\varphi_1, \psi_1, \varphi_2, \psi_2)$ is:

$$((1 + 2\varphi_1)t_1 - \psi_1 t_1^2, \alpha(1 + 2\varphi_2)t_2 + (\beta(1 + 4\varphi_2) - \alpha\psi_2)t_2^2),$$

from which we deduce:

$$\varphi_1 = \varphi_2 \quad \text{and} \quad 2\beta\varphi_2 - \alpha\psi_2 = -\alpha^2\psi_1. \quad (9)$$

Now let $\phi: \mathbf{k}^2 \rightarrow V^{\oplus 2}$ be given by $\begin{pmatrix} \varphi_{11} & \psi_{11} & \varphi_{12} & \psi_{12} \\ \varphi_{21} & \psi_{21} & \varphi_{22} & \psi_{22} \end{pmatrix}$, with Plücker coordinates w_{ij} for the minor of the i -th and j -th columns. The first condition in (9) immediately implies

$$w_{13} = 0 \quad \text{and} \quad w_{12} = -w_{23}, w_{14} = w_{34}. \quad (10)$$

The second condition in (9) implies

$$\alpha w_{12} = w_{14} \quad \text{and} \quad 2\beta w_{12} = \alpha w_{24} \quad (11)$$

so that $(\alpha, \beta) \in \mathbf{k}^\times \times \mathbf{k}$ is determined as soon as $w_{12}, w_{14} \neq 0$. Notice that (10) implies in particular the Plücker equation

$$w_{12}w_{34} - w_{13}w_{24} + w_{14}w_{23}.$$

It is easy to see that the condition for ϕ to be a sub-Lie algebra is

$$\text{rk} \begin{pmatrix} \varphi_{11} & \psi_{11} & \varphi_{12} & \psi_{12} \\ \varphi_{21} & \psi_{21} & \varphi_{22} & \psi_{22} \\ 0 & w_{12} & 0 & w_{34} \end{pmatrix} = 2,$$

translating into

$$\begin{aligned} w_{12}w_{13} &= 0 & w_{12}(w_{34} - w_{14}) &= 0 \\ w_{13}w_{34} &= 0 & (w_{12} + w_{23})w_{34} &= 0 \end{aligned}$$

automatically satisfied after (10) too. These equations cut inside $\mathbb{P}_{[w_{ij}]}^5$ the locus

$$(\mathbb{A}_{w_{14}/w_{12}}^1 \setminus \{0\}) \times \mathbb{A}_{w_{24}/w_{12}}^1.$$

More generally, given a type II subalgebra $R_{\alpha, \beta}$ of $W = \bigoplus_{i=1}^m \mathbf{k}[[t_i]]/(t_i^2)$, with generators of the form described in (1), the subalgebra of $V_{(\varphi_i, \psi_i)_{i=1, \dots, m}}^{\oplus m}$ preserving $R_{\alpha, \beta}$ is isomorphic to $\mathbf{k}^{\oplus m}$ with equations (see Lemma 3.1):

$$\begin{cases} 2\varphi_1 = \varphi_i = 2\varphi_m, & \text{for } i = 2, \dots, m-1, \\ 2\beta_{1,m}\varphi_m - \alpha_{1,m}\psi_m = -\alpha_{1,m}^2\psi_1; \end{cases}$$

it is easily seen that such a subalgebra of $V^{\oplus m}$ does not determine $R_{\alpha, \beta}$, but it does determine $(\alpha_{1,m}, \beta_{1,m})$. The case of type I is analogous.

We apply the preceding discussion to the study of automorphism groups of complete marked curves with a genus two singularity. The relevant category has been formalised in van der Wyck's thesis, see [vdW10, Proposition 1.102, Theorem 1.105 and Corollary 1.106], where he introduces the concept of type T reduced complete pointed curves with resolution, and the algebraic stack \mathcal{N}_T of such objects. The type encodes the number and isomorphism class of the singularities, the distribution of genus and markings among the components of the normalisation, and the adjacency data between components and singular points.

In the case that T has a unique singularity of genus two, with m one-marked rational branches, the stack \mathcal{N}_T is isomorphic to $[\mathbb{A}^1/\mathbb{G}_m]$ (see [vdW10, Examples 1.111-112]), so it has two points: one with \mathbb{G}_m , and the other with trivial stabiliser.

DEFINITION 3.5. The *atom* of type I_m (the name is borrowed from [AFSvdW17]) is obtained by gluing the subalgebra of $\mathbf{k}[t_1] \oplus \dots \oplus \mathbf{k}[t_m]$ generated by x_1, \dots, x_m as in (4) with m copies of $(\mathbf{k}[s], (s))$ under the identification $s_i = t_i^{-1}$. The multiplicative group \mathbb{G}_m acts on the atom by $\lambda.t_i = \lambda^3 t_i$ for $i = 1, \dots, m-1$ and $\lambda.t_i = \lambda t_i$ for $i = m$.

Similarly, the atom of type II_m is obtained by gluing the subalgebra of $\mathbf{k}[t_1] \oplus \dots \oplus \mathbf{k}[t_m]$ generated by x_1, \dots, x_{m-1} (and y) as in (2) (and following lines) with m copies of $(\mathbf{k}[s], (s))$ under the identification $s_i = t_i^{-1}$. There is a \mathbb{G}_m -action on the type II atom by $\lambda.t_i = \lambda t_i$ for $i = 1, m$ and $\lambda.t_i = \lambda^2 t_i$ for $i = 2, \dots, m-1$.

The curve with a genus two singularity and one-marked rational branches that has trivial automorphism group will be called the *non-atom*.

Again, here is a more geometric way to realise the dichotomy. The non-Gorenstein genus one singularity of type σ_0^{II} (resp. σ_0^I), with one-marked rational branches, has automorphism group \mathbb{G}_m^{m-1} (resp. \mathbb{G}_m^m). This acts on the tangent space at the singular point: of the lines fixed by this action, only one (call it ℓ') sits inside the open subset corresponding to the crimping space; all other lines in the crimping space are identified under the group action (call ℓ their equivalence class). Collapsing ℓ yields the non-atom, while collapsing ℓ' yields the atom.

As a third viewpoint, automorphisms can be studied by twisting the exact sequences of Lemma 3.1 by the ideal of the markings, and then taking global sections. The dicotomy arises then from the map ϕ : if the last condition imposed on infinitesimal automorphisms interweaves first and second order non-trivially (i.e. when $\beta_{1,m}$, resp. $\gamma_{m,m}$, are non-zero) then it is enough that automorphisms are trivial to second order on every branch for them to be trivial for good.

Finally, we shall rephrase the finiteness of automorphism groups explicitly in terms of types. Recall Smyth's description of genus one curves with no infinitesimal automorphisms [Smy11a, Proposition 2.3, Corollary 2.4].

DEFINITION 3.6. Let (C, p_1, \dots, p_n) be a reduced pointed curve. A connected subcurve $D \subseteq C$ is said to be *nodally attached* if $D \cap \overline{C \setminus D}$ consists of nodes only. We say that C is *residually DM* (rDM) if every nodal and nodally attached subcurve D of C , marked by $\{p_i \in D\} \cup (D \cap \overline{C \setminus D})$, is Deligne-Mumford stable. As usual, by *special points* we mean markings and nodes.

COROLLARY 3.7. Let (C, p_1, \dots, p_n) be a Gorenstein pointed curve of arithmetic genus two over a field of characteristic $\neq 2, 3$. $H^0(C, \Omega_C^\vee(-\sum_{i=1}^n p_i)) = 0$ is equivalent to either of the following:

- (i) C has a singularity of type $I_{m \geq 1}$: either all branches contain exactly one special point and C is the non-atom; or each of its axes contains at least one special point, and at least one branch has at least two. Furthermore C is rDM.

- (ii) C has a singularity of type $II_{m \geq 2}$: either all branches contain exactly one special point and C is the non-atom; or at least one of its twin branches contains a special point, each of its axes contains at least one, and at least one branch has at least two. Furthermore C is rDM.
- (iii) C has two elliptic m -fold points: each of their branches contains at least one special point or is shared, and at least one branch for each singular point contains at least one extra special point. Furthermore C is rDM.
- (iv) C has one elliptic m -fold point: one of its branches is a genus one curve, and every other branch contains at least one special point; otherwise, all branches contain at least one, and either two of its branches coincide, or at least one branch has at least two special points. Furthermore C is rDM.
- (v) C contains only nodes and is Deligne-Mumford stable.

4. Admissible covers and semistable tails

Given a family of prestable (pointed) curves of genus two over the spectrum of a discrete valuation ring $\mathcal{C} \rightarrow \Delta$, with smooth generic fibre \mathcal{C}_η and regular total space, we classify the subcurves of the central fibre \mathcal{C}_0 that can be contracted to yield a Gorenstein singularity of genus two.

In the genus one case, Smyth answered the analogous question by identifying the class of *balanced* subcurves [Smy11a, Definition 2.11]: subcurves of arithmetic genus one, such that, when breaking them into a *core* (minimal subcurve of genus one, not containing any separating node) and a number of rational trees (with root corresponding to the component adjacent to the core, and leaves corresponding to the components adjacent to the portion of \mathcal{C}_0 that is not contracted), the distance between any leaf and the root of any such tree is constant, not depending on the tree either.

In the case at hand, the answer turns out to be slightly more complicated: first, the special branch(es) of a type I (resp. II) singularity are connected through rational chains to a Weierstrass (resp. two conjugate) point(s) of the core. Second, the lengths of the rational trees may vary according to where their attaching points lie, but the special chains are always the shortest, and, together with the configuration of the attaching points on the core, they determine the length of any other chain.

4.1 A quick recap on admissible covers

While there are no special points on a smooth curve of genus zero or one, the simplest instance of Brill-Noether theory involves smooth curves of genus two. Every such C is *hyperelliptic*: it admits a unique (up to reparametrisation) two-fold cover $\phi: C \rightarrow \mathbb{P}^1$, induced by the complete canonical linear system, i.e. $|K_C|$ is the unique \mathfrak{g}_2^1 on C ; said otherwise, there is a unique element $\sigma \in \text{Aut}(C)$, called the *hyperelliptic involution*, such that $C/\langle \sigma \rangle \simeq \mathbb{P}^1$. A point $x \in C$ is called *Weierstrass* if it is a ramification point for ϕ (or, equivalently, a fixed point for σ); from the Riemann-Hurwitz formula it follows that there are six Weierstrass points on every smooth curve of genus two. Two points x_1, x_2 are said to be conjugate (write $x_2 = \bar{x}_1$) if there exists a point $z \in \mathbb{P}^1$ such that $\phi^{-1}(z) = \{x_1, x_2\}$ (or, equivalently, $\sigma(x_1) = x_2$). These notions may be extended to nodal curves by declaring (C, x) to be Weierstrass if its stabilisation lies in the closure of

$$\mathcal{W} = \{(C, x) \mid C \text{ smooth and } x \text{ Weierstrass}\} \subseteq \overline{\mathcal{M}}_{2,1},$$

and similarly for conjugate points. We then need to study the limiting behaviour of Weierstrass points when a smooth curve degenerates to a nodal one. This is a difficult problem when it

comes to higher genus curves; it has received considerable attention since the '70s, in work of E. Arbarello, D. Eisenbud, J. Harris, and many others. In our case it boils down to understanding admissible covers [HM82] of degree two with a branch locus of degree six.

DEFINITION 4.1. A family of *pointed hyperelliptic admissible covers* over S is a finite morphism $\psi: (C, D_R, p_1, \bar{p}_1, \dots, p_n, \bar{p}_n) \rightarrow (T, D_B, \psi(p_1), \dots, \psi(p_n))$ over S such that:

- (i) $(C, D_R, \mathbf{p}, \bar{\mathbf{p}})$ and $(T, D_B, \psi(\mathbf{p}))$ are prestable curves, with unordered smooth disjoint multisections D_R and D_B of length $2g+2$, and ordered smooth disjoint multisections $\mathbf{p}, \bar{\mathbf{p}}, \psi(\mathbf{p})$;
- (ii) C has arithmetic genus g , and $(T, D_B, \psi(\mathbf{p}))$ is a stable rational tree;
- (iii) ψ is a double cover on an open $U \subseteq T$ dense over S ;
- (iv) ψ is étale on $C^{\text{sm}} \setminus D_R$, and $\psi(p_i) = \psi(\bar{p}_i)$ for $i = 1, \dots, n$;
- (v) ψ maps D_R to D_B with simple ramification, and it maps nodes of C to nodes of T so that in local coordinates:

$$\psi^\#: \mathcal{O}_S[u, v]/(uv - s) \rightarrow \mathcal{O}_S[x, y]/(xy - t)$$

maps $u \mapsto x^i, v \mapsto y^i, s \mapsto t^i$ for $i = 1$ or 2 .

THEOREM 4.2 [HM82, Moc95]. *The moduli stack of pointed hyperelliptic admissible covers $\overline{\mathcal{H}}_{g,n}$ is a proper and smooth Deligne-Mumford stack with normal crossing boundary and forgetful morphisms $s: \overline{\mathcal{H}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}$ and $t: \overline{\mathcal{H}}_{g,n} \rightarrow \overline{\mathcal{M}}_{0,2g+2|n}$.*

$\overline{\mathcal{H}}_{g,n}$ provides a nice compactification of the locus of hyperelliptic curves in $\overline{\mathcal{M}}_{g,n}$. Besides the original sources, we have benefited from the exposition in [Dia85, Appendix 2], [Cuk89, Proposition (3.0.6)], and [HM98, Theorem 5.45].

We extract some information on the Weierstrass and conjugate loci in genus two: up to the involution action, the Weierstrass locus is isomorphic to $\overline{\mathcal{M}}_{0,6}/\mathfrak{S}_5$, and the conjugate locus is isomorphic to $\overline{\mathcal{M}}_{0,7}/\mathfrak{S}_6$. We remark that (C, x) being Weierstrass is an intrinsic notion if C is of compact type (or, more generally, tree-like), but it may depend on the smoothing otherwise (i.e. the fibre of $\overline{\mathcal{W}} \rightarrow \overline{\mathcal{M}}_2$ may have positive dimension). We spell out an explicit description.

- If x belongs to a component of genus one E , which is attached to another component of genus one at a node y , then x is Weierstrass iff $2x \sim 2y \in \text{Pic}(E)$; if instead E has a self-node that glues y_1 with y_2 , then x is Weierstrass iff $2x \sim y_1 + y_2 \in \text{Pic}(E)$.

If x is on a rational component R , x is Weierstrass if either R is attached to a genus one curve at two distinct points; or R has a self-node gluing y_1 and y_2 and is attached to a genus one tail at y_3 , in which case we require $\phi(y_1) = \phi(y_2)$ for a double cover $\phi: R \rightarrow \mathbb{P}^1$ ramified at x and y_3 ; or R has two self-nodes gluing y_1 with y_2 , and y_3 with y_4 , in which case we require x to be a ramification point for a double cover $\phi: R \rightarrow \mathbb{P}^1$ such that $\phi(y_1) = \phi(y_2)$ and $\phi(y_3) = \phi(y_4)$ - geometrically, if we embed \mathbb{P}^1 as a conic $C \subseteq \mathbb{P}^2$, the line through x and $\overline{y_1 y_2} \cap \overline{y_3 y_4}$ should be tangent to C at x . See Figure 2.

- If x_1 and x_2 are conjugate, they have to map to the same component of the target of the admissible cover. We may adapt the description of the previous point by replacing every condition on $2x$ by its analogue for $x_1 + x_2$. There are a few more situations to take into account: x_1 and x_2 could belong to a rational component R bubbling off from a Weierstrass point of a genus two curve; or bridging between two distinct curves of genus one; or x_1 and x_2 could lie on two distinct rational components R_1 and R_2 intersecting each other at one node

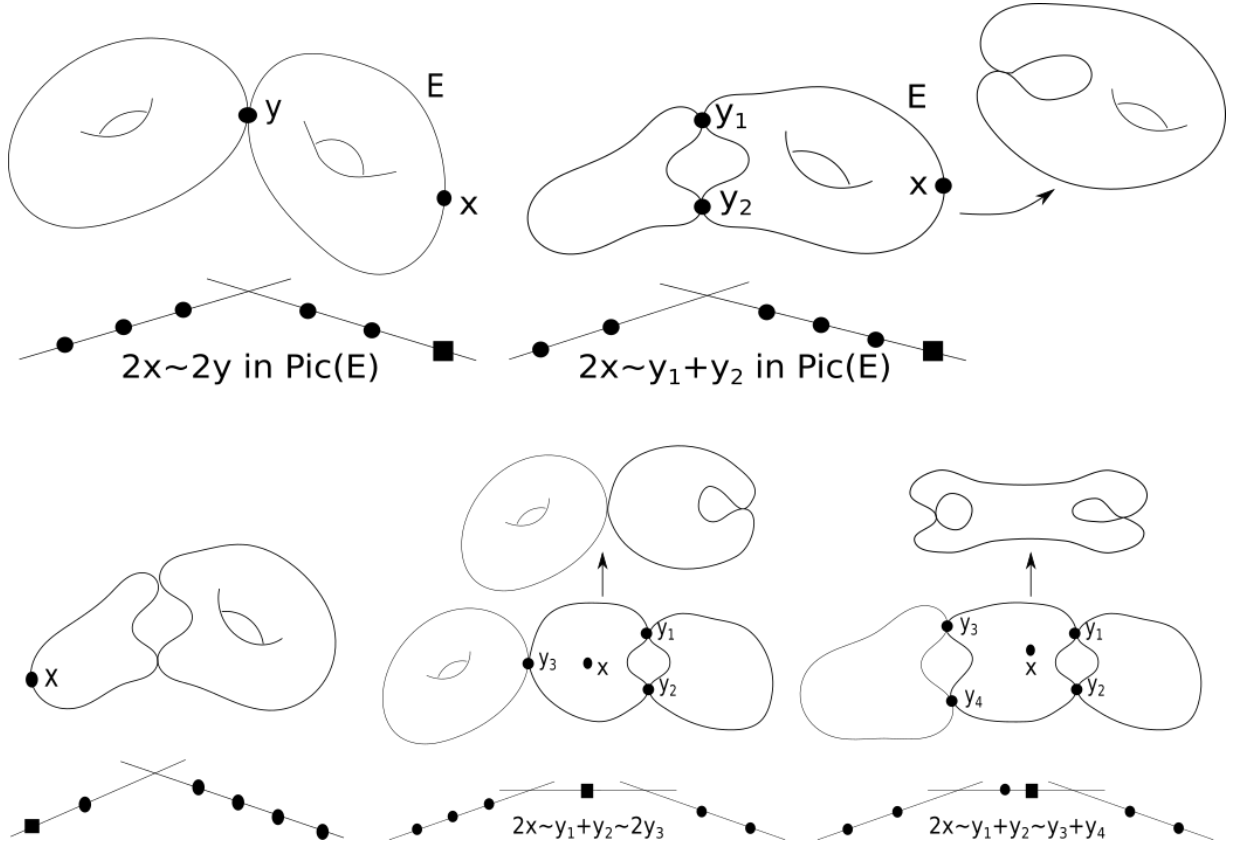


FIGURE 2. Admissible covers and Weierstrass points.

and meeting a curve of genus one in two distinct points (\dagger); or R_1 and R_2 intersecting each other in three points. See Figure 3.

Remark 4.3. In case (\dagger), the singularity of the total space of a smoothing $\mathcal{C} \rightarrow \Delta$ at the two distinguished nodes (separating the elliptic component from the rational chain) are both A_k for the same k , because they map to the same node of the target in the admissible cover. This consideration is stable under base change, and it therefore entails a symmetry of the rational chain in the model with regular total space.

4.2 A quick recap on logarithmically smooth curves

Logarithmically smooth curves are prestable curves endowed with a suitable logarithmic structure [Kat00]. There is a minimal [Gil12] such logarithmic structure $(C, M_C^{\min}) \rightarrow (S, M_S^{\min})$, determining all the others by pullback: this is the logarithmic structure on the moduli stack of prestable curves $\mathfrak{M}_{g,n}$ induced by its normal crossing boundary. More explicitly, M_S^{\min} is a locally free logarithmic structure, with generators of the characteristic sheaf \overline{M}_S^{\min} corresponding to nodes of the curve. The tropicalization \square of a logarithmically smooth curve (C, M_C) over a geometric point $(S = \text{Spec}(\mathbf{k}), M_S)$ consists of its dual graph $\Gamma(C)$ metrised in \overline{M}_S - the length of an edge is its smoothing parameter. In particular, a family of prestable curve over a trait Δ gives rise to a standard tropical curve (metrised in \mathbb{N}). After [GS13] and [CCUW20], piecewise-linear (PL) functions on \square with values in \overline{M}_S correspond to sections of $\Gamma(C, \overline{M}_C)$. The latter determine

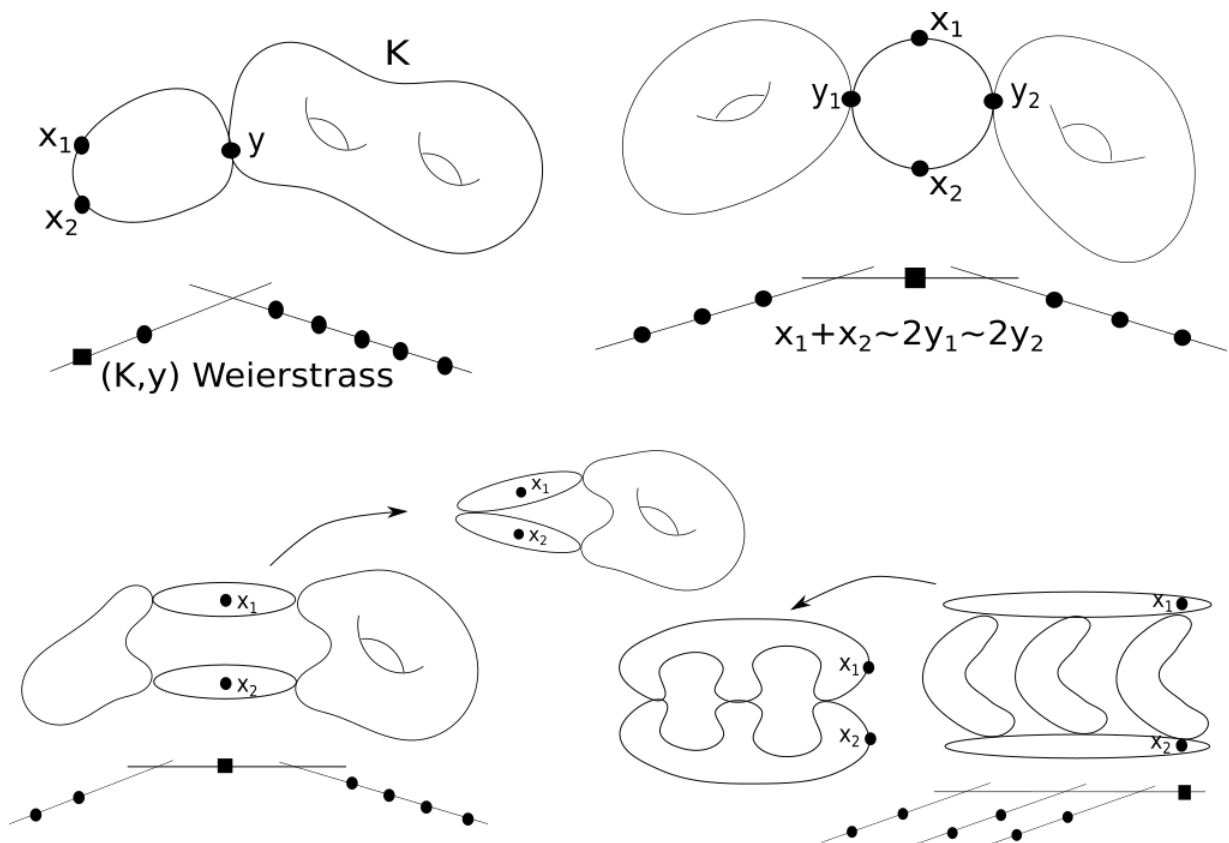


FIGURE 3. Admissible covers and conjugate points.

\mathcal{O}_C^* -torsors (and therefore line bundles) on C by the short exact sequence:

$$0 \rightarrow \mathcal{O}_C^* \rightarrow M_C^{\text{gp}} \rightarrow \overline{M}_C^{\text{gp}} \rightarrow 0.$$

A detailed analysis of this correspondence can be found in [RSPW19a, Proposition 2.4.1]. See also [Boz19, p.9] for a description in local charts.

The moduli space of pointed hyperelliptic admissible covers $\overline{\mathcal{H}}_{g,n}$ is also logarithmically smooth with locally free logarithmic structure induced by the normal crossing boundary [Moc95]. Generators of $\overline{M}_{\mathcal{H}}^{\min}$ correspond to the nodes of the source curve C , with two of them being identified if the cover is a local isomorphism around them. This allows us to exploit tropical geometry as above [CMR16].

4.3 Minimal curves

DEFINITION 4.4. A projective Gorenstein curve C is *minimal* if it contains no node x such that the normalisation of C at x consists of two connected components, one of which has genus zero.

When C is nodal, minimal is equivalent to semistable (no rational tails). Compare with [Cat82, Definition 3.2] for an even stronger notion. When C has arithmetic genus one, this is the same as saying that C contains no separating nodes. Recall [Smy11a, Lemma 3.3].

LEMMA 4.5. A minimal Gorenstein curve Z of arithmetic genus one can be: a smooth elliptic curve; a ring of $r \geq 1$ copies of \mathbb{P}^1 ; or an elliptic m -fold point whose normalisation is the disjoint

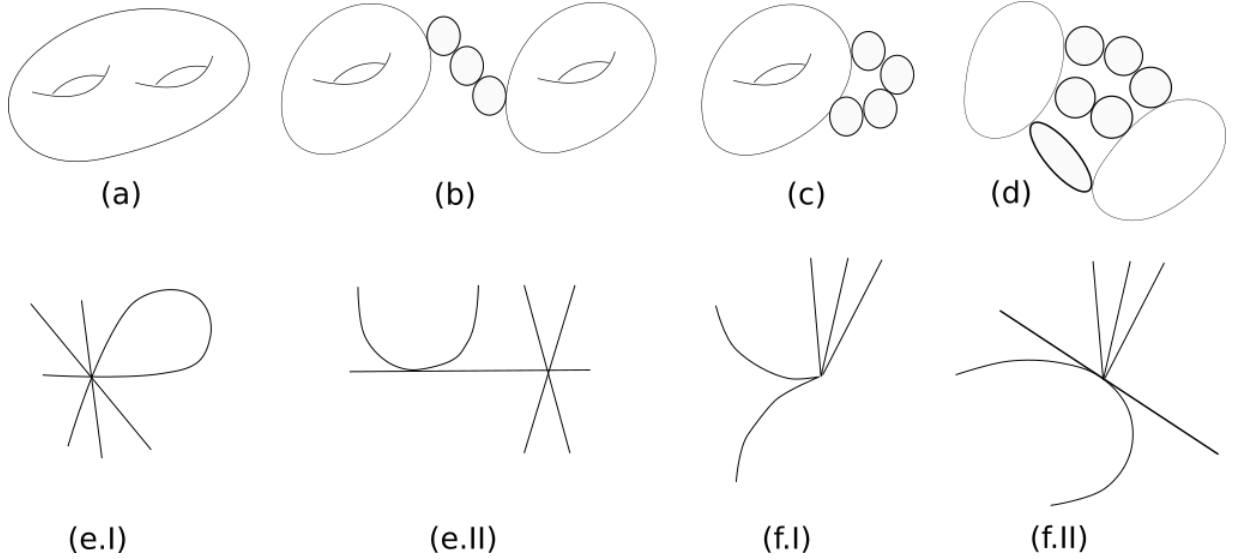


FIGURE 4. Minimal curves of genus two.

union of m copies of \mathbb{P}^1 . In any case $\omega_Z \simeq \mathcal{O}_Z$.

We provide a similar description of minimal curves of genus two; the proof is left to the reader.

LEMMA 4.6. *A minimal Gorenstein curve of genus two can be either of the following (Figure 4):*

- (a) *a smooth curve of genus two;*
- (b) *the union of two minimal Gorenstein curves of genus one, E_1 and E_2 , nodally separated by a (possibly empty) rational chain R ;*
- (c) *the union of a minimal Gorenstein curve of genus one E , and a (possibly empty) rational chain R , along two distinct nodes;*
- (d) *the union of two copies of $(\mathbb{P}^1, 0, 1, \infty)$ with three (possibly empty) rational chains R_0, R_1, R_∞ joining the homonymous points;*
- (e) *an elliptic m -fold point whose pointed normalisation is the disjoint union of either $m - 2$ copies of $(\mathbb{P}^1, 0)$ and a semistable rational chain $(R, 0, \infty)$, or $m - 1$ copies of $(\mathbb{P}^1, 0)$ and a 1-pointed minimal Gorenstein curve of genus one (if the latter is not irreducible and $m \neq 1$, there are two genus one subcurves sharing a rational branch);*
- (f) *or a singularity of genus two with m -branches, whose normalisation is the disjoint union of m copies of \mathbb{P}^1 .*

Remark 4.7. In both cases (e) and (f) there are *special* branches supporting the degree of ω_Z (compare with Definition 2.4 and Corollary 2.5; recall that the restriction of the dualising sheaf to a component introduces a twist by the conductor ideal, see [Cat82, Proposition 1.2]).

4.4 Semistable tails

Let $\overline{\mathcal{C}}_0$ be a minimal curve with a genus two singularity of type *I* (resp. *II*), and let $\overline{\mathcal{C}}$ be a one-parameter smoothing over a trait Δ . Let \mathcal{P} denote $\mathbb{P}(\pi_*\omega_{\overline{\mathcal{C}}/\Delta})$, which is a \mathbb{P}^1 -bundle over

Δ . It follows from an easy calculation (or from [Cat82, Theorem D]) that the canonical series is basepoint-free, and so there is a morphism:

$$\begin{array}{ccc} \overline{\mathcal{C}} & \xrightarrow{\quad} & \mathcal{P} \\ & \searrow \quad \swarrow & \\ & \Delta & \end{array}$$

such that, in the central fibre, it restricts to a double cover on the special branch (resp. an isomorphism on each of the special branches) and it contracts the axes. The general fibre is the hyperelliptic cover $\overline{\mathcal{C}}_{\bar{\eta}} \rightarrow \mathbb{P}_{\bar{\eta}}^1$, endowing $\mathbb{P}_{\bar{\eta}}^1$ with a length 6 branch divisor $B_{\bar{\eta}}$. Possibly after passing to a finite cover of Δ , $B_{\bar{\eta}}$ is actually defined over η , and we can take the stable model (\mathcal{T}, B) of $(\mathbb{P}_{\eta}^1, B_{\eta})$, together with its associated double cover \mathcal{C} . We thus have a diagram:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\psi} & \mathcal{T} \\ \downarrow \phi & & \downarrow \\ \overline{\mathcal{C}} & \longrightarrow & \mathcal{P} \end{array}$$

over Δ (by a slight abuse of notation), where the upper row is a family of admissible covers.

The line bundle $\mathcal{O}_{\mathcal{P}}(1)$ pulls back to ω_{π} on $\overline{\mathcal{C}}$. Its pullback $\mathcal{O}_{\mathcal{T}}(1)$ on \mathcal{T} has degree 1 on exactly one component of the tree. Pulling back further to \mathcal{C} , we gather the following information:

- (a) $\phi^*\omega_{\pi} = \omega_{\pi}(\mathcal{Z})$ for a vertical divisor \mathcal{Z} supported on the exceptional locus $\text{Exc}(\phi) =: Z$.
- (b) $\psi^*\mathcal{O}_{\mathcal{T}}(1) = \mathcal{O}_{\mathcal{C}}(q + \bar{q})$ for a choice of two conjugate points of \mathcal{C} lying over the same point of \mathcal{T} , belonging to the component on which $\mathcal{O}_{\mathcal{T}}(1)$ is ample.
- (c) \mathcal{Z} is the pullback of a vertical divisor on \mathcal{T} .

This description leads to the following simple observations:

- (I) If $\overline{\mathcal{C}}_0$ has a type *I* singularity, the branch of \mathcal{C}_0 corresponding to the singular branch of $\overline{\mathcal{C}}_0$ is attached to a Weierstrass point of \mathcal{Z} with respect to ψ .
- (II) If $\overline{\mathcal{C}}_0$ has a type *II* singularity, the branches of \mathcal{C}_0 corresponding to the twin branches of $\overline{\mathcal{C}}_0$ are attached to two conjugate points of \mathcal{Z} with respect to ψ .

Moreover, the distance of the special branch(es) from the core is always less than that of the axes; the ratio is roughly 1 : 3 in case *I*, and 1 : 2 in case *II*, but, more precisely, this depends on the relative position of the attaching points of the chains in the dual graph of the core. An elegant treatment uses the language of tropical geometry.

We consider the tropicalization $\square \rightarrow \mathbb{T}$ of ψ , as in Section 4.2. After further base-change and normalised blow-ups, we can assume that \mathcal{C} has regular total space; this only affects \square by subdividing edges, not changing their lengths. Now \square is nothing but the dual graph of the special fiber \mathcal{C}_0 , with edges of length 1.

The vertical divisor \mathcal{Z} can be represented by a piecewise-linear function on \square with integral slope along the edges; moreover, observation (c) above shows that λ is pulled back from a piecewise-linear function λ_T on \mathbb{T} - for this to be true we have to allow half-integral slopes along the edges. Finding λ becomes a simple matter of degree-matching on the tree \mathbb{T} ; this shows existence and uniqueness. Recall that the canonical divisor K_{\square} has degree:

$$2g(v) - 2 + \text{val}(v) \tag{12}$$

on a vertex v of \square , where $g: V(\square) \rightarrow \mathbb{Z}$ is the genus assignment, and $\text{val}(v)$ is the number of bounded edges adjacent to v ; (12) is also the degree of ω_π when restricted to the component of \mathcal{C}_0 corresponding to v .

Notice that \top is decorated with six unlabelled legs corresponding to the branch divisor B ; we call them *B-legs*. It follows from the Riemann-Hurwitz formula that the divisor $\mathcal{O}_\top(1)$ on \top pulling back to K_\square has degree:

$$\text{val}(v') - 2 + \frac{1}{2} \#\{f \text{ B-leg} \mid f \rightarrow v'\}$$

on a vertex v' of \top . Therefore, the equation that we have to solve is:

$$\text{val}(v') - 2 + \frac{1}{2} \#\{f \text{ B-leg} \mid f \rightarrow v'\} + \sum_{e \rightarrow v'} s(\bar{\lambda}_T, e) = 1 \quad (13)$$

on the vertex of \top corresponding to the special branch, and 0 otherwise; where $s(\bar{\lambda}_T, e)$ denotes the outgoing slope of $\bar{\lambda}_T$ along the edge e .

For the reader's benefit, we include Figure 5 to illustrate the shape of λ in the simplest possible case, namely when the core is smooth. The blue numbers represent the slope of λ along the corresponding edges. Figure 6 exhibits how the distance of the axes from the core can vary when the latter becomes more degenerate.

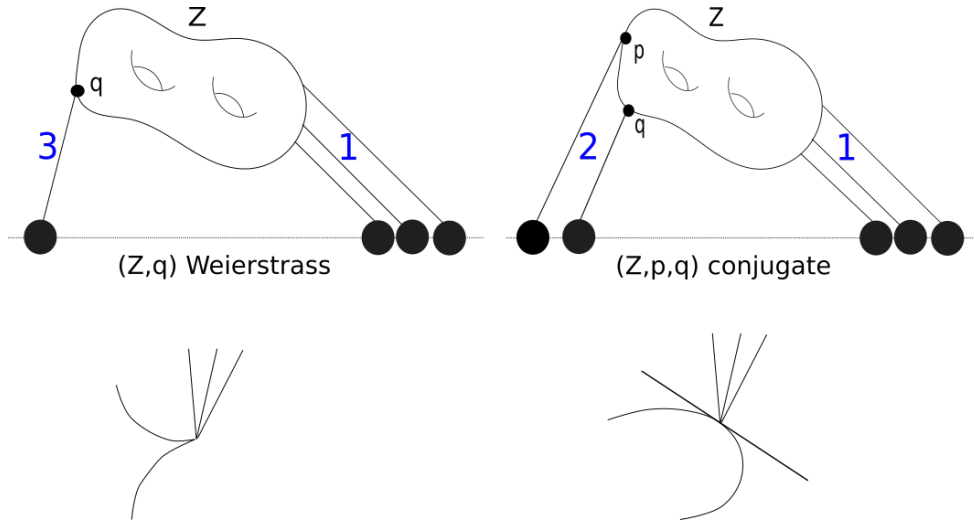


FIGURE 5. Semistable tail of a type I_4 (left), resp. type II_5 (right), singularity, generic case: the core is smooth, the singular branch is attached to a Weierstrass point (resp. the twin branches are attached to conjugate points), the other branches are attached to distinct points, and the corresponding edge-length is three (resp. two) times longer than the special one.

Two important observations allow us to write down λ explicitly in all possible situations:

- (i) The balancing equation (13) is unaffected by *tropical modifications*, i.e. growing a tree on which λ has constant slope 1.
- (ii) The balancing equation (13) is stable under edge contraction.

It follows that it is enough to study the case that the core consists of a configuration of rational curves; there are only two stable such configurations, named *dumbbell* and *theta*. Figure 7 (from

[BC20]) illustrates the situation: we draw both the source (above) and the target (below) of the tropical admissible cover; the blue numbers on the latter represent the slope of λ_T - that of λ can be recovered by multiplying with the expansion factor of ψ ; and the red vertices correspond to the special branches. The vertices corresponding to the axes of the genus two singularity do not appear in the picture: they lie at the same height as the red vertices, on an arbitrary configuration of trees emanating from the core, along which λ has slope 1.

PROPOSITION 4.8. *Let $\phi: \mathcal{C} \rightarrow \mathcal{C}$ be a birational contraction over the spectrum of a discrete valuation ring Δ , where: $\mathcal{C} \rightarrow \Delta$ is a family of prestable (reduced, nodal) curves of arithmetic genus two with smooth generic fibre \mathcal{C}_η ; $\overline{\mathcal{C}} \rightarrow \Delta$ is a family of Gorenstein curves with a genus two singularity of type I_m (resp. II_m) at $q \in \overline{\mathcal{C}}_0$. Denote by $(Z; q_1, \dots, q_m)$ the exceptional locus $\text{Exc}(\phi) = \phi^{-1}(q)$, marked with $Z \cap \overline{\mathcal{C}}_0 \setminus \overline{Z}$, where q_m corresponds to the singular branch of $\overline{\mathcal{C}}_0$ (resp. q_1, q_m correspond to the twin branches of $\overline{\mathcal{C}}_0$). Then:*

- Viceversa, every such genus two subcurve can be contracted to a Gorenstein singularity.

Proof. (of Proposition 4.9) By blowing down some rational tails outside Z , we can assume that $\mathcal{C}_0 \setminus Z = \sqcup_{i=1}^m T_i$ with each $T_i \simeq \mathbb{P}^1$. The image of $p_i(0)$ and $p_j(0)$ might now coincide for $i \neq j$.

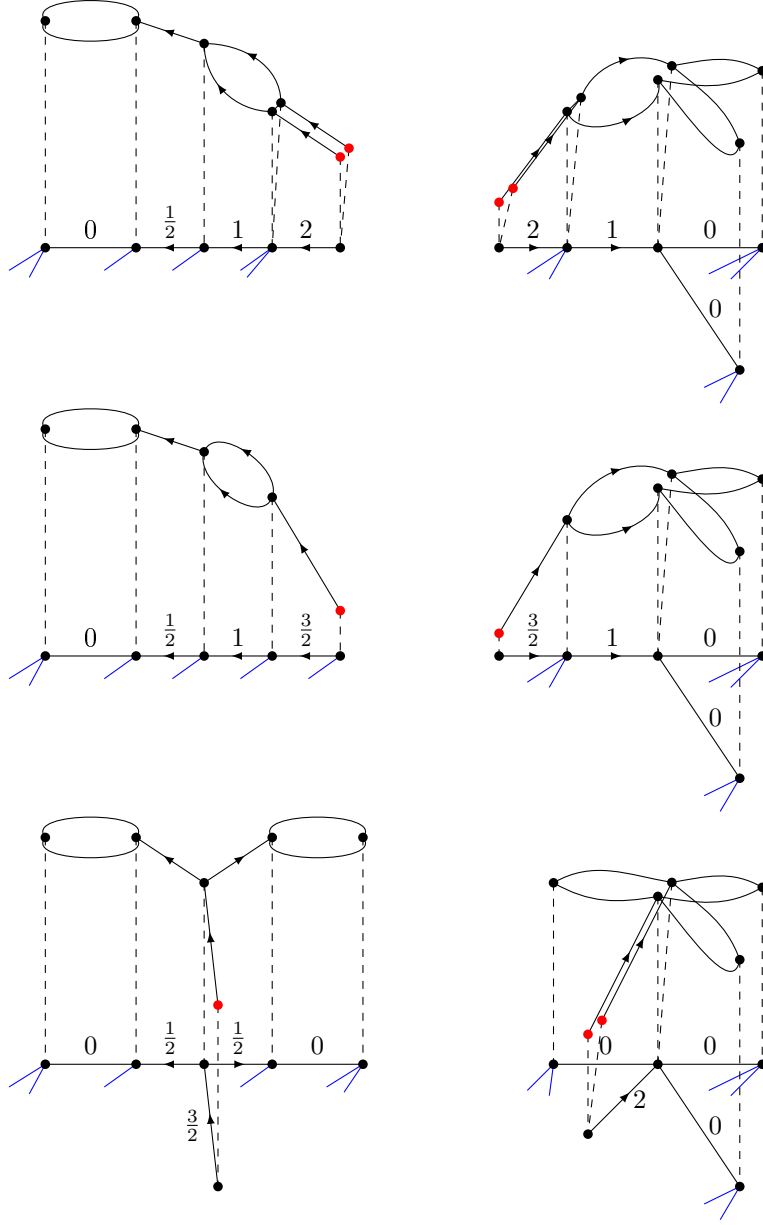


FIGURE 7. Tropicalization of semistable tails with maximally degenerated core: the dumbbell (l), and the theta graph (r). The red vertices correspond to the special branches.

The total space of the curve can still be assumed to be smooth by the Castelnuovo criterion. By abuse of notation, we denote the resulting family of pointed curves by $(\mathcal{C}, p_1, \dots, p_n)$. By assumption on the shape of Z , we can find an effective Cartier \mathcal{Z} supported on Z such that $\mathcal{L} = \omega_{\mathcal{C}/\Delta}(\mathcal{Z} + \sum p_i)$ is trivial on Z and relatively ample elsewhere (both on T_i and on the generic fibre). Consider a second line bundle $\mathcal{L}' = \mathcal{O}(2p_1 + \sum p_i)$ (resp. $\mathcal{O}(p_1 + \bar{p}_1 + \sum p_i)$). Since we assumed p_1 to be Weierstrass (resp. p_1 and \bar{p}_1 to be conjugate), $\mathcal{L}_\eta \simeq \mathcal{L}'_\eta$. On the other hand it is easy to see that the multi-degrees of \mathcal{L}_0 and \mathcal{L}'_0 coincide, as Z is unmarked and each rational tail is isomorphic to \mathbb{P}^1 ; it follows from the separatedness of $\text{Pic}_{\mathcal{C}/\Delta}^0 \rightarrow \Delta$ (see [Del85, p. 136] or [BLR90, §9.4]) that \mathcal{L} and \mathcal{L}' are isomorphic line bundles, so that, in particular, \mathcal{L} is trivial on a neighbourhood of Z . Observe now that

$$R^1\pi_*\mathcal{L}(-\mathcal{Z}) = R^1\pi_*\omega_{\mathcal{C}/\Delta}(\sum p_i) = 0$$

by semistability, hence $\pi_*\mathcal{L} \rightarrow \pi_*(\mathcal{L}|_Z) = \pi_*\mathcal{O}_Z$, which contains the constants, showing that \mathcal{L} is semiample along Z ; that it is along the T_i is easier.

We therefore have a well-defined morphism:

$$\mathcal{C} \xrightarrow{\phi} \overline{\mathcal{C}} = \text{Proj}_\Delta \left(\bigoplus_{n \geq 0} \pi_*\mathcal{L}^{\otimes n} \right) \rightarrow \Delta$$

associated to \mathcal{L} . The proof that $\overline{\mathcal{C}} \rightarrow \Delta$ is a flat family of Gorenstein curves goes along the lines of [Smy11a, Lemma 2.13] or [RSPW19a, Proposition 3.7.3.1]. It is then clear from the classification that it contains a type I_m (resp. II_m) singularity. \square

Remark 4.10. It follows that genus two Gorenstein singularities are smoothable.

It would be interesting to construct the contraction of Proposition 4.9 pointwise - as opposed to in a smoothing family - by extending the methods of [Boz19].

Caveat. Given a family of semistable curves $\pi: \mathcal{C} \rightarrow \Delta$ over a discrete valuation ring, and a line bundle \mathcal{L} on \mathcal{C} that is trivial on a higher genus subcurve Z of \mathcal{C}_0 and π -ample elsewhere, is \mathcal{L} π -semiample? is it relatively generated by global sections? In positive characteristic, a positive answer follows from results of S. Keel [Kee99]. If $\text{char}(\mathbf{k}) = 0$, the answer depends on the family; this should not come as a surprise, since we have seen how it relies on the tropicalization.

We construct a counterexample using the theory of limit linear series (l.l.s.): we produce a linear series that can be smoothed while having basepoints along a Weierstrass tail.³ For any such smoothing, the corresponding line bundle \mathcal{L} is not globally generated along Z - though we do not know how powers of \mathcal{L} behave, so we cannot conclude that \mathcal{L} is not semiample.

Let X_0 be the nodal curve obtained by attaching $R \simeq \mathbb{P}^1$ to a Weierstrass point q of a smooth genus two curve Z . Choose $d \gg 0$ ($d \geq 5$ is enough), and let us study the moduli space of complete linear systems of degree d on (smoothings of) X_0 ; with $r = d - 2$, the Brill-Noether number is $\rho = 2$ (the dimension of the Jacobian). On the other hand, assume that the R -aspect of the l.l.s. has $\mathcal{L}_{Y|Z} \simeq \mathcal{O}_Z$; then the Z -aspect has $\mathcal{L}_{Z|Z} \simeq \mathcal{O}_Z(dq)$, whose vanishing sequence is $\alpha_Z(q) = \{0, 1, \dots, d-5, d-4, d-2, d\}$, from which we deduce for the complementary aspect $\alpha_R(q) \geq \{0, 2, 4, 5, \dots, d-1, d\}$. We want to show that all such aspects are smoothable, by

³It was pointed out by F. Carocci that a similar computation can be carried out for a genus one tail as well. This shows that the regularity assumption of [Smy11a, Lemma 2.13] is necessary.

appealing to the Regeneration Theorem [HM98, Theorem 5.41]. If t denotes a coordinate on R near the node, we have a choice of a two-dimensional subspace of $\langle 1, t, t^2, t^3 \rangle_{\mathbf{k}} \subseteq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$ meeting the subspace $\langle t^2, t^3 \rangle_{\mathbf{k}}$ non-trivially; this can be described as the locus of lines in $\text{Gr}(1, \mathbb{P}^3)$ meeting a fixed line ℓ : a Schubert cycle of dimension 3. The Regeneration Theorem needs:

$$\dim(\mathcal{G}_d^r(X/B)) = \dim(B) + \rho.$$

We therefore have to put X_0 in a family over a base B of dimension 1 at least. We shall do so by considering the family X obtained by attaching R to a moving point of Z , so that X_0 is the fibre of X over $q \in Z$.

Let us start by examining the other possibilities for $\mathcal{G}_d^{d-2}(X_0)$: the R -aspect can in fact restrict to any line bundle of degree 0 on Z , which we are going to write as $\mathcal{O}_Z(p_1 + p_2 - 2q)$ for two moving points p_1, p_2 on Z (think of them as coordinates on $\text{Pic}(Z)$). Then $\mathcal{L}_Z = \mathcal{O}_Z((d-2)q + p_1 + p_2)$. In the table we collect: restrictions on p_1, p_2 (together with the dimension of the locus they cut in $\text{Pic}(Z)$) determining a fixed behaviour of α_Z (of which we record only the two largest values), the restrictions on α_R this imposes, and the corresponding locus of sections over R (in $\text{Gr}(1, \mathbb{P}^3)$, together with its dimension). We care about the sum of the two dimension terms.

$\subseteq \text{Pic}(Z)$	dim	$\alpha_{Z;d-3,d-2}$	$\alpha_{R;0,1}$	$\subseteq \mathbb{P}H^0(R, \mathcal{O}_R(d))$	dim
$p_1 + p_2 \sim 2q$	0	$\{d-2, d\}$	$\geq \{0, 2\}$	$\{\ell' \in \text{Gr}(1, \mathbb{P}^3) \ell' \cap \ell \neq \emptyset\}$	3
$2q \approx p_1 + p_2 \geq q$	1	$\{d-3, d-1\}$	$\geq \{1, 3\}$	\mathbb{P}^1	1
$p_1 + p_2 \not\geq q$	2	$\{d-3, d-2\}$	$\{2, 3\}$	pt	0

Letting q vary in $B = Z$, we may generically assume that it is not Weierstrass. The first column labelled “dim” now also accounts for the varying attaching point. We find:

$\subseteq \text{Pic}(X)$	dim	$\alpha_{Z;d-3,d-2}$	$\alpha_{R;0,1}$	$\subseteq \mathbb{P}H^0(R, \mathcal{O}_R(d))$	dim
$p_1 + p_2 \sim \omega_Z$	$0 + 1$	$\{d-2, d-1\}$	$\geq \{1, 2\}$	$(\mathbb{P}^2)^*$	2
$p_1 + p_2 \sim 2q$	$0 + 1$	$\{d-3, d\}$	$\geq \{0, 3\}$	\mathbb{P}^2	2
$\omega_Z, 2q \approx p_1 + p_2 \geq q$	$1 + 1$	$\{d-3, d-1\}$	$\geq \{1, 3\}$	\mathbb{P}^1	1
$p_1 + p_2 \not\geq q$	$2 + 1$	$\{d-3, d-2\}$	$\{2, 3\}$	pt	0

We conclude that $\mathcal{G}_d^{d-2}(X/B)$ has pure dimension 3. The Regeneration Theorem implies that all l.i.s. with $\mathcal{L}_{R|Z} \simeq \mathcal{O}_Z$ - in particular those containing Z in the base-locus - are smoothable.

5. The new moduli functors

The idea is to replace subcurves of positive genus with isolated singularities, the number of special points on the former bounding the number of branches of the latter. The following is a slight generalisation of [Smy11a, Definition 3.4].

DEFINITION 5.1. Let (C, p_1, \dots, p_n) be a reduced curve, marked by smooth points. For a connected subcurve $D \subseteq C$, we define its *level* to be:

$$\text{lev}(D) = |D \cap \overline{C \setminus D}| + |\{p_1, \dots, p_n\} \cap D|.$$

In this definition, the multiplicity of $D \cap \overline{C \setminus D}$ is not taken into account.

We omit the proof of the following lemma; compare with [Smy11a, Corollary 3.2, Lemma 3.5].

LEMMA 5.2. Let (C, p_1, \dots, p_n) be a pointed semistable curve of arithmetic genus two, with minimal genus two subcurve Z . For every subcurve $Z' \subseteq C$ of genus two, we have an inclusion $Z \subseteq Z'$ and $\text{lev}(Z) \leq \text{lev}(Z')$.

DEFINITION 5.3. We say that a point *cleaves* to a component of a curve if it is connected to it by a possibly empty chain of rational curves.

Remark 5.4. Singularities of genus zero and one appear in the miniversal family of singularities of genus two. Also, singularities of type *I* do appear in the miniversal family of singularities of type *II*, and viceversa. For low values of m , this follows from a neat result of Grothendieck ([CML13, p. 2277], see also [Arn72, Dem75]):

THEOREM 5.5. *Let (C, q) be a curve singularity of ADE type. The singularities appearing in the miniversal deformation of (C, q) are all and only the ADE singularities whose Dynkin diagram can be obtained as a full subgraph of the Dynkin diagram of (C, q) .*

We finally come to the definition of m -stability for curves of genus two.

DEFINITION 5.6. Fix positive integers $1 \leq m < n$. Let (C, p_1, \dots, p_n) be a connected, reduced, complete curve of arithmetic genus two, marked by smooth distinct points. We say that C is m -stable if:

- (i) C is Gorenstein with only: nodes; elliptic l -fold points, $l \leq m + 1$; type $I_{\leq m}$, type $II_{\leq m}$, and dangling II_{m+1} singularities of genus two, as singular points.
- (ii) If Z is a connected subcurve of arithmetic genus two, then $\text{lev}(Z) > m$.
- (iii) If E is a connected subcurve of arithmetic genus one, then either $\text{lev}(E) > m + 1$, or E is not nodally attached and p_1 cleaves to it.
- (iv) $H^0(C, \Omega_C^\vee(-\sum_{i=1}^n p_i)) = 0$.
- (v) If C contains a singularity of genus two, or an elliptic l -fold with a self-branch or a genus one branch, p_1 cleaves to one of the special branches (see Remark 4.7).

Remark 5.7. The definition is not \mathfrak{S}_n -symmetric. In the argument below, we exploit the asymmetry to write the dualising line bundle of a genus two (sub)curve Z as $\omega_Z \simeq \mathcal{O}_Z(q_1 + \bar{q}_1)$, where q_1 is the point of Z closest to p_1 , and \bar{q}_1 its conjugate, sometimes depending on a one-parameter smoothing. Compare with the situation in genus one, where the dualising line bundle of a minimal Gorenstein curve is trivial (all smooth points are non-special). We also refer to p_1 when deciding which genus one subcurve to contract first in case there are two of the same level.

Remark 5.8. If there is a nodally attached subcurve of genus one, condition (iii) and condition (iv) jointly imply condition (ii). Indeed, from Corollary 3.7 we have $\text{lev}(Z) \geq \text{lev}(E) - 1$. The only cases (up to relabelling) in which the level drops by one are: when $Z = (E, p_1, \dots, p_{l-2}, q_1, q_2) \sqcup_{\{q_1, q_2\}} (\mathbb{P}^1, q_1, q_2, p_{l-1})$; and when $Z = (E_1, p_1, \dots, p_{l-1}, q) \sqcup_q (E_2, q)$, where all the E have genus one.

We are going to prove the following:

THEOREM 5.9. *For $1 \leq m < n$, the moduli stack of n -pointed m -stable curves of genus two $\overline{\mathcal{M}}_{2,n}^{(m)}$ is a proper Deligne-Mumford stack of finite type over $\text{Spec}(\mathbb{Z}[\frac{1}{6}])$ - containing $\mathcal{M}_{2,n}$ as a dense open substack, and therefore irreducible.*

Proof. (i) -*Algebraicity (diagonal)*. The diagonal $\Delta: \overline{\mathcal{M}}_{2,n}^{(m)} \rightarrow \overline{\mathcal{M}}_{2,n}^{(m)} \times \overline{\mathcal{M}}_{2,n}^{(m)}$ is representable, quasiprojective, and of finite type. Since m -stable curves are canonically polarised, it follows from Grothendieck's theory of Hilbert schemes that the *Iso*-functor between two m -stable curves over S is representable by a quasiprojective scheme over S .

- (ii) -*Algebraicity (atlas) & irreducibility.* There exists an irreducible scheme H , of finite type over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{6}])$, with a smooth and surjective morphism $H \rightarrow \overline{\mathcal{M}}_{2,n}^{(m)}$. Fix an integer $N > 2 + 8(m + 1)$; let $d = N(2 + n)$ and $r = d - 2$. By Lemma 5.10 below, every n -pointed m -stable curve over a field \mathbf{k} admits a pluri-log-canonical embedding of degree d in \mathbb{P}^r . Let H_0 denote the Hilbert scheme of degree d , genus two curves in \mathbb{P}^r . Let $H_1 \subseteq H_0 \times (\mathbb{P}^r)^{\times n}$ denote the locally closed subscheme consisting of $([C], p_1, \dots, p_n)$ such that every p_i belongs to the smooth locus of C ; this is open in the incidence variety. By Lemma 5.11 below, there is an open subscheme $H_2 \subseteq H_1$ parametrising m -stable curves (note that H_1 is of finite type over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{6}])$, and in particular Noetherian). By the representability of the Picard scheme [MFK94, Proposition 5.1], there is a locally closed subscheme $H \subseteq H_2$ representing $([C], p_1, \dots, p_n)$ such that $\mathcal{O}_C(1) = (\omega_C(\sum_{i=1}^n p_i))^N$. Now there is a morphism $H \rightarrow \overline{\mathcal{M}}_{2,n}^{(m)}$ that is surjective by construction, and smooth because two different embeddings of an m -stable curve differ by the choice of a basis of $H^0(C, (\omega_C(\sum_{i=1}^n p_i))^N)$. Since every m -stable curve is smoothable (Remark 4.10 and [Kol96, II.6.3]), H is irreducible.
- (iii) -*DM.* The diagonal $\Delta: \overline{\mathcal{M}}_{2,n}^{(m)} \rightarrow \overline{\mathcal{M}}_{2,n}^{(m)} \times \overline{\mathcal{M}}_{2,n}^{(m)}$ is unramified. It is enough to show that, for an m -stable curve (C, p_1, \dots, p_n) over a field \mathbf{k} , the *Iso*-group scheme $\mathrm{Aut}_{\mathbf{k}}(C, p_1, \dots, p_n)$ is unramified. Its tangent space at the identity can be identified with the vector space $H^0(C, \Omega_C^\vee(-\sum_{i=1}^n p_i))$, which vanishes by Definition 5.6iv. Note that we need the assumption on the base characteristic in order to translate this vanishing into a combinatorial criterion on the pointed normalisation and singularity type of the curves (Corollary 3.7).
- (iv) -*Properness.* Follows from the valuative criterion (Proposition 5.12).

□

LEMMA 5.10 (boundedness). *If (C, p_1, \dots, p_n) is an m -stable curve of genus two, the N -th power of $A = \omega_C(\sum_{i=1}^n p_i)$ is very ample for every $N > 2 + 8(m + 1)$.*

Proof. It is enough to show that, for every pair of points $p, q \in C$ (possibly equal):

- (i) *basepoint-freeness:* $H^1(C, A^{\otimes N} \otimes I_p) = 0$;
- (ii) *separating points and tangent vectors:* $H^1(C, A^{\otimes N} \otimes I_p I_q) = 0$.

By Serre duality we may equivalently show that $H^0(C, \omega_C \otimes A^{-N} \otimes (I_p I_q)^\vee) = 0$. Let $\nu: \tilde{C} \rightarrow C$ be the normalisation, and let $\nu^{-1}(p) = \{p_1, \dots, p_h\}$, $\nu^{-1}(q) = \{q_1, \dots, q_k\}$, with $h, k \leq m + 1$. It follows from Proposition 2.2 (and [Smy11a, Proposition A.3]) that $\nu_* \mathcal{O}_{\tilde{C}}(-D) \subseteq I_p I_q$ for $D = 4(\sum_{i=1}^h p_i + \sum_{j=1}^k q_j)$ (note that $\deg(D) \leq 8(m + 1)$); furthermore, the quotient is torsion, therefore, by applying $\mathcal{H}om(-, \mathcal{O}_C)$, we find $(I_p I_q)^\vee \subseteq \nu_* \mathcal{O}_{\tilde{C}}(D)$. It is thus enough to show that $H^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(D) \otimes \nu^*(\omega_C \otimes A^{-N})) = 0$. Finally, $\nu^* \omega_C$ (resp. $\nu^* A$) has degree at most two (resp. at least one) on each component of \tilde{C} , hence it is enough to take $N > 2 + 8(m + 1)$. □

LEMMA 5.11 (deformation openness). *Let $(\mathcal{C}, \sigma_1, \dots, \sigma_n) \rightarrow S$ be a family of curves over a Noetherian base scheme with n sections. The locus*

$$\{s \in S \mid (\mathcal{C}_{\bar{s}}, \sigma_1(\bar{s}), \dots, \sigma_n(\bar{s})) \text{ is } m\text{-stable}\}$$

is Zariski-open in S .

Proof. Having connected fibres which are Gorenstein curves of arithmetic genus two is an open condition (see for example [Sta20, Tag 0E1M]). Only singularities of genus zero (nodes), one (elliptic l -folds), and two may then occur.

The case $m = 1$ deserves special attention. In this case, that condition (i) is open follows from acknowledging that $I_1 = A_4$, $I_2 = A_5$, while tacnodes, cusps, and nodes are A_3 , A_2 , and A_1 -singularities respectively, and from Grothendieck's result on the deformation theory of ADE singularities (see Theorem 5.5 above).

The case $m \geq 2$ simply follows from upper semicontinuity of embedded dimension and the fact that we have exhausted all possible Gorenstein singularities of genus ≤ 2 , and embedding dimension $\leq m + 1$.

Condition (iv) translates to: the locus where the automorphism group is unramified is open in the base. Homogeneity can be used to prove that being unramified, which is open in the source, is also open in the target, for the structural morphism of a group scheme; see the end of the proof of [Smy11a, Lemma 3.10].

The other conditions are topological, hence constructible. Under Noetherian assumptions, it is enough to check their openness over the spectrum of a discrete valuation ring. Assume that the geometric generic fibre $C_{\bar{\eta}}$ contains two genus one subcurves $E_{1,\bar{\eta}}$ and $E_{2,\bar{\eta}}$; their closures E_1 and E_2 in \mathcal{C} are then flat families of genus one curves over Δ . If $E_{1,\bar{\eta}}$ and $E_{2,\bar{\eta}}$ are disconnected, then so are E_1 and E_2 , by local constancy of the number of connected components (from the Zariski decomposition and [Sta20, Tag 0E0D]). If $E_{1,\bar{\eta}}$ and $E_{2,\bar{\eta}}$ are joined by a disconnecting node $q_{\bar{\eta}}$, then so are $E_{1,0}$ and $E_{2,0}$; indeed, the unique limit of $q_{\bar{\eta}}$ must be a singular point of the projection, but cannot be any worse than a node (because we have already used up all of our genus allowance). Finally, if $E_{1,\bar{\eta}}$ and $E_{2,\bar{\eta}}$ share a branch, then so do $E_{1,0}$ and $E_{2,0}$; on the other hand, if $E_{i,\bar{\eta}}$ has more than one branch, then so does E_i . Similarly, if $C_{\bar{\eta}}$ contains only one subcurve of genus one, with two nodes joined by a rational chain, so does C_0 . To summarise,

$$|E_{i,\bar{\eta}} \cap \overline{C_{\bar{\eta}} \setminus E_{i,\bar{\eta}}}| = |E_{i,0} \cap \overline{C_0 \setminus E_{i,0}}|.$$

The number of markings on E_i is also constant. Hence we can deduce condition (iii) for $C_{\bar{\eta}}$ from the same condition on C_0 . Condition (ii) follows in this case from Remark 5.8; it can be proven analogously when there is no subcurve of genus one.

Finally, suppose that $C_{\bar{\eta}}$ has a genus two singularity, then so does C_0 . The (union of the) distinguished branch(es) $E_{\bar{\eta}}$ of $C_{\bar{\eta}}$ is a genus one singularity, and so is its limit E_0 in C_0 . It has to contain the distinguished branch(es) of C_0 , because any subcurve not containing them has genus zero; therefore, by assumption, E_0 contains $p_{1,0}$. Then also $E_{\bar{\eta}}$ contains $p_{1,\bar{\eta}}$, because the markings are contained in the non-singular locus of the curve. Similarly, if $C_{\bar{\eta}}$ has a genus one singularity with a self-branch, the limit of such a branch is a genus one subcurve E_0 of C_0 ; the latter may very well acquire a genus two singularity, but E_0 will contain the special branches of it, so it will be connected to p_1 . We conclude as above. The case that $C_{\bar{\eta}}$ contains a genus one subcurve of low level is analogous. We have thus proved that condition (v) is open. \square

PROPOSITION 5.12 (Valuative criterion of properness for $\overline{\mathcal{M}}_{2,n}^{(m)}$). *Given a smooth n -pointed curve of genus two C_{η} over a discrete valuation field $\eta = \text{Spec}(K) \hookrightarrow \Delta$, there exists a finite base-change $\Delta' \rightarrow \Delta$ after which C_{η} can be completed to an m -stable curve over Δ' . Two such models are always dominated by a third one.*

Existence of limits. By properness of the moduli space of pointed admissible covers, after a finite base-change $\Delta' \rightarrow \Delta$ we can complete C_{η} to a prestable curve $\mathcal{C}' \rightarrow \Delta'$ together with an admissible cover $\mathcal{C}' \rightarrow \mathcal{T}'$ over Δ' . We drop the primes from the notation.

The position of p_1 determines a piecewise-linear function $\lambda: \square \rightarrow \mathbb{R}$. Roughly speaking, the subcurve to be contracted will be of the form $\lambda^{-1}(\mathbb{R}_{>\rho})$, where ρ is the value attained by λ on

a vertex of \sqsubset , such that there are $\leq m$ edges leaving $\lambda^{-1}(\{\rho\})$ in the upward direction, and $\geq m+1$ leaving it downwards. Cutting λ at level ρ , and subdividing \sqsubset accordingly, provides a partial destabilisation $\tilde{\mathcal{C}} \rightarrow \mathcal{C}$, and a new piecewise-linear function μ on $\tilde{\sqsubset}$, such that the line bundle $\omega_{\tilde{\mathcal{C}}}(\mathbf{p})(\mu)$ induces the desired contraction to the m -stable limit.

We construct a piecewise-linear function λ_T on \top first. We call *core* of \top the image of the core of \sqsubset . Up to a global translation by \mathbb{R} , the function λ_T is characterised by having:

- slope 2 or $\frac{3}{2}$ towards the core on the edges separating p_1 from the core, according to whether they are “conjugate” or “Weierstrass” (i.e. whether the admissible cover is ramified or not over them), and slope 2 along the infinite leg corresponding to p_1 ;
- slope 1 or $\frac{1}{2}$ towards the core on every other edge and infinite leg outside the core, according to whether they are “conjugate” or “Weierstrass”;
- slope on the core determined by balancing (13) as in Figure 7.

We fix the value of λ_T by saying its maximum is 0 (compare with the level graphs of [BCG⁺19]), although this choice is arbitrary. Let λ be the pullback piecewise-linear function on \sqsubset .

DEFINITION 5.13. Given $\rho \in \mathbb{R}_{\leq 0}$, we call $\sqsubset_{>\rho} := \lambda^{-1}(\{\rho\})$ the *overcurve* of ρ . We call *undervale* of $\sqsubset_{>\rho}$ the number of bounded edges and infinite legs leaving $\lambda^{-1}(\{\rho\})$ in the downward direction; *overvalence* the number of edges and (subdivided) legs leaving $\lambda^{-1}(\{\rho\})$ in the upward direction (take it to be 0 when $\rho = 0$).

In order to find the m -stable limit, there is a unique cutoff value ρ . We distinguish two cases.

- (i) Assume for simplicity that the core is irreducible, corresponding to a vertex $v \in \sqsubset$. There is a unique $\rho \in \mathbb{R}_{\leq 0}$ such that the undervale of $\sqsubset_{>\rho}$ is at least $m+1$, while the overvalence is at most m . Indeed, by stability (best to argue on \top), every vertex apart from v has a unique edge going up and at least two going down, and for $\rho \ll 0$ both values of valence coincide with n , the number of infinite legs, which is $\geq m+1$. The same argument applies, whatever \sqsubset may look like, as soon as ρ is low enough, so that $\sqsubset_{>\rho}$ is connected of genus two. Subdivide all the edges of \sqsubset at height ρ ; this produces a model $\tilde{\sqsubset}$ of \sqsubset , and consequently a blow-up $\tilde{\mathcal{C}} \rightarrow \mathcal{C}$. Let $\mu := \max\{\lambda - \rho, 0\}$ denote a new piecewise-linear function on $\tilde{\sqsubset}$. The curve:

$$\overline{\mathcal{C}} = \underline{\text{Proj}}_{\Delta} \left(\bigoplus_{d \geq 0} \omega_{\tilde{\mathcal{C}}/\Delta}(p_1 + \dots + p_n)(\mu)^{\otimes d} \right)$$

contains a genus two singularity with less than m branches (the overvalence) and more than $m+1$ special points (the undervale) in the central fibre; it is endowed with a birational contraction $\phi: \tilde{\mathcal{C}} \rightarrow \overline{\mathcal{C}}$, see Proposition 4.9. Upon contracting any rational tail away from the singularity, $\overline{\mathcal{C}}_0$ is the m -stable limit of C_η .

- (ii) Suppose that the core consists of two curves of genus one, separated by a rational bridge. Start from the one on which λ attains the maximum value - call it E_2 . Notice that λ has slope 1 on all edges departing from E_2 , so the situation looks like the one of [RSPW19a]. Consider only Δ_2 , the connected component of $\lambda^{-1}(\mathbb{R}_{>\rho})$ that contains E_2 . Sliding ρ towards $-\infty$, one of the following two situations occurs first:
- (a) the undervale of Δ_2 is at least $m+2$, or
 - (b) $\lambda^{-1}(\mathbb{R}_{>\rho})$ is a curve of genus two.

In the latter case, we proceed as in the previous point. Suppose instead that we can find ρ such that Δ_2 has genus one and undervalence at least $m + 2$. Subdivide \square at $\lambda^{-1}(\rho)$ to get $\tilde{\square}$; set $\mu := \max\{\lambda - \rho, 0\}$, and let \mathcal{C}' be constructed by applying the Proj construction to $\omega_{\tilde{\mathcal{C}}/\Delta}(p_1 + \dots + p_n)(\mu)$. The central fibre \mathcal{C}'_0 contains an elliptic l -fold point instead of E_2 ($l \leq m$ being the overvalence of Δ_2), by Smyth's contraction lemma [Smy11a, Lemma 2.13] or [RSPW19a, Proposition 3.7.6.1]. Now, contract Δ_2 formally to a single point $*$, producing a new metric graph \square' . Consider a piecewise-linear function λ' on \square' that has constant ingoing slope 1 towards E_1 ; set $\lambda'(*) = 0$. We look for a $\rho' > 0$ such that the overcurve Δ'_1 has undervalence at least $m + 2$. If it can be found, setting $\mu' := \max\{\lambda' - \rho', 0\}$ and subdividing \square' to make it piecewise-linear, we construct $\bar{\mathcal{C}}$ as above. $\bar{\mathcal{C}}_0$ contains two disjoint genus one subcurves, each one having level at least $m + 2$. If ρ' cannot be found, set it to be 0; proceeding as above, we construct $\bar{\mathcal{C}}$ with central fibre containing two elliptic singularities sharing a branch. Notice that p_1 cleaves to the one replacing E_1 , which is the only one with possibly few special points.

Finally, if both $\lambda(E_1) = \lambda(E_2) = 0$ (notice that in this case p_1 cleaves to the midpoint of the rational bridge separating them), then the construction should start by lowering ρ simultaneously for both subcurves. If any of the two reaches undervalence $m + 2$ before $\square_{\geq \rho}$ becomes of genus two, then we contract this subcurve first, and proceed as above to contract the other one.

□

Example 5.14. We illustrate the above proof by means of an example, see Figure 8. Suppose the central fibre has two elliptic subcurves separated by a rational bridge, each of them connected to a two-pointed rational tail. Suppose furthermore that the model has regular total space; thus, every finite edge of the tropicalization has length 1. The rational tail supporting p_1 is attached to a point of E_1 that is 2-torsion with respect to the other elliptic curve. The picture on the left displays the various cutoff levels ρ depending on a choice of m . On the right, a cartoon picture of the corresponding m -stable limits.

Uniqueness of limits. Suppose that $\mathcal{C} \rightarrow \Delta$ and $\mathcal{C}' \rightarrow \Delta'$ are m -stable limits of C_η . Up to a further base-change (and a slight abuse of notation), there is a diagram:

$$\begin{array}{ccc}
 & \mathcal{C}^{ss} & \\
 \phi \swarrow & & \searrow \phi' \\
 \mathcal{C} & & \mathcal{C}' \\
 & \searrow & \swarrow \\
 & \Delta &
 \end{array}$$

extending the isomorphism between the generic fibres, where \mathcal{C}^{ss} has semistable central fibre and regular total space, by the semistable reduction theorem.

Claim 1. If \mathcal{C}'_0 has only singularities of genus $\leq i$ ($i = 0, 1$), then so does \mathcal{C}_0 .

First, assume that \mathcal{C}'_0 has only nodes. If \mathcal{C}_0 has a singular point x of genus one, $E := \phi^{-1}(x)$ is an *unmarked* subcurve of arithmetic genus one and level $\leq m + 1$ of \mathcal{C}_0^{ss} . Then so is $\phi'(E)$: indeed, ϕ' being a contraction, it has connected fibres, which bars $\phi'|_E$ from being a finite cover of a \mathbb{P}^1 . This contradicts the m -stability of \mathcal{C}' . We may argue similarly if x is a genus two singularity

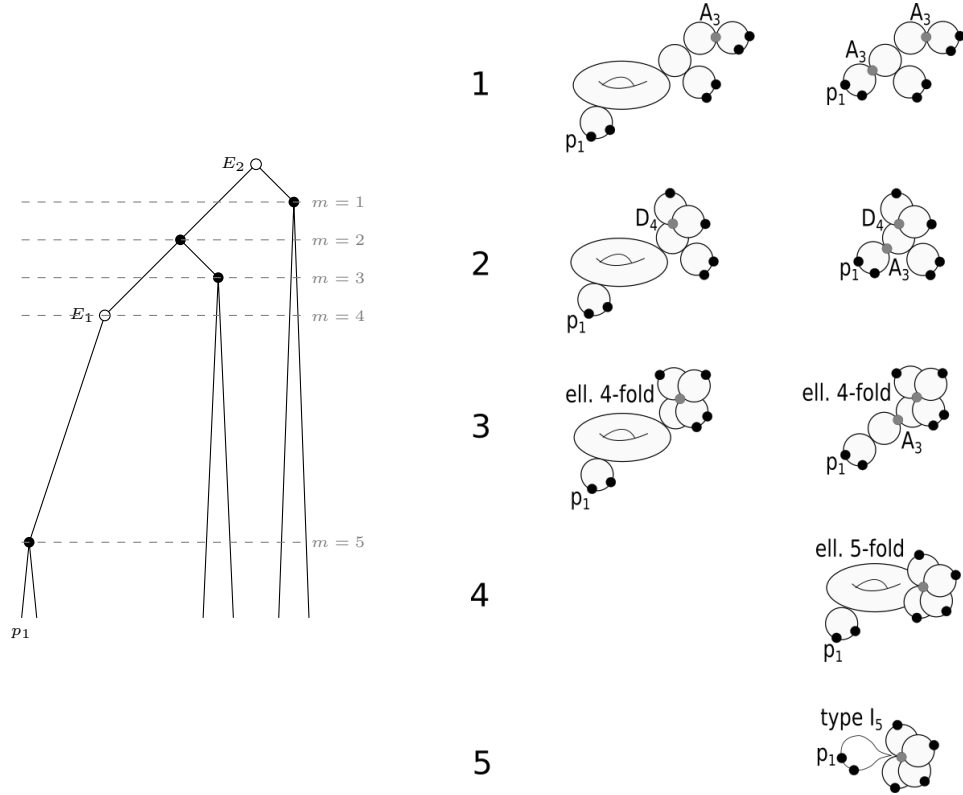


FIGURE 8. Valuative criterion: an example with two genus one subcurves. Left: the graph of λ with values of ρ depending on m . Right: m , the first contraction C' , the end result \bar{C} .

with $\leq m$ branches. On the other hand, if x is dangling II_{m+1} , there is a -1 -curve R adjacent to $\phi^{-1}(x)$; ϕ' must contract R by DM stability of \mathcal{C}' , hence $\phi'(\phi^{-1}(x))$ is again a genus two curve of level $\leq m$.

Assume now that \mathcal{C}'_0 has at worst singularities of genus one, while \mathcal{C}_0 has a singularity x of genus two; the case of a dangling II_{m+1} can be excluded as above. Then $\mathcal{C}_0^{ss} = Z \cup R_1 \cup \dots \cup R_l$, where $Z = \phi^{-1}(x)$ is a semistable tail of the genus two singularity x , and R_1, \dots, R_l , $l \leq m$, are rooted rational trees, the root components corresponding to the branches of x under ϕ . If Z has geometric genus two, or is irreducible of geometric genus one, $\phi'(Z)$ violates the m -stability of \mathcal{C}' . If Z contains a unique subcurve E of genus one, with a rational bridge R , then at least one of R_1, \dots, R_l must be connected to R , otherwise \mathcal{C}'_0 - which is obtained by contracting a balanced subcurve around E , not including the entire R - would have a positive dimensional automorphism group (scaling a semistable component of R). Therefore $\text{lev}(E) \leq (l-1) + 2 \leq m+1$. Similarly, if Z contains two subcurves of genus one E_1 and E_2 , then $(\text{lev}(E_1) - 1) + (\text{lev}(E_2) - 1) \leq l$, hence at least one of the two has level $\leq m+1$. In any case, $\phi'(E)$ contradicts the m -stability of \mathcal{C}' .

Claim 2. We may assume that \mathcal{C}^{ss} contains either no -1 -curve, or only one, which is contracted by neither ϕ nor ϕ' .

If there is a -1 -curve contracted by both, ϕ and ϕ' factor through a smaller regular model. Assume there is a -1 -curve not contracted by ϕ . Then, by condition (iv), its image has to be one of the special branches of a genus two singularity; on the other hand, by condition (v), the only special branch of a singularity of type I must contain some special points; therefore we conclude

that the singular point x of \mathcal{C}_0 is dangling of type II_{l+1} , $l \leq m$. If we let $Z = \phi^{-1}(x)$, we may write $\mathcal{C}_0 = Z \cup R_0 \cup \dots \cup R_l$, with $R_0 = R$, and R_1 the tail containing p_1 . By Claim 1, ϕ' has to contract a genus two subcurve Z' as well. If Z' is of the shape described in Proposition 4.9 and it contains R , then $Z \subsetneq Z'$ is easily seen, which implies \mathcal{C}'_0 has a singularity of type II with more than $m+1$ branches, by the level condition (ii) on \mathcal{C}_0 ; on the other hand, if $Z' \subsetneq Z$ were disjoint from R , then \mathcal{C}'_0 would not satisfy condition (ii). Similarly, if Z' were to be of the shape described in Proposition 4.9, then R_0 and R_1 would have to meet in a “trunk” T attached to a Weierstrass point of the core of \mathcal{C}_0^{ss} ; if Z' started down along the trunk closer to the core, then $Z' \subsetneq Z$ (so \mathcal{C}'_0 would not satisfy (ii)), while if Z' started at the top of the trunk or further away from the core, then $Z \subsetneq Z'$ (so \mathcal{C}'_0 would not satisfy (i)). We conclude that, if \mathcal{C}_0 has a dangling II_{l+1} singularity, then so does \mathcal{C}'_0 .

Claim 3. The exceptional loci of ϕ and ϕ' coincide.

If \mathcal{C}_0 has only nodes, then so does \mathcal{C}'_0 by Claim 1, and we can conclude by the uniqueness part of the stable reduction theorem.

If \mathcal{C}_0 has a genus one singularity x , it cannot have a genus two singularity as well, so neither can \mathcal{C}'_0 by Claim 1. If \mathcal{C}_0 has a second genus one singularity y , let $E_1 = \phi^{-1}(x)$ and $E_2 = \phi^{-1}(y)$; they are disjoint balanced subcurves of genus one and level $\leq m+1$ in \mathcal{C}_0^{ss} , therefore ϕ' must contract them. Enlarging the contraction radius of any one of them would yield a singularity with at least $m+2$ branches (by condition (iii) on \mathcal{C}_0), unless by enlarging we make them touch, in which case we would contract to a genus two singularity; but this is not possible, by Claim 1. The case of a single genus one singularity with a genus one branch, or with a disjoint subcurve of genus one, or with two branches joined by a (possibly empty) rational chain, is similar.

Finally, the case that \mathcal{C}_0 has a genus two singularity, has already been discussed at the end of Claim 2. To summarise, writing $\mathcal{C}_0^{ss} = Z \cup R_1 \cup \dots \cup R_l$, with $Z = \phi^{-1}(x)$ and $l \leq m$ - the case of a dangling II_{m+1} was dealt with before -, $\phi'(Z)$ must be a point x' , by stability considerations. Call $Z' = (\phi')^{-1}(x')$ and note that $Z \subseteq Z'$. If x and x' are singularities of the same type, $Z = Z'$ is easily deduced by level/singularity (i.e. outer/inner valence) considerations, the key point being that the shape of the curve has only one parameter (the “radius” of the circle), which is determined by m -stability. On the other hand, if x were of type II and x' of type I , the two special trees determined by x would have to share a trunk attached to a Weierstrass point of the core, and $Z \subseteq Z'$ would imply $Z \subsetneq Z'$, which together with condition (ii) for \mathcal{C}'_0 would make x' into a singularity with too many branches.

The claim follows from observing that the exceptional locus of ϕ (resp. ϕ') is the union of the fibres F over higher genus singularities of \mathcal{C}_0 (resp. \mathcal{C}'_0), and the rational components with only two special points that are disjoint from F .

Claim 4. The generic isomorphism between \mathcal{C} and \mathcal{C}' extends over Δ .

Follows from [Deb01, Lemma 1.15]. □

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