Lecture notes - Introduction to moduli theory

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Chapter 1

The functorial point of view in algebraic geometry and examples

1.1 Yoneda's lemma

Motivation: Suppose we start with a variety X defined over $\mathbb{Q}$. It may very well have no rational points (i.e. sections $\mathrm{Spec}(\mathbb{Q}) \to X$) but have points over a number field $\mathbb{Q} \subseteq \mathbf{k}$ (i.e. $\mathbb{Q}$-morphisms $\mathrm{Spec}(\mathbf{k}) \to X$). More generally, it is interesting to "test" an S-scheme X by looking at S-morphisms from S-schemes T to X - in fact, fat points and curves already tell you a lot about the geometry of X. We will see that testing X with all morphisms from S-schemes allows you to recover X completely.

Example 1.1.1. The $\mathbb{R}$-scheme $\mathrm{Spec}\,\mathbb{R}[x]/(x^2+1)$ has no $\mathbb{R}$-points (or *rational points*), but it has two $\mathbb{C}$-points.

We will often assume that $S = \mathrm{Spec}(R)$ is an affine scheme. If we also suppose that $X = \mathrm{Spec}\,R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$, then looking for R-points of X is the same as looking for solutions of the system of polynomial equations

$$\begin{cases} f_1(x_1, \ldots, x_n) & = 0 \\ \cdots & \\ f_m(x_1, \ldots, x_n) & = 0 \end{cases}$$

It is often the case that one has to pass to an extension of R (i.e. a finitely generated R-algebra A) in order to find any solution - which is to say a diagram

$$\begin{tikzcd} & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ur, dashed] & \mathrm{Spec}(R) \end{tikzcd}$$

The set of morphisms from $T = \mathrm{Spec}(A)$ to X is denoted by $h_X(T)$, $h_X(A)$, $X(T)$, or $X(A)$. For a morphism of R-algebras $A \to B$, corresponding to $T' = \mathrm{Spec}(B) \to T$, there is an induced map $X(A) \to X(B)$ - or $X(T) \to X(T')$ - by composition

$$\begin{tikzcd} & & X \arrow[d] \\ T' = \mathrm{Spec}(B) \arrow[r] \arrow[urr, dashed] & T = \mathrm{Spec}(A) \arrow[r] \arrow[ur] & S = \mathrm{Spec}(R) \end{tikzcd}$$

Thus, h_X determines a contravariant functor $(Sch/S) \to (Set)$ - or, restricting to affine schemes, a covariant functor $(Alg/R) \to (Set)$.

More generally, given a category $\mathscr{C}$ and an object X of $\mathscr{C}$, $h_X = \mathrm{Hom}_{\mathscr{C}}(-, X)$ defines an object of the category $\widehat{\mathscr{C}}$ of functors $\mathscr{C}^{\mathrm{op}} \to (Set)$ - with natural transformations as arrows. Given morphisms $f\colon X \to Y$ and $\phi\colon T' \to T$, there is a commutative diagram

$$\begin{tikzcd} h_X(T') \arrow[r, "f\circ"] \arrow[d, "\circ\phi"] & h_Y(T') \arrow[d, "\circ\phi"] \\ h_X(T) \arrow[r, "f\circ"] & h_Y(T) \end{tikzcd}$$

In particular, there is an induced map $\mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\widehat{\mathscr{C}}}(h_X, h_Y)$.

Lemma 1.1.2 (weak Yoneda)**.** $\mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\widehat{\mathscr{C}}}(h_X, h_Y)$ *is a bijection.*

Thus the association $X \mapsto h_X$ embeds $\mathscr{C}$ as a full subcategory of $\widehat{\mathscr{C}}$. Recall

Lemma 1.1.3. *A functor $F\colon \mathscr{A} \to \mathscr{B}$ is an* equivalence *of categories if and only if it is* fully faithful *and* essentially surjective*.*

An object of $\widehat{\mathscr{C}}$ is called *representable* if it is isomorphic to one of the form h_X for some $X \in \mathrm{ob}(\mathscr{C})$. Therefore $\mathscr{C}$ is equivalent to the full subcategory of representable objects in $\widehat{\mathscr{C}}$.

Let $F \in \mathrm{ob}(\widehat{\mathscr{C}})$, $X \in \mathrm{ob}(\mathscr{C})$, and $\xi \in F(X)$. Then ξ induces $h_X \to F$ by associating to $\phi\colon T \to X$ the element $(F\phi)(\xi)$ of $F(T)$. On the other hand, given a natural transformation $h_X \to F$, we can produce the object of $F(X)$ associated to id_X.

Lemma 1.1.4 (Yoneda)**.** *With notation as above, these functions are inverse to each other, thus establishing a bijection between $F(X)$ and $\mathrm{Hom}_{\widehat{\mathscr{C}}}(h_X, F)$.*

Exercise 1.1.5. Yoneda implies weak Yoneda.

Suppose that $F \simeq h_X$ is representable; then the element $\xi \in F(X)$ corresponding to id_X is called the *universal object*: it has the property that for every $\tau \in F(T)$ there exists a unique $\phi\colon T \to X$ such that $\tau = (F\phi)(\xi)$, and it is therefore unique up to unique isomorphism.

Example 1.1.6. $\mathscr{P}\colon (Set) \to (Set)$ the power set is represented by $X = \{0,1\}$ with universal object $\{1\}$.

Example 1.1.7. $\mathscr{P}^{\mathrm{open}}\colon (Top) \to (Set)$ associating to T the set of open subsets in the topology of T is represented by the same object as in the previous example, endowed with the topology $\{\emptyset, \{1\}, X\}$. Note that arrows in (Top) are continuous maps.

Example 1.1.8. Let $\mathbf{k}$ be a field. $\mathscr{P}^{\mathrm{open}}\colon (Sch/\mathbf{k}) \to (Set)$ associating to a $\mathbf{k}$-scheme the set of its open subschemes is not representable. Suppose it were representable by a $\mathbf{k}$-scheme X with universal open subscheme U. Then for every $\mathbf{k}$-scheme T, there would be a unique morphism $\phi\colon T \to X$ such that $\phi^{-1}(U) = T$; but then there would be a unique morphism $T \to U$, and therefore $U = \operatorname{Spec}(\mathbf{k})$. Since $\mathbf{k}$-points are closed, this would imply that all open subschemes of a $\mathbf{k}$-scheme are also closed, which is false. (I learned this, as many other things, from Angelo Vistoli.)

Example 1.1.9. The functor associating to an S-scheme X its global regular functions $\Gamma(X, \mathscr{O}_X)$ is represented by $\mathbb{A}^1_S$. It is in fact a functor in rings; when we think of it as a functor in groups, or *group scheme*, we usually denote it by $\mathbb{G}_{a,S}$.

Example 1.1.10. The functor associating to an S-scheme X its invertible functions $\Gamma(X, \mathscr{O}_X^*)$ is represented by the group scheme $\mathbb{G}_{m,S} = \underline{\operatorname{Spec}}_S \mathscr{O}_S[z, z^{-1}]$.

1.2 Example: projective bundles

In this section we introduce some more useful general concepts: fibre products, representable maps, open and closed subfunctors.

Example 1.2.1. The functor of points of projective space is

$$h_{\mathbb{P}^r_S}(T) = \{(L, s_0, \ldots, s_r) : L \in \operatorname{Pic}(T), s_i \in \Gamma(T, L) : \forall \mathfrak{p} \in T, \exists i : s_i(\mathfrak{p}) \neq 0\}/\sim$$

The equivalence is given by associating to an S-map $\phi\colon T \to \mathbb{P}^r_S$ the isomorphism class of $(\phi^*\mathscr{O}_{\mathbb{P}^r}(1), \phi^*x_0, \ldots, \phi^*x_r)$ for a choice of coordinates on $\mathbb{P}^r$. There is a map $h_{\mathbb{A}^{r+1}_S \setminus S} \to h_{\mathbb{P}^r_S}$ given by $(f_0, \ldots, f_r) \mapsto [\mathscr{O}, f_0, \ldots, f_r]$. In the following, we are going to give a description by gluing charts.

Exercise 1.2.2. Prove that $\operatorname{Aut}(\mathbb{P}^r) = PGL_{r+1}$.

Definition 1.2.3. Let $F_1, F_2, G \in \operatorname{ob}(\widehat{\mathscr{C}} = \operatorname{Fun}\colon \mathscr{C}^{\mathrm{op}} \to (Set))$ be functors, with natural transformations $\phi_1\colon F_1 \to G$ and $\phi_2\colon F_2 \to G$. Let $F_1 \times_G F_2$ be defined by

$$F_1 \times_G F_2(T) = \{(\xi_1, \xi_2) \in F_1(T) \times F_2(T) : \phi_1(\xi_1) = \phi_2(\xi_2)\}.$$

Exercise 1.2.4. $F_1 \times_G F_2$ is a fiber product in $\widehat{\mathscr{C}}$.

Exercise 1.2.5. If $F_1 = h_{X_1}, F_2 = h_{X_2}, G = h_Y$, then $F_1 \times_G F_2 = h_{X_1 \times_Y X_2}$.

Definition 1.2.6. An arrow $\phi \in \mathrm{Hom}_{\widehat{\mathscr{C}}}(F, G)$ is called *representable* if for every $Y \in \mathscr{C}$ and every $h_Y \to G$, the fiber product $F \times_G h_Y$ is representable ($\simeq h_X$).

Remark 1.2.7. If G is representable, then $F \to G$ is representable iff F is.

Definition 1.2.8. Let $\mathscr{P}$ be a property of arrows in $\mathscr{C}$, which is stable under base-change. A representable arrow $F \to G$ in $\widehat{\mathscr{C}}$ is said to have property $\mathscr{P}$ if, for every $h_Y \to G$ and $h_X = h_Y \times_G F$, $X \to Y$ has it.

Example 1.2.9. Open and closed subfunctors $F \subseteq G \colon (Sch) \to (Set)$.

Exercise 1.2.10. Consider the functor on S-schemes $U_j(T) = \{(L, s_0, \dots, s_r) : L \in \mathrm{Pic}(T), s_i \in \Gamma(T, L) : \forall \mathfrak{p} \in T, s_i(\mathfrak{p}) \neq 0\}/\sim$. Then $U_j \to h_{\mathbb{P}^r_S}$ is an open subfunctor. Hint: for every $f \colon Y \to \mathbb{P}^r$, $U_j \times_{h_{\mathbb{P}^r}} h_Y$ is represented by the open subscheme of Y where $f^*(s_j) \in \Gamma(Y, f^*\mathscr{O}_{\mathbb{P}^r}(1))$ is non-zero. Besides, $U_j \simeq h_{\mathbb{A}^r_S}$ for every j. What is $U_i \times_{h_{\mathbb{P}^r}} U_j$?

Consider now the following more general problem: Let $S = \mathrm{Spec}(R)$ and E an R-module (or equivalently a quasi-coherent sheaf $\mathscr{E}$ on S). Define $Q_E \colon (Sch)^{\mathrm{op}} \to (Set)$ by $Q_E(f \colon T \to S) = \{f^*\mathscr{E} \twoheadrightarrow \mathscr{L} : \mathscr{L} \in \mathrm{Pic}(T)\}/\sim$.

Proposition 1.2.11. *Q_E is represented by an S-scheme $\mathbb{P}(\mathscr{E})$. Furthermore, $\mathbb{P}(\mathscr{E}) \to S$ is projective when E is a finitely generated R-module.*

We shall prove this in two steps.

Step I: *when E is finitely generated.* Observe that, if $E = R^{r+1}$, then $Q_E = h_{\mathbb{P}^r}$. Indeed, the condition that for every $\mathfrak{p} \in T$ at least one of the s_i does not vanish at $\mathfrak{p}$ is equivalent to the surjectivity of $\mathscr{O}_T^{r+1} \to L$. Now, for every finitely generated E, we can find an exact sequence (presentation)

$$R^I \to F := R^{r+1} \to E \to 0$$

with I possibly infinite. The representability claim follows from the following

Lemma 1.2.12. *With notation as above, $Q_E \subseteq Q_F = h_{\mathbb{P}^r}$ is a closed subfunctor.*

Proof. Suppose we are given $f \colon T \to \mathbb{P}^r$. A surjection $\mathscr{O}_T^{r+1} \twoheadrightarrow \mathscr{L}$ factors through $\mathscr{E} \twoheadrightarrow \mathscr{L}$ if and only if the composite $\mathscr{O}_T^I \to \mathscr{L}$ vanishes; this is an intersection of closed conditions. (Or one could say that the closed subscheme of T induced by $Q_E \subseteq h_{\mathbb{P}^r}$ is cut by the ideal $(\mathscr{L}^\vee)^I \twoheadrightarrow \mathscr{I} \subseteq \mathscr{O}_T$.) $\square$

Step II: *in general.* Recall the following construction: to E we can associate the symmetric algebra $\mathscr{S}_E^\bullet := \bigoplus_{n\in\mathbb{N}} \mathrm{Sym}^n E$. This is an $\mathbb{N}$-graded algebra generated in degree one ($\mathscr{S}_E^1 = E$) and such that $\mathscr{S}_E^0 = R$. It has the universal property

$$\mathrm{Hom}_{R-Alg}(\mathscr{S}_E^\bullet, A) = \mathrm{Hom}_{R-Mod}(E, A)$$

To every quasi-coherent sheaf of graded $\mathscr{O}_S$-algebras $\mathscr{S}^\bullet$ we can associate an S-scheme by the *relative Proj* construction, $P = \underline{\mathrm{Proj}}_S(\mathscr{S}^\bullet)$, which is constructed as follows: for an open affine $U \subseteq S$, take $P_{|U} = \mathrm{Proj}(\mathscr{S}^\bullet(U)) \to U = \mathrm{Proj}(\mathscr{S}^0(U))$; and notice that for an inclusion of open affine subsets $V \subseteq U$ we get $\mathscr{S}^\bullet(V) = \mathscr{S}^\bullet(U) \otimes_{\mathscr{O}_S(U)} \mathscr{O}_S(V)$, therefore we can glue. It comes with an invertible sheaf $\mathscr{O}_P(1)$: locally on $U \subseteq S$, $\mathscr{O}_P(1)$ is the sheaf associated by the $\tilde{}$ construction to the graded module $\mathscr{S}^\bullet(1)(U)$, whose degree d piece is $\mathscr{S}^{d+1}(U)$.

All of these constructions are functorial on the base S.

Going back to our situation, we let $\mathbb{P}(\mathscr{E}) = \underline{\mathrm{Proj}}(\mathscr{S}_E^\bullet)$.

Remark 1.2.13. If E is free of rank $r+1$, $\mathscr{S}_E^\bullet$ is isomorphic to a polynomial algebra in $r+1$ variables over R, and $\mathbb{P}(\mathscr{E}) = \mathbb{P}^r_S$. If $\mathscr{E} \twoheadrightarrow \mathscr{F}$ is a surjective homomorphism of sheaves, then $\mathbb{P}(\mathscr{F}) \subseteq \mathbb{P}(\mathscr{E})$ is a closed S-subscheme, and the $\mathscr{O}(1)$ is preserved under restriction. If E is finitely generated, for every point $\mathfrak{p}$ of S, $E_\mathfrak{p} = E \otimes_R \mathbf{k}(\mathfrak{p})$ is a finite dimensional $\mathbf{k}(\mathfrak{p})$-vector space, hence $\mathbb{P}(\mathscr{E})_{|\mathfrak{p}} = \mathbb{P}^{n_\mathfrak{p}}_{\mathbf{k}(\mathfrak{p})}$, and the $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ restricts to the usual $\mathscr{O}(1)$. If E is finitely generated, by choosing a surjection $R^{r+1} \twoheadrightarrow E$ we can show that $\mathbb{P}(\mathscr{E}) \to S$ is projective (locally on the base). In fact, we could have started the discussion by taking a quasi-coherent sheaf $\mathscr{E}$ on any scheme S. Finally, notice that tensoring$\mathscr{E}$ by a line bundle does not change $\mathbb{P}(\mathscr{E})$ but does change $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$.

Lemma 1.2.14. *$\mathbb{P}(\mathscr{E})$ represents the functor Q_E.*

Proof. Let p denote the projection $\mathbb{P}(\mathscr{E}) \to S$. The morphism of graded modules

$$E \otimes_R \mathscr{S}_E^\bullet \twoheadrightarrow \mathscr{S}_E^\bullet(1)$$

corresponds to a surjection $p^*\mathscr{E} \twoheadrightarrow \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$, which we can take to be the universal object, inducing $h_{\mathbb{P}(\mathscr{E})} \to Q_E$. On the other hand, for an S-scheme $f\colon T \to S$, and an object $f^*\mathscr{E} \twoheadrightarrow \mathscr{L}$ of $Q_E(T)$, we get

$$T = \mathbb{P}(\mathscr{L}) \subseteq \mathbb{P}(f^*\mathscr{E}) = \mathbb{P}(\mathscr{E}) \times ST$$

which is the same as an S-morphism $T \to \mathbb{P}(\mathscr{E})$. □

Exercise 1.2.15. Show that the transformations above are inverse to one another, thus determining $Q_E \simeq h_{\mathbb{P}(\mathscr{E})}$. Fix all the details in the previous remark. Show that if $\mathscr{E}$ and $\mathscr{F}$ are locally free and $\mathbb{P}(\mathscr{E}) \simeq_\phi \mathbb{P}(\mathscr{F})$, then there is a line bundle $\mathscr{M}$ on S such that $\mathscr{E} \simeq \mathscr{F} \otimes \mathscr{M}$. Hint: $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1) \otimes \phi^*\mathscr{O}_{\mathbb{P}(\mathscr{F})}(-1)$ is pulled back from S.

1.3 Example: Grassmannians

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