A QUANTUM LEFSCHETZ THEOREM FOR QUASIMAP INVARIANTS VIA RELATIVE QUASIMAPS

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ABSTRACT. We define moduli spaces of relative stable quasimaps in the spirit of A. Gathmann. When X is a smooth toric variety and Y is a smooth very ample hypersurface we obtain a virtual class on the moduli space, which is used to define relative quasimap invariants. We obtain a recursion formula which expresses each relative invariant in terms of invariants of lower multiplicity. Finally we apply this formula to obtain a quantum Lefschetz theorem expressing the absolute quasimap invariants of Y in terms of those of X. We include several appendices collecting proofs of standard results in quasimap theory.

CONTENTS

1. INTRODUCTION

The results of this paper arise from a fusion of two theories: stable quasimaps and relative stable maps. In this introductory section we briefly summarise these, providing the context for our work.

1.1. **Stable quasimaps.** The moduli space of STABLE TORIC QUASIMAPS

$$\overline{Q}_{g,n}(X,\beta)$$

was constructed by Ciocan-Fontanine and Kim [CFK10] as an alternative compactification of the moduli space of smooth curves in a toric variety. It is

a Deligne–Mumford stack of finite type, and is proper if X is proper. Moreover, when X is smooth it admits a perfect obstruction theory and hence a virtual fundamental class, which one can use to define curve-counting invariants for X, called QUASIMAP INVARIANTS.

This theory agrees with the theory of stable quotients [MOP11] when both are defined, namely when X is a projective space. There is a common generalisation given by the theory of stable quasimaps to GIT quotients [CFKM14]. However for simplicity we will work in the toric setting (though this restriction is probably not essential for our arguments). Thus when we say "quasimap" we are implicitly talking about toric quasimaps.

The quasimap invariants are expected to coincide with the Gromov–Witten invariants when X is a toric Fano variety [CM]; this has been proven in a number of examples, including $X = \mathbb{P}^N$ [Man12b, §5.4].

In general, however, the invariants differ, the difference being encoded by certain wall-crossing formulas [CFK14]. The motivation for this comes from mirror symmetry: the idea is that the quasimap invariants of X should correspond to the B-side theory of X (this is in contrast to the Gromov–Witten invariants, which live on the A-side); see [CFK10, §7].

1.2. **Relative stable maps.** In [Gat02] Gathmann consructs a moduli space of relative stable maps to the pair (X, Y) as a closed substack of the moduli space of (absolute) stable maps to X:

$$\overline{\mathcal{M}}_{g,\alpha}(X|Y,\beta) \hookrightarrow \overline{\mathcal{M}}_{g,n}(X,\beta)$$

Unfortunately this space does not admit a natural perfect obstruction theory. Nevertheless in the case where Y is very ample it is still possible to construct a virtual fundamental class by intersection-theoretic methods, and hence one can define relative Gromov–Witten invariants.

Gathmann then proves a recursion formula which in particular allows one to recover the relative Gromov–Witten invariants from the absolute ones. This is applied in [Gat03] to obtain a quantum Lefschetz theorem for $Y \subseteq X$.

1.3. **Relative stable quasimaps.** In this paper we combine the two stories above, constructing moduli spaces of relative stable quasimaps. We prove a recursion relation similar to Gathmann's formula, and use this to derive a quantum Lefschetz formula for quasimap invariants.

The plan of the paper is as follows. In §§2.1-2.2 we provide a brief review of the theories of stable quasimaps and relative stable maps mentioned above. Then in §2.3 we define the moduli spaces of relative stable quasimaps

$$\overline{Q}_{g,\alpha}(X|Y,\beta)$$

where X is a smooth toric variety and Y is a smooth hypersurface. We *do not* require that Y is toric.

In §3 we examine the special case of $H \subseteq \mathbb{P}^N$. We find that, although the moduli space is not in general smooth, it is irreducible of the expected dimension (in fact, more than this: it is the closure of the so-called "nice locus"). Thus it admits a fundamental class which we can use to define relative quasimap invariants.

Also for $\mathbb{P}^N$ there exists a comparison morphism from the moduli space of stable maps to the moduli space of quasimaps, which is birational. We use this morphism to push down Gathmann's recursion formula for relative stable maps to obtain a recursion formula for relative stable quasimaps. The stronger stability condition for quasimaps significantly simplifies the correction terms which appear.

In §4 we extend the recursion formula to arbitrary pairs (X, Y) where Y is very ample, by taking the embedding $X \hookrightarrow \mathbb{P}^N$ defined by $\mathcal{O}(Y)$ and pulling back the formula for $(\mathbb{P}^N, H)$. This of course requires some comparison theorems for virtual classes, for which we have to examine the perfect obstruction theories.

In §5 we apply the recursion formula obtained in §4 to obtain a quantum Lefschetz theorem for quasimap invariants. This recovers [REFERENCE] in [CFK14].

We also include several appendices, collecting together results which are presumably well-known to experts, but for which we could not find references in the literature.

Appendix [REF] contains foundational lemmas of quasimap theory, including functoriality, the existence of relative perfect obstruction theories and the splitting theorem.

Appendix [REF] discusses a well-known intersection-theoretic construction – the so-called "diagonal pull-back" – and shows that it agrees with the virtual pull-back of [Man12a] (when both are defined) and that it commutes with Gysin maps and virtual pull-backs.

Finally Appendix [REF] discusses the comparison morphism from maps to quasimaps (used in the proof of the recursion relation in §3).

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1.4. Table of notation.

2. Relative stable quasimaps

2.1. Review of absolute stable quasimaps.

We briefly recall the definition and basic properties of the moduli space of toric quasimaps; see [CFK10] for more details.

Definition 2.1 ([CFK10, Definition 3.1.1])**.** Let $X = X_\Sigma$ be a smooth and projective toric variety with fan $\Sigma \subseteq N_\mathbb{Q}$ and let $\mathcal{O}_{X_\Sigma}(1)$ be a fixed polarisation, which we can write (non-uniquely) in terms of the T-invariant divisors as:

$$\mathcal{O}_{X_\Sigma}(1) = \otimes_{\rho\in\Sigma(1)} \mathcal{O}_{X_\Sigma}(D_\rho)^{\otimes \alpha_\rho}$$

for some $\alpha_\rho \in \mathbb{Z}$. Given a fixed genus $g \geq 0$, number of marked points $n \geq 0$ and curve class $\beta \in \mathrm{H}_2^+(X)$ a **stable (toric) quasimap** is given by the data

$$((C, x_1, \ldots, x_n), (L_\rho, u_\rho)_{\rho\in\Sigma(1)}, (\varphi_m)_{m\in M})$$

where:

(1) $(C, x_1, \ldots, x_n)$ is a prestable curve of genus g with n marked points;
(2) the L_ρ are line bundles on C of degree $d_\rho = D_\rho \cdot \beta$;
(3) the u_ρ are global sections of L_ρ;
(4) $\varphi_m : \otimes_\rho L_\rho^{\otimes\langle\rho,m\rangle} \to \mathcal{O}_C$ are isomorphisms, such that $\varphi_m \otimes \varphi_{m'} = \varphi_{m+m'}$ for all $m, m' \in M$.

These are required to satisfy the following two conditions:

(1) **nondegeneracy**: there is a finite (possibly empty) set of smooth and non-marked points $B \subseteq C$, called the **basepoints** of the quasimap, such that for all $x \in C \setminus B$ there exists a maximal cone $\sigma \in \Sigma_{\max}$ with $u_\rho(x) \neq 0$ for all $\rho \not\subset \sigma$;
(2) **stability**: if we let $L = \otimes_\rho L_\rho^{\otimes \alpha_\rho}$ then the following $\mathbb{Q}$-divisor is ample

$$\omega_C(x_1 + \ldots + x_n) \otimes L^{\otimes\epsilon}$$

for every rational $\epsilon > 0$.

Remark 2.2. This definition is motivated by the Σ-collections of D. Cox [Cox95a]; see also Appendix A. The point is that a quasimap defines a rational morphism $C \dashrightarrow X$ with base locus equal to B; in particular a quasimap without any basepoints defines a morphism $C \to X$. Thus the basepoints appear in the boundary of the moduli space, in much the same way as the locus of stable maps with rational tails appears in the boundary of the moduli space of stable maps (this is actually more than just a vague analogy; these loci correspond to each other under the comparison morphism; see Appendix C).

More generally, we can define the notion of a family of quasimaps over a base scheme S, and what it means for two such families to be isomorphic; we thus obtain a moduli space

$$\overline{\mathcal{Q}}_{g,n}(X, \beta)$$

of stable (toric) quasimaps to X, which is a proper Deligne–Mumford stack of finite type. It can be shown that this definition does not depend on the choice of polarisation.

As with the case of stable maps, there is a combinatorial characterisation of stability which is much easier to check in practice; a prestable quasimap is stable if and only if the following conditions hold:

(1) the line bundle L defined above must have strictly positive degree on any rational component with fewer than three special points, and on any elliptic component with no special points;

(2) C cannot have any rational components with fewer than two special points.

Condition (1) is analogous to the ordinary stability condition for stable maps. Condition (2) is new, however, and gives quasimaps a distinctly different flavour to stable maps; we shall sometimes refer to it as the **strong stability condition**.

The moduli space $\overline{Q}_{g,n}(X,\beta)$ admits a perfect obstruction theory relative to the moduli space $\mathfrak{M}_{g,n}$ of source curves, and hence one can construct a virtual class

$$[\overline{Q}_{g,n}(X,\beta)]^{\mathrm{virt}} \in \mathrm{A}_{\mathrm{vdim}\,\overline{Q}_{g,n}(X,\beta)}\left(\overline{Q}_{g,n}(X,\beta)\right)$$

where the virtual dimension is the same as for stable maps:

$$\mathrm{vdim}\,\overline{Q}_{g,n}(X,\beta) = (\dim X - 3)(1-g) - (K_X \cdot \beta) + n$$

Since the markings are not basepoints there exist evaluation maps

$$\mathrm{ev}_i : \overline{Q}_{g,n}(X,\beta) \to X$$

and there are ψ-classes defined in the usual way by pulling back the relative dualising sheaf of the universal curve

$$\psi_i = \mathrm{c}_1(x_i^* \omega_{C/\overline{Q}})$$

where $C \to \overline{Q} = \overline{Q}_{g,n}(X,\beta)$ is the universal curve and $x_i : \overline{Q} \to C$ is the section defining the ith marked point. Putting all these pieces together, we can define **quasimap invariants**:

$$\langle \gamma_1 \psi^{a_1}, \ldots, \gamma_n \psi^{a_n} \rangle^X_{g,n,\beta} = \int_{[\overline{Q}_{g,n}(X,\beta)]^{\mathrm{virt}}} \prod_{i=1}^n \mathrm{ev}_i^* \gamma_i \psi_i^{a_i}$$

(We use the same correlator notation as in Gromov–Witten theory; since we will never talk about Gromov–Witten invariants this should not cause any confusion.)

[EXAMPLES.]

2.2. **Review of relative stable maps.** Given a smooth projective variety X and a smooth divisor Y, the moduli space of relative stable maps parametrises stable maps to X with specified tangencies to Y at the marked points; [Gat02] for details.

Definition 2.3 ([Gat02, Definition 1.1 and Remark 1.4])**.** Let X be a smooth projective variety and $Y \subseteq X$ a smooth divisor. Fix a number $n \geq 0$ of marked points, a curve class $\beta \in \mathrm{H}_2^+(X)$ and an n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non-negative integers such that $\Sigma_i \alpha_i \leq Y \cdot \beta$. Then the moduli space

$$\overline{\mathcal{M}}_{0,\alpha}(X|Y,\beta)$$

of relative stable maps to (X,Y) is defined to be the locus in $\overline{\mathcal{M}}_{0,n}(X,\beta)$ of stable maps $(C, x_1, \ldots, x_n, f)$ such that, if Z is a connected component of $f^{-1}(Y) \subseteq C$, then

(1) if Z is a point and is equal to a marked point x_i, then the multiplicity of f to Y at x_i is greater than or equal to α_i;
(2) if Z is one-dimensional (hence a union of irreducible components of C) and if we let $C^{(i)}$ for $1 \leq i \leq r$ denote the irreducible components of C adjacent to Z, and $m^{(i)}$ denote the multiplicity of $f|_{C^{(i)}}$ to Y at the node $Z \cap C^{(i)}$, then we must have:

$$Y \cdot f_*[Z] + \sum_{i=1}^r m^{(i)} \geq \sum_{x_i \in Z} \alpha_i$$

Remark 2.4. In the case of maximal multiplicity $\Sigma_i \alpha_i = Y \cdot \beta$, all the inequalities in the above definition must actually be equalities.

This forms a proper closed substack of $\overline{\mathcal{M}}_{0,n}(X,\beta)$. In the case $X = \mathbb{P}^N$, $Y = H$ one can show that it is irreducible of the correct dimension, and hence admits a fundamental class from which one can define relative Gromov–Witten invariants.

In general if $Y \subseteq X$ is very ample one can use the embedding $X \hookrightarrow \mathbb{P}^N$ to obtain a cartesian diagram:

$$\begin{tikzcd} \overline{\mathcal{M}}_{0,\alpha}(X|Y,\beta) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{\mathcal{M}}_{0,\alpha}(\mathbb{P}^N|H,d) \arrow[d] \\ \overline{\mathcal{M}}_{0,n}(X,\beta) \arrow[r, "\varphi"] & \overline{\mathcal{M}}_{0,n}(\mathbb{P}^N,d) \end{tikzcd}$$

Then the fact that $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^N,d)$ is smooth allows us to define a virtual class on $\overline{\mathcal{M}}_{0,\alpha}(X|Y,\beta)$ by virtual (or diagonal) pull-back (see Appendix B of the current paper):

$$[\overline{\mathcal{M}}_{0,\alpha}(X|Y,\beta)]^{\mathrm{virt}} = \varphi^![\overline{\mathcal{M}}_{0,\alpha}(\mathbb{P}^N|H,d)]$$

Thus one can define relative Gromov–Witten invariants. In §§2-4 Gathmann proves a recursion relation inside the Chow group of $\overline{\mathcal{M}}_{0,\alpha}(X|Y,\beta)$

$$(\alpha_k \psi_k + \mathrm{ev}_k^* Y)[\overline{\mathcal{M}}_{0,\alpha}(X|Y,\beta)]^{\mathrm{virt}} = [\overline{\mathcal{M}}_{0,\alpha+e_k}(X|Y,\beta)]^{\mathrm{virt}} + [D_{\alpha,k}(X,\beta)]^{\mathrm{virt}}$$

where $D_{\alpha,k}(X,\beta)$ is an appropriate **comb locus**. Repeated application of this result shows that the relative Gromov–Witten invariants of (X,Y) and the Gromov–Witten invariants of Y are completely determined by the Gromov–Witten invariants of X. This relation is then worked out explicitly in cases of particular interest in [Gat03] to obtain a new proof of the mirror theorem.

2.3. **Definition of relative stable quasimaps.** We now give the main definition of the paper. From here on X will denote a smooth projective toric variety and $Y \subseteq X$ a smooth very ample hypersurface. We do not require that Y is toric.

Consider the line bundle $\mathcal{O}(Y)$ and the section s_Y cutting out Y. By [Cox95b] we have a natural isomorphism

$$\mathrm{H}^0(X, \mathcal{O}(Y)) = \Bbbk\left\langle \prod_\rho z_\rho^{a_\rho} : \Sigma_\rho a_\rho D_\rho = Y \right\rangle$$

where the z_ρ for $\rho \in \Sigma_X(1)$ are the generators of the Cox ring of X and the a_ρ are non-negative integers. We can therefore write s_Y as

$$s_Y = \sum_{\underline{a}=(a_\rho)} \lambda_{\underline{a}} \prod_\rho z_\rho^{a_\rho}$$

for some scalars $\lambda_{\underline{a}} \in \Bbbk$. The idea then is that a quasimap

$$((C, x_1, \ldots, x_n), (L_\rho, u_\rho)_{\rho \in \Sigma_X(1)}, (\varphi_m)_{m \in M})$$

maps to Y at $x \in C$ if and only if the section

$$u_Y := \sum_{\underline{a}} \lambda_{\underline{a}} \prod_\rho u_\rho^{a_\rho}$$

vanishes at x. We now explain how to make sense of the expression above. For each $\underline{a}$ we have a well-defined section

$$u_{\underline{a}} := \lambda_{\underline{a}} \prod_\rho u_\rho^{a_\rho} \in \mathrm{H}^0(C, \otimes_\rho L_\rho^{\otimes a_\rho})$$

and if we have $\underline{a}$ and $\underline{b}$ such that $\sum_\rho a_\rho D_\rho = Y = \sum_\rho b_\rho D_\rho$ then these differ by an element m of M. Thus the isomorphism φ_m allows us to view the sections $u_{\underline{a}}$ and $u_{\underline{b}}$ as sections of the same bundle, which we denote by L_Y. Then we can sum these together to obtain u_Y. There is a choice involved here, but up to isomorphism it does not matter; see the proof of functoriality in Appendix A for more details.

The upshot is that we obtain a line bundle L_Y on C (which plays the role of the "pull-back" of $\mathcal{O}(Y)$) and a global section

$$u_Y \in \mathrm{H}^0(C, L_Y)$$

which plays the role of the "pull-back" of s.

Definition 2.5. With notation as above, let $n \geq 0$ be number of marked points, $\beta \in \mathrm{H}_2^+(X)$ be a curve class and $\alpha = (\alpha_1, \ldots, \alpha_n)$ a collection of non-negative integers such that $\Sigma_i \alpha_i \leq Y \cdot \beta$. Then we define the **moduli space of relative stable quasimaps**

$$\overline{\mathcal{Q}}_{0,\alpha}(X|Y, \beta) \subseteq \overline{\mathcal{Q}}_{0,n}(X, \beta)$$

to be the locus of quasimaps such that, if Z is a connected component of the vanishing locus of u_Y in C, then

(1) if Z is a point and is equal to a marked point x_i, then the order of vanishing of u_Y at x_i is greater than or equal to α_i;

(2) if Z is one-dimensional (hence a union of irreducible components) and if we let $C^{(i)}$ for $1 \leq i \leq r$ denote the irreducible components of C adjacent to Z, and $m^{(i)}$ the order of vanishing of u_Y at the node $Z \cap C^{(i)}$, then we must have:

$$\deg L_Y|_Z + \sum_{i=1}^r m^{(i)} \geq \sum_{x_i \in Z} \alpha_i \tag{1}$$

Remark 2.6. In the second case above we call Z an **internal** component and the $C^{(i)}$ **external** components.

As it stands we do not know much about this locus. In the following section we will examine the case $X = \mathbb{P}^N$ and $Y = H$ a hyperplane in detail. We will then apply the results obtained there to deduce facts about the general case.

3. Recursion formula for $\mathbb{P}^N$ relative H

3.1. **Basic properties of the moduli space.** In this section we will show that the moduli space

$$\overline{\mathcal{Q}}_{0,\alpha}(\mathbb{P}^N|H, d)$$

is irreducible of the expected dimension, and thus admits a fundamental class. We then prove a recursion formula for these fundamental classes by pushing forward Gathmann's recursion formula along the comparison morphism χ.

We set $X = \mathbb{P}^N$ and $Y = H = \{z_0 = 0\}$. Given a quasimap

$$(C, x_1, \ldots, x_n, L, u_0, \ldots, u_N) \in \overline{\mathcal{Q}}_{0,n}(\mathbb{P}^N, d)$$

the line bundle L_Y of the previous section is equal to L and the section u_Y is equal to u_0. Let us denote by

$$\mathcal{Q}_{0,\alpha}(\mathbb{P}^N|H, d) \subseteq \overline{\mathcal{Q}}_{0,n}(\mathbb{P}^N, d)$$

(without the bar) the **nice locus**, consisting of those quasimaps with irreducible source curve (i.e. a $\mathbb{P}^1$), no basepoints (so we get an actual map), with no component of the curve mapping inside H and with the map having tangency at least α_i to H at the marking x_i.

This is an irreducible, locally closed substack of $\overline{\mathcal{Q}}_{0,n}(\mathbb{P}^N, d)$ of codimension $\Sigma_i \alpha_i$, by essentially the same argument as in [Gat02, Lemma 1.8]. In fact it is isomorphic to the nice locus inside the stable map space, denoted $\mathcal{M}_{0,\alpha}(\mathbb{P}^N|H, d)$ by Gathmann (see [Gat02, Def. 1.6]; the stricter stability condition has no effect when the source curve is irreducible, provided of course that $n \geq 2$). We thus obtain:

Lemma 3.1. *The comparison morphism restricts to a morphism*

$$\chi : \overline{\mathcal{M}}_{0,\alpha}(\mathbb{P}^N|H, d) \to \overline{\mathcal{Q}}_{0,\alpha}(\mathbb{P}^N|H, d)$$

Proof. We need to verify that a relative stable map is sent to a relative stable quasimap by χ. Since the contraction of a rational tail R always occurs away from the markings, we only need to examine the internal components Z of the quasimap.

Consider then Z; for each basepoint x on Z there is a rational tail R of the stable map attached to Z at x. This is either internal (mapped into H) or external (not mapped into H).

If R is internal then both R and Z live inside the same connected component Z' of $f^{-1}(H)$. Applying χ has the effect of contracting R and adding a line bundle to Z of degree equal to $H \cdot f_*[R]$. Thus the left hand side of the inequality (1) is left unchanged, and since the right hand side is also unaltered the inequality is satisfied.

On the other hand if R is external then the multiplicity $m^{(R)}$ of $R \cap Z$ satisfies:

$$m^{(R)} \leq H \cdot f_*[R]$$

Since applying χ has the effect of replacing $m^{(R)}$ by $H \cdot f_*[R]$ in the left hand side of (1), the inequality still holds for the quasimap. Thus we obtain a morphism from the relative stable map space to the relative quasimap space, as claimed. □

Lemma 3.2. *$\overline{Q}_{0,\alpha}(\mathbb{P}^r|H,d)$ is equal to the closure of the nice locus $Q_{0,\alpha}(\mathbb{P}^r|H,d)$ inside $\overline{Q}_{0,n}(\mathbb{P}^r,d)$.*

Proof. $\overline{Q}_{0,\alpha}(\mathbb{P}^r|H,d) \subseteq \overline{Q_{0,\alpha}(\mathbb{P}^r|H,d)}$: we show that any relative stable quasimap can be infinitesimally deformed to a relative stable quasimap with no basepoints. This is in particular a relative stable map; we then appeal to [Gat02, Prop. 1.14] to deform this stable map and obtain a point in the nice locus. Since this deformation does not introduce any rational tails, this is also a deformation of quasimaps, and the statement follows.

We induct on the number of components containing at least one basepoint. Suppose this number is non-zero (otherwise there is nothing to prove) and pick such a component C_0, with base-points $y_1,\ldots,y_k$. Recall that this means that $u_i(y_j)=0$ for all i and j. We will deform the section $u_N|_{C_0}$ to a new section $u'_N|_{C_0}$ in such a way that $u'_N|_{C_0}(y_j)\neq 0$ and in such a way that we do not introduce any new basepoints. Notice that since the relative condition only depends on u_0, the resulting deformed quasimap will still be a relative quasimap.

Now, by the nondegeneracy condition we must have $\deg(L|_{C_0})>0$, and since $C_0 \cong \mathbb{P}^1$ we can find a section w_0 of $L|_{C_0} \cong \mathcal{O}_{\mathbb{P}^1}(d_0)$ not vanishing at any of the base-points p_i.

We then set

$$u'_N|_{C_0} := u_N|_{C_0} + \epsilon w_0$$

and $u'_i|_{C_0} = u_i|_{C_0}$ for all other i. Notice that $u'_N|_{C_0}(y_j)\neq 0$ for all j as claimed. It is also clear that we do not introduce any new basepoints, since

$u'_N|_{C_0}(y)=0$ implies $u_N|_{C_0}(y)=0$ (put differently: being a basepoint is a close condition).

It remains to extend the section $u'_N|_{C_0}$ to a section u'_N on the whole curve. Let $C_1,\ldots,C_r$ be the components of C adjacent to C_0 and let $q_i = C_0 \cap C_i$. We need to modify the sections $u_N|_{C_i}$ in such a way that $u'_N|_{C_i}(q_i) = u'_N|_{C_0}(q_i)$.

By nondegeracy, we can choose a section w_i of $L|_{C_i}$ such that $w(q_i)\neq 0$. Then set:

$$u'_{N\,C_i} := u_N|_{C_i} + \epsilon\left(\frac{w_0(q_i)}{w_i(q_i)}\right)\cdot w_i$$

Then indeed we have:

$$u'_N|_{C_i}(q_i) = u_N(q_i) + \epsilon\left(\frac{w_0(q_i)}{w_i(q_i)}\right)\cdot w_i(q_i) = u_N(q_i) + \epsilon w_0(q_i) = u'_N|_{C_0}(q_i)$$

We can continue this process, replacing C_0 by C_i; since the genus of the curve is zero, we will never come to the same curve twice. In this way we obtain a new quasimap

$$(C, x_1,\ldots,x_n, L, u_0,\ldots,u_{N-1},u'_N)$$

over $\operatorname{Spec}\Bbbk[\epsilon]/(\epsilon^2)$ which has no basepoints on C_0. We can repeat this process for all the components of C (using higher powers of ϵ each time in order to ensure that we never introduce additional basepoints) and thus we obtain an infinitesimal deformation of our original quasimap which has no basepoints, as required.

$\overline{Q_{0,\alpha}(\mathbb{P}^r|H,d)} \subseteq \overline{Q}_{0,\alpha}(\mathbb{P}^r|H,d)$: consider a family of stable quasimaps over a smooth curve S, such that the generic fibre lies in the nice locus. We may blow-up the source curve (a fibered surface over S) in the locus of basepoints (which consists of finitely many smooth points of the central fiber) in order to obtain an actual morphism to $\mathbb{P}^N$. This has the effect of adding rational tails at the basepoints in the central fibre. If the morphism is constant on any of these rational tails we may contract them, and thus we obtain a family of stable maps which pushes down along χ to our original family of quasimaps.

The general fibre is not modified at all, and so is still in the nice locus. By [Gat02, Lemma 1.9] it follows that the central fibre is a relative stable map, and then by applying χ and appealing to Lemma 3.1 it follows that the same is true for the central fibre of the family of quasimaps. □

Corollary 3.3. *The moduli space $\overline{Q}_{0,\alpha}(\mathbb{P}^N|H,d)$ is irreducible of the expected dimension. Hence it admits a fundamental class.*

Proof. This holds because the moduli space is equal to the closure of the nice locus, which is irreducible of the expected dimension. □

Corollary 3.4. *The comparison morphism is a birational map from the moduli space of relative stable maps to the moduli space of stable maps. In particular it sends the fundamental class to the fundamental class.*

Proof. This follows because the comparison morphism restricts to an isomorphism on the nice locus, which by the lemma above is a dense open subset of both spaces. □

3.2. **The recursion formula.** We wish to obtain a recursion formula relating the quasimap invariants of multiplicity α with the quasimap invariants of multiplicity $\alpha + e_k$, as in [Gat02, Theorem 2.6]. This process of "increasing the multiplicities" can be naively performed in the same way as Gathmann: for $m = \alpha_k + 1$ the following section (of the pull-back of the jet bundle of the universal line bundle)

$$\sigma_k^m := x_k^* d^m_{C/\overline{\mathcal{Q}}}(u_0) \in H^0(\overline{\mathcal{Q}}, x_k^* \mathcal{P}^m_{C/\overline{\mathcal{Q}}}(\mathcal{L}))$$

cuts out $\overline{\mathcal{Q}}_{0,\alpha+e_k}(\mathbb{P}^r|H,d)$ inside $\overline{\mathcal{Q}}_{0,\alpha}(\mathbb{P}^r|H,d)$, along with a number of degenerate contributions (called the **comb loci**) parametrising quasimaps for which x_k belongs to an internal component $Z \subseteq C$ (a component on which u_0 vanishes), such that

$$\deg(L|_Z) + \sum_{i=1}^r m^{(i)} = \sum_{x_i \in Z} \alpha_i$$

(here by "component" we really mean "connected component of the vanishing locus of u_0"). Quasimap stability means that these degenerate contributions cannot contain any rational tails; this is really the only difference with the case of stable maps.

Indeed, we can actually push forward Gathmann's formula along the comparison morphism

$$\chi\colon \overline{\mathcal{M}}_{0,\alpha}(\mathbb{P}^N|H,d) \to \overline{\mathcal{Q}}_{0,\alpha}(\mathbb{P}^N|H,d)$$

and due to Corollary 3.2 above, the only terms which change are the comb loci containing rational tails. In fact these disappear, since the restriction of the comparison map to these loci has positive-dimensional fibres:

Lemma 3.5. *Consider a rational tail component in the comb locus of the moduli space of stable maps, i.e. a moduli space of the form:*

$$\overline{\mathcal{M}}_{0,(m^{(i)})}(\mathbb{P}^N|H,d)$$

Then (assuming that $Nd > 1$) we have

$$\dim\left([\overline{\mathcal{M}}_{0,(m^{(i)})}(\mathbb{P}^N|H,d)] \cap \mathrm{ev}_1^*(\mathrm{pt}_H)\right) > 0$$

where $\mathrm{pt}_H \in \mathrm{A}^{N-1}(H)$ is a point class. Thus the pushforward along χ of any comb locus with a rational tail is 0.

Proof. This is a simple dimension count. We have

$$\begin{aligned}\dim\left([\overline{\mathcal{M}}_{0,(m^{(i)})}(\mathbb{P}^N|H,d)] \cap \mathrm{ev}_1^*(\mathrm{pt}_H)\right) &= (N-3) + d(N+1) + (1-m^{(i)}) - (N-1)\\ &= (Nd-1) + (d-m^{(i)})\end{aligned}$$

from which the lemma follows because $m^{(i)} \leq d$. □

Remark 3.6. With an eye to the future, we remark that these rational tail components contribute nontrivially to the Gromov–Witten invariants of a Calabi–Yau hypersurface in projective space, and so their disappearance in our recursion formula may account for the divergence between Gromov–Witten and quasimap invariants in the Calabi–Yau case [Gat03, Rmk. 1.6].

Since we wish to apply the projection formula to Gathmann's recursion relation, we should express the cohomological terms which appears as pull-backs:

Lemma 3.7. *We have:*

$$\chi^*(\psi_k) = \psi_k$$
$$\chi^*(\mathrm{ev}_k^* H) = \mathrm{ev}_k^* H$$

Proof. We will actually show that:

$$\chi^* x_k^* \omega_{C/\overline{\mathcal{Q}}} = x_k^* \omega_{C/\overline{\mathcal{M}}}$$
$$\chi^* x_k^* \mathcal{L} = \mathrm{ev}_k^* \mathcal{O}_{\mathbb{P}^N}(H)$$

This follows by considering the following diagram:

$$\begin{tikzcd}
 & & \mathbb{P}^N & \\
C_{\overline{\mathcal{M}}} \arrow[urr, bend left, "f"] \arrow[r, "\sigma^{ss}"'] \arrow[dr] & \chi^* C_{\overline{\mathcal{Q}}} \arrow[ur, dashed] \arrow[rr] \arrow[d] \arrow[drr, phantom, "\square"] & & C_{\overline{\mathcal{Q}}} \arrow[ul, dashed] \arrow[d] \\
 & \overline{\mathcal{M}}_{0,\alpha}(\mathbb{P}^N|H,d) \arrow[u, bend right, "x_k"'] \arrow[ul, bend left, "x_k"] \arrow[rr, "\chi"] & & \overline{\mathcal{Q}}_{0,\alpha}(\mathbb{P}^N|H,d) \arrow[u, bend right, "x_k"']
\end{tikzcd}$$

where σ^{ss} is the strong stabilisation map which contracts the rational tails, and so is an isomorphism near the markings. □

Proposition 3.8. *Define the comb locus $D^{\mathcal{Q}}_{\alpha,k}(\mathbb{P}^r|H,d)$ as the union of the moduli spaces*

$$\overline{\mathcal{Q}}_{0,|\alpha^{(0)}|+r}(H,d_0) \times_{H^r} \prod_{i=1}^r \overline{\mathcal{Q}}_{0,(m^{(i)})\cup\alpha^{(i)}}(\mathbb{P}^N|H,d_i)$$

where the sum runs over all splittings $d = \sum d_i$ and $\alpha = \bigcup \alpha^{(i)}$ such that the above spaces are all well-defined (in particular $|\alpha^{(0)}| + k$ and $|\alpha^{(i)}| + 1$ are all ≥ 2) and such that

$$d_0 + \sum_{i=1}^r m^{(i)} = \sum \alpha^{(0)}$$

Equip this with the sum of the (product) fundamental classes. Then the following formula holds

$$(\alpha_k \psi_k + \mathrm{ev}_k^* H) \cdot [\overline{\mathcal{Q}}_{0,\alpha}(\mathbb{P}^r|H,d)] = [\overline{\mathcal{Q}}_{0,\alpha+e_k}(\mathbb{P}^r|H,d)] + [D^{\mathcal{Q}}_{\alpha,k}(\mathbb{P}^r|H,d)].$$