# A QUANTUM LEFSCHETZ THEOREM FOR QUASIMAP INVARIANTS VIA RELATIVE QUASIMAPS

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Abstract. We define moduli spaces of relative toric quasimaps in genus 0 in the spirit of A. Gathmann. When $X$ is a smooth toric variety and $Y$ is a very ample hypersurface we construct a virtual class on the moduli space, which is used to define relative quasimap invariants. We obtain a recursion formula which expresses each relative invariant in terms of invariants of lower multiplicity. Finally we apply this formula to derive a quantum Lefschetz theorem expressing the absolute quasimap invariants of $Y$ in terms of those of $X$. We include several appendices collecting proofs of standard results in quasimap theory.

Contents



## 1. Introduction

The results of this paper arise from a fusion of two theories: stable quasimaps and relative stable maps. In this introductory section we briefly summarise these, providing the context for our work.

### 1.1. Stable quasimaps.

The moduli space of **stable toric quasimaps**

$$\overline{Q}_{g,n}(X, \beta)$$

was constructed by Ciocan-Fontanine and Kim [CFK10] as an alternative compactification of the moduli space of smooth curves in a toric variety. It is a Deligne–Mumford stack of finite type, and is proper if $X$ is proper. Moreover, when $X$ is smooth (and in fact more generally: see [CFKM14, §4.5]) it admits a perfect obstruction theory and hence a virtual fundamental class, which one can use to define curve-counting invariants for $X$, called **quasimap invariants**.

This theory agrees with the theory of stable quotients [MOP11] when both are defined, namely when $X$ is a projective space. There is a common generalisation given by the theory of stable quasimaps to GIT quotients [CFKM14]. However for simplicity we will work mostly in the toric setting (though this restriction is probably not essential for our arguments). Thus when we say "quasimap" we are implicitly talking about toric quasimaps.

1.2. **Relative stable maps.** In [Gat02] Gathmann consructs a moduli space of relative stable maps to the pair $(X, Y)$ as a closed substack of the moduli space of (absolute) stable maps to $X$:

$$\overline{\mathcal{M}}_{0,\alpha}(X|Y, \beta) \hookrightarrow \overline{\mathcal{M}}_{0,n}(X, \beta)$$

Unfortunately this space does not admit a natural perfect obstruction theory. Nevertheless in the case where $Y$ is very ample it is still possible to construct a virtual fundamental class by intersection-theoretic methods, and hence one can define relative Gromov–Witten invariants.

Gathmann then proves a recursion formula which in particular allows one to recover the relative Gromov–Witten invariants from the absolute ones. This is applied in [Gat03] to obtain a quantum Lefschetz theorem for $Y \subseteq X$.

1.3. **Relative stable quasimaps.** In this paper we combine the two stories above, constructing moduli spaces of relative stable quasimaps in genus 0. We prove a recursion relation similar to Gathmann's formula, and use this to derive a quantum Lefschetz formula for quasimap invariants.

The plan of the paper is as follows. In §§2.1-2.2 we provide a brief review of the theories of stable quasimaps and relative stable maps mentioned above. Then in §2.3 we define the moduli spaces of relative stable quasimaps

$$\overline{\mathcal{Q}}_{g,\alpha}(X|Y, \beta)$$

where $X$ is a smooth toric variety and $Y$ is a hypersurface. We *do not* require that $Y$ is toric.

In §3 we examine the special case of $H \subseteq \mathbb{P}^N$. We find that, although the moduli space is not in general smooth, it is irreducible of the expected dimension (in fact, more than this: it is the closure of the so-called "nice locus" consisting of maps from a smooth curve which do not land inside the hypersurface). Thus it has an actual fundamental class which we can use to define relative quasimap invariants.

Also for $\mathbb{P}^N$ there exists a comparison morphism from the moduli space of stable maps to the moduli space of quasimaps, which is birational. We use this morphism to push down Gathmann's recursion formula for relative stable maps to obtain a recursion formula for relative stable quasimaps. The stronger stability condition for quasimaps significantly simplifies the correction terms which appear.

In §4 we extend the recursion formula to arbitrary pairs $(X, Y)$ where $Y$ is very ample, by taking the embedding $X \hookrightarrow \mathbb{P}^N$ defined by $\mathcal{O}_X(Y)$ and pulling back the formula for $(\mathbb{P}^N, H)$. This of course requires some comparison theorems for virtual classes, for which we have to examine the perfect obstruction theories. This recursion formula can then be applied to obtain a quantum Lefschetz theorem for quasimap invariants, which recovers Corollary 5.5.1 in [CFK14].

We also include a couple of appendices collecting together results which are presumably well-known to experts, but for which we could not find references in the literature.

Appendix §**??** discusses the comparison morphism from maps to quasimaps (used in the proof of the recursion relation in §3).

Appendix §A contains foundational lemmas of quasimap theory, including functoriality and the splitting axiom. Appendix B discusses the DIAGONAL PULL-BACK along a morphism whose target is unobstructed.

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1.4. **Table of notation.** We will use the following notation, most of which is introduced in the main body of the paper.

| | |
|---|---|
| $X$ | a smooth projective toric variety |
| $Y$ | a very ample hypersurface in $X$ |
| $\Sigma$ | the fan of $X$ |
| $\Sigma(1)$ | the set of 1-dimensional cones of $\Sigma$ |
| $\rho$ | an element of $\Sigma(1)$ |
| $D_\rho$ | the toric divisor in $X$ corresponding to $\rho$ |
| $\overline{\mathcal{M}}_{g,n}(X,\beta)$ | the moduli space of stable maps |
| $\mathcal{M}_{0,\alpha}(X\|Y,\beta)$ | the nice locus of relative stable maps |
| $\overline{\mathcal{M}}_{0,n}(X\|Y,\beta)$ | the moduli space of relative stable maps (§2.2) |
| $\overline{\mathcal{Q}}_{g,n}(X,\beta)$ | the moduli space of toric quasimaps (§2.1) |
| $\mathcal{Q}_{0,\alpha}(X\|Y,\beta)$ | the nice locus of relative quasimaps (§3.1) |
| $\overline{\mathcal{Q}}_{0,\alpha}(X\|Y,\beta)$ | the moduli space of relative quasimaps (§2.3) |
| $\mathcal{D}^{\mathcal{Q}}_{\alpha,k}(X\|Y,\beta)$ | the quasimap comb locus (§3.2) |
| $\mathcal{D}^{\mathcal{Q}}(X\|Y,A,B,M)$ | (a component of) the (quasimap) comb locus (§3.2) |
| $\mathcal{E}^{\mathcal{Q}}(X\|Y,A,B,M)$ | the total product for the comb locus |
| $\mathcal{D}^{\mathcal{Q}}(X,A,B)$ | the quasimap centipede locus |
| $\mathcal{E}^{\mathcal{Q}}(X,A,B)$ | the total product for the cenitepede locus |
| $\mathfrak{M}_{g,n}$ | the moduli stack of prestable curves |
| $\mathfrak{Pic}$ | the relative Picard stack of the universal curve over $\mathfrak{M}_{g,n}$ |
| $\mathfrak{Bun}_G$ | the moduli stack of principal $G$-bundles on the universal curve over $\mathfrak{M}_{g,n}$ |
| $\overline{\mathcal{Q}}(f)$ | the push-forward between quasimap spaces (§A.1) |
| $\chi$ | the comparison map from stable maps to quasimaps (§**??**) |
| $f^!$ | Gysin morphism for $f$ an l.c.i. embedding |
| $f^!_{\mathrm{v}}$ | virtual pull-back for $f$ virtually smooth |
| $f^!_\Delta$ | diagonal virtual pull-back |

## 2. Relative stable quasimaps

2.1. **Review of absolute stable quasimaps.** We briefly recall the definition and basic properties of the moduli space of toric quasimaps; see [CFK10] for more details.

**Definition 2.1** ([CFK10, Definition 3.1.1]). Let $X = X_\Sigma$ be a smooth and projective toric variety with fan $\Sigma \subseteq N_{\mathbb{Q}}$ and let $\mathcal{O}_{X_\Sigma}(1)$ be a fixed polarisation, which we can write (non-uniquely) in terms of the $T$-invariant divisors as:

$$\mathcal{O}_{X_\Sigma}(1) = \otimes_{\rho \in \Sigma(1)} \mathcal{O}_{X_\Sigma}(D_\rho)^{\otimes \alpha_\rho}$$

for some $\alpha_\rho \in \mathbb{Z}$. We fix the following numerical invariants: a genus $g \geq 0$, number of marked points $n \geq 0$ and curve class $\beta \in \mathrm{H}_2^+(X)$. Then a **stable (toric) quasimap** is given by the data

$$((C, x_1, \ldots, x_n), (L_\rho, u_\rho)_{\rho \in \Sigma(1)}, (\varphi_m)_{m \in M})$$

where:

(1) $(C, x_1, \ldots, x_n)$ is a prestable curve of genus $g$ with $n$ marked points;
(2) the $L_\rho$ are line bundles on $C$ of degree $d_\rho = D_\rho \cdot \beta$;
(3) the $u_\rho$ are global sections of $L_\rho$;
(4) $\varphi_m \colon \otimes_\rho L_\rho^{\otimes \langle \rho, m \rangle} \to \mathcal{O}_C$ are isomorphisms, such that $\varphi_m \otimes \varphi_{m'} = \varphi_{m+m'}$ for all $m, m' \in M$.

These are required to satisfy the following two conditions:

(1) **nondegeneracy**: there is a finite (possibly empty) set of smooth and non-marked points $B \subseteq C$, called the **basepoints** of the quasimap, such that for all $x \in C \setminus B$ there exists a maximal cone $\sigma \in \Sigma_{\max}$ with $u_\rho(x) \neq 0$ for all $\rho \not\subset \sigma$;
(2) **stability**: if we let $L = \otimes_\rho L_\rho^{\otimes \alpha_\rho}$ then the following $\mathbb{Q}$-divisor is ample

$$\omega_C(x_1 + \ldots + x_n) \otimes L^{\otimes \epsilon}$$

for every rational $\epsilon > 0$.

**Remark 2.2.** This definition is motivated by the $\Sigma$-collections of D. Cox [Cox95a]; see also Appendix A.1. The point is that a quasimap defines a rational morphism $C \dashrightarrow X$ with base locus equal to $B$. (This can be expressed in a more generalisable manner as follows: a quasimap is a map to the stack quotient $[\mathbb{A}^{\Sigma(1)}/(\mathbb{G}_{\mathrm{m}})^r]$ such that $B$ is the preimage of the unstable locus.)

In particular a quasimap without any basepoints defines a morphism $C \to X$. Thus the basepoints appear in the (virtual) boundary of the moduli space, in much the same way as the locus of stable maps with rational tails appears in the boundary of the moduli space of stable maps (this is something more than just a vague analogy; these loci correspond to each other under the comparison morphism when $X \simeq \mathbb{P}^N$; see Appendix **??**).

More generally, we can define the notion of a family of quasimaps over a base scheme $S$, and what it means for two such families to be isomorphic; we thus obtain a moduli space

$$\overline{Q}_{g,n}(X,\beta)$$

of stable (toric) quasimaps to $X$, which is a proper Deligne–Mumford stack of finite type. It can be shown that this definition does not depend on the choice of polarisation.

As with the case of stable maps, there is a combinatorial characterisation of stability which is much easier to check in practice; a prestable quasimap is stable if and only if the following conditions hold:

(1) the line bundle $L$ defined above must have strictly positive degree on any rational component with fewer than three special points, and on any elliptic component with no special points;
(2) $C$ cannot have any rational components with fewer than two special points.

Condition (1) is analogous to the ordinary stability condition for stable maps. Condition (2) is new, however, and gives quasimaps a distinctly different flavour to stable maps; we shall sometimes refer to it as the STRONG STABILITY CONDITION.

**Remark 2.3.** Notice that quasimap spaces are "smaller", so if a comparison morphism exists it should be from stable maps to quasimaps; in fact a morphism between the moduli spaces comes with a morphism of the universal curves, and this ought to be a contraction of the rational tails (the other way round, one could sprout a rational tail from any base points, and maybe even the degree of the line bundles would be determined, but a fundamental indeterminacy remains as to what the sections should be, i.e. where does the rational tail map).

The moduli space $\overline{Q}_{g,n}(X,\beta)$ admits a perfect obstruction theory relative to the moduli space $\mathfrak{M}_{g,n}$ of source curves, and hence one can construct a virtual class

$$[\overline{Q}_{g,n}(X,\beta)]^{\text{virt}} \in \mathrm{A}_{\operatorname{vdim}\overline{Q}_{g,n}(X,\beta)}\left(\overline{Q}_{g,n}(X,\beta)\right)$$

where the virtual dimension is the same as for stable maps:

$$\operatorname{vdim}\overline{Q}_{g,n}(X,\beta) = (\dim X - 3)(1-g) - (K_X \cdot \beta) + n$$

Since the markings are not basepoints there exist evaluation maps

$$\mathrm{ev}_i : \overline{Q}_{g,n}(X,\beta) \to X$$

and there are $\psi$-classes defined in the usual way by pulling back the relative dualising sheaf of the universal curve

$$\psi_i = \mathrm{c}_1(x_i^* \omega_{C/\overline{Q}})$$

where $C \to \overline{Q} = \overline{Q}_{g,n}(X,\beta)$ is the universal curve and $x_i : \overline{Q} \to C$ is the section defining the $i$th marked point. Putting all these pieces together, we can define

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$$\langle \gamma_1 \psi^{a_1}, \ldots, \gamma_n \psi^{a_n} \rangle^X_{g,n,\beta} = \int_{[\overline{Q}_{g,n}(X,\beta)]^{\mathrm{virt}}} \prod_{i=1}^{n} \mathrm{ev}_i^*(\gamma_i) \psi_i^{a_i}$$

(We use the same correlator notation as in Gromov–Witten theory; since we will never talk about Gromov–Witten invariants this should not cause any confusion.)

2.2. **Review of relative stable maps.** Given a smooth projective variety $X$ and a smooth divisor $Y$, the moduli space of relative stable maps parametrises stable maps to $X$ with specified tangencies to $Y$ at the marked points; see [Gat02] for details.

**Definition 2.4** ([Gat02, Definition 1.1])**.** Let $X$ be a smooth projective variety and $Y \subseteq X$ a smooth divisor. Fix a number $n \geq 0$ of marked points, a curve class $\beta \in \mathrm{H}_2^+(X)$ and an $n$-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non-negative integers such that $\Sigma_i \alpha_i \leq Y \cdot \beta$. Then the moduli space

$$\overline{\mathcal{M}}_{0,\alpha}(X|Y, \beta)$$

of relative stable maps to $(X, Y)$ is defined to be the locus in $\overline{\mathcal{M}}_{0,n}(X, \beta)$ of stable maps $(C \to S, (x_i : S \to C)_{i=1}^n, f : C \to X)$ satisfying the following two conditions:

(1) if $x_i$ is a marked point such that $\alpha_i > 0$ then $f(x_i) \in Y$;
(2) if we consider $f^*[Y] \in \mathrm{A}_0(f^{-1}Y)$ then the difference $f^*[Y] - \Sigma_i \alpha_i x_i$ is an effective class.

This forms a closed substack of $\overline{\mathcal{M}}_{0,n}(X, \beta)$. Condition (1) is required in order for the class $\Sigma_i \alpha_i x_i$ to make sense in $\mathrm{A}_0(f^{-1}Y)$.

**Remark 2.5.** The definition above works in families; however there is a more combinatorial definition for individual maps which is more useful in practice (see [Gat02, Remark 1.4]): a stable map $(C, x_1, \ldots, x_n, f)$ is a relative stable map if and only if, if $Z$ is a connected component of $f^{-1}(Y) \subseteq C$, then

(1) if $Z$ is a point and is equal to a marked point $x_i$, then the multiplicity of $f$ to $Y$ at $x_i$ is greater than or equal to $\alpha_i$;
(2) if $Z$ is one-dimensional (hence a union of irreducible components of $C$) and if we let $C^{(i)}$ for $1 \leq i \leq r$ denote the irreducible components of $C$ adjacent to $Z$, and $m^{(i)}$ denote the multiplicity of $f|_{C^{(i)}}$ to $Y$ at the node $Z \cap C^{(i)}$, then we must have:

$$Y \cdot f_*[Z] + \sum_{i=1}^{r} m^{(i)} \geq \sum_{x_i \in Z} \alpha_i \tag{1}$$

**Remark 2.6.** In the case of maximal multiplicity $\Sigma_i \alpha_i = Y \cdot \beta$, all the inequalities in the above definition must actually be equalities.

In the case $X = \mathbb{P}^N$, $Y = H$ one can show that $\overline{\mathcal{M}}_{0,\alpha}(\mathbb{P}^N|H, d)$ is irreducible with dimension equal to the expected dimension:

$$\operatorname{vdim} \overline{\mathcal{M}}_{0,\alpha}(X|Y, \beta) = \operatorname{vdim} \overline{\mathcal{M}}_{0,n}(X, \beta) - \sum_i \alpha_i$$

Hence it has a fundamental class from which one can define relative Gromov–Witten invariants.

In general if $Y \subseteq X$ is very ample one can use the embedding $X \hookrightarrow \mathbb{P}^N$ to obtain a cartesian diagram:

$$\begin{tikzcd} \overline{\mathcal{M}}_{0,\alpha}(X|Y,\beta) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{\mathcal{M}}_{0,\alpha}(\mathbb{P}^N|H,d) \arrow[d] \\ \overline{\mathcal{M}}_{0,n}(X,\beta) \arrow[r, "\varphi"] & \overline{\mathcal{M}}_{0,n}(\mathbb{P}^N,d) \end{tikzcd}$$

Then the fact that $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^N,d)$ is smooth allows us to define a virtual class on $\overline{\mathcal{M}}_{0,\alpha}(X|Y,\beta)$ by virtual (or diagonal) pull-back (see Appendix **??** of the current paper):

$$[\overline{\mathcal{M}}_{0,\alpha}(X|Y,\beta)]^{\mathrm{virt}} = \varphi^![\overline{\mathcal{M}}_{0,\alpha}(\mathbb{P}^N|H,d)]$$

Thus one can define relative Gromov–Witten invariants. In §§2-4 Gathmann proves a recursion relation inside the Chow group of $\overline{\mathcal{M}}_{0,\alpha}(X|Y,\beta)$

$$(\alpha_k\psi_k + \mathrm{ev}_k^* Y)[\overline{\mathcal{M}}_{0,\alpha}(X|Y,\beta)]^{\mathrm{virt}} = [\overline{\mathcal{M}}_{0,\alpha+e_k}(X|Y,\beta)]^{\mathrm{virt}} + [\mathcal{D}_{\alpha,k}(X,\beta)]^{\mathrm{virt}}$$

where $\mathcal{D}_{\alpha,k}(X,\beta)$ is an appropriate **comb locus**. Repeated application of this result shows that the relative Gromov–Witten invariants of $(X,Y)$ and the Gromov–Witten invariants of $Y$ are completely determined by the Gromov–Witten invariants of $X$. This relation is then worked out explicitly in cases of particular interest in [Gat03] to obtain a new proof of the mirror theorem.

2.3. **Definition of relative stable quasimaps.** We now give the main definition of the paper. From here on $X$ will denote a smooth projective toric variety and $Y \subseteq X$ a very ample hypersurface. We do not require that $Y$ is toric.

Consider the line bundle $\mathcal{O}_X(Y)$ and the section $s_Y$ cutting out $Y$. By [Cox95b] we have a natural isomorphism

$$\mathrm{H}^0(X,\mathcal{O}_X(Y)) = \Bbbk\left\langle \prod_\rho z_\rho^{a_\rho} : \Sigma_\rho a_\rho D_\rho = Y \right\rangle$$

where the $z_\rho$ for $\rho \in \Sigma(1)$ are the generators of the Cox ring of $X$ and the $a_\rho$ are non-negative integers. We can therefore write $s_Y$ as

$$s_Y = \sum_{\underline{a}=(a_\rho)} \lambda_{\underline{a}} \prod_\rho z_\rho^{a_\rho}$$

for some scalars $\lambda_{\underline{a}} \in \Bbbk$. The idea then is that a quasimap

$$((C,x_1,\ldots,x_n),(L_\rho,u_\rho)_{\rho\in\Sigma(1)},(\varphi_m)_{m\in M})$$

maps to $Y$ at $x \in C$ if and only if the section

$$u_Y := \sum_{\underline{a}} \lambda_{\underline{a}} \prod_\rho u_\rho^{a_\rho}$$

vanishes at $x$. We now explain how to make sense of the expression above. For each $\underline{a}$ we have a well-defined section

$$u_{\underline{a}} := \lambda_{\underline{a}} \prod_{\rho} u_\rho^{a_\rho} \in \mathrm{H}^0(C, \otimes_\rho L_\rho^{\otimes a_\rho})$$

and if we have $\underline{a}$ and $\underline{b}$ such that $\sum_\rho a_\rho D_\rho = Y = \sum_\rho b_\rho D_\rho$ then these differ by an element $m$ of $M$. Thus the isomorphism $\varphi_m$ allows us to view the sections $u_{\underline{a}}$ and $u_{\underline{b}}$ as sections of the same bundle, which we denote by $L_Y$. Then we can sum these together to obtain $u_Y$. There is a choice involved here, but up to isomorphism it does not matter; see the proof of functoriality in Appendix A.1 for more details.

The upshot is that we obtain a line bundle $L_Y$ on $C$ (which plays the role of the "pull-back" of $\mathcal{O}_X(Y)$) and a global section

$$u_Y \in \mathrm{H}^0(C, L_Y)$$

which plays the role of the "pull-back" of $s_Y$.

**Definition 2.7.** With notation as above, let $n \geq 0$ be a number of marked points, $\beta \in \mathrm{H}_2^+(X)$ be a curve class and $\alpha = (\alpha_1, \ldots, \alpha_n)$ a collection of non-negative integers such that $\Sigma_i \alpha_i \leq Y \cdot \beta$. Then we define the MODULI SPACE OF RELATIVE STABLE QUASIMAPS

$$\overline{Q}_{0,\alpha}(X|Y, \beta) \subseteq \overline{Q}_{0,n}(X, \beta)$$

to be the locus of quasimaps

$$(C \to S, (x_i : S \to C)_{i=1}^n, (L_\rho, u_\rho)_{\rho \in \Sigma(1)}, (\varphi_m)_{m \in M})$$

such that:

(1) if $x_i$ is a marking such that $\alpha_i > 0$, then $x_i^* u_Y = 0$;
(2) if we let $u_Y^*(0) \in \mathrm{A}_0(u_Y^{-1}(0))$ denote the class defined by the Gysin map for $L_Y$, then the difference $u_Y^*(0) - \Sigma_i \alpha_i x_i$ is an effective class.

**Remark 2.8.** As in the case of relative stable maps (see §2.2) there is an alternative definition which is easier to check in practice: a quasimap is a relative quasimap if and only if, if $Z$ is a connected component of $u_Y^{-1}(0)$, then:

(1) if $Z$ is a point and is equal to a marked point $x_i$, then the order of vanishing of $u_Y$ at $x_i$ is greater than or equal to $\alpha_i$;
(2) if $Z$ is one-dimensional (hence a union of irreducible components) and if we let $C^{(i)}$ for $1 \leq i \leq r$ denote the irreducible components of $C$ adjacent to $Z$, and $m^{(i)}$ the order of vanishing of $u_Y$ at the node $Z \cap C^{(i)}$, then we must have:

$$\deg L_Y|_Z + \sum_{i=1}^{r} m^{(i)} \geq \sum_{x_i \in Z} \alpha_i \tag{2}$$

**Remark 2.9.** In the second case above we call $Z$ an INTERNAL component and the $C^{(i)}$ EXTERNAL components.

As it stands we do not know much about this locus. In the following section we will examine the case $X = \mathbb{P}^N$ and $Y = H$ a hyperplane in detail. We will then apply the results obtained there to deduce facts about the general case.

## 3. Recursion formula for $\mathbb{P}^N$ relative $H$

3.1. **Basic properties of the moduli space.** In this section we will show that the moduli space

$$\overline{\mathcal{Q}}_{0,\alpha}(\mathbb{P}^N|H, d)$$

is irreducible of the expected dimension, and thus admits a fundamental class. We then prove a recursion formula for these fundamental classes by pushing forward Gathmann's recursion formula along the comparison morphism

$$\chi : \overline{\mathcal{M}}_{0,n}(\mathbb{P}^N, d) \to \overline{\mathcal{Q}}_{0,n}(\mathbb{P}^N, d)$$

(see Appendix **??**). For the remainder of this section we set $X = \mathbb{P}^N$ and $Y = H = \{z_0 = 0\}$. Given a quasimap

$$(C, x_1, \ldots, x_n, L, u_0, \ldots, u_N) \in \overline{\mathcal{Q}}_{0,n}(\mathbb{P}^N, d)$$

the line bundle $L_Y$ of the previous section is equal to $L$ and the section $u_Y$ is equal to $u_0$.

**Lemma 3.1.** *The comparison morphism restricts to a morphism*

$$\chi : \overline{\mathcal{M}}_{0,\alpha}(\mathbb{P}^N|H, d) \to \overline{\mathcal{Q}}_{0,\alpha}(\mathbb{P}^N|H, d)$$

*Proof.* We need to verify that a relative stable map is sent to a relative stable quasimap by $\chi$. Since the contraction of a rational tail $R$ always occurs away from the markings, we only need to examine the internal components $Z$ of the quasimap.

Consider then $Z$; for each basepoint $x$ on $Z$ there is a rational tail $R$ of the stable map attached to $Z$ at $x$. This is either internal (mapped into $H$) or external (not mapped into $H$).

If $R$ is internal then both $R$ and $Z$ live inside the same connected component $Z'$ of $f^{-1}(H)$. Applying $\chi$ has the effect of contracting $R$ and increasing the degree of the line bundle on $Z$ by $H \cdot f_*[R]$. Thus the left hand side of the inequality (2) is left unchanged, and since the right hand side is also unaltered the inequality is satisfied.

On the other hand if $R$ is external then the multiplicity $m^{(R)}$ of $R \cap Z$ satisfies:

$$m^{(R)} \leq H \cdot f_*[R]$$

Since applying $\chi$ has the effect of replacing $m^{(R)}$ by $H \cdot f_*[R]$ in the left hand side of (2), the inequality still holds for the quasimap. Thus we obtain a morphism from the relative stable map space to the relative quasimap space, as claimed. □

Let us denote by

$$\mathcal{Q}_{0,\alpha}(\mathbb{P}^N|H, d) \subseteq \overline{\mathcal{Q}}_{0,\alpha}(\mathbb{P}^N|H, d)$$

(without the bar) the **nice locus**, consisting of those quasimaps with irreducible source curve (i.e. a $\mathbb{P}^1$), no basepoints (so we get an actual map), with no component of the curve mapping inside $H$ and with the map having tangency at least $\alpha_i$ to $H$ at the marking $x_i$.

This is an irreducible, locally closed substack of $\overline{Q}_{0,n}(\mathbb{P}^N, d)$ of codimension $\Sigma_i \alpha_i$, by essentially the same argument as in [Gat02, Lemma 1.8]. In fact it is isomorphic to the nice locus inside the stable map space, denoted $\mathcal{M}_{0,\alpha}(\mathbb{P}^N|H, d)$ by Gathmann (see [Gat02, Def. 1.6]); the stricter stability condition has no effect when the source curve is irreducible, provided of course that $n \geq 2$).

**Lemma 3.2.** *$\overline{Q}_{0,\alpha}(\mathbb{P}^N|H, d)$ is equal to the closure of the nice locus $Q_{0,\alpha}(\mathbb{P}^N|H, d)$ inside $\overline{Q}_{0,n}(\mathbb{P}^N, d)$.*

*Proof.* $\overline{Q}_{0,\alpha}(\mathbb{P}^N|H, d) \subseteq \overline{Q_{0,\alpha}(\mathbb{P}^N|H, d)}$: we show that any relative stable quasimap can be infinitesimally deformed to a relative stable quasimap with no basepoints. This is in particular a relative stable map; we then appeal to [Gat02, Prop. 1.14] to deform this stable map and obtain a point in the nice locus. Since this deformation does not introduce any rational tails, this is also a deformation of quasimaps, and the statement follows.

We induct on the number of components containing at least one base-point. Suppose this number is non-zero (otherwise there is nothing to prove) and pick such a component $C_0$, with base-points $y_1, \ldots, y_k$. Recall that this means that $u_i(y_j) = 0$ for all $i$ and $j$. We will deform the section $u_N|_{C_0}$ to a new section $u'_N|_{C_0}$ in such a way that $u'_N|_{C_0}(y_j) \neq 0$ and in such a way that we do not introduce any new basepoints. Notice that since the relative condition only depends on $u_0$, the resulting deformed quasimap will still be a relative quasimap.

Now, by nondegeneracy and the fact that there exists a basepoint, we must have $\deg(L|_{C_0}) > 0$, and since $C_0 \cong \mathbb{P}^1$ we can find a section $w_0$ of $L|_{C_0} \cong \mathcal{O}_{\mathbb{P}^1}(d_0)$ not vanishing at any of the base-points $p_i$.

We then set

$$u'_N|_{C_0} := u_N|_{C_0} + \epsilon w_0$$

and $u'_i|_{C_0} = u_i|_{C_0}$ for all other $i$. Notice that $u'_N|_{C_0}(y_j) \neq 0$ for all $j$ as claimed. It is also clear that we do not introduce any new basepoints, since $u'_N|_{C_0}(y) = 0$ implies $u_N|_{C_0}(y) = 0$ (put differently: being a basepoint is a close condition).

It remains to extend the section $u'_N|_{C_0}$ to a section $u'_N$ on the whole curve. Let $C_1, \ldots, C_r$ be the components of $C$ adjacent to $C_0$ and let $q_i = C_0 \cap C_i$. We need to modify the sections $u_N|_{C_i}$ in such a way that $u'_N|_{C_i}(q_i) = u'_N|_{C_0}(q_i)$.

By nondegeracy, we can choose a section $w_i$ of $L|_{C_i}$ such that $w(q_i) \neq 0$. Then set:

$$u'_{N\,C_i} := u_N|_{C_i} + \epsilon \left( \frac{w_0(q_i)}{w_i(q_i)} \right) \cdot w_i$$

Then indeed we have:

$$u'_N|_{C_i}(q_i) = u_N(q_i) + \epsilon \left( \frac{w_0(q_i)}{w_i(q_i)} \right) \cdot w_i(q_i) = u_N(q_i) + \epsilon w_0(q_i) = u'_N|_{C_0}(q_i)$$

We can continue this process, replacing $C_0$ by $C_i$; since the genus of the curve is zero there are no cycles in the dual intersection graph, and so we will never come to the same curve twice. In this way we obtain a new quasimap

$$(C, x_1, \ldots, x_n, L, u_0, \ldots, u_{N-1}, u'_N)$$

over $\operatorname{Spec} \Bbbk[\epsilon]/(\epsilon^2)$ which has no basepoints on $C_0$. We can repeat this process for all the components of $C$ (using higher powers of $\epsilon$ each time in order to ensure that we never introduce additional basepoints) and thus we obtain an infinitesimal deformation of our original quasimap which has no basepoints, as required.

$\overline{\mathcal{Q}_{0,\alpha}(\mathbb{P}^N|H,d)} \subseteq \overline{\mathcal{Q}}_{0,\alpha}(\mathbb{P}^N|H,d)$: consider a family of stable quasimaps over a smooth curve $S$, such that the generic fibre lies in the nice locus. We may blow-up the source curve (a fibered surface over $S$) in the locus of basepoints (which consists of finitely many smooth points of the central fiber) in order to obtain an actual morphism to $\mathbb{P}^N$. This has the effect of adding rational tails at the basepoints in the central fibre. If the morphism is constant on any of these rational tails we may contract them, and thus we obtain a family of stable maps which pushes down along $\chi$ to our original family of quasimaps.

The general fibre is not modified at all, and so is still in the nice locus. By [Gat02, Lemma 1.9] it follows that the central fibre is a relative stable map, and then by applying $\chi$ and appealing to Lemma 3.1 it follows that the same is true for the central fibre of the family of quasimaps. □

**Corollary 3.3.** *The moduli space $\overline{\mathcal{Q}}_{0,\alpha}(\mathbb{P}^N|H,d)$ is irreducible of the expected dimension. Hence it has a fundamental class.*

*Proof.* This holds because the moduli space is equal to the closure of the nice locus, which is irreducible of the expected dimension. □

**Corollary 3.4.** *The comparison morphism from relative stable maps to relative quasimaps is birational. In particular it sends the fundamental class to the fundamental class.*

*Proof.* This follows because the comparison morphism restricts to an isomorphism on the nice locus, which by the lemma above is a dense open subset of both spaces. □

3.2. **Proof of the recursion formula.** We wish to obtain a recursion formula relating the quasimap invariants of multiplicity $\alpha$ with the quasimap invariants of multiplicity $\alpha + e_k$, as in [Gat02, Theorem 2.6]. This process of "increasing the multiplicities" can be naively performed in the same way as Gathmann: for $m = \alpha_k + 1$ the following section (of the pull-back of the jet bundle of the universal line bundle)

$$\sigma_k^m := x_k^* d^m_{C/\overline{\mathcal{Q}}}(u_0) \in \mathrm{H}^0(\overline{\mathcal{Q}}, x_k^* \mathcal{P}^m_{C/\overline{\mathcal{Q}}}(\mathcal{L}))$$

cuts out $\overline{\mathcal{Q}}_{0,\alpha+e_k}(\mathbb{P}^N|H,d)$ inside $\overline{\mathcal{Q}}_{0,\alpha}(\mathbb{P}^N|H,d)$, along with a number of degenerate contributions (called the COMB LOCI) parametrising quasimaps for which $x_k$ belongs to an internal component $Z \subseteq C$ (a component on which $u_0$ vanishes), such that

$$\deg(L|_Z) + \sum_{i=1}^{r} m^{(i)} = \sum_{x_i \in Z} \alpha_i$$

(here by "component" we really mean "connected component of the vanishing locus of $u_0$"). Quasimap stability means that these degenerate contributions cannot contain any rational tails; this is really the only difference with the case of stable maps.

Indeed, we can actually push forward Gathmann's formula along the comparison morphism

$$\chi \colon \overline{\mathcal{M}}_{0,\alpha}(\mathbb{P}^N|H,d) \to \overline{Q}_{0,\alpha}(\mathbb{P}^N|H,d)$$

and due to Corollary 3.4 above, the only terms which change are the comb loci containing rational tails. In fact these disappear, since the restriction of the comparison map to these loci has positive-dimensional fibres:

**Lemma 3.5.** *Consider a rational tail component in the comb locus of the moduli space of stable maps, i.e. a moduli space of the form:*

$$\overline{\mathcal{M}}_{0,(m^{(i)})}(\mathbb{P}^N|H,d)$$

*Then (assuming that $Nd > 1$) we have*

$$\dim\left([\overline{\mathcal{M}}_{0,(m^{(i)})}(\mathbb{P}^N|H,d)] \cap \mathrm{ev}_1^*(\mathrm{pt}_H)\right) > 0$$

*where $\mathrm{pt}_H \in \mathrm{A}^{N-1}(H)$ is a point class. Thus the pushforward along $\chi$ of any comb locus with a rational tail is* 0*.*

*Proof.* This is a simple dimension count. We have

$$\begin{aligned}\dim\left([\overline{\mathcal{M}}_{0,(m^{(i)})}(\mathbb{P}^N|H,d)] \cap \mathrm{ev}_1^*(\mathrm{pt}_H)\right) &= (N-3) + d(N+1) + (1-m^{(i)}) - (N-1) \\ &= (Nd-1) + (d-m^{(i)})\end{aligned}$$

from which the lemma follows because $m^{(i)} \leq d$. □

**Remark 3.6.** With an eye to the future, we remark that these rational tail components contribute nontrivially to the Gromov–Witten invariants of a Calabi–Yau hypersurface in projective space, and so their disappearance in our recursion formula may account for the divergence between Gromov–Witten and quasimap invariants in the Calabi–Yau case [Gat03, Rmk. 1.6].

Since we wish to apply the projection formula to Gathmann's recursion relation, we should express the cohomological terms which appears as pull-backs:

**Lemma 3.7.** *We have:*

$$\begin{aligned}\chi^*(\psi_k) &= \psi_k \\ \chi^*(\mathrm{ev}_k^* H) &= \mathrm{ev}_k^* H\end{aligned}$$

*Proof.* We will actually show that:

$$\begin{aligned}\chi^* x_k^* \omega_{C/\overline{Q}} &= x_k^* \omega_{C/\overline{\mathcal{M}}} \\ \chi^* x_k^* \mathcal{L} &= \mathrm{ev}_k^* \mathcal{O}_{\mathbb{P}^N}(H)\end{aligned}$$

This follows by considering the following diagram:

$$\begin{tikzcd}
 & & \mathbb{P}^N & \\
C_{\overline{\mathcal{M}}} \arrow[urr, bend left, "f"] \arrow[r, "\sigma^{ss}"'] \arrow[dr] & \chi^* C_{\overline{Q}} \arrow[ur, dashed] \arrow[rr] \arrow[d, shift right] \arrow[drr, phantom, "\square"] & & C_{\overline{Q}} \arrow[ul, dashed] \arrow[d, shift right] \\
 & \overline{\mathcal{M}}_{0,\alpha}(\mathbb{P}^N|H,d) \arrow[rr, "\chi"] \arrow[ul, bend left, "x_k"] \arrow[u, shift right, bend right, "x_k"'] & & \overline{Q}_{0,\alpha}(\mathbb{P}^N|H,d) \arrow[u, shift right, bend right, "x_k"']
\end{tikzcd}$$

where $\sigma^{ss}$ is the strong stabilisation map which contracts the rational tails, and so is an isomorphism near the markings. $\square$

**Proposition 3.8.** *Define the* QUASIMAP COMB LOCUS $\mathcal{D}^{Q}_{\alpha,k}(\mathbb{P}^N|H,d)$ *as the union of the moduli spaces*

$$\mathcal{D}^{Q}(\mathbb{P}^N|H,A,B,M) := \overline{Q}_{0,|\alpha^{(0)}|+r}(H,d_0) \times_{H^r} \prod_{i=1}^{r} \overline{Q}_{0,(m^{(i)})\cup\alpha^{(i)}}(\mathbb{P}^N|H,d_i)$$

*where the union runs over all splittings $A = (\alpha^{(0)},\ldots,\alpha^{(r)})$ of the markings, $B = (d_0,\ldots,d_r)$ of the degree and all valid multiplicites $M = (m^{(1)},\ldots,m^{(r)})$ such that the above spaces are all well-defined (in particular $|\alpha^{(0)}|+r$ and $|\alpha^{(i)}|+1$ are all $\geq 2$) and such that*

$$d_0 + \sum_{i=1}^{r} m^{(i)} = \sum \alpha^{(0)}$$

*Equip this with the sum of the (product) fundamental classes. Then the following formula holds*

$$(\alpha_k \psi_k + \mathrm{ev}_k^* H) \cdot [\overline{Q}_{0,\alpha}(\mathbb{P}^N|H,d)] = [\overline{Q}_{0,\alpha+e_k}(\mathbb{P}^N|H,d)] + [\mathcal{D}^{Q}_{\alpha,k}(\mathbb{P}^N|H,d)].$$

*Proof.* This follows from [Gat02, Thm. 2.6] by pushforward along $\chi$, using the projection fomula and Lemmas 3.4, 3.5 and 3.7 . $\square$

## 4. Recursion formula in the general case

We now move on to the general case. Let $X$ be an arbitrary toric variety (smooth and proper) and $Y \subseteq X$ a very ample hypersurface (not necessarily toric). The complete linear system associated to $\mathcal{O}_X(Y)$ defines an embedding $i : X \hookrightarrow \mathbb{P}^N$ such that $i^{-1}(H) = Y$ (for a certain hyperplane $H$). By the functoriality property of quasimap spaces (see Appendix A.1) we have a map:

$$k := \overline{Q}(i) : \overline{Q}_{0,n}(X,\beta) \to \overline{Q}_{0,n}(\mathbb{P}^N,d)$$

where $d = i_*\beta$. Since $i$ is a closed embedding it follows that $k$ is as well. Furthermore $k$ admits a compatible perfect obstruction theory (see Appendix A.2), so we have a notion of virtual pull-back along $k$ (which coincides with the diagonal pull-back according to Lemma B.1).

It is easy to show that $k$ restricts to a morphism between the relative spaces, and thus we have a diagram of embeddings

$$
\begin{tikzcd}
\overline{Q}_{0,\alpha}(X|Y,\beta) \arrow[r, hook, "g"] \arrow[d, hook, "f"] \arrow[dr, phantom, "\square"] & \overline{Q}_{0,\alpha}(\mathbb{P}^N|H,d) \arrow[d, hook, "j"] \\
\overline{Q}_{0,n}(X,\beta) \arrow[r, hook, "k"] & \overline{Q}_{0,n}(\mathbb{P}^N,d)
\end{tikzcd}
$$

which one can show is cartesian. As such we can define a virtual class on $\overline{Q}_{0,\alpha}(X|Y,\beta)$ by (virtual or diagonal) pullback.

The idea is to prove the recursion formula for $(X,Y)$ by pulling back the formula for $(\mathbb{P}^N,H)$ along $k$. In order to do this, we need to understand how the various virtual classes involved in the formula pull back along this map. The first two terms of the recursion formula pull back by the very definition of the virtual class:

**Lemma 4.1.** $k^![\overline{Q}_{0,\alpha}(\mathbb{P}^N|H,d)] = [\overline{Q}_{0,\alpha}(X|Y,\beta)]^{\mathrm{virt}}$

It remains to consider the third term, namely the virtual class of the comb locus. This is the technical heart of the proof.

4.1. **Comb loci pull back.** Recall that we can write $\mathcal{D}^{Q}_{\alpha,k}(X|Y,\beta)$ as a union of comb loci

$$\mathcal{D}^{Q}(X|Y,A,B,M) := \overline{Q}_{0,A_0\cup\{q_1,\ldots,q_r\}}(Y,\beta_0) \times_{Y^r} \prod_{i=1}^{r} \overline{Q}_{0,\alpha^{(i)}\cup(m_i)}(X|Y,\beta_i)$$

where $A$ and $B$ are partitions of the marked points and curve class respectively, and $M=(m_1,\ldots,m_r)$ records the intersection multiplicity with $Y$ at the nodes connecting the internal component to the external components (the spine of the comb to the teeth). Since the virtual class on $\mathcal{D}^{Q}_{\alpha,k}(X|Y,\beta)$ is equal to the sum of the virtual classes of the $\mathcal{D}^{Q}(X|Y,A,B,M)$, we can deal with each of these comb loci separately.

**Remark 4.2.** Note that $Y$ is not toric, and so we should clarify what we mean by:

$$\overline{Q}(Y) = \overline{Q}_{0,A_0\cup\{q_1,\ldots,q_n\}}(Y,\beta_0)$$

There are two possibilities here: one is to *define* this space as the cartesian product:

$$
\begin{tikzcd}
\overline{Q}(Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{Q}(H) \arrow[d] \\
\overline{Q}(X) \arrow[r] & \overline{Q}(\mathbb{P}^N)
\end{tikzcd}
$$

and equip it with the pullback virtual class (using the fact that the base is smooth).

This has obvious advantages from the point of view of our computations, but is conceptually unsatisfying. On the other hand, $Y \subseteq X$ defines a $(\mathbb{C}^*)^r$-invariant subvariety in the prequotient of $X$, which we refer to (by analogy with the case $X=\mathbb{P}^r$) as the **cone of Y**:

$$C(Y) \subseteq \mathbb{A}_k^{\Sigma_X(1)}$$

Then $Y$ is equal to the GIT quotient

$$Y = C(Y) /\!\!/ (\mathbb{C}^*)^r$$

and so we may use the more general theory of quasimaps to GIT quotients ([CFKM14]) to define $\overline{Q}(Y)$ and its virtual class.

We should then check that these two definitions of $\overline{Q}(Y)$ agree (i.e. that there exists an isomorphism between these moduli spaces which preserves the virtual class). This is carried out in Appendix **??**.

The comb locus sits inside the full product

$$\mathcal{E}^Q(X|Y,A,B,M) := \overline{Q}_{0,A_0\cup\{q_1,\ldots,q_r\}}(Y,\beta_0) \times \prod_{i=1}^{r} \overline{Q}_{0,\alpha^{(i)}\cup(m_i)}(X|Y,\beta_i)$$

which we may endow with the product virtual class

$$[\mathcal{E}^Q(X|Y,A,B,M)]^{\mathrm{virt}} := [\overline{Q}_{0,A_0\cup\{q_1,\ldots,q_r\}}(Y,\beta_0)]^{\mathrm{virt}} \times \prod_{i=1}^{r} [\overline{Q}_{0,\alpha^{(i)}\cup(m_i)}(X|Y,\beta_i)]^{\mathrm{virt}}$$

We have the following cartesian diagram

$$\begin{tikzcd}
\mathcal{D}^Q(X|Y,A,B,M) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{E}^Q(X|Y,A,B,M) \arrow[d] \\
X^r \arrow[r, "\Delta_{X^r}"] & X^r \times X^r
\end{tikzcd}$$

and we can use this to define a **product virtual class** on the comb locus:

$$[\mathcal{D}^Q(X|Y,A,B,M)]^{\mathrm{virt}} := \Delta_{X^r}^! [\mathcal{E}^Q(X|Y,A,B,M)]^{\mathrm{virt}}$$

**Remark 4.3.** This is the same definition of the virtual class of the comb locus that we gave in §3.2 in the case $(X,Y) = (\mathbb{P}^N, H)$.

On the other hand, there is another cartesian diagram defining the comb locus:

$$\begin{tikzcd}
\mathcal{D}^Q(X|Y,A,B,M) \arrow[r, "k"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{D}^Q(\mathbb{P}^N|H,A,i_*B,M) \arrow[d] \\
\overline{Q}_{0,n}(X,\beta) \arrow[r, "k"] & \overline{Q}_{0,n}(\mathbb{P}^N,d)
\end{tikzcd}$$

**Remark 4.4.** Technically this is not quite correct: really the fibre product is the union of comb loci over all partitions $B'$ such that $i_*B' = i_*B$ . But this subtlety makes no difference to the aguments.

**Lemma 4.5.** *For any $\alpha$ we have:*

$$k^![\mathcal{D}^Q(\mathbb{P}^N|H,A,i_*B,M)]^{\mathrm{virt}} = [\mathcal{D}^Q(X|Y,A,B,M)]^{\mathrm{virt}}$$

Let us introduce the following shorthand notation: we fix the the data of $A$, $B$, $M$ and set:

$$\begin{aligned}
\mathcal{D}(X|Y) &:= \mathcal{D}^{Q}(X|Y, A, B, M) \\
\mathcal{E}(X|Y) &:= \mathcal{E}^{Q}(X|Y, A, B, M) \\
\mathcal{D}(X) &:= \mathcal{D}^{Q}(X, A, B) \\
\mathcal{E}(X) &:= \mathcal{E}^{Q}(X, A, B) \\
\overline{Q}(X) &:= \overline{Q}_{0,n}(X, \beta)
\end{aligned}$$

and similarly for $(\mathbb{P}^N, H)$; see Appendix A.3 for the definition of $\mathcal{D}(X)$ and $\mathcal{D}(Y)$. We have a cartesian diagram

$$\begin{tikzcd}
\mathcal{E}(X|Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{E}(\mathbb{P}^N|H) \arrow[d, "\theta"] \\
\mathcal{E}(X) \arrow[r] & \mathcal{E}(\mathbb{P}^N)
\end{tikzcd}$$

and since $\mathcal{E}(\mathbb{P}^N)$ is smooth and there is a natural fundamental class on $\mathcal{E}(\mathbb{P}^N|H)$, we have a diagonal pull-back morphism $\theta^! = \theta^!_\Delta$ (see Appendix B).

**Lemma 4.6.** $[\mathcal{E}(X|Y)]^{\mathrm{virt}} = \theta^![\mathcal{E}(X)]^{\mathrm{virt}}$

*Proof.* It suffices to check that in the following cartesian diagram

$$\begin{tikzcd}
\overline{Q}(Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{Q}(H) \arrow[d, "\theta"] \\
\overline{Q}(X) \arrow[r] & \overline{Q}(\mathbb{P}^N)
\end{tikzcd}$$

we have $\theta^![\overline{Q}(X)]^{\mathrm{virt}} = [\overline{Q}(Y)]^{\mathrm{virt}}$; this is carried out in Appendix **??**. $\square$

Now consider the following cartesian diagram

$$\begin{tikzcd}
\mathcal{D}(X) \arrow[r] \arrow[d, "\varphi_X"] \arrow[dr, phantom, "\square"] & \mathcal{D}(\mathbb{P}^N) \arrow[r] \arrow[d, "\varphi_{\mathbb{P}^N}"] \arrow[dr, phantom, "\square"] & \mathfrak{M}^{\mathrm{wt}}_{A,B} \arrow[d, "\psi"] \\
\overline{Q}(X) \arrow[r, "k"] & \overline{Q}(\mathbb{P}^N) \arrow[r] & \mathfrak{M}^{\mathrm{wt}}_{0,n}
\end{tikzcd}$$

from which we see that

$$\psi^![\overline{Q}(X)]^{\mathrm{virt}} = \psi^! k^! [\overline{Q}(\mathbb{P}^N)] = k^! \psi^! [\overline{Q}(\mathbb{P}^N)]$$

by commutativity of virtual pullbacks. Note that we have:

$$\psi^![\overline{Q}(X)]^{\mathrm{virt}} = \Delta^!_{X^r}[\mathcal{E}(X)]^{\mathrm{virt}}$$

by the splitting axiom (see Lemma A.7).

*Proof of Lemma 4.5.* Putting all the preceding results together, we consider the cartesian digram:

$$\begin{tikzcd}
\mathcal{D}(X|Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{E}(X|Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{E}(\mathbb{P}^N|H) \arrow[d, "\theta"] \\
\mathcal{D}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{E}(X) \arrow[r] \arrow[d] & \mathcal{E}(\mathbb{P}^N) \\
X^r \arrow[r, "\Delta_{X^r}"] & X^r \times X^r &
\end{tikzcd}$$

We then have:

$$\begin{aligned}
[\mathcal{D}(X|Y)]^{\mathrm{virt}} &= \Delta^!_{X^r}[\mathcal{E}(X|Y)]^{\mathrm{virt}} & \\
&= \Delta^!_{X^r}\theta^![\mathcal{E}(X)]^{\mathrm{virt}} & \text{by Lemma 4.6} \\
&= \theta^!\Delta^!_{X^r}[\mathcal{E}(X)]^{\mathrm{virt}} & \text{by commutativity} \\
&= \theta^!\psi^![\overline{Q}(X)]^{\mathrm{virt}} & \text{by the splitting axiom} \\
&= \theta^! k^! \psi^![\overline{Q}(\mathbb{P}^N)] & \text{by the above} \\
&= \theta^! k^! \Delta^!_{(\mathbb{P}^N)^r}[\mathcal{E}(\mathbb{P}^N)]^{\mathrm{virt}} & \text{by the splitting axiom} \\
&= k^! \Delta^!_{(\mathbb{P}^N)^r}\theta^![\mathcal{E}(\mathbb{P}^N)]^{\mathrm{virt}} & \text{by commutativity} \\
&= k^![\mathcal{D}(\mathbb{P}^N|H)]^{\mathrm{virt}} &
\end{aligned}$$

Summing over all the components of $\mathcal{D}^Q_{\alpha,k}(\mathbb{P}^N|H,d)$ we obtain the result. □

**Theorem 4.7.** *Let $X$ be a smooth and proper toric variety and let $Y \subseteq X$ be a very ample hypersurface (not necessarily toric). Then, with the set-up as in the preceding discussion, we have an equality*

$$(\alpha_k\psi_k + ev_k^*[Y])[\overline{Q}_{0,\alpha}(X|Y,\beta)]^{\mathrm{virt}} = [\overline{Q}_{0,\alpha+e_k}(X|Y,\beta)]^{\mathrm{virt}} + [\mathcal{D}^Q_{\alpha,k}(X|Y,\beta)]^{\mathrm{virt}}$$

*in the Chow group of $\overline{Q}_{0,\alpha}(X|Y,\beta)$.*

*Proof.* Apply $k^!$ to Proposition 3.8, using Lemmas 4.1 and 4.5. □

## Appendix A. Notes on quasimaps

A.1. **Functoriality.** In the case of stable maps, a morphism $f : X \to Y$ induces a morphism between the corresponding moduli spaces

$$\overline{\mathcal{M}}_{g,n}(X,\beta) \to \overline{\mathcal{M}}_{g,n}(Y, f_*\beta)$$

given by composition with $f$ (in general this induced morphism may involve stabilisation of the source curve). Because of this, the construction of the moduli space of stable maps is said to be functorial.

It is natural to ask whether the same holds for the moduli space of quasimaps. Since here the objects of the moduli space are not maps, we cannot simply compose

with $f$, and indeed it is not immediately clear how we should proceed. In [CFK14, Section 3.1] a definition is given when $f$ is an embedding into a projective space; however, this uses the more general language of GIT quotients which we seek to avoid here. As such, we will provide an alternative (but entirely equivalent) construction in the setting of toric varieties, which also relaxes the conditions on the map $f$ and the target $Y$.
[1]

Our approach uses the language of $\Sigma$–collections introduced by D. Cox. This approach is natural insofar as a quasimap is a generalisation of a $\Sigma$–collection. We will refer extensively to [Cox95b] and [Cox95a], which we recommend as an introduction for any readers unfamiliar with the theory.

Let $X$ and $Y$ be smooth and proper toric varieties with fans $\Sigma_X \subseteq N_X$ and $\Sigma_Y \subseteq N_Y$. Suppose we are given $f : Y \to X$ (which we do not assume to be a toric morphism). By [Cox95a, Theorem 1.1] the data of such a map is equivalent to a $\Sigma_X$–collection on $Y$:

$$((L_\rho, u_\rho)_{\rho\in\Sigma_X(1)}, (\varphi_{m_x})_{m_x\in M_X})$$

In addition, [Cox95b] allows us to describe line bundles on $Y$ and their global sections in terms of the homogeneous coordinates $(z_\tau)_{\tau\in\Sigma_Y(1)}$. All of these observations are combined into the following theorem, which is so useful that we will state it here in its entirety:

**Theorem A.1.** [Cox95a, Theorem 3.2] *The data of a morphism $f : Y \to X$ is the same as the data of homogeneous polynomials*

$$P_\rho \in S^Y_{\beta_\rho}$$

*for $\rho \in \Sigma_X(1)$, where $\beta_\rho \in \operatorname{Pic} Y$ and $S^Y_{\beta_\rho}$ is the corresponding graded piece of the Cox ring*

$$S^Y = k[z_\tau : \tau \in \Sigma_Y(1)]$$

*This data is required to satisfy the following two conditions:*

*(1)* $\sum_{\rho\in\Sigma_X(1)} \beta_\rho \otimes n_\rho = 0$ *in* $\operatorname{Pic} Y \otimes N_X$.

*(2)* $(P_\rho(z_\tau)) \notin Z(\Sigma_X) \subseteq \mathbb{A}_k^{\Sigma_X(1)}$ *whenever* $(z_\tau) \notin Z(\Sigma_Y) \subseteq \mathbb{A}_k^{\Sigma_Y(1)}$.

*Furthermore, two such sets of data $(P_\rho)$ and $(P'_\rho)$ correspond to the same morphism if and only if there exists a $\lambda \in \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Pic} X, \mathbb{G}_m)$ such that*

$$\lambda(D_\rho) \cdot P_\rho = P'_\rho$$

*for all $\rho \in \Sigma_X(1)$. Finally, if we define $\tilde{f}(z_\tau) = (P_\rho(z_\tau))$ then this defines a lift of $f$ to the prequotients:*

$$\begin{tikzcd} \mathbb{A}_k^{\Sigma_Y(1)} \setminus Z(\Sigma_Y) \arrow[r, "\tilde{f}"] \arrow[d, "\pi"] & \mathbb{A}_k^{\Sigma_X(1)} \setminus Z(\Sigma_X) \arrow[d, "\pi"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

[1]We should probably look a bit harder to see if the definition exists elsewhere.

**Aside A.2.** Throughout this section we will stick to the notation established above; in particular we will use $\rho$ to denote a ray in $\Sigma_X(1)$ and $\tau$ to denote a ray in $\Sigma_Y(1)$.

Recall our goal: given a map $f : Y \to X$ we wish to define a "push-forward" map:

$$f_* : \overline{Q}_{g,n}(Y, \beta) \to \overline{Q}_{g,n}(X, f_*\beta)$$

Consider therefore a quasimap $(C, (L_\tau, u_\tau)_{\tau \in \Sigma_Y(1)}, (\varphi_{m_Y})_{m_Y \in M_Y})$ with target $Y$. Pick data $(P_\rho)_{\rho \in \Sigma_X(1)}$ corresponding to the map $f$, as in the theorem above; we will later see that our construction does not depend on this choice.

The idea of the construction is as follows. Let us pretend for a moment that $C$ is toric and that the quasimap is without basepoints, so that we have an actual morphism $C \to Y$. Then we can lift this morphism to the prequotient as in the following diagram

$$\begin{tikzcd}
\mathbb{A}_k^{\Sigma_C(1)} \setminus Z(\Sigma_C) \arrow[r, "(u_\tau)"] \arrow[d] & \mathbb{A}_k^{\Sigma_Y(1)} \setminus Z(\Sigma_Y) \arrow[r, "(P_\rho)"] \arrow[d] & \mathbb{A}_k^{\Sigma_X(1)} \setminus Z(\Sigma_X) \arrow[d] \\
C \arrow[r] & Y \arrow[r] & X
\end{tikzcd}$$

from which it follows that the composition $C \to Y \to X$ is given in homogeneous coordinates by:

$$(P_\rho((u_\tau)_{\tau \in \Sigma_Y(1)}))_{\rho \in \Sigma_X(1)}$$

In general of course $C$ is not a toric variety and the quasimap is not basepoint-free. Nevertheless, as we will see, we can still make sense of the expression $P_\rho(u_\tau)$ as a section of a line bundle on $C$. This will allow us to define the pushforward of our quasimap.

Let us begin. For each $\rho$, $P_\rho$ is a polynomial in the $z_\tau$; we can write it as

$$P_\rho(z_\tau) = \sum_{\underline{a}} P_\rho^{\underline{a}}(z_\tau) = \sum_{\underline{a}} \mu_{\underline{a}} \prod_\tau z_\tau^{a_\tau} \tag{3}$$

where the sum is over a finite number of multindices $\underline{a} = (a_\tau) \in \mathbb{N}^{\Sigma_Y(1)}$ and the $\mu_{\underline{a}}$ are nonzero scalars. For each $\underline{a}$ consider the following line bundle on $C$:

$$\tilde{L}_\rho^{\underline{a}} = \bigotimes_\tau L_\tau^{\otimes a_\tau}$$

Then we may take the following section of $\tilde{L}_\rho^{\underline{a}}$:

$$\tilde{u}_\rho^{\underline{a}} = P_\rho^{\underline{a}}(u_\tau) = \mu_{\underline{a}} \prod_\tau u_\tau^{a_\tau}$$

Thus each of the terms $P_\rho^{\underline{a}}$ of $P_\rho$ defines a section $\tilde{u}_\rho^{\underline{a}}$ of a line bundle $\tilde{L}_\rho^{\underline{a}}$. But what we want is a single section $\tilde{u}_\rho$ of a single line bundle $\tilde{L}_\rho$. This is where the isomorphisms $\varphi_{m_Y}$ come in.

Recall that we have a short exact sequence:

$$0 \longrightarrow M_Y \xrightarrow{\theta} \mathbb{Z}^{\Sigma_Y(1)} \longrightarrow \operatorname{Pic} Y \longrightarrow 0 \tag{4}$$

Let $\underline{a}$ and $\underline{b}$ be multindices appearing in the sum (3) above. By the homogeneity of $P_\rho$ we have

$$\sum_\tau a_\tau D_\tau = \beta_\rho = \sum_\tau b_\tau D_\tau$$

which is precisely the statement that in the above sequence $\underline{a}$ and $\underline{b}$ map to the same element of $\operatorname{Pic} Y$ (namely $\beta_\rho$). Hence there exists an $m_Y \in M_Y$ such that:

$$\theta(m_Y) = \underline{a} - \underline{b}$$

Now, the isomorphism $\varphi_{m_Y}$ (contained in the data of our original quasimap) is a map:

$$\varphi_{m_Y} : \bigotimes_\tau L_\tau^{\otimes \langle m_Y, n_\tau \rangle} \cong \mathcal{O}_C$$

By definition, $\theta(m_Y) = (\langle m_Y, n_\tau \rangle)_{\tau \in \Sigma_Y(1)}$. But also $\theta(m_Y) = (a_\tau - b_\tau)_{\tau \in \Sigma_Y(1)}$. Hence we have:

$$\varphi_{m_Y} : \bigotimes_\tau L_\tau^{\otimes a_\tau} \cong \bigotimes_\tau L_\tau^{\otimes b_\tau}$$

In other words, we have well-defined canonical isomorphisms

$$\tilde{L}_\rho^{\underline{a}} \cong \tilde{L}_\rho^{\underline{b}}$$

for all $\underline{a}$ and $\underline{b}$. Let us choose one such $\underline{a}$ (it doesn't matter which); call it $\underline{a}^\rho$. We define:

$$\tilde{L}_\rho = \tilde{L}_\rho^{\underline{a}^\rho}$$

Then for all $\underline{b}$ we can use the above isomorphism to view $\tilde{u}_\rho^{\underline{b}}$ as a section of $\tilde{L}_\rho$. Summing all of these together we obtain a section $\tilde{u}_\rho$ of $\tilde{L}_\rho$, which we can write (with abuse of notation) as:

$$\tilde{u}_\rho = \sum_{\underline{a}} \mu_{\underline{a}} \prod_\tau u_\tau^{a_\tau}$$

Note that if we had made a different choice of $\underline{a}^\rho$ above the result would have been isomorphic.

Thus far we have constructed line bundles and sections $(\tilde{L}_\rho, \tilde{u}_\rho)_{\rho \in \Sigma_X(1)}$ on $C$. It remains to define the isomorphisms

$$\tilde{\varphi}_{m_X} : \otimes_\rho \tilde{L}_\rho^{\otimes \langle m_X, n_\rho \rangle} \cong \mathcal{O}_C$$

for all $m_X \in M_X$. The left hand side is:

$$\otimes_\rho \tilde{L}_\rho^{\otimes \langle m_X, n_\rho \rangle} = \otimes_\rho \left( \otimes_\tau L_\tau^{\otimes a_\tau^\rho} \right)^{\otimes \langle m_X, n_\rho \rangle} = \otimes_\tau L_\tau^{\otimes \left( \sum_\rho a_\tau^\rho \langle m_X, n_\rho \rangle \right)}$$

Now, for $m_Y \in M_Y$ we have isomorphisms $\varphi_{m_Y} : \otimes_\tau L_\tau^{\otimes \langle m_Y, n_\tau \rangle} \cong \mathcal{O}_C$. Hence, in order to construct $\tilde{\varphi}_{m_X}$ we need to find an $m_Y$ such that

$$\langle m_Y, n_\tau \rangle = \sum_\rho a_\tau^\rho \langle m_X, n_\rho \rangle$$

for all $\tau \in \Sigma_Y(1)$ (we will then set $\tilde{\varphi}_{m_X} = \varphi_{m_Y}$). Consider therefore the short exact sequence (4). Recall that $\theta(m_Y) = (\langle m_Y, n_\tau \rangle)_{\tau\in\Sigma_Y(1)}$. Hence we need to show that

$$\left( \sum_\rho a_\tau^\rho \langle m_X, n_\rho \rangle \right)_{\tau\in\Sigma_Y(1)}$$

belongs to the image of $\theta$, i.e. that it belongs to the kernel of the second map (notice that $m_Y$ is then unique because $\theta$ is injective). This is equivalent to saying that

$$\sum_\tau \sum_\rho a_\tau^\rho \langle m_X, n_\rho \rangle D_\tau = 0 \in \operatorname{Pic} Y$$

Now, we have

$$\sum_\tau a_\tau^\rho D_\tau = \beta_\rho$$

so that the above sum becomes

$$\sum_\rho \langle m_X, n_\rho \rangle \beta_\rho = \left\langle m_X, \sum_\rho \beta_\rho \otimes n_\rho \right\rangle = \langle m_X, 0 \rangle = 0$$

where $\sum_\rho \beta_\rho \otimes n_\rho = 0$ by Condition (1) in Theorem A.1. So there does indeed exist a (unique) $m_Y \in M_Y$ such that $\langle m_Y, n_\tau \rangle = \sum_\rho a_\tau^\rho \langle m_X, n_\rho \rangle$, so that we can set:

$$\tilde{\varphi}_{m_X} = \varphi_{m_Y} : \bigotimes_\rho \tilde{L}_\rho^{\otimes \langle m_X, n_\rho \rangle} \cong \mathcal{O}_C$$

Thus, we have produced a quasimap with target $X$:

$$(C, (\tilde{L}_\rho, \tilde{u}_\rho)_{\rho\in\Sigma_X(1)}, (\tilde{\varphi}_{m_X})_{m_X \in M_X})$$

The proof that this construction does not depend on the choice of $(P_\rho)$ is straightforward and is left to the reader.

It remains to demonstrate that the quasimap thus constructed is nondegenerate and stable. Nondegeneracy follows immediately from Condition (2) in Theorem A.1. Put differently: the original quasimap defined a rational map $C \dashrightarrow Y$, whereas the new quasimap defines a rational map which is simply the composition $C \dashrightarrow Y \to X$. Therefore the set of basepoints is exactly the same.

Stability is a bit more tricky: it is here that we will end up having to put some extra conditions on the map $f$. First, notice that there are no rational tails because the source curve is unchanged.

Next let $C' \subseteq C$ be a component with exactly 2 special points. Then we need to show (see [CFK10, Definition 3.1.1]) that the following line bundle has positive degree on $C'$:

$$\tilde{\mathcal{L}} = \bigotimes_\rho \tilde{L}_\rho^{\otimes \tilde{\alpha}_\rho}$$

Here the $\tilde{\alpha}_\rho$ are defined by fixing a polarisation on $X$:

$$\mathcal{O}_X(1) = \bigotimes_\rho \mathcal{O}_X(\tilde{\alpha}_\rho D_\rho)$$

The choice of polarisation makes no difference: a quasimap is stable with respect to one polarisation if and only if it is stable with respect to all others. In order to make use of the fact that the original quasimap to $Y$ was stable, we will make the following assumption on $f$:

(1) there exists an ample line bundle $\mathcal{O}_X(1)$ on $X$ such that $f^*\mathcal{O}_X(1)$ is ample on $Y$

This is satisfied if, for example, $f$ is an embedding (which is the only case we will need in this paper). Given this assumption, we can set $\mathcal{O}_Y(1) = f^*\mathcal{O}_X(1)$. We then have:

$$\begin{aligned}\mathcal{O}_Y(1) &= \bigotimes_\rho f^*\mathcal{O}_X(D_\rho)^{\otimes \tilde{\alpha}_\rho} = \bigotimes_\rho \mathcal{O}_Y(\sum_\tau a^\rho_\tau D_\tau)^{\otimes \tilde{\alpha}_\rho} \\ &= \bigotimes_\rho \bigotimes_\tau \mathcal{O}_Y(a^\rho_\tau \tilde{\alpha}_\rho D_\tau) = \bigotimes_\tau \mathcal{O}_Y(D_\tau)^{\otimes \sum_\rho a^\rho_\tau \tilde{\alpha}_\rho}\end{aligned}$$

Thus for $\tau \in \Sigma_Y(1)$ we have $\alpha_\tau = \sum_\rho a^\rho_\tau \tilde{\alpha}_\rho$ and by stability of the original quasimap the line bundle $\mathcal{L} = \otimes_\tau L_\tau^{\otimes \alpha_\tau}$ has positive degree on $C'$. But:

$$\mathcal{L} = \bigotimes_\tau L_\tau^{\otimes \alpha_\tau} = \bigotimes_\rho \bigotimes_\tau \left(L_\tau^{\otimes a^\rho_\tau}\right)^{\otimes \tilde{\alpha}_\rho} = \bigotimes_\rho \tilde{L}_\rho^{\otimes \tilde{\alpha}_\rho} = \tilde{\mathcal{L}}$$

We have thus proven that $\tilde{\mathcal{L}}$ has positive degree on $C'$, so the pushed-forward quasimap is stable. This completes the proof of the following.

**Theorem A.3.** *Let $X$ and $Y$ be smooth proper toric varieties and $f : Y \to X$ a morphism. Assume that $f$ satisfies Condition (1) above. Then there exists a natural push-forward map*

$$\mathcal{Q}(f) \colon \overline{\mathcal{Q}}_{g,n}(Y, \beta) \to \overline{\mathcal{Q}}_{g,n}(X, f_*\beta)$$

*which does not modify the underlying prestable curves.*

**Aside A.4.** We expect that such a map exists even if $f$ does not satisfy Condition (1). However, in this case we will need to modify the underlying prestable curves by contracting unstable components. The same is true in the stable maps case.

Finally, let us describe how this push-forward morphism behaves when $f$ is a nonconstant map $\mathbb{P}^r \to \mathbb{P}^N$, since we will make use of this later. Write $f$ in homogeneous coordinates as:

$$f[z_0, \ldots, z_r] = [f_0(z_0, \ldots, z_r), \ldots, f_N(z_0, \ldots, z_r)]$$

where the $f_i$ are all homogeneous of degree $a$. Then given a quasimap with target $\mathbb{P}^r$

$$(C, L, u_o, \ldots, u_r)$$

the pushed-forward quasimap with target $\mathbb{P}^N$ is:

$$(C, L^{\otimes a}, f_0(u_0, \dots, u_r), \dots, f_N(u_0, \dots, u_r))$$

(This is stable as long as $a > 0$, which is precisely when $f$ satisfies Condition (1) above.)

A.2. **Relative obstruction theories for $Q(Y) \to Q(X)$.** Assume now that $f\colon Y \to X$ is a morphism satisfying Condition (1) above, so that it induces

$$k = Q(f)\colon \overline{Q}_{g,n}(Y, \beta) \to \overline{Q}_{g,n}(X, f_*\beta).$$

Even in the easiest possible case when $Y \subseteq X$ is an l.c.i. subscheme, $k$ is not necessarily a regular embedding, so the Gysin map in the sense of [Ful98] does not necessarily exist. Yet, when $\overline{Q}_{g,n}(X, f_*\beta)$ is a smooth stack (or rather its standard obstruction theory w.r.t. the moduli stack of prestable curves is unobstructed, which happens e.g. in the cases $X = \mathbb{P}^r$ and $(g, n) = (0, n)$ or $(1, 0)$), we may "pull back along $k$", and we are going to explain why.

In [Man12a] a generalisation of the Gysin map (called the **virtual pull-back**) is defined for morphisms endowed with a relative perfect obstruction theory. Moreover, a sufficient condition is given (Corollary 4.9) for this map to respect the virtual classes.

**Lemma A.5.** *There exists a relative obstruction theory $E_k$ for the morphism*

$$k : \overline{Q}_{g,n}(Y, \beta) \to \overline{Q}_{g,n}(X, f_*\beta)$$

*which fits into a compatible triple with the standard obstruction theories for the quasimap spaces over $\mathfrak{M}_{g,n}$. Furthermore, $E_k$ is perfect as soon as $\overline{Q}_{g,n}(X, f_*\beta)$ is unobstructed, so that:*

$$k_v^![\overline{Q}_{g,n}(X, f_*\beta)] = [\overline{Q}_{g,n}(Y, \beta)]^{\mathrm{virt}}$$

*Proof.* Note first that, since $k$ does not change the source curve of a quasimap, we indeed have a commuting triangle:

$$\begin{tikzcd} \overline{Q}_{g,n}(Y, \beta) \arrow[rr, "k"] \arrow[dr] & & \overline{Q}_{g,n}(X, f_*\beta) \arrow[dl] \\ & \mathfrak{M}_{g,n} & \end{tikzcd}$$

We have perfect obstruction theories $E_{\overline{Q}(Y)/\mathfrak{M}}$ and $E_{\overline{Q}(X)/\mathfrak{M}}$ and we want to find a perfect obstruction theory $E_k$. Consider the diagram of universal curves

$$\begin{tikzcd} C_Y \arrow[r, "\alpha"] \arrow[d, "\pi"] \arrow[dr, phantom, "\square"] & C_X \arrow[d, "\rho"] \\ \overline{Q}_{g,n}(Y, \beta) \arrow[r, "k"] & \overline{Q}_{g,n}(X, f_*\beta) \end{tikzcd}$$

which is cartesian because $k$ does not alter the source curve of any quasimap. We have sheaves $\mathcal{F}_Y$ and $\mathcal{F}_X$ on $\mathcal{C}_Y$ and $\mathcal{C}_X$ respectively such that:

$$E^{\vee}_{\overline{\mathcal{Q}}(Y)/\mathfrak{M}} = \mathrm{R}^{\bullet}\,\pi_*\mathcal{F}_Y$$
$$E^{\vee}_{\overline{\mathcal{Q}}(X)/\mathfrak{M}} = \mathrm{R}^{\bullet}\,\rho_*\mathcal{F}_X$$

It follows (by flatness of $\rho$) that when we pull back the latter obstruction theory to $\overline{\mathcal{Q}}(Y)$ we obtain:

$$k^*E^{\vee}_{\overline{\mathcal{Q}}(X)/\mathfrak{M}} = \mathrm{R}^{\bullet}\,\pi_*\alpha^*\mathcal{F}_X$$

To construct a compatible triple, we require a morphism $k^*E_{\overline{\mathcal{Q}}(X)/\mathfrak{M}} \to E_{\overline{\mathcal{Q}}(Y)/\mathfrak{M}}$. Dually, it is therefore enough to construct a morphism of sheaves on $\mathcal{C}_Y$

$$\mathcal{F}_Y \to \alpha^*\mathcal{F}_X$$

and then apply $\mathrm{R}^{\bullet}\,\pi_*$. This is analogous to the morphism $f^*T_Y \to f^*T_X|_Y$ which is used in the stable maps setting. However the construction for quasimaps requires a little more ingenuity, because we do not quite have access to a universal map $f$.

The sheaf $\mathcal{F}_Y$ is defined on $\mathcal{C}_Y$ by the short exact sequence

$$0 \to \mathcal{O}_{\mathcal{C}_Y}^{\oplus r_Y} \to \oplus_\tau \mathcal{L}_\tau \to \mathcal{F}_Y \to 0$$

where $r_Y = \operatorname{rk}\operatorname{Pic} X$ (implicitly we have chosen a basis for this $\mathbb{Z}$-module). Similarly $\mathcal{F}_X$ is defined on $\mathcal{C}_X$ by:

$$0 \to \mathcal{O}_{\mathcal{C}_X}^{\oplus r_X} \to \oplus_\rho \mathcal{L}_\rho \to \mathcal{F}_X \to 0$$

We will construct our morphism by first constructing a morhism:

$$\oplus_\tau \mathcal{L}_\tau \to \alpha^* \oplus_\rho \mathcal{L}_\rho$$

Recall that $f\colon Y \to X$ is given by homogeneous polynomials

$$P_\rho \in S^Y_{\beta_\rho} \subset S^Y = k[z_\tau : \tau \in \Sigma_Y(1)]$$

in the Cox ring of $Y$, where $\beta_\rho = f^*[D_\rho] \in \operatorname{Pic} Y$. For all monomials appearing in $P_\rho$, if we look at their exponents $(a_\tau)_{\tau\in\Sigma_Y(1)}$, we have $\sum_{\tau\in\Sigma_Y(1)} a_\tau[D_\tau] = \beta_\rho$ by homogeneity, hence we can use the isomorphisms parametrised by $M_Y$ as above in order to interpret

$$(P_\rho)_{\rho\in\Sigma_X(1)}\colon \bigoplus_{\tau\in\Sigma_Y(1)} L_\tau \to \bigoplus_{\rho\in\Sigma_X(1)} \beta_\rho = \alpha^*\left(\bigoplus_{\rho\in\Sigma_X(1)} L_\rho\right).$$

On the other hand, $f\colon Y \to X$ induces a pullback map on line bundles $\operatorname{Pic}(X) \to \operatorname{Pic}(Y)$ (for which $\mathbb{Z}$-modules we have implicitly chosen bases above), the dual (or transpose) to which gives us a matrix

$$Q \in \mathcal{M}_{r_X\times r_Y}(\mathbb{Z})$$

It is now clear by the very functoriality construction that the square in the following diagram is commutative, hence it induces the (dashed) map of sheaves that we were hoping for

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_{C_Y}^{\oplus r_Y} \arrow[r] \arrow[d, "Q"] & \oplus_\tau \mathcal{L}_\tau \arrow[r] \arrow[d, "(P_\rho)"] & \mathcal{F}_Y \arrow[r] \arrow[d, dashed] & 0 \\ & \mathcal{O}_{C_Y}^{\oplus r_X} \arrow[r] & \alpha^*\left(\oplus_\rho \mathcal{L}_\rho\right) \arrow[r] & \alpha^*\mathcal{F}_X \arrow[r] & 0 \end{tikzcd} \tag{5}$$

Applying $\mathrm{R}^\bullet \pi_*$ and dualising we obtain a morphism between the obstruction theories for the quasimap spaces, and we can complete this to obtain an exact triangle

$$k^* E_{\overline{Q}(X)/\mathfrak{M}} \to E_{\overline{Q}(Y)/\mathfrak{M}} \to E_k \xrightarrow{[1]}$$

on $\overline{Q}(Y)$. The complex $E_k$ is perfect (locally isomorphic to a bounded complex of vector bundles) because the other two are, and the axioms of a triangulated category give a morphism of exact triangles

$$\begin{tikzcd} k^* E_{\overline{Q}(X)/\mathfrak{M}} \arrow[r] \arrow[d] & E_{\overline{Q}(Y)/\mathfrak{M}} \arrow[r] \arrow[d] & E_k \arrow[r, "{[1]}"] \arrow[d] & {} \\ k^* L_{\overline{Q}(X)/\mathfrak{M}} \arrow[r] & L_{\overline{Q}(Y)/\mathfrak{M}} \arrow[r] & L_k \arrow[r, "{[1]}"] & {} \end{tikzcd}$$

It follows from a simple diagram chase that $E_k \to L_k$ is a relative obstruction theory. On the other hand, assuming that $\overline{Q}_{g,n}(X, f_*\beta)$ is unobstructed, we may look at the long exact sequence in cohomology and find

$$0 \to \mathrm{h}^{-2}(E_k) \to \mathrm{h}^{-1}(k^* E_{\overline{Q}(X)/\mathfrak{M}}) = 0$$

Hence $\mathrm{h}^{-2}(E_k) = 0$ and it is easy to show using similar arguments that $E_k$ is of perfect amplitude contained in $[-1, 0]$.

$\square$

In particular, for every smooth projective variety $i\colon X \hookrightarrow \mathbb{P}^r$, we have thus produced a virtual pull-back morphism

$$k^!_{\mathrm{v}} : A_*(\overline{Q}_{0,n}(\mathbb{P}^r, d)) \to A_*(\overline{Q}_{0,n}(X, \beta))$$

where $d = i_*\beta$, and more generally for any cartesian diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & G \arrow[d] \\ \overline{Q}_{0,n}(X, \beta) \arrow[r, "k"] & \overline{Q}_{0,n}(\mathbb{P}^N, d) \end{tikzcd}$$

we get an associated virtual pull-back morphism:

$$k^!_{\mathrm{v}} : A_*(G) \to A_*(F)$$

A.3. **Splitting principle.** Consider boundary strata of the space of quasimaps, i.e. where the underlying curve is reducible and has a prescribed profile, by which we mean the dual graph decorated by the degree of the universal line bundle on each component: there are two natural virtual classes on such a stratum, namely the one induced by the splitting type of the curve and the product one. We are going to show that these virtual classes coincide; this works pretty much like it does in Gromov-Witten theory.

Fix a smooth projective toric variety $X$, and numerical invariants $g, n, \beta$ such that $\overline{Q}_{g,n}(X,\beta)$ is defined. Now fix a partition $A = (A_0, \ldots, A_r)$ of the genus and marked points, and a partition $B = (\beta_0, \ldots, \beta_r)$ of the curve class, such that every factor in the following product makes sense, and consider the space (which we call the CENTIPEDE LOCUS):

$$\mathcal{D}^Q(X,A,B) := \overline{Q}_{g_0,A_0\cup\{q_1,\ldots,q_r\}}(X,\beta_0) \times_{X^r} \prod_{i=1}^{r} \overline{Q}_{g_i,A_i\cup\{q_i\}}(X,\beta_i)$$

We can equip the centipede locus with the product virtual class in the following way. Set

$$\mathcal{E}^Q(X,A,B) := \overline{Q}_{g_0,A_0\cup\{q_1,\ldots,q_r\}}(X,\beta_0) \times \prod_{i=1}^{r} \overline{Q}_{g_i,A_i\cup\{q_i\}}(X,\beta_i)$$

which we endow with the product class:

$$[\mathcal{E}^Q(X,A,B)]^{\mathrm{virt}} := [\overline{Q}_{g_0,A_0\cup\{q_1,\ldots,q_r\}}(X,\beta_0)]^{\mathrm{virt}} \times \prod_{i=1}^{r} [\overline{Q}_{g_i,A_i\cup\{q_i\}}(X,\beta_i)]^{\mathrm{virt}}$$

We then consider the cartesian diagram

$$\begin{tikzcd} \mathcal{D}^Q(X,A,B) \arrow[r, "h"] \arrow[d, "\mathrm{ev}_q"] \arrow[dr, phantom, "\square"] & \mathcal{E}^Q(X,A,B) \arrow[d, "\mathrm{ev}_q"] \\ X^r \arrow[r, "\Delta_{X^r}"] & X^r \times X^r \end{tikzcd} \tag{6}$$

and, since $X$ is smooth so $\Delta_{X^r}$ is a regular embedding, define:

$$[\mathcal{D}^Q(X,A,B)]^{\mathrm{virt}} := \Delta^!_{X^r}([\mathcal{E}^Q(X,A,B)]^{\mathrm{virt}})$$

Notice that, by defining

$$\mathfrak{M}^{\mathrm{wt}}_{A,B} := \mathfrak{M}^{\mathrm{wt}}_{g_0,A_0\cup\{q_1,\ldots,q_r\},\beta_0} \times \prod_{i=1}^{r} \mathfrak{M}^{\mathrm{wt}}_{g_i,A_i\cup\{q_i\},\beta_i}$$

there is a triangle

$$\begin{tikzcd} \mathcal{D}^Q(X,A,B) \arrow[rr, "h"] \arrow[dr, "\rho_D"'] & & \mathcal{E}^Q(X,A,B) \arrow[dl, "\rho_E"] \\ & \mathfrak{M}^{\mathrm{wt}}_{A,B} & \end{tikzcd} \tag{7}$$

and the product virtual class on $\mathcal{E}^Q(X, A, B)$ corresponds to the product of the standard obstruction theories for each factor $\overline{Q}_{g_i, A_i \cup \{q_i\}}(X, \beta_i) \to \mathfrak{M}^{\mathrm{wt}}_{A_i, B_i}$ (the latter is étale over the usual moduli space of prestable curves by forgetting the weight, hence they have isomorphic cotangent complexes).

On the other hand, we have the following cartesian diagram

$$\begin{tikzcd} \mathcal{D}^Q(X, A, B) \arrow[r, "\varphi"] \arrow[d, "\rho_D"] \arrow[dr, phantom, "\square"] & \overline{Q}_{0,n}(X, \beta) \arrow[d, "\rho_Q"] \\ \mathfrak{M}^{\mathrm{wt}}_{A,B} \arrow[r, "\psi"] & \mathfrak{M}^{\mathrm{wt}}_{g,n,\beta} \end{tikzcd} \tag{8}$$

**Remark A.6.** The bottom horizontal map is not a closed immersion: due to the existence of degree–0 rational components, there may be many possible equally valid ways of breaking up a nodal curve.

Yet $\psi$ has a natural perfect obstruction theory, given by $L_\psi$: we only need to show that it is supported in $[-1, 0]$. Consider the exact triangle:

$$\psi^* L_{\mathfrak{M}^{\mathrm{wt}}_{g,n,\beta}} \to L_{\mathfrak{M}^{\mathrm{wt}}_{A,B}} \to L_\psi \xrightarrow{[1]}$$

The first two terms are concentrated in degrees $[0, 1]$, because they are the cotangent complexes of smooth Artin stacks. Therefore $L_\psi$ is concentrated in degrees $[-1, 1]$. Furthermore, if we examine the long exact cohomology sequence near $\mathrm{h}^1(L_\psi)$ we find

$$\mathrm{h}^1(\psi^* L_{\mathfrak{M}^{\mathrm{wt}}_{g,n,\beta}}) \to \mathrm{h}^1(L_{\mathfrak{M}^{\mathrm{wt}}_{A,B}}) \to \mathrm{h}^1(L_\psi) \to 0$$

and hence we must show that the first map is surjective. But this is dual to the map which takes an infinitesimal automorphism of the disconnected curve to an infinitesimal automorphism of the corresponding connected curve (obtained by glueing together the "nodal" marked points). The requirement of preserving the markings translates into that of fixing the node after the gluing operation, so the (infinitesimal) automorphism groups coincide. Hence $\mathrm{h}^1(L_\psi) = 0$ as claimed. (This also descends from the fact that the fibres of $\psi$ are Deligne–Mumford.)

**Lemma A.7.** *$(h^* E_{\mathcal{E}^Q(A,B,X)}, \phi^* E_{\rho_Q}, \mathrm{ev}_q^* E_{\Delta_{X^r}})$ is a compatible triple for the triangle (7), hence*

$$\psi^! [\overline{Q}_{g,n}(X, \beta)] = \Delta^!_{X^r} [\mathcal{E}^Q(A, B, X)] \in A_*(\mathcal{D}^Q(A, B, X)).$$

*Proof.* We need to construct a morphism of triangles

$$\begin{tikzcd} h^* E_{\mathcal{E}^Q(A,B,X)} \arrow[r] \arrow[d] & \phi^* E_{\rho_Q} \arrow[r] \arrow[d] & \mathrm{ev}_q^* E_{\Delta_{X^r}} \arrow[r, "{[1]}"] \arrow[d] & {} \\ h^* L_{\rho_E} \arrow[r] & L_{\rho_D} \arrow[r] & L_h \arrow[r, "{[1]}"] & {} \end{tikzcd}$$

Consider the following diagram:

$$\begin{tikzcd} h^*\tilde{C} \arrow[r, "\nu"] \arrow[dr, "\eta"'] & \varphi^*C \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & C \arrow[d, "\pi"] \\ & \mathcal{D}^Q(X,A,B) \arrow[r, "\varphi"] & \overline{Q}_{0,n}(X,\beta) \end{tikzcd}$$

Here $\tilde{C}$ is the universal (disconnected) curve over $\mathcal{E}^Q(X,A,B)$, which we have pulled back to $\mathcal{D}^Q(X,A,B)$, while $\varphi^*C$ is the universal curve over $\mathcal{D}^Q(X,A,B)$. Therefore the map $\nu : h^*\tilde{C} \to \varphi^*C$ is (fiberwise) a partial normalisation map given by detaching the nodes which connect the "trunk" of the centipede to the "legs."

There are natural sheaves $\mathcal{F}$ and $\tilde{\mathcal{F}}$ on $C$ and $h^*\tilde{C}$ respectively, such that

$$\varphi^* E^\vee_{\rho_Q} = \mathrm{R}^\bullet\, \pi_*\mathcal{F}$$
$$h^* E^\vee_{\rho_E} = \mathrm{R}^\bullet\, \eta_*\tilde{\mathcal{F}}$$

Furthermore $\nu^*\mathcal{F} \simeq \tilde{\mathcal{F}}$, hence by tensoring the partial normalisation short exact sequence

$$0 \to \mathcal{O}_{\varphi^*C} \to \nu_*\mathcal{O}_{h^*\tilde{C}} \to \mathcal{O}_q \to 0$$

with $\mathcal{F}$ and applying the projection formula, we obtain

$$0 \to \mathcal{F} \to \nu_*\tilde{\mathcal{F}} \to \mathcal{F}_q \to 0$$

on $\varphi^*C$, where $q$ is the locus of nodes connecting the trunk to the spine. (The fact that the morphism on the left is injective follows by applying the Snake Lemma to the short exact sequence defining $\mathcal{F}$.) To this we can apply $\mathrm{R}^\bullet\, \pi_*$ to obtain an exact triangle

$$\mathrm{R}^\bullet\, \pi_*\mathcal{F} \to \mathrm{R}^\bullet\, \eta_*\tilde{\mathcal{F}} \to \mathrm{R}^\bullet\, \pi_*\mathcal{F}_q \xrightarrow{[1]} \tag{9}$$

Finally, notice that, since quasimaps are required not to have base-points at the nodes, the fibre of the sheaf $\mathcal{F}$ at each of the nodes $q$ can actually be identified with the tangent to the toric variety $X$ at the image of the node itself, i.e. $\mathrm{R}^\bullet\, \pi_*\mathcal{F}_q \simeq \mathrm{ev}_q^*\, T_{X^r} = T_{\Delta_{X^r}}[-1]$.

The statement now follows from functoriality of virtual pull-backs. $\square$

## Appendix B. Some intersection-theoretic lemmas

Consider a morphism of DM stacks $f\colon Y \to X$ over a smooth base $\mathfrak{M}$, such that $X$ is *smooth* over $\mathfrak{M}$ and $Y$ carries a virtual class given by a perfect obstruction theory $E^\bullet_{Y/\mathfrak{M}}$. Then, for every Cartesian diagram

$$\begin{tikzcd} G \arrow[r, "g"] \arrow[d, "q"] \arrow[dr, phantom, "\square"] & F \arrow[d, "p"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

and every class $\alpha \in A_*(F)$, we may define

$$f^!_\Delta(\alpha) = \Delta^!_X([Y]^{\mathrm{vir}} \times \alpha) \in A_*(G)$$

which we call a *diagonal* virtual pull-back. We first claim that it coincides with the usual virtual pull-back along $f$ in the presence of a compatible perfect obstruction theory relative to $f$.

**Lemma B.1.** *Assume that there exists a relative obstruction theory $E_f^\bullet$ compatible with $E^\bullet_{Y/\mathfrak{M}}$ and the standard (unobstructed) obstruction theory for $X$, i.e.*

$$\begin{tikzcd} f^*L^\bullet_{X/\mathfrak{M}} \arrow[r] \arrow[d, equal] & E^\bullet_{Y/\mathfrak{M}} \arrow[r] \arrow[d] & E^\bullet_f \arrow[r, "{[1]}"] \arrow[d] & {} \\ f^*L^\bullet_{X/\mathfrak{M}} \arrow[r] & L^\bullet_{Y/\mathfrak{M}} \arrow[r] & L^\bullet_f \arrow[r, "{[1]}"] & {} \end{tikzcd}$$

*Then for every Cartesian diagram and every class $\alpha \in A_*(F)$ as above,*

$$f^!_E(\alpha) = f^!_\Delta(\alpha).$$

Secondly, we claim that the *diagonal* virtual pull-back behaves similarly to an ordinary virtual pull-back (e.g. commutes with other virtual pull-backs) even in the absence of a compatible perfect obstruction theory.

**Lemma B.2.** *The* diagonal *virtual pull-back morphism as defined above commutes with ordinary Gysin maps and with virtual pull-backs.*

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