A QUANTUM LEFSCHETZ THEOREM FOR QUASIMAP INVARIANTS VIA RELATIVE QUASIMAPS

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Abstract. We define moduli spaces of relative stable quasimaps in the spirit of A. Gathmann. When X is a smooth toric variety and Y is a smooth very ample hypersurface we obtain a virtual class on the moduli space, which is used to define relative quasimap invariants. We obtain a recursion formula which expresses each relative invariant in terms of invariants of lower multiplicity. Finally we apply this formula to obtain a quantum Lefschetz theorem expressing the absolute quasimap invariants of Y in terms of those of X. We include several appendices collecting proofs of standard results in quasimap theory.

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1. Introduction

The results of this paper arise from a fusion of two theories: stable quasimaps and relative stable maps. In this introductory section we briefly summarise these, providing the context for our work.

1.1. **Stable quasimaps.** The moduli space of stable toric quasimaps

$$\overline{Q}_{g,n}(X,\beta)$$

was constructed by Ciocan-Fontanine and Kim [CFK10] as an alternative compactification of the moduli space of smooth curves in a toric variety. It is a Deligne–Mumford stack of finite type, and is proper if X is proper. Moreover, when X is smooth it admits a perfect obstruction theory and hence a virtual fundamental class, which one can use to define curve-counting invariants for X, called **QUASIMAP INVARIANTS**.

This theory agrees with the theory of stable quotients [MOP11] when both are defined, namely when X is a projective space. There is a common generalisation given by the theory of stable quasimaps to GIT quotients [CFKM14]. However for simplicity we will work in the toric setting (though this restriction is probably not essential for our arguments). Thus when we say "quasimap" we are implicitly talking about toric quasimaps.

The quasimap invariants are expected to coincide with the Gromov–Witten invariants when X is a toric Fano variety [CM]; this has been proven in a number of examples, including $X = \mathbb{P}^N$ [Man12b, §5.4].

In general, however, the invariants differ, the difference being encoded by certain wall-crossing formulas [CFK14]. The motivation for this comes from mirror symmetry: the idea is that the quasimap invariants of X should correspond to the B-side theory of X (this is in contrast to the Gromov–Witten invariants, which live on the A-side); see [CFK10, §7].

1.2. **Relative stable maps.** In [Gat02] Gathmann consructs a moduli space of relative stable maps to the pair (X, Y) as a closed substack of the moduli space of (absolute) stable maps to X:

$$\overline{\mathcal{M}}_{g,\alpha}(X|Y, \beta) \hookrightarrow \overline{\mathcal{M}}_{g,n}(X, \beta)$$

Unfortunately this space does not admit a natural perfect obstruction theory. Nevertheless in the case where Y is very ample it is still possible to construct a virtual fundamental class by intersection-theoretic methods, and hence one can define relative Gromov–Witten invariants.

Gathmann then proves a recursion formula which in particular allows one to recover the relative Gromov–Witten invariants from the absolute ones. This is applied in [Gat03] to obtain a quantum Lefschetz theorem for $Y \subseteq X$.

1.3. **Relative stable quasimaps.** In this paper we combine the two stories above, constructing moduli spaces of relative stable quasimaps. We prove a recursion relation similar to Gathmann's formula, and use this to derive a quantum Lefschetz formula for quasimap invariants.

The plan of the paper is as follows. In §§2.1-2.2 we provide a brief review of the theories of stable quasimaps and relative stable maps mentioned above. Then in §2.3 we define the moduli spaces of relative stable quasimaps

$$\overline{\mathcal{Q}}_{g,\alpha}(X|Y, \beta)$$

where X is a smooth toric variety and Y is a smooth hypersurface. We *do not* require that Y is toric.

In §3 we examine the special case of $H \subseteq \mathbb{P}^N$. We find that, although the moduli space is not in general smooth, it is irreducible of the expected dimension (in fact, more than this: it is the closure of the so-called "nice locus"). Thus it admits a fundamental class which we can use to define relative quasimap invariants.

Also for $\mathbb{P}^N$ there exists a comparison morphism from the moduli space of stable maps to the moduli space of quasimaps, which is birational. We use this morphism to push down Gathmann's recursion formula for relative stable maps to obtain a recursion formula for relative stable quasimaps. The stronger stability condition for quasimaps significantly simplifies the correction terms which appear.

In §**??** we extend the recursion formula to arbitrary pairs (X, Y) where Y is very ample, by taking the embedding $X \hookrightarrow \mathbb{P}^N$ defined by $\mathcal{O}(Y)$ and pulling back the formula for $(\mathbb{P}^N, H)$. This of course requires some comparison theorems for virtual classes, for which we have to examine the perfect obstruction theories.

In §5 we apply the recursion formula obtained in §**??** to obtain a quantum Lefschetz theorem for quasimap invariants. This recovers [REFERENCE] in [CFK14].

We also include several appendices, collecting together results which are presumably well-known to experts, but for which we could not find references in the literature.

Appendix [REF] contains foundational lemmas of quasimap theory, including functoriality, the existence of relative perfect obstruction theories and the splitting theorem.

Appendix [REF] discusses a well-known intersection-theoretic construction – the so-called "diagonal pull-back" – and shows that it agrees with the virtual pull-back of [Man12a] (when both are defined) and that it commutes with Gysin maps and virtual pull-backs.

Finally Appendix [REF] discusses the comparison morphism from maps to quasimaps (used in the proof of the recursion relation in §3).

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1.4. **Table of notation.** We will use the following notation, most of which is introduced in the main body of the paper.

X	a smooth projective toric variety
Y	a smooth very ample hypersurface in X
Σ	the fan of X
$\Sigma(1)$	the set of 1-dimensional cones of Σ
ρ	an element of $\Sigma(1)$
D_ρ	the toric divisor in X corresponding to ρ
$\overline{\mathcal{M}}_{g,n}(X,\beta)$	the moduli space of stable maps
$\mathcal{M}_{0,\alpha}(X\|Y,\beta)$	the nice locus of relative stable maps
$\overline{\mathcal{M}}_{0,n}(X\|Y,\beta)$	the moduli space of relative stable maps (§2.2)
$\mathfrak{M}_{g,n}$	the moduli space of prestable curves
$\overline{\mathcal{Q}}_{g,n}(X,\beta)$	the moduli space of stable toric quasimaps (§2.1)
$\mathcal{Q}_{0,\alpha}(X\|Y,\beta)$	the nice locus of relative quasimaps (§3.1)
$\overline{\mathcal{Q}}_{0,\alpha}(X\|Y,\beta)$	the moduli space of relative stable quasimaps (§2.3)
$\overline{\mathcal{Q}}(f)$	the push-forward morphism between quasimap moduli spaces (§B.1)
χ	the comparison morphism from stable maps to quasimaps (§A)
$\mathcal{D}^{\mathcal{Q}}_{\alpha,k}(X\|Y,\beta)$	the quasimap comb locus (§3.2)
$\mathcal{D}^{\mathcal{Q}}(X\|Y,A,B,M)$	a component of the quasimap comb locus (§3.2)
$\mathcal{E}^{\mathcal{Q}}(X\|Y,A,B,M)$	the total product from which the virtual class of the quasimap comb locus is
$\mathcal{D}^{\mathcal{Q}}(X,A,B)$	the quasimap centipede locus (§[REF])
$\mathcal{E}^{\mathcal{Q}}(X,A,B)$	the total product from which the virtual class of the quasimap centipede loc
$f^!$	Gysin morphism when f is an l.c.i. embedding
$f^!_{\mathrm{v}}$	virtual pull-back when f is virtually smooth (§[REF])
$f^!_\Delta$	diagonal virtual pull-back (§[REF])

2. Relative stable quasimaps

2.1. **Review of absolute stable quasimaps.** We briefly recall the definition and basic properties of the moduli space of toric quasimaps; see [CFK10] for more details.

Definition 2.1 ([CFK10, Definition 3.1.1])**.** Let $X = X_\Sigma$ be a smooth and projective toric variety with fan $\Sigma \subseteq N_{\mathbb{Q}}$ and let $\mathcal{O}_{X_\Sigma}(1)$ be a fixed polarisation, which we can write (non-uniquely) in terms of the T-invariant divisors as:

$$\mathcal{O}_{X_\Sigma}(1) = \otimes_{\rho\in\Sigma(1)}\mathcal{O}_{X_\Sigma}(D_\rho)^{\otimes\alpha_\rho}$$

for some $\alpha_\rho \in \mathbb{Z}$. Given a fixed genus $g \geq 0$, number of marked points $n \geq 0$ and curve class $\beta \in \mathrm{H}_2^+(X)$ a **stable (toric) quasimap** is given by the data

$$((C,x_1,\ldots,x_n),(L_\rho,u_\rho)_{\rho\in\Sigma(1)},(\varphi_m)_{m\in M})$$

where:

(1) $(C,x_1,\ldots,x_n)$ is a prestable curve of genus g with n marked points;
(2) the L_ρ are line bundles on C of degree $d_\rho = D_\rho\cdot\beta$;
(3) the u_ρ are global sections of L_ρ;

(4) $\varphi_m \colon \otimes_\rho L_\rho^{\otimes \langle \rho, m \rangle} \to \mathcal{O}_C$ are isomorphisms, such that $\varphi_m \otimes \varphi_{m'} = \varphi_{m+m'}$ for all $m, m' \in M$.

These are required to satisfy the following two conditions:

(1) **NONDEGENERACY**: there is a finite (possibly empty) set of smooth and non-marked points $B \subseteq C$, called the **BASEPOINTS** of the quasimap, such that for all $x \in C \setminus B$ there exists a maximal cone $\sigma \in \Sigma_{\max}$ with $u_\rho(x) \neq 0$ for all $\rho \not\subset \sigma$;
(2) **STABILITY**: if we let $L = \otimes_\rho L_\rho^{\otimes \alpha_\rho}$ then the following $\mathbb{Q}$-divisor is ample

$$\omega_C(x_1 + \ldots + x_n) \otimes L^{\otimes \epsilon}$$

for every rational $\epsilon > 0$.

Remark 2.2. This definition is motivated by the Σ-collections of D. Cox [Cox95a]; see also Appendix B.1. The point is that a quasimap defines a rational morphism $C \dashrightarrow X$ with base locus equal to B; in particular a quasimap without any basepoints defines a morphism $C \to X$. Thus the basepoints appear in the boundary of the moduli space, in much the same way as the locus of stable maps with rational tails appears in the boundary of the moduli space of stable maps (this is actually more than just a vague analogy; these loci correspond to each other under the comparison morphism; see Appendix A).

More generally, we can define the notion of a family of quasimaps over a base scheme S, and what it means for two such families to be isomorphic; we thus obtain a moduli space

$$\overline{\mathcal{Q}}_{g,n}(X, \beta)$$

of stable (toric) quasimaps to X, which is a proper Deligne–Mumford stack of finite type. It can be shown that this definition does not depend on the choice of polarisation.

As with the case of stable maps, there is a combinatorial characterisation of stability which is much easier to check in practice; a prestable quasimap is stable if and only if the following conditions hold:

(1) the line bundle L defined above must have strictly positive degree on any rational component with fewer than three special points, and on any elliptic component with no special points;
(2) C cannot have any rational components with fewer than two special points.

Condition (1) is analogous to the ordinary stability condition for stable maps. Condition (2) is new, however, and gives quasimaps a distinctly different flavour to stable maps; we shall sometimes refer to it as the **STRONG STABILITY CONDITION.**

The moduli space $\overline{\mathcal{Q}}_{g,n}(X, \beta)$ admits a perfect obstruction theory relative to the moduli space $\mathfrak{M}_{g,n}$ of source curves, and hence one can construct a virtual class

$$[\overline{\mathcal{Q}}_{g,n}(X,\beta)]^{\mathrm{virt}} \in \mathrm{A}_{\mathrm{vdim}\,\overline{\mathcal{Q}}_{g,n}(X,\beta)}\left(\overline{\mathcal{Q}}_{g,n}(X,\beta)\right)$$

where the virtual dimension is the same as for stable maps:

$$\mathrm{vdim}\,\overline{\mathcal{Q}}_{g,n}(X,\beta) = (\dim X - 3)(1-g) - (K_X \cdot \beta) + n$$

Since the markings are not basepoints there exist evaluation maps

$$\mathrm{ev}_i : \overline{\mathcal{Q}}_{g,n}(X,\beta) \to X$$

and there are ψ-classes defined in the usual way by pulling back the relative dualising sheaf of the universal curve

$$\psi_i = \mathrm{c}_1(x_i^* \omega_{\mathcal{C}/\overline{\mathcal{Q}}})$$

where $\mathcal{C} \to \overline{\mathcal{Q}} = \overline{\mathcal{Q}}_{g,n}(X,\beta)$ is the universal curve and $x_i : \overline{\mathcal{Q}} \to \mathcal{C}$ is the section defining the ith marked point. Putting all these pieces together, we can define **QUASIMAP INVARIANTS**:

$$\langle \gamma_1 \psi^{a_1}, \ldots, \gamma_n \psi^{a_n} \rangle^X_{g,n,\beta} = \int_{[\overline{\mathcal{Q}}_{g,n}(X,\beta)]^{\mathrm{virt}}} \prod_{i=1}^{n} \mathrm{ev}_i^* \gamma_i \psi_i^{a_i}$$

(We use the same correlator notation as in Gromov–Witten theory; since we will never talk about Gromov–Witten invariants this should not cause any confusion.)

[EXAMPLES.]

2.2. **Review of relative stable maps.** Given a smooth projective variety X and a smooth divisor Y, the moduli space of relative stable maps parametrises stable maps to X with specified tangencies to Y at the marked points; [Gat02] for details.

Definition 2.3 ([Gat02, Definition 1.1 and Remark 1.4])**.** Let X be a smooth projective variety and $Y \subseteq X$ a smooth divisor. Fix a number $n \geq 0$ of marked points, a curve class $\beta \in \mathrm{H}_2^+(X)$ and an n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non-negative integers such that $\Sigma_i \alpha_i \leq Y \cdot \beta$. Then the moduli space

$$\overline{\mathcal{M}}_{0,\alpha}(X|Y,\beta)$$

of relative stable maps to (X,Y) is defined to be the locus in $\overline{\mathcal{M}}_{0,n}(X,\beta)$ of stable maps $(C, x_1, \ldots, x_n, f)$ such that, if Z is a connected component of $f^{-1}(Y) \subseteq C$, then

(1) if Z is a point and is equal to a marked point x_i, then the multiplicity of f to Y at x_i is greater than or equal to α_i;
(2) if Z is one-dimensional (hence a union of irreducible components of C) and if we let $C^{(i)}$ for $1 \leq i \leq r$ denote the irreducible components

of C adjacent to Z, and $m^{(i)}$ denote the multiplicity of $f|_{C^{(i)}}$ to Y at the node $Z \cap C^{(i)}$, then we must have:

$$Y \cdot f_*[Z] + \sum_{i=1}^{r} m^{(i)} \geq \sum_{x_i \in Z} \alpha_i$$

Remark 2.4. In the case of maximal multiplicity $\Sigma_i \alpha_i = Y \cdot \beta$, all the inequalities in the above definition must actually be equalities.

This forms a proper closed substack of $\overline{\mathcal{M}}_{0,n}(X, \beta)$. In the case $X = \mathbb{P}^N$, $Y = H$ one can show that it is irreducible of the correct dimension, and hence admits a fundamental class from which one can define relative Gromov–Witten invariants.

In general if $Y \subseteq X$ is very ample one can use the embedding $X \hookrightarrow \mathbb{P}^N$ to obtain a cartesian diagram:

$$\begin{tikzcd} \overline{\mathcal{M}}_{0,\alpha}(X|Y,\beta) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{\mathcal{M}}_{0,\alpha}(\mathbb{P}^N|H,d) \arrow[d] \\ \overline{\mathcal{M}}_{0,n}(X,\beta) \arrow[r, "\varphi"] & \overline{\mathcal{M}}_{0,n}(\mathbb{P}^N,d) \end{tikzcd}$$

Then the fact that $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^N, d)$ is smooth allows us to define a virtual class on $\overline{\mathcal{M}}_{0,\alpha}(X|Y, \beta)$ by virtual (or diagonal) pull-back (see Appendix **??** of the current paper):

$$[\overline{\mathcal{M}}_{0,\alpha}(X|Y,\beta)]^{\mathrm{virt}} = \varphi^![\overline{\mathcal{M}}_{0,\alpha}(\mathbb{P}^N|H,d)]$$

Thus one can define relative Gromov–Witten invariants. In §§2-4 Gathmann proves a recursion relation inside the Chow group of $\overline{\mathcal{M}}_{0,\alpha}(X|Y, \beta)$

$$(\alpha_k\psi_k + \mathrm{ev}_k^* Y)[\overline{\mathcal{M}}_{0,\alpha}(X|Y,\beta)]^{\mathrm{virt}} = [\overline{\mathcal{M}}_{0,\alpha+e_k}(X|Y,\beta)]^{\mathrm{virt}} + [\mathcal{D}_{\alpha,k}(X,\beta)]^{\mathrm{virt}}$$

where $\mathcal{D}_{\alpha,k}(X, \beta)$ is an appropriate COMB LOCUS. Repeated application of this result shows that the relative Gromov–Witten invariants of (X, Y) and the Gromov–Witten invariants of Y are completely determined by the Gromov–Witten invariants of X. This relation is then worked out explicitly in cases of particular interest in [Gat03] to obtain a new proof of the mirror theorem.

2.3. **Definition of relative stable quasimaps.** We now give the main definition of the paper. From here on X will denote a smooth projective toric variety and $Y \subseteq X$ a smooth very ample hypersurface. We do not require that Y is toric.

Consider the line bundle $\mathcal{O}(Y)$ and the section s_Y cutting out Y. By [Cox95b] we have a natural isomorphism

$$\mathrm{H}^0(X, \mathcal{O}(Y)) = \Bbbk\left\langle \prod_\rho z_\rho^{a_\rho} : \Sigma_\rho a_\rho D_\rho = Y \right\rangle$$

where the z_ρ for $\rho \in \Sigma_X(1)$ are the generators of the Cox ring of X and the a_ρ are non-negative integers. We can therefore write s_Y as

$$s_Y = \sum_{\underline{a}=(a_\rho)} \lambda_{\underline{a}} \prod_\rho z_\rho^{a_\rho}$$

for some scalars $\lambda_{\underline{a}} \in \Bbbk$. The idea then is that a quasimap

$$((C, x_1, \ldots, x_n), (L_\rho, u_\rho)_{\rho \in \Sigma_X(1)}, (\varphi_m)_{m \in M})$$

maps to Y at $x \in C$ if and only if the section

$$u_Y := \sum_{\underline{a}} \lambda_{\underline{a}} \prod_\rho u_\rho^{a_\rho}$$

vanishes at x. We now explain how to make sense of the expression above. For each $\underline{a}$ we have a well-defined section

$$u_{\underline{a}} := \lambda_{\underline{a}} \prod_\rho u_\rho^{a_\rho} \in \mathrm{H}^0(C, \otimes_\rho L_\rho^{\otimes a_\rho})$$

and if we have $\underline{a}$ and $\underline{b}$ such that $\sum_\rho a_\rho D_\rho = Y = \sum_\rho b_\rho D_\rho$ then these differ by an element m of M. Thus the isomorphism φ_m allows us to view the sections $u_{\underline{a}}$ and $u_{\underline{b}}$ as sections of the same bundle, which we denote by L_Y. Then we can sum these together to obtain u_Y. There is a choice involved here, but up to isomorphism it does not matter; see the proof of functoriality in Appendix B.1 for more details.

The upshot is that we obtain a line bundle L_Y on C (which plays the role of the "pull-back" of $\mathcal{O}(Y)$) and a global section

$$u_Y \in \mathrm{H}^0(C, L_Y)$$

which plays the role of the "pull-back" of s.

Definition 2.5. With notation as above, let $n \geq 0$ be number of marked points, $\beta \in \mathrm{H}_2^+(X)$ be a curve class and $\alpha = (\alpha_1, \ldots, \alpha_n)$ a collection of non-negative integers such that $\Sigma_i \alpha_i \leq Y \cdot \beta$. Then we define the **moduli space of relative stable quasimaps**

$$\overline{\mathcal{Q}}_{0,\alpha}(X|Y, \beta) \subseteq \overline{\mathcal{Q}}_{0,n}(X, \beta)$$

to be the locus of quasimaps such that, if Z is a connected component of the vanishing locus of u_Y in C, then

(1) if Z is a point and is equal to a marked point x_i, then the order of vanishing of u_Y at x_i is greater than or equal to α_i;
(2) if Z is one-dimensional (hence a union of irreducible components) and if we let $C^{(i)}$ for $1 \leq i \leq r$ denote the irreducible components of C adjacent to Z, and $m^{(i)}$ the order of vanishing of u_Y at the node $Z \cap C^{(i)}$, then we must have:

$$\deg L_Y|_Z + \sum_{i=1}^r m^{(i)} \geq \sum_{x_i \in Z} \alpha_i \tag{1}$$

Remark 2.6. In the second case above we call Z an **internal** component and the $C^{(i)}$ **external** components.

As it stands we do not know much about this locus. In the following section we will examine the case $X = \mathbb{P}^N$ and $Y = H$ a hyperplane in detail. We will then apply the results obtained there to deduce facts about the general case.

3. Recursion formula for $\mathbb{P}^N$ relative H

3.1. Basic properties of the moduli space.

In this section we will show that the moduli space

$$\overline{Q}_{0,\alpha}(\mathbb{P}^N|H,d)$$

is irreducible of the expected dimension, and thus admits a fundamental class. We then prove a recursion formula for these fundamental classes by pushing forward Gathmann's recursion formula along the comparison morphism χ.

We set $X = \mathbb{P}^N$ and $Y = H = \{z_0 = 0\}$. Given a quasimap

$$(C, x_1, \ldots, x_n, L, u_0, \ldots, u_N) \in \overline{Q}_{0,n}(\mathbb{P}^N, d)$$

the line bundle L_Y of the previous section is equal to L and the section u_Y is equal to u_0. Let us denote by

$$Q_{0,\alpha}(\mathbb{P}^N|H,d) \subseteq \overline{Q}_{0,n}(\mathbb{P}^N, d)$$

(without the bar) the **nice locus**, consisting of those quasimaps with irreducible source curve (i.e. a $\mathbb{P}^1$), no basepoints (so we get an actual map), with no component of the curve mapping inside H and with the map having tangency at least α_i to H at the marking x_i.

This is an irreducible, locally closed substack of $\overline{Q}_{0,n}(\mathbb{P}^N, d)$ of codimension $\Sigma_i \alpha_i$, by essentially the same argument as in [Gat02, Lemma 1.8]. In fact it is isomorphic to the nice locus inside the stable map space, denoted $\mathcal{M}_{0,\alpha}(\mathbb{P}^N|H,d)$ by Gathmann (see [Gat02, Def. 1.6]; the stricter stability condition has no effect when the source curve is irreducible, provided of course that $n \geq 2$). We thus obtain:

Lemma 3.1. *The comparison morphism restricts to a morphism*

$$\chi : \overline{\mathcal{M}}_{0,\alpha}(\mathbb{P}^N|H,d) \to \overline{Q}_{0,\alpha}(\mathbb{P}^N|H,d)$$

Proof. We need to verify that a relative stable map is sent to a relative stable quasimap by χ. Since the contraction of a rational tail R always occurs away from the markings, we only need to examine the internal components Z of the quasimap.

Consider then Z; for each basepoint x on Z there is a rational tail R of the stable map attached to Z at x. This is either internal (mapped into H) or external (not mapped into H).

If R is internal then both R and Z live inside the same connected component Z' of $f^{-1}(H)$. Applying χ has the effect of contracting R and adding a line bundle to Z of degree equal to $H \cdot f_*[R]$. Thus the left hand side of the inequality (1) is left unchanged, and since the right hand side is also unaltered the inequality is satisfied.

On the other hand if R is external then the multiplicity $m^{(R)}$ of $R \cap Z$ satisfies:

$$m^{(R)} \leq H \cdot f_*[R]$$

Since applying χ has the effect of replacing $m^{(R)}$ by $H \cdot f_*[R]$ in the left hand side of (1), the inequality still holds for the quasimap. Thus we obtain a morphism from the relative stable map space to the relative quasimap space, as claimed. □

Lemma 3.2. *$\overline{Q}_{0,\alpha}(\mathbb{P}^N|H,d)$ is equal to the closure of the nice locus $Q_{0,\alpha}(\mathbb{P}^N|H,d)$ inside $\overline{Q}_{0,n}(\mathbb{P}^N,d)$.*

Proof. $\overline{Q}_{0,\alpha}(\mathbb{P}^N|H,d) \subseteq \overline{Q_{0,\alpha}(\mathbb{P}^N|H,d)}$: we show that any relative stable quasimap can be infinitesimally deformed to a relative stable quasimap with no basepoints. This is in particular a relative stable map; we then appeal to [Gat02, Prop. 1.14] to deform this stable map and obtain a point in the nice locus. Since this deformation does not introduce any rational tails, this is also a deformation of quasimaps, and the statement follows.

We induct on the number of components containing at least one basepoint. Suppose this number is non-zero (otherwise there is nothing to prove) and pick such a component C_0, with base-points $y_1, \ldots, y_k$. Recall that this means that $u_i(y_j) = 0$ for all i and j. We will deform the section $u_N|_{C_0}$ to a new section $u'_N|_{C_0}$ in such a way that $u'_N|_{C_0}(y_j) \neq 0$ and in such a way that we do not introduce any new basepoints. Notice that since the relative condition only depends on u_0, the resulting deformed quasimap will still be a relative quasimap.

Now, by the nondegeneracy condition we must have $\deg(L|_{C_0}) > 0$, and since $C_0 \cong \mathbb{P}^1$ we can find a section w_0 of $L|_{C_0} \cong \mathcal{O}_{\mathbb{P}^1}(d_0)$ not vanishing at any of the base-points p_i.

We then set

$$u'_N|_{C_0} := u_N|_{C_0} + \epsilon w_0$$

and $u'_i|_{C_0} = u_i|_{C_0}$ for all other i. Notice that $u'_N|_{C_0}(y_j) \neq 0$ for all j as claimed. It is also clear that we do not introduce any new basepoints, since $u'_N|_{C_0}(y) = 0$ implies $u_N|_{C_0}(y) = 0$ (put differently: being a basepoint is a close condition).

It remains to extend the section $u'_N|_{C_0}$ to a section u'_N on the whole curve. Let $C_1, \ldots, C_r$ be the components of C adjacent to C_0 and let $q_i = C_0 \cap C_i$. We need to modify the sections $u_N|_{C_i}$ in such a way that $u'_N|_{C_i}(q_i) = u'_N|_{C_0}(q_i)$.

By nondegeracy, we can choose a section w_i of $L|_{C_i}$ such that $w(q_i) \neq 0$. Then set:

$$u'_{N C_i} := u_N|_{C_i} + \epsilon \left(\frac{w_0(q_i)}{w_i(q_i)} \right) \cdot w_i$$

Then indeed we have:

$$u'_N|_{C_i}(q_i) = u_N(q_i) + \epsilon \left(\frac{w_0(q_i)}{w_i(q_i)} \right) \cdot w_i(q_i) = u_N(q_i) + \epsilon w_0(q_i) = u'_N|_{C_0}(q_i)$$

We can continue this process, replacing C_0 by C_i; since the genus of the curve is zero, we will never come to the same curve twice. In this way we obtain a new quasimap

$$(C, x_1, \ldots, x_n, L, u_0, \ldots, u_{N-1}, u'_N)$$

over $\operatorname{Spec} \mathbb{k}[\epsilon]/(\epsilon^2)$ which has no basepoints on C_0. We can repeat this process for all the components of C (using higher powers of ϵ each time in order to ensure that we never introduce additional basepoints) and thus we obtain an infinitesimal deformation of our original quasimap which has no basepoints, as required.

$\overline{\mathcal{Q}_{0,\alpha}(\mathbb{P}^N|H,d)} \subseteq \overline{\mathcal{Q}}_{0,\alpha}(\mathbb{P}^N|H,d)$: consider a family of stable quasimaps over a smooth curve S, such that the generic fibre lies in the nice locus. We may blow-up the source curve (a fibered surface over S) in the locus of basepoints (which consists of finitely many smooth points of the central fiber) in order to obtain an actual morphism to $\mathbb{P}^N$. This has the effect of adding rational tails at the basepoints in the central fibre. If the morphism is constant on any of these rational tails we may contract them, and thus we obtain a family of stable maps which pushes down along χ to our original family of quasimaps.

The general fibre is not modified at all, and so is still in the nice locus. By [Gat02, Lemma 1.9] it follows that the central fibre is a relative stable map, and then by applying χ and appealing to Lemma 3.1 it follows that the same is true for the central fibre of the family of quasimaps. □

Corollary 3.3. *The moduli space $\overline{\mathcal{Q}}_{0,\alpha}(\mathbb{P}^N|H,d)$ is irreducible of the expected dimension. Hence it admits a fundamental class.*

Proof. This holds because the moduli space is equal to the closure of the nice locus, which is irreducible of the expected dimension. □

Corollary 3.4. *The comparison morphism is a birational map from the moduli space of relative stable maps to the moduli space of stable maps. In particular it sends the fundamental class to the fundamental class.*

Proof. This follows because the comparison morphism restricts to an isomorphism on the nice locus, which by the lemma above is a dense open subset of both spaces. □

3.2. **Proof of the recursion formula.** We wish to obtain a recursion formula relating the quasimap invariants of multiplicity α with the quasimap invariants of multiplicity $\alpha + e_k$, as in [Gat02, Theorem 2.6]. This process of "increasing the multiplicities" can be naively performed in the same way as Gathmann: for $m = \alpha_k + 1$ the following section (of the pull-back of the jet bundle of the universal line bundle)

$$\sigma_k^m := x_k^* d^m_{C/\overline{Q}}(u_0) \in \mathrm{H}^0(\overline{Q}, x_k^* \mathcal{P}^m_{C/\overline{Q}}(\mathcal{L}))$$

cuts out $\overline{Q}_{0,\alpha+e_k}(\mathbb{P}^N|H,d)$ inside $\overline{Q}_{0,\alpha}(\mathbb{P}^N|H,d)$, along with a number of degenerate contributions (called the **comb loci**) parametrising quasimaps for which x_k belongs to an internal component $Z \subseteq C$ (a component on which u_0 vanishes), such that

$$\deg(L|_Z) + \sum_{i=1}^r m^{(i)} = \sum_{x_i \in Z} \alpha_i$$

(here by "component" we really mean "connected component of the vanishing locus of u_0"). Quasimap stability means that these degenerate contributions cannot contain any rational tails; this is really the only difference with the case of stable maps.

Indeed, we can actually push forward Gathmann's formula along the comparison morphism

$$\chi \colon \overline{\mathcal{M}}_{0,\alpha}(\mathbb{P}^N|H,d) \to \overline{Q}_{0,\alpha}(\mathbb{P}^N|H,d)$$

and due to Corollary 3.2 above, the only terms which change are the comb loci containing rational tails. In fact these disappear, since the restriction of the comparison map to these loci has positive-dimensional fibres:

Lemma 3.5. *Consider a rational tail component in the comb locus of the moduli space of stable maps, i.e. a moduli space of the form:*

$$\overline{\mathcal{M}}_{0,(m^{(i)})}(\mathbb{P}^N|H,d)$$

Then (assuming that $Nd > 1$) we have

$$\dim\left([\overline{\mathcal{M}}_{0,(m^{(i)})}(\mathbb{P}^N|H,d)] \cap \mathrm{ev}_1^*(\mathrm{pt}_H)\right) > 0$$

where $\mathrm{pt}_H \in \mathrm{A}^{N-1}(H)$ is a point class. Thus the pushforward along χ of any comb locus with a rational tail is 0.

Proof. This is a simple dimension count. We have

$$\begin{aligned}\dim\left([\overline{\mathcal{M}}_{0,(m^{(i)})}(\mathbb{P}^N|H,d)] \cap \mathrm{ev}_1^*(\mathrm{pt}_H)\right) &= (N-3) + d(N+1) + (1-m^{(i)}) - (N-1) \\ &= (Nd-1) + (d-m^{(i)})\end{aligned}$$

from which the lemma follows because $m^{(i)} \leq d$. ∎

Remark 3.6. With an eye to the future, we remark that these rational tail components contribute nontrivially to the Gromov–Witten invariants of a Calabi–Yau hypersurface in projective space, and so their disappearance in our recursion formula may account for the divergence between Gromov–Witten and quasimap invariants in the Calabi–Yau case [Gat03, Rmk. 1.6].

Since we wish to apply the projection formula to Gathmann's recursion relation, we should express the cohomological terms which appears as pull-backs:

Lemma 3.7. *We have:*

$$\begin{aligned} \chi^*(\psi_k) &= \psi_k \\ \chi^*(\operatorname{ev}_k^* H) &= \operatorname{ev}_k^* H \end{aligned}$$

Proof. We will actually show that:

$$\begin{aligned} \chi^* x_k^* \omega_{C/\overline{Q}} &= x_k^* \omega_{C/\overline{\mathcal{M}}} \\ \chi^* x_k^* \mathcal{L} &= \operatorname{ev}_k^* \mathcal{O}_{\mathbb{P}^N}(H) \end{aligned}$$

This follows by considering the following diagram:

$$\begin{tikzcd}
 & & \mathbb{P}^N & \\
C_{\overline{\mathcal{M}}} \arrow[r, "\sigma^{ss}"'] \arrow[urr, bend left, "f"] \arrow[dr] & \chi^* C_{\overline{Q}} \arrow[ur, dashed] \arrow[rr] \arrow[d, shift right] \arrow[drr, phantom, "\square"] & & C_{\overline{Q}} \arrow[ul, dashed] \arrow[d, shift right] \\
 & \overline{\mathcal{M}}_{0,\alpha}(\mathbb{P}^N|H,d) \arrow[rr, "\chi"] \arrow[u, shift right, bend right, "x_k"'] \arrow[ul, bend left, "x_k"] & & \overline{Q}_{0,\alpha}(\mathbb{P}^N|H,d) \arrow[u, shift right, bend right, "x_k"']
\end{tikzcd}$$

where σ^{ss} is the strong stabilisation map which contracts the rational tails, and so is an isomorphism near the markings. □

Proposition 3.8. *Define the* QUASIMAP COMB LOCUS $\mathcal{D}^{Q}_{\alpha,k}(\mathbb{P}^N|H,d)$ *as the union of the moduli spaces*

$$\mathcal{D}^{Q}(\mathbb{P}^N|H,A,B,M) = \overline{Q}_{0,|\alpha^{(0)}|+r}(H,d_0) \times_{H^r} \prod_{i=1}^{r} \overline{Q}_{0,(m^{(i)})\cup\alpha^{(i)}}(\mathbb{P}^N|H,d_i)$$

where the union runs over all splittings $A = (\alpha^{(0)}, \ldots, \alpha^{(r)})$ of the markings, $B = (d_0, \ldots, d_r)$ of the degree and all valid multiplicites $M = (m^{(1)}, \ldots, m^{(r)})$ such that the above spaces are all well-defined (in particular $|\alpha^{(0)}| + k$ and $|\alpha^{(i)}| + 1$ are all ≥ 2) and such that

$$d_0 + \sum_{i=1}^{r} m^{(i)} = \sum \alpha^{(0)}$$

Equip this with the sum of the (product) fundamental classes. Then the following formula holds

$$(\alpha_k \psi_k + \mathrm{ev}_k^* H) \cdot [\overline{Q}_{0,\alpha}(\mathbb{P}^N|H,d)] = [\overline{Q}_{0,\alpha+e_k}(\mathbb{P}^N|H,d)] + [\mathcal{D}^Q_{\alpha,k}(\mathbb{P}^N|H,d)].$$

Proof. This follows from [Gat02, Thm. 2.6] by pushforward along χ, using the projection fomula and Lemmas , 3.5 and 3.7 . ◻

4. Recursion formula in the general case

We now move on to the general case. Let X be an arbitrary toric variety (smooth and proper) and $Y \subseteq X$ a very ample hypersurface (not necessarily toric). The complete linear system associated to $\mathcal{O}_X(Y)$ defines an embedding $i : X \hookrightarrow \mathbb{P}^N$ such that $i^{-1}(H) = Y$ (for a certain hyperplane H). By the functoriality property of quasimap spaces (see Appendix B.1) we have a map:

$$k := Q(i) : \overline{Q}_{0,n}(X,\beta) \to \overline{Q}_{0,n}(\mathbb{P}^N, d)$$

where $d = i_*\beta$. Since i is a closed embedding it follows that k is as well. Furthermore k admits a compatible perfect obstruction theory - see Section B.2 -, so we have a notion of virtual pull-back along k (which coincides with the *diagonal* pull-back according to Lemma C.1).

It is easy to show that k restricts to a morphism between the relative spaces, and thus we have a diagram of embeddings

$$\begin{tikzcd}
\overline{Q}_{0,\alpha}(X|Y,\beta) \arrow[r, hook, "g"] \arrow[d, hook, "f"] \arrow[dr, phantom, "\square"] & \overline{Q}_{0,\alpha}(\mathbb{P}^N|H,d) \arrow[d, hook, "j"] \\
\overline{Q}_{0,n}(X,\beta) \arrow[r, hook, "k"] & \overline{Q}_{0,n}(\mathbb{P}^N,d)
\end{tikzcd}$$

which one can show is cartesian. As such we can define a virtual class on $\overline{Q}_{0,\alpha}(X|Y,\beta)$ by (virtual or diagonal) pullback.

The idea is to prove the recursion formula for (X,Y) by pulling back the formula for $(\mathbb{P}^N,H)$ along $k = Q(i)$. In order to do this, we need to understand how the various virtual classes involved in the formula pull back along this map. Note that the first two terms of the recursion formula pull back trivially along k, i.e. $[\overline{Q}_{0,\alpha}(X|Y,\beta)]^{\mathrm{virt}} = k^![\overline{Q}_{0,\alpha}(\mathbb{P}^N|H,d)]^{\mathrm{virt}}$ by the very definition. It remains to consider the third term, namely the virtual class of the comb locus. This is the technical heart of the proof.

4.1. **Comb loci pull back.**

Recall that we can write $\mathcal{D}^Q_{\alpha,k}(X|Y,\beta)$ as the disjoint union of spaces

$$\mathcal{D}^Q(X|Y,A,B,M) = \overline{Q}_{0,A_0\cup\{q_1,\ldots,q_r\}}(Y,\beta_0) \times_{Y^r} \prod_{i=1}^r \overline{Q}_{0,\alpha^{(i)}\cup(m_i)}(X|Y,\beta_i)$$

where A and B are partitions of the marked points and curve class respectively, and $M = (m_1,\ldots,m_r)$ records the intersection multiplicity with Y

at the nodes which connect the internal component to the external components (the spine of the comb to the teeth). It is convenient to deal with each $\mathcal{D}^Q(X|Y, A, B, M)$ separately.

The comb locus sits inside the full product

$$\mathcal{E}^Q(X|Y, A, B, M) = \overline{Q}_{0,A_0\cup\{q_1,\dots,q_r\}}(Y, \beta_0) \times \prod_{i=1}^{r} \overline{Q}_{0,\alpha^{(i)}\cup(m_i)}(X|Y, \beta_i)$$

which we may endow with the product virtual class

$$[\mathcal{E}^Q(X|Y, A, B, M)]^{\mathrm{virt}} = [\overline{Q}_{0,A_0\cup\{q_1,\dots,q_r\}}(Y, \beta_0)]^{\mathrm{virt}} \times \prod_{i=1}^{r} [\overline{Q}_{0,\alpha^{(i)}\cup(m_i)}(X|Y, \beta_i)]^{\mathrm{virt}}$$

and the cartesian diagram

$$\begin{tikzcd} \mathcal{D}^Q(X|Y, A, B, M) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{E}^Q(X|Y, A, B, M) \arrow[d] \\ X^r \arrow[r, "\Delta_{X^r}"] & X^r \times X^r \end{tikzcd}$$

let us define the following *product* virtual class

$$[\mathcal{D}^Q(X|Y, A, B, M)]^{\mathrm{prod}} = \Delta^!_{X^r}[\mathcal{E}^Q(X|Y, A, B, M)]^{\mathrm{virt}}.$$

The same we can do at the level of $\mathbb{P}^N$ relative to H.

On the other hand, there is another cartesian diagram

$$\begin{tikzcd} \mathcal{D}^Q(X|Y, A, B, M) \arrow[r, "k_{|D}"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{D}^Q(\mathbb{P}^N|H, A, i_*B, M) \arrow[d] \\ \overline{Q}_{0,n}(X, \beta) \arrow[r, "k"] & \overline{Q}_{0,n}(\mathbb{P}^N, d) \end{tikzcd}$$

We wish to prove the following

Lemma 4.1. *For any α we have:*

$$k^![\mathcal{D}^Q(\mathbb{P}^N|H, A, i_*B, M)]^{\mathrm{prod}} = [\mathcal{D}^Q(X|Y, A, B, M)]^{\mathrm{prod}}.$$

Introduce the following shorthand notation: assuming the data of A (with $n = |A|$), B (with $\beta = \sum_{\beta_i \in B} \beta_i$) and M have been fixed for $X|Y$ (respectively, A, i_*B and M for $\mathbb{P}^N|H$), set:

$$\mathcal{D}(X|Y) := \mathcal{D}^Q(X|Y, A, B, M) \quad \text{and similarly for the absolute space}$$
$$\mathcal{E}(X|Y) := \mathcal{E}^Q(X|Y, A, B, M) \quad \text{and similarly for the absolute space}$$
$$Q(X) := \overline{Q}_{0,n}(X, \beta)$$

and similarly for $\mathbb{P}^N|H$.

Consider the cartesian diagram

$$\begin{tikzcd}
\mathcal{E}(X|Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{E}(\mathbb{P}^N|H) \arrow[d, "\theta"] \\
\mathcal{E}(X) \arrow[r] & \mathcal{E}(\mathbb{P}^N)
\end{tikzcd}$$

Lemma 4.2. $[\mathcal{E}(X|Y)]^{\mathrm{prod}} = \theta^![\mathcal{E}(X)]^{\mathrm{prod}}$, *where* $\theta^!$ *is the diagonal pullback introduced in Appendix C.*

Proof. The only thing to check is that

$$\mathcal{Q}(Y) \equiv \mathcal{Q}(H) \times_{\mathcal{Q}(\mathbb{P}^N)} \mathcal{Q}(X)$$

This is done in Appendix B.3. $\square$

Consider the cartesian diagram

$$\begin{tikzcd}
\mathcal{D}(X) \arrow[r] \arrow[d, "\varphi_X"] \arrow[dr, phantom, "\square"] & \mathcal{D}(\mathbb{P}^N) \arrow[r] \arrow[d, "\varphi_{\mathbb{P}^N}"] \arrow[dr, phantom, "\square"] & \mathfrak{M}^{\mathrm{wt}}_{A,B} \arrow[d, "\psi"] \\
\mathcal{Q}(X) \arrow[r, "k"] & \mathcal{Q}(\mathbb{P}^N) \arrow[r] & \mathfrak{M}^{\mathrm{wt}}_{0,n}
\end{tikzcd}$$

from which we see that

$$\psi^![\mathcal{Q}(X)]^{\mathrm{virt}} = k^!\psi^![\mathcal{Q}(\mathbb{P}^N)]$$

by commutativity of virtual pullbacks. Note that these classes are related to the product ones by the Splitting Principle (see Lemma B.8).

Finally, the relevant cartesian diagram is

$$\begin{tikzcd}
\mathcal{D}(X|Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{E}(X|Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{E}(\mathbb{P}^N|H) \arrow[d, "\theta"] \\
\mathcal{D}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{E}(X) \arrow[r] \arrow[d] & \mathcal{E}(\mathbb{P}^N) \\
X^r \arrow[r, "\Delta_{X^r}"] & X^r \times X^r &
\end{tikzcd}$$

The proof of Lemma 4.1 now follows from

$$\begin{aligned}
[\mathcal{D}(X|Y)]^{\mathrm{prod}} &= \Delta^!_{X^r}[\mathcal{E}(X|Y)]^{\mathrm{virt}} \\
&= \Delta^!_{X^r}\theta^![\mathcal{E}(X)]^{\mathrm{virt}} && \text{by Lemma 4.2} \\
&= \theta^!\Delta^!_{X^r}[\mathcal{E}(X)]^{\mathrm{virt}} && \text{by commutativity} \\
&= \theta^!\psi^![\mathcal{Q}(X)]^{\mathrm{virt}} && \text{by the splitting principle} \\
&= \theta^!k^!\psi^![\mathcal{Q}(\mathbb{P}^N)] && \text{by the above} \\
&= \theta^!k^!\Delta^!_{(\mathbb{P}^N)^r}[\mathcal{E}(\mathbb{P}^N)]^{\mathrm{virt}} && \text{by the splitting principle} \\
&= k^!\Delta^!_{(\mathbb{P}^N)^r}\theta^![\mathcal{E}(\mathbb{P}^N)]^{\mathrm{virt}} && \text{by commutativity} \\
&= k^![\mathcal{D}(\mathbb{P}^N|H)]^{\mathrm{prod}}.
\end{aligned}$$

Since the ψ-classes pull back naturally slong k, we have all the ingredients necessary to prove the following

Theorem 4.3. *Let X be a smooth and proper toric variety and let $Y \subseteq X$ be a very ample hypersurface (not necessarily toric). Then, with the set-up as in the preceding discussion, we have an equality*

$$(\alpha_k\psi_k + ev_k^*[Y])[\overline{Q}_{0,\alpha}(X|Y,\beta)]^{\mathrm{virt}} = [\overline{Q}_{0,\alpha+e_k}(X|Y,\beta)]^{\mathrm{virt}} + [\mathcal{D}^{Q}_{\alpha,k}(X|Y,\beta)]^{\mathrm{virt}}$$

in the Chow group of $\overline{Q}_{0,n}(X,\beta)$.

5. The quasimap mirror theorem

We are going to reproduce Gathmann's proof of the Mirror Theorem with relative stable maps [Gat03] in the context of quasimaps, thanks to the extension of his formula to this setting that we have proved in the previous sections. We have chosen to work with unparametrised quasimaps, hence the minimum number of markings is two; this minimal choice turns out to be extremely convenient because it determines the shape of the source curve to a high degree, so to grant a great level of control on degenerate contributions appearing in Gathmann's algorithm. The absence of rational tails in the quasimap moduli space makes the recursion much simpler, even in the CY case.

We would like to think of this as a Lefschetz-type theorem, in that it expresses certain (restricted) quasimap invariants of a hypersurface Y in terms of those of the ambient space X. As it turns out, we have also retrieved a formula of Ciocan-Fontanine and Kim [CFK14, Corollary 5.5.1] (but with more restrictive assumptions on the target); under this new light, the formula can be simply interpreted as a relation between some residues for the $\mathbb{G}_m$-action on the space of 0-pointed and 1-pointed *parametrised* quasimap invariants of the hypersurface Y. It is remarkable how, knowing only about a small sector (i.e. invariants with few insertions), it is possible to formally reconstruct the full quasimap potential; a point which was greatly clarified to us by the discussion in [CFK14, Section 5.5].

We are going to be interested in the following **setup**: X is a smooth projective toric variety and Y is a smooth *very ample* hypersurface in it, satisfying the following *semi-positivity assumption*, that $-K_Y$ is nef. Notice that, by adjunction, it follows from our hypotheses that $-K_X$ is positive (at least) on every effective curve class *coming from Y*. Let us denote by r the dimension of X and assume it is *at least 3*. Then, in fact, every curve class on X comes from Y (by Lefschetz's hyperplane theorem) and X *is Fano*.

Denote dual bases for $H^*(X;\mathbb{Q})$ by η^i and η_i ($i = 0,\dots,k$), with $\eta^0 = \mathbb{1}_X$ and $\eta^1 = Y$, which induce bases $\rho_i = i^*\eta_i$ for $i^*H^*(X)$ (extend it to a basis of $H^*(Y)$ by adding $\rho_{k+1}\dots,\rho_{k'}$) and dually $\rho^i, i = 1,\dots,k'$; notice that the class of a point on Y is given by restricting the dual of η^1, i.e. it is ρ_1, while the class of a point on X is annihilated when restricted to Y, i.e. $\rho_0 = 0$. Furthermore, remark that, everytime we look at a relative space

$\overline{Q}_{0,\{m,0\}}(X|Y,\beta)$ with $m>0$, the evaluation map $\mathrm{ev}_1\colon \overline{Q}_{0,\{m,0\}}(X|Y,\beta)\to X$ factors through Y (so all the insertions can be first pulled back to Y).

Definition 5.1. Let X be a smooth projective toric variety (or a complete intersection in a toric variety, or more generally any GIT quotient for which the quasimap spaces are defined), and consider

$$S_0^X(z,\beta)=(\mathrm{ev}_1)_*\left(\frac{1}{z-\psi_1}[\overline{Q}_{0,2}(X,\beta)]^{\mathrm{vir}}\right)$$

for every effective curve class $\beta\in H_2^+(X,\mathbb{Z})$. Set $S_0^X(z,0)=\mathbb{1}_X$ and

$$S_0^X(z,q)=\sum_{\beta\geq 0}S_0^X(z,\beta)q^\beta.$$

Theorem 5.2. *Let X be a toric Fano variety of dimension at least 3, and $i\colon Y\subseteq X$ a very ample hypersurface such that $-K_Y$ is nef. Then*

$$\frac{\sum_{\beta\geq 0}q^\beta\prod_{j=0}^{Y.\beta}(Y+jz)S_0^X(z,\beta)}{P_0(q)}=i_*S_0^Y(z,q) \tag{2}$$

where

$$\begin{aligned}P_0(q)=&1+\sum_{\substack{\beta>0:\\K_Y.\beta=0}}(Y.\beta)q^\beta\langle[pt_Y],\mathbb{1}_X\rangle_{\overline{Q}_{0,\{Y.\beta,0\}}(X|Y,\beta)}\\=&1+\sum_{\substack{\beta>0:\\K_Y.\beta=0}}q^\beta(Y.\beta)!\langle\psi_1^{Y.\beta-1}[pt_X],\mathbb{1}_X\rangle_{\overline{Q}_{0,2}(X,\beta)}.\end{aligned}$$

Proof. Define

$$S_{0,(m)}^{X|Y}(z,\beta)=(\mathrm{ev}_1)_*\left(\frac{1}{z-\psi_1}[\overline{Q}_{0,\{m,0\}}(X|Y,\beta)]^{\mathrm{vir}}\right),$$

which coincides with the absolute S_0-function defined above for $m=0$, and

$$T_{(m)}^{X|Y}(z,\beta)=(\mathrm{ev}_1)_*\left(m[\overline{Q}_{0,\{m,0\}}(X|Y,\beta)]^{\mathrm{vir}}+\frac{1}{z-\psi_1}[D_m^Q(X|Y,\beta)]^{\mathrm{vir}}\right).$$

Then, by Gathmann's formula, we can prove that

$$(Y+mz)S_{0,(m)}^{X|Y}(z,\beta)=S_{0,(m+1)}^{X|Y}(z,\beta)+T_{(m)}^{X|Y}(z,\beta), \tag{3}$$

from which it follows that

$$\prod_{j=0}^{Y.\beta}(Y+jz)S_0^X(z,\beta)=\sum_{m=0}^{Y.\beta}\prod_{j=m+1}^{Y.\beta}(Y+jz)T_{(m)}^{X|Y}(z,\beta).$$

It is now a matter of evaluating the RHS. Notice that $T_{(m)}^{X|Y}(z,\beta)$ is made of two parts:

- the *boundary terms*: since there are only two markings and the first one is required to lie in Y, the strong stability condition for quasimaps forces the shape of the source curve to be that of a snake which the hypersurface cuts into two pieces, the internal one of degree $\beta^{(0)}$, and the external one of degree $\beta^{(1)}$ and multiplicity $m^{(1)}$ of contact with Y, with the first marked point belonging to the internal component and the second to the external one.

  The invariants which we need to consider will hence be of the form

  $$\langle i^*\eta_i\psi_1^j, \rho^h\rangle_{\overline{Q}_{0,2}(Y,\beta^{(0)})}\langle \rho_h, \mathbb{1}_X\rangle_{\overline{Q}_{0,\{m^{(1)},0\}}(X|Y,\beta^{(1)})}, \quad h \in \{1,\dots,k'\}$$

  Consider the following dimensional computation:

  $$\begin{aligned} 0 \le \operatorname{codim}\rho^h &= \dim Y - \operatorname{codim}\rho_h \\ &= \dim Y - \operatorname{vdim}\overline{Q}_{0,\{m^{(1)},0\}}(X|Y,\beta^{(1)}) \\ &= \dim Y - (\dim X - 3 - K_X\cdot\beta^{(1)} + 2 - m^{(1)}) \\ &= K_Y.\beta^{(1)} - Y.\beta^{(1)} + m^{(1)} \le 0 \end{aligned}$$

  where the last equality follows from adjunction, and the inequality follows from $K_Y \le 0$ and $m^{(1)} \le Y.\beta^{(1)}$. This shows that the only non-trivial contributions are due to the classes $\beta^{(1)}$ such that $K_Y.\beta^{(1)} = 0$, and the order of tangency is forced to be maximal, i.e. $m^{(1)} = Y.\beta^{(1)}$. Furthermore, the only relevant insertions are $\rho^1 = \mathbb{1}_Y$ and $\rho_1 = [pt_Y]$. Finally, $m^{(1)} = Y.\beta^{(1)}$ implies that

  $$m = \alpha_1 = Y.\beta^{(0)} + m^{(1)} = Y.\beta,$$

  hence the boundary contributions do not show up until the very end of the process of "increasing the multiplicity".
- The remaining term in $T_{(m)}^{X|Y}(z,\beta)$ is $m(\mathrm{ev}_1)_*[\overline{Q}_{0,\{m,0\}}(X|Y,\beta)]^{\mathrm{vir}}$; notice that it only gets insertions from the cohomology of X (restricted to Y). On the other hand

  $$\operatorname{vdim}\overline{Q}_{0,\{m,0\}}(X|Y,\beta) = \dim X - 3 - K_X.\beta + 2 - m \ge r - 1$$

  because $m \le Y.\beta$ and $-(K_X+Y).\beta \ge 0$, by adjunction, projection formula, and for every effective curve class β (coming from Y, but saying this is superfluous by Lefschetz's hyperplane theorem as we have already remarked); since the restriction of the class $[pt_X]$ to Y vanishes, the only insertion that contributes is η_1 (by definition of a dual basis, all other dimension 1 classes vanish when restricted to Y), forcing the equality $m = Y.\beta$, so that again this correction term is non-trivial only in the last step of the algorithm.

So, in the end, we see that equation 3 reduces to

$$\begin{aligned}
&\prod_{j=0}^{Y.\beta}(Y+jz)S_0^X(z,\beta)=T^{X|Y}_{(Y.\beta)}(z,\beta)\\
&=\sum_{i=1,\dots,k;j\geq 0} z^{j+1}\eta^i\langle\rho_i\psi_1^j,\mathbb{1}_Y\rangle_{\overline{Q}_{0,2}(Y,\beta)}\\
&+\sum_{\substack{0<\beta^{(0)}<\beta\\ \beta^{(0)}+\beta^{(1)}=\beta}} z^{j+1}\eta^i\langle\rho_i\psi_1^j,\mathbb{1}_Y\rangle_{\overline{Q}_{0,2}(Y,\beta^{(0)})}(Y.\beta^{(1)})\langle[pt_Y],\mathbb{1}_X\rangle_{\overline{Q}_{0,\{Y.\beta^{(1)},0\}}(X|Y,\beta^{(1)})}\\
&+\eta^1(Y.\beta)\langle[pt_Y],\mathbb{1}_X\rangle_{\overline{Q}_{0,\{Y.\beta,0\}}(X|Y,\beta)}
\end{aligned}$$

if β is such that $K_Y.\beta=0$ (which implies $K_Y.\beta^{(1)}=0$ as well, for every effective decomposition $\beta=\beta^{(0)}+\beta^{(1)}$, due to the semi-positivity assumption on Y); while, if $K_Y.\beta<0$, it simply reduces to

$$\prod_{j=0}^{Y.\beta}(Y+jz)S_0^X(z,\beta)=\sum_{i=1,\dots,k;j\geq 0} z^{j+1}\eta^i\langle\rho_i\psi_1^j,\mathbb{1}_Y\rangle_{\overline{Q}_{0,2}(Y,\beta)}=i_*S_0^Y(z,\beta).$$

The proof of the first claim is now evident. We are left with evaluating $P(q)$.

In order to do that, we use again Gathmann's algorithm, this time in the opposite direction, to go all the way back to X; so it starts:

$$[\overline{Q}_{0,\{Y.\beta,0\}}(X|Y,\beta)]^{\mathrm{vir}}=(Y+(Y.\beta-1)\psi_1)[\overline{Q}_{0,\{Y.\beta-1,0\}}(X|Y,\beta)]^{\mathrm{vir}}-[D^Q_{Y.\beta}(X|Y,\beta)]^{\mathrm{vir}}$$

When looking at the boundary, the invariants that come into play are of the form

$$\langle[pt_Y],\rho^h\rangle_{\overline{Q}_{0,2}(Y,\beta^{(0)})}\langle\rho_h,\mathbb{1}_X\rangle_{\overline{Q}_{0,\{Y.(\beta-\beta^{(0)})-1,0\}}(X|Y,\beta-\beta^{(0)})}$$

but notice that they must vanish by dimensional reasons, since

$$\mathrm{codim}(\rho^h)=\dim Y-3+2-K_Y.\beta^{(0)}-\dim Y=-1.$$

So

$$\begin{aligned}
&(Y.\beta)\langle[pt_Y],\mathbb{1}_X\rangle_{\overline{Q}_{0,\{Y.\beta,0\}}(X|Y,\beta)}=\\
&=(Y.\beta)\int_{[\overline{Q}_{0,2}(X,\beta)]^{\mathrm{vir}}}\mathrm{ev}_1^*(\eta_1)\prod_{j=0}^{Y.\beta-1}(\mathrm{ev}_1^*Y+j\psi_1)=\\
&=(Y.\beta)!\langle[pt_X]\psi_1^{Y.\beta-1},\mathbb{1}_X\rangle_{\overline{Q}_{0,2}(X,\beta)}.
\end{aligned}$$

the second equality because $Y.\eta_1=[pt_X]$ and $Y^2.\eta_1=0$. ◻

Corollary 5.3. *If Y is itself Fano, then there is no correction term*

$$\sum_{\beta\geq 0} q^\beta \prod_{j=0}^{Y.\beta}(Y+jz)S_0^X(z,\beta) = i_*S_0^Y(z,q)$$

Corollary 5.4. *Let Y_5 be the quintic three-fold in $\mathbb{P}^4$. Then*

$$i_*S_0^{Y_5}(z,q) = \frac{I^{Y_5}_{small}(z,q)}{P^{Y_5}(q)},$$

where

$$I^{Y_5}_{small}(z,q) = 5H + \sum_{d>0}\frac{\prod_{j=0}^{5d}(H+jz)}{\prod_{j=0}^{d}(H+jz)^5}q^d$$

and

$$P^{Y_5}(q) = 1 + \sum_{d>0}\frac{(5d)!}{(d!)^5}q^d.$$

Remark 5.5. This formula (and, more generally, formulae for concavex bundles over products of projective spaces) was already obtained in [CZ14, Theorem 1] via equivariant localisation.

5.1. **Comparison with the work of Ciocan-Fontanine and Kim.** We would like to compare our formula to [CFK14, Corollary 5.5.1].

In [CFK14, Section 5] they introduce (in the more general context of ϵ-stable quasimaps to GIT quotients)

- the J^ϵ-function:

$$J^\epsilon(\mathbf{t},z) = \sum_{k\geq 0,\beta\geq 0} q^\beta (\mathrm{ev}_\bullet)_*\left(\frac{\prod_{i=1}^k ev_i^*(\mathbf{t})}{k!}\cap \mathrm{Res}_{F_0}[\overline{QG}^\epsilon_{0,k}(Y,\beta)]^{\mathrm{vir}}\right)$$

- the S^ϵ-operator

$$S^\epsilon(z)(\gamma) = \sum_{m\geq 0,\beta\geq 0}\frac{q^\beta}{m!}(ev_1)_*\left(\frac{[\overline{Q}^\epsilon_{0,2+m}(Y,\beta)]^{\mathrm{vir}}}{z-\psi}ev_2^*(\gamma)\prod_{j=3}^{2+m}ev_j^*(\mathbf{t})\right)$$

- the P^ϵ-series

$$P^\epsilon(\mathbf{t},z) = \sum_h \rho^h \sum_{m\geq 0,\beta\geq 0}\frac{q^\beta}{m!}[\overline{QG}^\epsilon_{0,1+m}(Y,\beta)]\cap \mathrm{ev}_1^*(\rho_h p_\infty)$$

  where $p_\infty \in H^*_{\mathbb{G}_m}(\mathbb{P}^1)$ is defined via its restrictions to the $\mathbb{G}_m$-fixed points: $p_{\infty|0}=0, p_{\infty|\infty}=-z$.

They prove by localisation that [CFK14, Theorem 5.4.1]

$$J^\epsilon(z) = S^\epsilon(z)(P^\epsilon).$$

Furthermore, they prove that, restricting to $\mathbf{t}=0$ and semi-positive targets, the only class that matches non-trivially with $P^{\epsilon}_{|\mathbf{t}=0}$ is $[pt_Y]$, and the above formula takes the simpler form of a product [CFK14, Corollary 5.5.1]

$$\frac{J^{\epsilon}|_{\mathbf{t}=0}}{\langle [pt_Y], P^{\epsilon}|_{\mathbf{t}=0}\rangle} = \mathbb{1}_Y + \sum_h \rho^h (\sum_{\beta\neq 0} q^{\beta} \langle \frac{\rho_h}{z-\psi}, \mathbb{1}_Y\rangle^{\epsilon}_{0,2,\beta}).$$

Notice that the restriction of $S^{\epsilon}(z)(\mathbb{1}_Y)$ to $\mathbf{t}=0$ that appears on the RHS of this formula coincides with what we have called $S_0^Y(z,q)$ above.

They also observe that, if we write the $\frac{1}{z}$-expansion of $J^{\epsilon}_{\mathbf{t}=0}$ as

$$J^{\epsilon}_{\mathbf{t}=0} = J_0^{\epsilon}(q)\mathbb{1}_Y + O(\frac{1}{z})$$

then $\langle [pt_Y], P^{\epsilon}|_{\mathbf{t}=0}\rangle = J_0^{\epsilon}(q)$.

Let us look more closely at $J^{\epsilon}_{\mathbf{t}=0} = \sum_{\beta\geq 0} q^{\beta}(\mathrm{ev}_{\bullet})_* \left(\mathrm{Res}_{F_0}[\overline{QG}^{\epsilon}_{0,0}(Y,\beta)]^{\mathrm{vir}}\right)$. Recall that in our context $Y\subseteq X$ is a very ample hypersurface and X is toric Fano. Furthermore, set $\epsilon = 0^+$. We have the following diagram:

$$\begin{tikzcd}
\overline{QG}_{0,0}(Y,\beta) \arrow[d, hook, "\iota"] \arrow[dr, phantom, "\square"] & F_0^Y \arrow[l, hook'] \arrow[r, "\mathrm{ev}_\bullet"] \arrow[d] & Y \arrow[d, hook, "i"] \\
\overline{QG}_{0,0}(X,\beta) & F_0^X \arrow[l, hook'] \arrow[r, "\mathrm{ev}_\bullet"] & X
\end{tikzcd}$$

- By a slight generalisation of [CFKM14, Propositions 6.2.2 and 6.2.3], $\iota_*[\overline{QG}_{0,0}(Y,\beta)]^{\mathrm{vir}} = e(\pi_* E^Y_{0,0,\beta}(z)) \cap [\overline{QG}_{0,0}(X,\beta)]^{\mathrm{vir}}$ as $\mathbb{G}_m$-equivariant classes, where π is the universal curve on $\overline{QG}_{0,0}(X,\beta)$ and $E^Y_{0,0,\beta}(z)$ is the equivariant line bundle on it associated to $\mathcal{O}_X(Y)$.
- Since the fibers of π are irreducible (by the stability condition and the fact that there are no markings, there can only be the parametrised component), the following splitting holds:

$$e(\pi_* E^Y_{0,0,\beta}(z)) = \prod_{j=0}^{Y.\beta} c_1(\sigma_0^* E^Y_{0,0,\beta}(z)\otimes \omega_\pi^{\otimes j})$$

  coming from evaluating at (the j-th order infinitesimal thickening of) the zero section σ_0 and the jet bundles exact sequence:

$$0 \longrightarrow \pi_*(E^Y_{0,0,\beta}(-j\sigma_0)) \longrightarrow \pi_* E^Y_{0,0,\beta} \longrightarrow \sigma_0^* \mathrm{P}^{j-1}(E^Y_{0,0,\beta}) \longrightarrow 0$$

$$0 \longrightarrow \Omega_\pi^{\otimes j}\otimes E^Y_{0,0,\beta} \longrightarrow \mathrm{P}^j(E^Y_{0,0,\beta}) \longrightarrow \mathrm{P}^{j-1}(E^Y_{0,0,\beta}) \longrightarrow 0$$

which, restricting to F_0^X, gives:

$$\iota_*[F_0^Y]^{\mathrm{vir}} = \prod_{j=0}^{Y.\beta}(Y+iz)[F_0^X]^{\mathrm{vir}}.$$

- The small J^{0^+}-function for toric varieties has been evaluated by Givental [Giv96][CFK10, Definition 7.2.8]:

$$(\mathrm{ev}_\bullet)_*\frac{[F_0^X]^{\mathrm{vir}}}{e(N_{F_0/\overline{QG}_{0,0}(X,\beta)})} = \prod_{\rho\in\Sigma_X(1)}\frac{\prod_{j=-\infty}^{0}(D_\rho+jz)}{\prod_{j=-\infty}^{\int_\beta D_\rho}(D_\rho+jz)} = \frac{\prod_{\substack{\rho\in\Sigma_X(1)\colon D_\rho.\beta\le 0\\ j=\int_\beta D_\rho,\dots,0}}(D_\rho+jz)}{\prod_{\substack{\rho\in\Sigma_X(1)\colon D_\rho.\beta>0\\ j=1,\dots,\int_\beta D_\rho}}(D_\rho+jz)}$$

So, using $\sum_{\rho\in\Sigma_X(1)} D_\rho = -K_X$ and $(Y+K_X).\beta = 0$, we see that

$$J_0^Y(q) = \sum_{\beta\ge 0} q^\beta (Y.\beta)!\frac{\prod_{\rho\in\Sigma_X(1)\colon D_\rho.\beta<0}(-1)^{-D_\rho.\beta}(-D_\rho.\beta)!}{\prod_{\rho\in\Sigma_X(1)\colon D_\rho.\beta>0}(D_\rho.\beta)!}$$

- Since X is Fano, $J^X_{|\mathbf{t}=0} = S^X_{|\mathbf{t}=0}(\mathbb{1}_X)$.
- The coefficient $\langle [pt_X]\psi_1^{Y.\beta-1}, \mathbb{1}_X\rangle_{\overline{Q}_{0,2}(X,\beta)}$ that appears in our P-series (multiplied by $(Y.\beta)!$), can be deduced from the expansion of $S^X_{|\mathbf{t}=0}(\mathbb{1}_X)$ given above, and it turns out to be

$$\langle [pt_X], S^X_{|\mathbf{t}=0}(\mathbb{1}_X)\rangle[z^{Y.\beta}] = \frac{\prod_{\rho\in\Sigma_X(1)\colon D_\rho.\beta<0}(-1)^{-D_\rho.\beta}(-D_\rho.\beta)!}{\prod_{\rho\in\Sigma_X(1)\colon D_\rho.\beta>0}(D_\rho.\beta)!}.$$

So we may conclude that the i_* of [CFK14, Corollary 5.5.1] coincides with our Equation 2.

Appendix A. The comparison morphism

We summarise the existence of the comparison morphism for $\mathbb{P}^r$ and how it implies that GW and quasimap invariants of projective space coincide. This has been proven in [MOP11, Theorem 3] and [Man12b, Section 4.3] (but see also [Ber00, Proposition 4.1] and [PR03, Theorem 7.1] for inspiration). We shall try to clarify as many details as possible, for our own benefit and, hopefully, that of the novice reader.

In order to give a morphism $\chi\colon \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)\to\overline{Q}_{g,n}(\mathbb{P}^r,d)$ we need to be able to canonically associate a family of quasimaps on a base S to any family of stable maps on the same base.

The pointwise construction is the following: a stable map has no base points, so the only thing that might prevent it from being a stable quasimap is the presence of rational tails (of positive degree, by the stable maps stability condition). Let $C = C^{(0)}\sqcup_{q_i} R_i$ be the source curve; the rational tail R_i has degree d_i and is joined to the permanent curve $C^{(0)}$ at the node q_i, which is the only special point on R_i; hence all the markings belong to

$C^{(0)}$. The map to $\mathbb{P}^r$ is equivalent to the data of a line bundle $L = f^*\mathcal{O}_{\mathbb{P}^r}(1)$ on C and $r+1$ sections $s_0, \ldots, s_r$ thereof. We associate to such a stable map the quasimap $(C^{(0)}, \mathbf{x}; L_{|C^{(0)}} \otimes \mathcal{O}_{C^{(0)}}(\sum_i d_i q_i); \hat{s}_0, \ldots, \hat{s}_r)$, where $\hat{s}_j$ is the restriction of s_j to $C^{(0)}$, seen as a section of $L_{|C^{(0)}} \otimes \mathcal{O}_{|C^{(0)}}(\sum_i d_i q_i)$ through the inclusion $L_{|C^{(0)}} \hookrightarrow L_{|C^{(0)}} \otimes \mathcal{O}_{C^{(0)}}(\sum_i d_i q_i)$. Notice that the resulting quasimap has a base-point of order d_i at q_i.

The construction in families requires us to find a line bundle on the universal curve that is trivial on the rational tails and relatively ample elsewhere. This can be performed at the level of Picard stacks: let $\mathfrak{Pic}^{d,\mathrm{st}}_{g,n}$ be the open substack of $\mathfrak{Pic}(\pi\colon \mathfrak{C}_{g,n} \to \mathfrak{M}_{g,n})$ obtained by requiring that the total degree of the line bundle is d, the multi-degree is nonnegative and $\mathcal{L} \otimes \omega_\pi^{\log}$ is ample relative to π, where $\mathcal{L}$ is the universal line bundle. Let T^δ be the locus in the universal curve over $\mathfrak{Pic}^{d,\mathrm{st}}_{g,n}$ spanned by rational tails on which $\mathcal{L}$ has degree δ; this is a Cartier divisor by deformation theory and smoothness of the stack $\mathfrak{C}_{\mathfrak{Pic}}$. Notice that T^{δ_0} and T^{δ_1} (say $\delta_0 < \delta_1$) do intersect in a stratum of codimension 1 in both of them, where the rational tail splits into two rational components, the furthest from $C^{(0)}$ having degree δ_0.

[FIGURE]

Claim: the line bundle $\mathcal{M} = \mathcal{L} \otimes \omega_\pi^{\log} \otimes \bigotimes_{0<\delta\leq d} \mathcal{O}_{\mathfrak{C}}((\delta-1)T^\delta)$ on $\mathfrak{C}_{\mathfrak{Pic}}$ has degree 0 on every component of every rational tail, and is π-relatively ample elsewhere.

Proof. Consider a curve $C^{(0)} \sqcup_q R$ with a rational tail of degree δ, such that R consists of n many components $R^{(1)}, \ldots, R^{(n)}$, each of degree $\delta^{(1)}, \ldots, \delta^{(n)}$ respectively, numbered from the closest to the farthest from $C^{(0)}$; set $T_i = \bigcup_{j=i}^n R_j$ and $\epsilon_i = \delta - 1 - \sum_{j=1}^{i-1} \delta_j$.

[FIGURE]

A general one-parameter family in $\mathfrak{Pic}^{d,\mathrm{st}}_{g,n}$ will give us a smoothing of such a curve; the universal curve over such a family is a normal surface S; we can compute the degree of the restriction of $\mathcal{M}$ to components of the central fiber of this family by first restricting $\mathcal{M}$ to S, and then using intersection theory on this normal surface.

Notice that restricting $\bigotimes_{0<\delta\leq d} \mathcal{O}_{\mathfrak{C}}((\delta-1)T^\delta)$ to this family gives $\mathcal{O}_S(\sum_{j=1}^n \epsilon_j T_j)$. Since $R^{(i)}$ is a (-2)-curve for $i = 1, \ldots, n-1$, and $R^{(n)}$ is a (-1)-curve, we get

$$R^{(i)}.T_j = \begin{cases} 0, & \text{for } j < i \\ -1, & \text{for } j = i \\ 1, & \text{for } j = i+1 \\ 0 & \text{for } j > i+1 \end{cases}$$

hence $\deg(\mathcal{M}_{|R^{(i)}}) = \delta^{(i)} - \epsilon_i + \epsilon_{i+1} = 0$ for $i = 1 \dots, n-1$, while for $i = n$ it is $\delta^{(n)} - 1 - \epsilon_n = 0$, as $\omega^{\log}$ is trivial on the (-2) curves and has degree -1 on $R^{(n)}$. The last assertion of the claim follows from the stability condition and the fact that $O_{\mathfrak{C}}(T^\delta)$ is effective when restricted to $C^{(0)}$. □

By taking the relative Proj construction we obtain another curve $\hat{\mathfrak{C}} = \underline{\mathrm{Proj}}_{\mathfrak{Pic}}\left(\bigoplus_{k\geq 0} \pi_*\mathcal{M}^{\otimes k}\right)$ over $\mathfrak{Pic}^{d,\mathrm{st}}_{g,n}$, with a map ρ that contracts the rational tails

$$\begin{tikzcd} \mathfrak{C}_{\mathfrak{Pic}} \arrow[r, "\rho"] \arrow[dr, "\pi"'] & \hat{\mathfrak{C}} \arrow[d, "\pi'"] \\ & \mathfrak{Pic}^{d,\mathrm{st}}_{g,n} \end{tikzcd}$$

It is flat because it is a family of genus g curves over a reduced base. Furthermore, it can be checked by cohomology and base-change [Har77, Theorem 12.11][Knu83, Corollary 1.5] (notice that the fibers of ρ are either points or rational curves) that $\hat{\mathcal{L}} = \rho_*\left(\mathcal{L} \otimes \bigotimes_{0<\delta\leq d} O_{\mathfrak{C}}(\delta T^\delta)\right)$ is a line bundle on $\hat{\mathfrak{C}}$ of degree d relative to π' (such that $\rho^*\hat{\mathcal{L}} \simeq \mathcal{L} \otimes \bigotimes_{0<\delta\leq d} O_{\mathfrak{C}}(\delta T^\delta)$), hence the universal property gives us a commutative diagram (with Cartesian square)

$$\begin{tikzcd} \mathfrak{C}_{\mathfrak{Pic}} \arrow[r, "\rho"] \arrow[dr, "\pi"'] & \hat{\mathfrak{C}} \arrow[r] \arrow[d, "\pi'"] \arrow[dr, phantom, "\square"] & \mathfrak{C}_{\mathfrak{Pic}} \arrow[d, "\pi"] \\ & \mathfrak{Pic}^{d,\mathrm{st}}_{g,n} \arrow[r, "\chi'"] & \mathfrak{Pic}^{d,\mathrm{st}}_{g,n} \end{tikzcd}$$

The very same construction, with the line bundles pulled back from the Picard stack, and the sections of $\mathcal{L}$ seen as sections of $\mathcal{L} \otimes \bigotimes_{0<\delta\leq d} O_{\mathfrak{C}}(\delta T^\delta)$ through the inclusion of line bundles ($O_{\mathfrak{C}}(T^\delta)$ is effective), and descended to sections of $\hat{\mathcal{L}}$ on $\hat{\mathfrak{C}}$ gives us the comparison morphism $\chi\colon \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \to \overline{Q}_{g,n}(\mathbb{P}^r, d)$, fitting in a commutative diagram

$$\begin{tikzcd} \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \arrow[r, "\chi"] \arrow[d, "\nu_{\mathcal{M}}"] & \overline{Q}_{g,n}(\mathbb{P}^r, d) \arrow[d, "\nu_Q"] \\ \mathfrak{Pic}^{d,\mathrm{st}}_{g,n} \arrow[r, "\chi'"] & \mathfrak{Pic}^{d,\mathrm{st}}_{g,n} \end{tikzcd}$$

and, as before,

$$\begin{tikzcd} C_{\mathcal{M}} \arrow[r, "\rho"] \arrow[dr, "\pi_{\mathcal{M}}"'] & \hat{C} = \chi^* C_Q \arrow[r] \arrow[d, "\hat{\pi}"] \arrow[dr, phantom, "\square"] & C_Q \arrow[d, "\pi_Q"] \\ & \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \arrow[r, "\chi"] & \overline{Q}_{g,n}(\mathbb{P}^r, d) \end{tikzcd}$$

The comparison between virtual fundamental classes is best outlined in the arXiv version of [Man12b, Remark 5.20]. Call $\nu'_{\mathcal{M}} = \chi' \circ \nu_{\mathcal{M}}$. We may endow it with an obstruction theory by means of

$$\begin{tikzcd}
\nu_{\mathcal{M}}^* \mathbb{L}_{\chi'} \arrow[r] \arrow[d] & \mathbb{E}_{\nu'_{\mathcal{M}}} \arrow[r] \arrow[d] & \mathbb{E}_{\nu_{\mathcal{M}}} \arrow[r, "{[1]}"] \arrow[d] & {} \\
\nu_{\mathcal{M}}^* \mathbb{L}_{\chi'} \arrow[r] & \mathbb{L}_{\nu'_{\mathcal{M}}} \arrow[r] & \mathbb{L}_{\nu_{\mathcal{M}}} \arrow[r, "{[1]}"] & {}
\end{tikzcd}$$

Notice that χ' is a morphism (not of DM type) between smooth Artin stacks, hence we can only deduce that $\mathbb{L}_{\chi'}$ is supported in $[-1,1]$. It is therefore easily seen that $\mathbb{E}_{\nu'_{\mathcal{M}}}$ is also supported in $[-1,1]$; in order to show that it is actually a perfect obstruction theory, consider the long exact sequence

$$\begin{aligned}
0 \to h^{-1}\nu_{\mathcal{M}}^* \mathbb{L}_{\chi'} &\to h^{-1}\mathbb{E}_{\nu'_{\mathcal{M}}} \to h^{-1}\mathbb{E}_{\nu_{\mathcal{M}}} \\
\to h^{0}\nu_{\mathcal{M}}^* \mathbb{L}_{\chi'} &\to h^{0}\mathbb{E}_{\nu'_{\mathcal{M}}} \to h^{0}\mathbb{E}_{\nu_{\mathcal{M}}} \\
\to h^{1}\nu_{\mathcal{M}}^* \mathbb{L}_{\chi'} &\to h^{1}\mathbb{E}_{\nu'_{\mathcal{M}}} \to 0
\end{aligned}$$

and observe that, dually, $h^{-1}\nu_{\mathcal{M}}^* \mathbb{T}_{\chi'}$ injects into $h^0\mathbb{E}^\vee_{\nu_{\mathcal{M}}} \simeq h^0\mathbb{T}_{\nu_{\mathcal{M}}}$, because every infinitesimal automorphism of the rational tail induces a nontrivial deformation of the stable map (since the degree of the latter is positive on every component of the rational tail); we conclude that $h^1\mathbb{E}_{\nu'_{\mathcal{M}}} = 0$.

Claim: there is a morphism of obstruction theories $\chi^*\mathbb{E}_{\nu_Q} \to \mathbb{E}_{\nu_{\mathcal{M}}}$ [Man12b, Lemma 4.19].

Dually, $\mathbb{E}^\vee_{\nu_{\mathcal{M}}} = R^\bullet \pi_{\mathcal{M}*}\mathcal{L}^{\oplus r+1} = R^\bullet \hat{\pi}_*(\rho_*\mathcal{L}^{\oplus r+1})$, while, by cohomology and base-change, $\chi^*\mathbb{E}^\vee_{\nu_Q} = R^\bullet\hat{\pi}_*(\hat{\mathcal{L}}^{\oplus r+1})$, where $\hat{\mathcal{L}} = \rho_*\left(\mathcal{L} \otimes \bigotimes_{0<\delta\leq d} \mathcal{O}_{\mathfrak{C}}(\delta T^\delta)\right)$, so $\mathbb{E}^\vee_{\nu_{\mathcal{M}}} \to \chi^*\mathbb{E}^\vee_{\nu_Q}$ comes from the inclusion of line bundles on $\mathcal{C}_{\mathcal{M}}$

$$\mathcal{L} \hookrightarrow \mathcal{L} \otimes \bigotimes_{0<\delta\leq d} \mathcal{O}_{\mathfrak{C}}(\delta T^\delta).$$

Claim: this morphism factors through $\mathbb{E}_{\nu'_{\mathcal{M}}}$.

$$\begin{tikzcd}
& \chi^*\mathbb{E}_{\nu_Q} \arrow[dl, dashed, "\exists ?"'] \arrow[d] \arrow[dr, "\phi"] & \\
\mathbb{E}_{\nu'_{\mathcal{M}}} \arrow[r] & \mathbb{E}_{\nu_{\mathcal{M}}} \arrow[r] & \nu_{\mathcal{M}}^*\mathbb{L}_{\chi'}[1]
\end{tikzcd}$$

In order to prove that the dashed arrow exists, we need to show that ϕ is the zero map. Dually, we look at $\nu_{\mathcal{M}}^*\mathbb{T}_{\chi'}[-1] \xrightarrow{\phi^\vee} R^\bullet\hat{\pi}_*(\hat{\mathcal{L}}^{\oplus r+1})$. Notation: call R the rational tail, joined at the rest of the curve (which we denote by $(C^{(0)}, \mathbf{p})$ as a marked curve), at the node q, which we may occasionally think of as a (smooth) point on $C^{(0)}$. We claim that:

- $h^0(\phi^\vee)$ is zero because: the LHS involves automorphisms of the rational tail that leave $C^{(0)}$ fixed, while the RHS involves deformations of $C^{(0)}$, so there is no possible interference.
- $h^1(\phi^\vee)$ is zero because: this is slightly awkward. There are two types of possible contributions to the LHS. They correspond to either moving the node q along $C^{(0)}$, or smoothing it. The former appears in the relative tangent of χ' only if the marked curve $(C^{(0)}, \mathbf{p})$ has no automorphisms that may "move q back", i.e. $(C^{(0)}, \mathbf{p})$ is a stable pointed curve. The latter matters only if $(C^{(0)}, q, \mathbf{p})$ has no moduli, i.e. $(C^{(0)}, \mathbf{p})$ is a rational tail with less than 3 markings. I will try to justify why the first type vanishes under $h^1(\phi^\vee)$, and leave the second type because I do not understand it as yet. Look at the long exact sequence

$$
\begin{aligned}
0 \to \mathrm{Hom}(\Omega_{C^{(0)}}, \mathcal{O}_{C^{(0)}}(-q - \textstyle\sum p_i)) &\to \mathrm{Hom}(\Omega_{C^{(0)}}, \mathcal{O}_{C^{(0)}}(-\textstyle\sum p_i)) \to \\
T_{C^{(0)},q} \to \mathrm{Ext}^1(\Omega_{C^{(0)}}, \mathcal{O}_{C^{(0)}}(-q - \textstyle\sum p_i)) &\to \mathrm{Ext}^1(\Omega_{C^{(0)}}, \mathcal{O}_{C^{(0)}}(-\textstyle\sum p_i)) \to 0
\end{aligned}
$$

We are interested in what happens to

$$
\frac{T_{C^{(0)},q}}{\mathrm{Im}\left(\mathrm{Hom}(\Omega_{C^{(0)}}, \mathcal{O}_{C^{(0)}}(-\sum p_i))\right)}
$$

under $h^1(\phi^\vee)$. If we can show that $h^1(\phi^\vee)$ factors through $\mathrm{Ext}^1(\Omega_{C^{(0)}}, \mathcal{O}_{C^{(0)}}(-\sum p_i))$ we are in business. Indeed the natural maps

$$
\begin{tikzcd}
\mathrm{Def}_L \arrow[r] \arrow[d] & \mathrm{Def}_{(C,L)} \arrow[r] \arrow[d] & \mathrm{Def}_C \arrow[d] \\
H^1(\mathcal{O}_C) \arrow[r] & H^1(L^{\oplus r+1}) \arrow[r] & H^1(f^*T_{\mathbb{P}^r})
\end{tikzcd}
$$

show that $h^1(\phi^\vee)$ factors through

$$
\mathrm{Ext}^1(\Omega_{C^{(0)}}, \mathcal{O}_{C^{(0)}}(-q - \textstyle\sum p_i)) \to \mathrm{Ext}^1(\Omega_{C^{(0)}}, \mathcal{O}_{C^{(0)}}) \to \mathrm{Ext}^1(f^*\Omega_{\mathbb{P}^r}, \mathcal{O}_{C^{(0)}}) \simeq H^1(f^*T_{\mathbb{P}^r}).
$$

- $h^2(\phi^\vee)$ is zero because: $\mathbb{E}^\vee_{\nu'_{\mathcal{M}}}$ is supported in $[0,1]$.

Now the cone $C(\phi)$ gives an obstruction theory relative to χ. A priori, it is supported in $[-2, 0]$. By the octahedral axiom

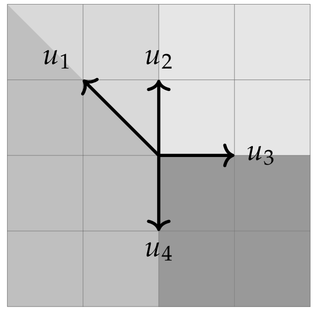

FIGURE 1. Toric fan for $\mathbb{F}_1$.

$$\begin{tikzcd}
\chi^*\mathbb{E}_{\nu_Q} \arrow[d, "\phi"] \arrow[dr, "\phi'"] & & & \\
\mathbb{E}_{\nu'_{\mathcal{M}}} \arrow[r] \arrow[dd] & \mathbb{E}_{\nu_{\mathcal{M}}} \arrow[rr] \arrow[dr] & & \nu^*_{\mathcal{M}}\mathbb{L}_{\chi'}[1] \\
& & C(\phi') \arrow[ur] & \\
C(\phi) \arrow[urr] & & &
\end{tikzcd}$$

it is enough to observe that $C(\phi')$ is supported in $[-1, 0]$ [Man12b, Lemma 4.20] and that $\nu^*_{\mathcal{M}}\mathbb{L}_{\chi'}[1]$ is supported in degrees $[-2, 0]$, in order to conclude that $C(\phi) = \mathbb{E}_\chi$ is a perfect obstruction theory. The conclusion that

$$\chi_*[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)]^{\mathrm{vir}} = [\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d)]^{\mathrm{vir}}$$

follows from the connectedness of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ [KP01] (hence of $\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d)$) and an application of the virtual push-forward theorem [Man12b, Proposition 4.21].

We shall now explain with an example the reason why a naive attempt to extend the comparison morphism to a general toric variety fails. The problem in a nutshell is that not all toric divisors are nef: a rational tail contained in a divisor which is not nef may have negative degree $-d$ with respect to the corresponding line bundle; when contracting such a rational tail, we shall take the line bundle $L(-dq)$, but what to do with the sections? We would like to divide them by z^d, where z is a local coordinate around q, but no condition forces such a divisibility to happen. Otherwise said, there is now an inclusion $L_{|C^{(0)}}(-dq) \hookrightarrow L_{|C^{(0)}}$, but the (restriction of the) given sections of L do not necessarily live in the image of $H^0(C^{(0)}, L_{|C^{(0)}}(-dq)) \hookrightarrow H^0(C^{(0)}, L_{|C^{(0)}})$.

A concrete example is found when looking at the Hirzebruch surface $\mathbb{F}_1 = \mathrm{Bl}_p\,\mathbb{P}^1$.

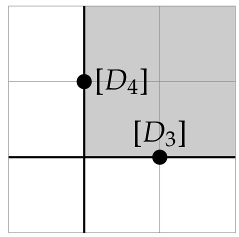

FIGURE 2. Nef cone $\mathrm{Nef}(\mathbb{F}_1)$.

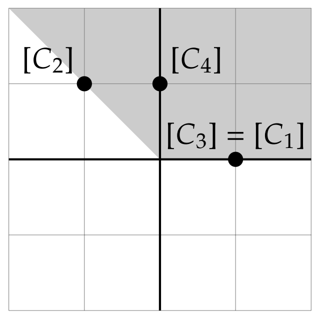

FIGURE 3. Mori cone $\overline{\mathrm{NE}}(\mathbb{F}_1)$.

$\mathrm{Pic}(\mathbb{F}_1)$ is generated by $[D_3]$ and $[D_4]$, with relations $[D_1] = [D_3]$ and $[D_2] = [D_4] - [D_3]$, and the intersection table is given by

$$\begin{cases} D_3^2 = 0 \\ D_3.D_4 = 0 \\ D_4^2 = 1 \end{cases}$$

When thinking of $\mathbb{F}_1$ as a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$, C_1 and C_3 represent the fibers of the bundle (over the toric points of $\mathbb{P}^1$), while C_4 (resp. C_2) is the zero/positive (resp. infinity/negative) section; when thinking of $\mathbb{F}_1$ as $\mathrm{Bl}_p\,\mathbb{P}^1$, C_2 is the exceptional divisor, C_4 is the toric line not passing through p, and C_1, C_3 are the strict transforms of the toric lines through p.

Let us look at $\overline{\mathcal{M}}_{0,2}(\mathbb{F}_1, [C_4])$. Since $[C_4] = [C_2] + [C_3]$, there are going to be maps of the following sort: the source curve is reducible $R_1 \sqcup_q R_2$, R_1 is mapped isomorphically to a fiber (i.e. in class $[C_3]$) and R_2 is mapped isomorphically to C_2, all the markings belong to R_1. So R_2 is a rational tail and deserves to be contracted. Notice that the line bundle $\mathcal{O}(D_2)$ has degree -1 on R_2 (and 1 on R_1). In this case everything works well because the corresponding section $u_{2|R_1}$ must vanish at the node, so we can divide it by a chosen (once for all toric line bundles) section of $\mathcal{O}_{R_1}(q)$.

Consider now $\overline{\mathcal{M}}_{0,2}(\mathbb{F}_1, 2[C_2] + [C_3])$. Certainly there are going to be maps similar to the ones described above, with R_2 now covering C_2 2: 1. The point is that $\mathcal{O}(D_2)$ has degree -2 on R_2, but $u_{2|R_1}$ doesn't have to vanish at the node of order 2, so we are in trouble. Something is going on here: in this case there is a boundary component where the map is of the type that we have just described, and the requirement that $u_{2|R_1}$ vanishes of order 2 at the node defines precisely the intersection with the main component. Check this. Could we possibly exploit this phenomenon to define a smaller compactification, possibly even smaller than quasimaps?

Appendix B. Notes on quasimaps

B.1. **Functoriality.** In the case of stable maps, a morphism $f : X \to Y$ induces a morphism between the corresponding moduli spaces

$$\overline{\mathcal{M}}_{g,n}(X, \beta) \to \overline{\mathcal{M}}_{g,n}(Y, f_*\beta)$$

given by composition with f (in general this induced morphism may involve stabilisation of the source curve). Because of this, the construction of the moduli space of stable maps is said to be **functorial**.

It is natural to ask whether the same holds for the moduli space of quasimaps. Since here the objects of the moduli space are not maps, we cannot simply compose with f, and indeed it is not immediately clear how we should proceed. In [CFK14, Section 3.1] a definition is given when f is an embedding into a projective space; however, this uses the more general language of GIT quotients which we seek to avoid here. As such, we will provide an alternative (but entirely equivalent) construction in the setting of toric varieties, which also relaxes the conditions on the map f and the target Y.
[1]

Our approach uses the language of Σ–collections introduced by D. Cox. This approach is natural insofar as a quasimap is a generalisation of a Σ–collection. We will refer extensively to [Cox95b] and [Cox95a], which we recommend as an introduction for any readers unfamiliar with the theory.

Let X and Y be smooth and proper toric varieties with fans $\Sigma_X \subseteq N_X$ and $\Sigma_Y \subseteq N_Y$. Suppose we are given $f : Y \to X$ (which we do not assume to be a toric morphism). By [Cox95a, Theorem 1.1] the data of such a map is equivalent to a Σ_X–collection on Y:

$$((L_\rho, u_\rho)_{\rho \in \Sigma_X(1)}, (\varphi_{m_x})_{m_x \in M_X})$$

In addition, [Cox95b] allows us to describe line bundles on Y and their global sections in terms of the homogeneous coordinates $(z_\tau)_{\tau \in \Sigma_Y(1)}$. All of these observations are combined into the following theorem, which is so useful that we will state it here in its entirety:

Theorem B.1. [Cox95a, Theorem 3.2] *The data of a morphism $f : Y \to X$ is the same as the data of homogeneous polynomials*

$$P_\rho \in S^Y_{\beta_\rho}$$

[1] We should probably look a bit harder to see if the definition exists elsewhere.

for $\rho \in \Sigma_X(1)$, where $\beta_\rho \in \operatorname{Pic} Y$ and $S^Y_{\beta_\rho}$ is the corresponding graded piece of the Cox ring

$$S^Y = k[z_\tau : \tau \in \Sigma_Y(1)]$$

This data is required to satisfy the following two conditions:

(1) $\sum_{\rho \in \Sigma_X(1)} \beta_\rho \otimes n_\rho = 0$ *in* $\operatorname{Pic} Y \otimes N_X$.

(2) $(P_\rho(z_\tau)) \notin Z(\Sigma_X) \subseteq \mathbb{A}_k^{\Sigma_X(1)}$ *whenever* $(z_\tau) \notin Z(\Sigma_Y) \subseteq \mathbb{A}_k^{\Sigma_Y(1)}$.

Furthermore, two such sets of data (P_ρ) and (P'_ρ) correspond to the same morphism if and only if there exists a $\lambda \in \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Pic} X, \mathbb{G}_m)$ such that

$$\lambda(D_\rho) \cdot P_\rho = P'_\rho$$

for all $\rho \in \Sigma_X(1)$. Finally, if we define $\tilde{f}(z_\tau) = (P_\rho(z_\tau))$ then this defines a lift of f to the prequotients:

$$\begin{tikzcd} \mathbb{A}_k^{\Sigma_Y(1)} \setminus Z(\Sigma_Y) \arrow[r, "\tilde{f}"] \arrow[d, "\pi"] & \mathbb{A}_k^{\Sigma_X(1)} \setminus Z(\Sigma_X) \arrow[d, "\pi"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

Aside B.2. Throughout this section we will stick to the notation established above; in particular we will use ρ to denote a ray in $\Sigma_X(1)$ and τ to denote a ray in $\Sigma_Y(1)$.

Recall our goal: given a map $f : Y \to X$ we wish to define a "push-forward" map:

$$f_* : \overline{Q}_{g,n}(Y, \beta) \to \overline{Q}_{g,n}(X, f_*\beta)$$

Consider therefore a quasimap $(C, (L_\tau, u_\tau)_{\tau \in \Sigma_Y(1)}, (\varphi_{m_Y})_{m_Y \in M_Y})$ with target Y. Pick data $(P_\rho)_{\rho \in \Sigma_X(1)}$ corresponding to the map f, as in the theorem above; we will later see that our construction does not depend on this choice.

The idea of the construction is as follows. Let us pretend for a moment that C is toric and that the quasimap is without basepoints, so that we have an actual morphism $C \to Y$. Then we can lift this morphism to the prequotient as in the following diagram

$$\begin{tikzcd} \mathbb{A}_k^{\Sigma_C(1)} \setminus Z(\Sigma_C) \arrow[r, "(u_\tau)"] \arrow[d] & \mathbb{A}_k^{\Sigma_Y(1)} \setminus Z(\Sigma_Y) \arrow[r, "(P_\rho)"] \arrow[d] & \mathbb{A}_k^{\Sigma_X(1)} \setminus Z(\Sigma_X) \arrow[d] \\ C \arrow[r] & Y \arrow[r] & X \end{tikzcd}$$

from which it follows that the composition $C \to Y \to X$ is given in homogeneous coordinates by:

$$(P_\rho((u_\tau)_{\tau \in \Sigma_Y(1)}))_{\rho \in \Sigma_X(1)}$$

In general of course C is not a toric variety and the quasimap is not basepoint-free. Nevertheless, as we will see, we can still make sense of the expression $P_\rho(u_\tau)$ as a section of a line bundle on C. This will allow us to define the pushforward of our quasimap.

Let us begin. For each ρ, P_ρ is a polynomial in the z_τ; we can write it as

$$P_\rho(z_\tau) = \sum_{\underline{a}} P_\rho^{\underline{a}}(z_\tau) = \sum_{\underline{a}} \mu_{\underline{a}} \prod_\tau z_\tau^{a_\tau} \tag{4}$$

where the sum is over a finite number of multindices $\underline{a} = (a_\tau) \in \mathbb{N}^{\Sigma_Y(1)}$ and the $\mu_{\underline{a}}$ are nonzero scalars. For each $\underline{a}$ consider the following line bundle on C:

$$\tilde{L}_\rho^{\underline{a}} = \bigotimes_\tau L_\tau^{\otimes a_\tau}$$

Then we may take the following section of $\tilde{L}_\rho^{\underline{a}}$:

$$\tilde{u}_\rho^{\underline{a}} = P_\rho^{\underline{a}}(u_\tau) = \mu_{\underline{a}} \prod_\tau u_\tau^{a_\tau}$$

Thus each of the terms $P_\rho^{\underline{a}}$ of P_ρ defines a section $\tilde{u}_\rho^{\underline{a}}$ of a line bundle $\tilde{L}_\rho^{\underline{a}}$. But what we want is a single section $\tilde{u}_\rho$ of a single line bundle $\tilde{L}_\rho$. This is where the isomorphisms φ_{m_Y} come in.

Recall that we have a short exact sequence:

$$0 \longrightarrow M_Y \xrightarrow{\theta} \mathbb{Z}^{\Sigma_Y(1)} \longrightarrow \operatorname{Pic} Y \longrightarrow 0 \tag{5}$$

Let $\underline{a}$ and $\underline{b}$ be multindices appearing in the sum (4) above. By the homogeneity of P_ρ we have

$$\sum_\tau a_\tau D_\tau = \beta_\rho = \sum_\tau b_\tau D_\tau$$

which is precisely the statement that in the above sequence $\underline{a}$ and $\underline{b}$ map to the same element of $\operatorname{Pic} Y$ (namely β_ρ). Hence there exists an $m_Y \in M_Y$ such that:

$$\theta(m_Y) = \underline{a} - \underline{b}$$

Now, the isomorphism φ_{m_Y} (contained in the data of our original quasimap) is a map:

$$\varphi_{m_Y} : \bigotimes_\tau L_\tau^{\otimes \langle m_Y, n_\tau \rangle} \cong \mathcal{O}_C$$

By definition, $\theta(m_Y) = (\langle m_Y, n_\tau \rangle)_{\tau \in \Sigma_Y(1)}$. But also $\theta(m_Y) = (a_\tau - b_\tau)_{\tau \in \Sigma_Y(1)}$. Hence we have:

$$\varphi_{m_Y} : \bigotimes_\tau L_\tau^{\otimes a_\tau} \cong \bigotimes_\tau L_\tau^{\otimes b_\tau}$$

In other words, we have well-defined canonical isomorphisms

$$\tilde{L}_\rho^{\underline{a}} \cong \tilde{L}_\rho^{\underline{b}}$$

for all $\underline{a}$ and $\underline{b}$. Let us choose one such $\underline{a}$ (it doesn't matter which); call it $\underline{a}^\rho$. We define:

$$\tilde{L}_\rho = \tilde{L}_\rho^{\underline{a}^\rho}$$

Then for all $\underline{b}$ we can use the above isomorphism to view $\tilde{u}_\rho^{\underline{b}}$ as a section of $\tilde{L}_\rho$. Summing all of these together we obtain a section $\tilde{u}_\rho$ of $\tilde{L}_\rho$, which we can write (with abuse of notation) as:

$$\tilde{u}_\rho = \sum_{\underline{a}} \mu_{\underline{a}} \prod_\tau u_\tau^{a_\tau}$$

Note that if we had made a different choice of $\underline{a}^\rho$ above the result would have been isomorphic.

Thus far we have constructed line bundles and sections $(\tilde{L}_\rho, \tilde{u}_\rho)_{\rho\in\Sigma_X(1)}$ on C. It remains to define the isomorphisms

$$\tilde{\varphi}_{m_X} : \otimes_\rho \tilde{L}_\rho^{\otimes\langle m_X, n_\rho\rangle} \cong \mathcal{O}_C$$

for all $m_X \in M_X$. The left hand side is:

$$\otimes_\rho \tilde{L}_\rho^{\otimes\langle m_X, n_\rho\rangle} = \otimes_\rho \left(\otimes_\tau L_\tau^{\otimes a_\tau^\rho}\right)^{\otimes\langle m_X, n_\rho\rangle} = \otimes_\tau L_\tau^{\otimes\left(\sum_\rho a_\tau^\rho\langle m_X, n_\rho\rangle\right)}$$

Now, for $m_Y \in M_Y$ we have isomorphisms $\varphi_{m_Y} : \otimes_\tau L_\tau^{\otimes\langle m_Y, n_\tau\rangle} \cong \mathcal{O}_C$. Hence, in order to construct $\tilde{\varphi}_{m_X}$ we need to find an m_Y such that

$$\langle m_Y, n_\tau\rangle = \sum_\rho a_\tau^\rho \langle m_X, n_\rho\rangle$$

for all $\tau \in \Sigma_Y(1)$ (we will then set $\tilde{\varphi}_{m_X} = \varphi_{m_Y}$). Consider therefore the short exact sequence (5). Recall that $\theta(m_Y) = (\langle m_Y, n_\tau\rangle)_{\tau\in\Sigma_Y(1)}$. Hence we need to show that

$$\left(\sum_\rho a_\tau^\rho \langle m_X, n_\rho\rangle\right)_{\tau\in\Sigma_Y(1)}$$

belongs to the image of θ, i.e. that it belongs to the kernel of the second map (notice that m_Y is then unique because θ is injective). This is equivalent to saying that

$$\sum_\tau \sum_\rho a_\tau^\rho \langle m_X, n_\rho\rangle D_\tau = 0 \in \operatorname{Pic} Y$$

Now, we have

$$\sum_\tau a_\tau^\rho D_\tau = \beta_\rho$$

so that the above sum becomes

$$\sum_\rho \langle m_X, n_\rho\rangle \beta_\rho = \left\langle m_X, \sum_\rho \beta_\rho \otimes n_\rho \right\rangle = \langle m_X, 0\rangle = 0$$

where $\sum_\rho \beta_\rho \otimes n_\rho = 0$ by Condition (1) in Theorem B.1. So there does indeed exist a (unique) $m_Y \in M_Y$ such that $\langle m_Y, n_\tau\rangle = \sum_\rho a_\tau^\rho \langle m_X, n_\rho\rangle$, so that we can set:

$$\tilde{\varphi}_{m_X} = \varphi_{m_Y} : \bigotimes_\rho \tilde{L}_\rho^{\otimes\langle m_X, n_\rho\rangle} \cong \mathcal{O}_C$$

Thus, we have produced a quasimap with target X:

$$(C, (\tilde{L}_\rho, \tilde{u}_\rho)_{\rho\in\Sigma_X(1)}, (\tilde{\varphi}_{m_X})_{m_X\in M_X})$$

The proof that this construction does not depend on the choice of (P_ρ) is straightforward and is left to the reader.

It remains to demonstrate that the quasimap thus constructed is nondegenerate and stable. Nondegeneracy follows immediately from Condition (2) in Theorem B.1. Put differently: the original quasimap defined a rational map $C \dashrightarrow Y$, whereas the new quasimap defines a rational map which is simply the composition $C \dashrightarrow Y \to X$. Therefore the set of basepoints is exactly the same.

Stability is a bit more tricky: it is here that we will end up having to put some extra conditions on the map f. First, notice that there are no rational tails because the source curve is unchanged.

Next let $C' \subseteq C$ be a component with exactly 2 special points. Then we need to show (see [CFK10, Definition 3.1.1]) that the following line bundle has positive degree on C':

$$\tilde{\mathcal{L}} = \bigotimes_\rho \tilde{L}_\rho^{\otimes\tilde{\alpha}_\rho}$$

Here the $\tilde{\alpha}_\rho$ are defined by fixing a polarisation on X:

$$\mathcal{O}_X(1) = \bigotimes_\rho \mathcal{O}_X(\tilde{\alpha}_\rho D_\rho)$$

The choice of polarisation makes no difference: a quasimap is stable with respect to one polarisation if and only if it is stable with respect to all others. In order to make use of the fact that the original quasimap to Y was stable, we will make the following assumption on f:

(1) there exists an ample line bundle $\mathcal{O}_X(1)$ on X such that $f^*\mathcal{O}_X(1)$ is ample on Y

This is satisfied if, for example, f is an embedding (which is the only case we will need in this paper). Given this assumption, we can set $\mathcal{O}_Y(1) = f^*\mathcal{O}_X(1)$. We then have:

$$\begin{aligned}\mathcal{O}_Y(1) &= \bigotimes_\rho f^*\mathcal{O}_X(D_\rho)^{\otimes\tilde{\alpha}_\rho} = \bigotimes_\rho \mathcal{O}_Y(\sum_\tau a_\tau^\rho D_\tau)^{\otimes\tilde{\alpha}_\rho}\\ &= \bigotimes_\rho \bigotimes_\tau \mathcal{O}_Y(a_\tau^\rho \tilde{\alpha}_\rho D_\tau) = \bigotimes_\tau \mathcal{O}_Y(D_\tau)^{\otimes \sum_\rho a_\tau^\rho \tilde{\alpha}_\rho}\end{aligned}$$

Thus for $\tau \in \Sigma_Y(1)$ we have $\alpha_\tau = \sum_\rho a_\tau^\rho \tilde{\alpha}_\rho$ and by stability of the original quasimap the line bundle $\mathcal{L} = \otimes_\tau L_\tau^{\otimes\alpha_\tau}$ has positive degree on C'. But:

$$\mathcal{L} = \bigotimes_\tau L_\tau^{\otimes\alpha_\tau} = \bigotimes_\rho \bigotimes_\tau \left(L_\tau^{\otimes a_\tau^\rho}\right)^{\otimes\tilde{\alpha}_\rho} = \bigotimes_\rho \tilde{L}_\rho^{\otimes\tilde{\alpha}_\rho} = \tilde{\mathcal{L}}$$

We have thus proven that $\tilde{\mathcal{L}}$ has positive degree on C', so the pushed-forward quasimap is stable. This completes the proof of the following.

Theorem B.3. *Let X and Y be smooth proper toric varieties and $f : Y \to X$ a morphism. Assume that f satisfies Condition (1) above. Then there exists a natural push-forward map*

$$Q(f)\colon \overline{Q}_{g,n}(Y,\beta) \to \overline{Q}_{g,n}(X, f_*\beta)$$

which does not modify the underlying prestable curves.

Aside B.4. We expect that such a map exists even if f does not satisfy Condition (1). However, in this case we will need to modify the underlying prestable curves by contracting unstable components. The same is true in the stable maps case.

Finally, let us describe how this push-forward morphism behaves when f is a nonconstant map $\mathbb{P}^r \to \mathbb{P}^N$, since we will make use of this later. Write f in homogeneous coordinates as:

$$f[z_0,\ldots,z_r] = [f_0(z_0,\ldots,z_r),\ldots,f_N(z_0,\ldots,z_r)]$$

where the f_i are all homogeneous of degree a. Then given a quasimap with target $\mathbb{P}^r$

$$(C, L, u_o, \ldots, u_r)$$

the pushed-forward quasimap with target $\mathbb{P}^N$ is:

$$(C, L^{\otimes a}, f_0(u_0,\ldots,u_r),\ldots,f_N(u_0,\ldots,u_r))$$

(This is stable as long as $a > 0$, which is precisely when f satisfies Condition (1) above.)

B.2. **Relative obstruction theories for $Q(Y) \to Q(X)$.** Assume now that $f\colon Y \to X$ is a morphism satisfying Condition (1) above, so that it induces

$$k = Q(f)\colon \overline{Q}_{g,n}(Y,\beta) \to \overline{Q}_{g,n}(X, f_*\beta).$$

Even in the easiest possible case when $Y \subseteq X$ is an l.c.i. subscheme, k is not necessarily a regular embedding, so the Gysin map in the sense of [Ful98] does not necessarily exist. Yet, when $\overline{Q}_{g,n}(X, f_*\beta)$ is a smooth stack (or rather its standard obstruction theory w.r.t. the moduli stack of prestable curves is unobstructed, which happens e.g. in the cases $X = \mathbb{P}^r$ and $(g,n) = (0,n)$ or $(1,0)$), we may "pull back along k", and we are going to explain why.

In [Man12a] a generalisation of the Gysin map (called the VIRTUAL PULLBACK) is defined for morphisms endowed with a relative perfect obstruction theory. Moreover, a sufficient condition is given (Corollary 4.9) for this map to respect the virtual classes.

Lemma B.5. *There exists a relative obstruction theory E_k for the morphism*

$$k : \overline{Q}_{g,n}(Y, \beta) \to \overline{Q}_{g,n}(X, f_*\beta)$$

which fits into a compatible triple with the standard obstruction theories for the quasimap spaces over $\mathfrak{M}_{g,n}$. Furthermore, E_k is perfect as soon as $\overline{Q}_{g,n}(X, f_*\beta)$ is unobstructed, so that:

$$k_v^![\overline{Q}_{g,n}(X, f_*\beta)] = [\overline{Q}_{g,n}(Y, \beta)]^{\mathrm{virt}}$$

Proof. Note first that, since k does not change the source curve of a quasimap, we indeed have a commuting triangle:

$$\begin{tikzcd}
\overline{Q}_{g,n}(Y,\beta) \arrow[rr, "k"] \arrow[dr] & & \overline{Q}_{g,n}(X, f_*\beta) \arrow[dl] \\
& \mathfrak{M}_{g,n} &
\end{tikzcd}$$

We have perfect obstruction theories $E_{\overline{Q}(Y)/\mathfrak{M}}$ and $E_{\overline{Q}(X)/\mathfrak{M}}$ and we want to find a perfect obstruction theory E_k. Consider the diagram of universal curves

$$\begin{tikzcd}
C_Y \arrow[r, "\alpha"] \arrow[d, "\pi"] \arrow[dr, phantom, "\square"] & C_X \arrow[d, "\rho"] \\
\overline{Q}_{g,n}(Y,\beta) \arrow[r, "k"] & \overline{Q}_{g,n}(X, f_*\beta)
\end{tikzcd}$$

which is cartesian because k does not alter the source curve of any quasimap. We have sheaves $\mathcal{F}_Y$ and $\mathcal{F}_X$ on C_Y and C_X respectively such that:

$$E^\vee_{\overline{Q}(Y)/\mathfrak{M}} = \mathrm{R}^\bullet\, \pi_*\mathcal{F}_Y$$
$$E^\vee_{\overline{Q}(X)/\mathfrak{M}} = \mathrm{R}^\bullet\, \rho_*\mathcal{F}_X$$

It follows (by flatness of ρ) that when we pull back the latter obstruction theory to $\overline{Q}(Y)$ we obtain:

$$k^*E^\vee_{\overline{Q}(X)/\mathfrak{M}} = \mathrm{R}^\bullet\, \pi_*\alpha^*\mathcal{F}_X$$

To construct a compatible triple, we require a morphism $k^*E_{\overline{Q}(X)/\mathfrak{M}} \to E_{\overline{Q}(Y)/\mathfrak{M}}$. Dually, it is therefore enough to construct a morphism of sheaves on C_Y

$$\mathcal{F}_Y \to \alpha^*\mathcal{F}_X$$

and then apply $\mathrm{R}^\bullet\, \pi_*$. This is analogous to the morphism $f^*T_Y \to f^*T_X|_Y$ which is used in the stable maps setting. However the construction for quasimaps requires a little more ingenuity, because we do not quite have access to a universal map f.

The sheaf $\mathcal{F}_Y$ is defined on C_Y by the short exact sequence

$$0 \to \mathcal{O}_{C_Y}^{\oplus r_Y} \to \oplus_\tau \mathcal{L}_\tau \to \mathcal{F}_Y \to 0$$

where $r_Y = \operatorname{rk}\operatorname{Pic} X$ (implicitly we have chosen a basis for this $\mathbb{Z}$-module). Similarly $\mathcal{F}_X$ is defined on C_X by:

$$0 \to \mathcal{O}_{C_X}^{\oplus r_X} \to \oplus_\rho \mathcal{L}_\rho \to \mathcal{F}_X \to 0$$

We will construct our morphism by first constructing a morhism:

$$\oplus_\tau \mathcal{L}_\tau \to \alpha^* \oplus_\rho \mathcal{L}_\rho$$

Recall that $f\colon Y \to X$ is given by homogeneous polynomials

$$P_\rho \in S^Y_{\beta_\rho} \subset S^Y = k[z_\tau : \tau \in \Sigma_Y(1)]$$

in the Cox ring of Y, where $\beta_\rho = f^*[D_\rho] \in \operatorname{Pic} Y$. For all monomials appearing in P_ρ, if we look at their exponents $(a_\tau)_{\tau\in\Sigma_Y(1)}$, we have $\sum_{\tau\in\Sigma_Y(1)} a_\tau[D_\tau] = \beta_\rho$ by homogeneity, hence we can use the isomorphisms parametrised by M_Y as above in order to interpret

$$(P_\rho)_{\rho\in\Sigma_X(1)}\colon \bigoplus_{\tau\in\Sigma_Y(1)} L_\tau \to \bigoplus_{\rho\in\Sigma_X(1)} \beta_\rho = \alpha^*\left(\bigoplus_{\rho\in\Sigma_X(1)} L_\rho\right).$$

On the other hand, $f\colon Y \to X$ induces a pullback map on line bundles $\operatorname{Pic}(X) \to \operatorname{Pic}(Y)$ (for which $\mathbb{Z}$-modules we have implicitly chosen bases above), the dual (or transpose) to which gives us a matrix

$$Q \in \mathcal{M}_{r_X\times r_Y}(\mathbb{Z})$$

It is now clear by the very functoriality construction that the square in the following diagram is commutative, hence it induces the (dashed) map of sheaves that we were hoping for

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_{C_Y}^{\oplus r_Y} \arrow[r] \arrow[d, "Q"] & \oplus_\tau \mathcal{L}_\tau \arrow[r] \arrow[d, "(P_\rho)"] & \mathcal{F}_Y \arrow[r] \arrow[d, dashed] & 0 \\ & \mathcal{O}_{C_Y}^{\oplus r_X} \arrow[r] & \alpha^*\left(\oplus_\rho \mathcal{L}_\rho\right) \arrow[r] & \alpha^*\mathcal{F}_X \arrow[r] & 0 \end{tikzcd} \tag{6}$$

Applying $\mathrm{R}^\bullet\,\pi_*$ and dualising we obtain a morphism between the obstruction theories for the quasimap spaces, and we can complete this to obtain an exact triangle

$$k^*E_{\overline{Q}(X)/\mathfrak{M}} \to E_{\overline{Q}(Y)/\mathfrak{M}} \to E_k \xrightarrow{[1]}$$

on $\overline{Q}(Y)$. The complex E_k is perfect (locally isomorphic to a bounded complex of vector bundles) because the other two are, and the axioms of a triangulated category give a morphism of exact triangles

$$\begin{tikzcd} k^*E_{\overline{Q}(X)/\mathfrak{M}} \arrow[r] \arrow[d] & E_{\overline{Q}(Y)/\mathfrak{M}} \arrow[r] \arrow[d] & E_k \arrow[r, "{[1]}"] \arrow[d] & {} \\ k^*L_{\overline{Q}(X)/\mathfrak{M}} \arrow[r] & L_{\overline{Q}(Y)/\mathfrak{M}} \arrow[r] & L_k \arrow[r, "{[1]}"] & {} \end{tikzcd}$$

It follows from a simple diagram chase that $E_k \to L_k$ is a relative obstruction theory. On the other hand, assuming that $\overline{Q}_{g,n}(X, f_*\beta)$ is unobstructed, we may look at the long exact sequence in cohomology and find

$$0 \to \mathrm{h}^{-2}(E_k) \to \mathrm{h}^{-1}(k^* E_{\overline{Q}(X)/\mathfrak{M}}) = 0$$

Hence $\mathrm{h}^{-2}(E_k) = 0$ and it is easy to show using similar arguments that E_k is of perfect amplitude contained in $[-1, 0]$.

□

Remark B.6. The short exact sequence that defines F should be thought of as the pullback of

$$0 \to [V/T] \times \mathfrak{t} \to V \times_T V \to T_{[V/T]} \to 0$$

where $T = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}(V /\!\!/ T), \mathbb{G}_m) \simeq \mathbb{G}_m^r$ is the torus acting on the vector space V, and $\mathfrak{t}$ its Lie algebra. Compare with [CFKM14, Equation 5.1.1]. In fact, F fails to be a vector bundle precisely at the base-points. Note also that the commutativity of the diagram 6 comes from the fact that the lift of $f \colon Y \to X$ to $V_Y^s \to V_X^s$ in Theorem B.1 is equivariant with respect to the action of the tori according to the homomorphism $T_Y = \mathrm{Pic}(Y)^\vee \to T_X = \mathrm{Pic}(X)^\vee$.

In particular, for every smooth projective variety $i \colon X \hookrightarrow \mathbb{P}^r$, we have thus produced a virtual pull-back morphism

$$k_{\mathrm{v}}^! : A_*(\overline{Q}_{0,n}(\mathbb{P}^r, d)) \to A_*(\overline{Q}_{0,n}(X, \beta))$$

where $d = i_*\beta$, and more generally for any cartesian diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & G \arrow[d] \\ \overline{Q}_{0,n}(X, \beta) \arrow[r, "k"] & \overline{Q}_{0,n}(\mathbb{P}^N, d) \end{tikzcd}$$

we get an associated virtual pull-back morphism:

$$k_{\mathrm{v}}^! : A_*(G) \to A_*(F)$$

B.3. **Comparison with the GIT construction.** Let $Y \subseteq X$ be a very ample hypersurface in a smooth projective toric variety, which is cut by a homogeneous polynomial (of degree $\mathcal{O}_X(Y)$) $P_Y \in k[z_\rho : \rho \in \Sigma_X(1)]$. The complete linear system associated to Y gives an embedding $X \hookrightarrow \mathbb{P}^N$ such that Y is the intersection of X and a certain hyperplane $H \subseteq \mathbb{P}^N$. Consider the following cartesian diagram

$$\begin{tikzcd} \overline{Q}_{g,n}(Y, \beta) \arrow[r] \arrow[d] & \overline{Q}_{g,n}(X, \beta) \arrow[d, "k"] \\ \overline{Q}_{g,n}(H, d) \arrow[r] & \overline{Q}_{g,n}(\mathbb{P}^N, d) \end{tikzcd}$$

where $d\ell$ is the push-forward of the curve class β. Here $\overline{Q}_{g,n}(Y,\beta)$ is seen as the closed substack of $\overline{Q}_{g,n}(X,\beta)$ representing those quasimaps $(C,\mathbf{x};L_\rho\colon \rho\in\Sigma_X(1),u_\rho\in H^0(C,L_\rho))$ such that $P_Y(\mathbf{u})=0$. This diagram can be used to endow $\overline{Q}_{g,n}(Y,\beta)$ with a virtual class.

We wish to compare this with the GIT approach of [CFKM14]. Here Y is seen as the GIT quotient of the affine cone $C_Y\subseteq\mathbb{A}^{|\Sigma_X(1)|}$ with respect to the "diagonal" action of $G:=\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}(X),\mathbb{G}_m)\simeq\mathbb{G}_m^{\rho_X}\to\mathbb{G}_m^{|\Sigma_X(1)|}$ (C_Y is invariant because it is cut by a homogeneous equation). Objects of $\overline{Q}_{g,n}(Y,\beta)^{\mathrm{GIT}}$ are diagrams of the form

$$\begin{tikzcd} P \arrow[r] \arrow[d, "G"] & C_Y \\ C & \end{tikzcd} \qquad \text{or, equivalently,} \qquad \begin{tikzcd} P\times_G C_Y \arrow[d, shift right, "\rho"'] \\ C \arrow[u, bend right, "u"'] \end{tikzcd}$$

and the dual perfect obstruction theory with respect to $\mathfrak{Bun}_G$ is given by $R^\bullet\pi_*(u^*\mathbb{T}_\rho^\bullet)$, where $\pi\colon\mathcal{C}_{\mathfrak{Bun}}\to\mathfrak{Bun}_G$ is the universal curve.

Notice that $\mathfrak{Bun}_G\simeq\times^r_{\mathfrak{M}_{g,n}}\mathfrak{Pic}$ by taking the line bundles $\bigoplus_{i=1}^{\rho_X}L^{(i)}=P\times_G\mathbb{A}^{\rho_X}\to C$ associated to the G-torsor $P\to C$. Furthermore, the G-equivariant embedding in a smooth stack

$$\begin{tikzcd} P\times_G C_Y \arrow[r, hook] \arrow[d, shift right, "\rho"'] & P\times_G\mathbb{A}^{|\Sigma_X(1)|}\simeq\bigoplus_{\rho\in\Sigma_X(1)}L_\rho \arrow[dl] \\ C \arrow[u, bend right, "u"'] & \end{tikzcd}$$

gives us $u^*T_\rho^\bullet\simeq[\bigoplus_{\rho\in\Sigma_X(1)}\mathcal{L}_\rho\to E^Y_{g,n,\beta}]$, where $E^Y_{g,n,\beta}$ is the line bundle associated to the universal ones $(\mathcal{L}_\rho)$ by the same rule that takes $(\mathcal{O}_X(D_\rho))$ to $\mathcal{O}_X(Y)$, and the arrow is induced by P_Y. This shows that both the modular interpretation and the obstruction theory coincide.

B.4. **Splitting principle.** Consider boundary strata of the space of quasimaps, i.e. where the underlying curve is reducible and has a prescribed profile, by which we mean the dual graph decorated by the degree of the universal line bundle on each component: there are two natural virtual classes on such a stratum, namely the one induced by the splitting type of the curve and the product one. We are going to show that these virtual classes coincide; this works pretty much like it does in Gromov-Witten theory.

Fix a smooth projective toric variety X, and numerical invariants g,n,β such that $\overline{Q}_{g,n}(X,\beta)$ is defined. Now fix a partition $A=(A_0,\dots,A_r)$ of the genus and marked points, and a partition $B=(\beta_0,\dots,\beta_r)$ of the curve class, such that every factor in the following product makes sense, and consider the space (which we call the **centipede locus**):

$$\mathcal{D}^{Q}(X,A,B) := \overline{Q}_{g_0,A_0\cup\{q_1,\ldots,q_r\}}(X,\beta_0)\times_{X^r}\prod_{i=1}^{r}\overline{Q}_{g_i,A_i\cup\{q_i\}}(X,\beta_i)$$

We can equip the centipede locus with the product virtual class in the following way. Set

$$\mathcal{E}^{Q}(X,A,B) := \overline{Q}_{g_0,A_0\cup\{q_1,\ldots,q_r\}}(X,\beta_0)\times\prod_{i=1}^{r}\overline{Q}_{g_i,A_i\cup\{q_i\}}(X,\beta_i)$$

which we endow with the product class:

$$[\mathcal{E}^{Q}(X,A,B)]^{\mathrm{virt}} := [\overline{Q}_{g_0,A_0\cup\{q_1,\ldots,q_r\}}(X,\beta_0)]^{\mathrm{virt}}\times\prod_{i=1}^{r}[\overline{Q}_{g_i,A_i\cup\{q_i\}}(X,\beta_i)]^{\mathrm{virt}}$$

We then consider the cartesian diagram

$$\begin{tikzcd} \mathcal{D}^{Q}(X,A,B) \arrow[r, "h"] \arrow[d, "\mathrm{ev}_q"] \arrow[dr, phantom, "\square"] & \mathcal{E}^{Q}(X,A,B) \arrow[d, "\mathrm{ev}_q"] \\ X^r \arrow[r, "\Delta_{X^r}"] & X^r\times X^r \end{tikzcd} \tag{7}$$

and, since X is smooth so Δ_{X^r} is a regular embedding, define:

$$[\mathcal{D}^{Q}(X,A,B)]^{\mathrm{virt}} := \Delta^{!}_{X^r}([\mathcal{E}^{Q}(X,A,B)]^{\mathrm{virt}})$$

Notice that, by defining

$$\mathfrak{M}^{\mathrm{wt}}_{A,B} := \mathfrak{M}^{\mathrm{wt}}_{g_0,A_0\cup\{q_1,\ldots,q_r\},\beta_0}\times\prod_{i=1}^{r}\mathfrak{M}^{\mathrm{wt}}_{g_i,A_i\cup\{q_i\},\beta_i}$$

there is a triangle

$$\begin{tikzcd} \mathcal{D}^{Q}(X,A,B) \arrow[rr, "h"] \arrow[dr, "\rho_D"'] & & \mathcal{E}^{Q}(X,A,B) \arrow[dl, "\rho_E"] \\ & \mathfrak{M}^{\mathrm{wt}}_{A,B} & \end{tikzcd} \tag{8}$$

and the product virtual class on $\mathcal{E}^{Q}(X,A,B)$ corresponds to the product of the standard obstruction theories for each factor $\overline{Q}_{g_i,A_i\cup\{q_i\}}(X,\beta_i)\to\mathfrak{M}^{\mathrm{wt}}_{A_i,B_i}$ (the latter is étale over the usual moduli space of prestable curves by forgetting the weight, hence they have isomorphic cotangent complexes).

On the other hand, we have the following cartesian diagram

$$\begin{tikzcd} \mathcal{D}^{Q}(X,A,B) \arrow[r, "\varphi"] \arrow[d, "\rho_D"] \arrow[dr, phantom, "\square"] & \overline{Q}_{0,n}(X,\beta) \arrow[d, "\rho_Q"] \\ \mathfrak{M}^{\mathrm{wt}}_{A,B} \arrow[r, "\psi"] & \mathfrak{M}^{\mathrm{wt}}_{g,n,\beta} \end{tikzcd} \tag{9}$$

Remark B.7. The bottom horizontal map is not a closed immersion: due to the existence of degree–0 rational components, there may be many possible equally valid ways of breaking up a nodal curve. For instance, consider the following example of two elements which map to the same curve under ψ. [FIGURE]

Yet ψ has a natural perfect obstruction theory, given by L_ψ: we only need to show that it is supported in $[-1,0]$. Consider the exact triangle:

$$\psi^* L_{\mathfrak{M}^{\mathrm{wt}}_{g,n,\beta}} \to L_{\mathfrak{M}^{\mathrm{wt}}_{A,B}} \to L_\psi \xrightarrow{[1]}$$

The first two terms are concentrated in degrees $[0,1]$, because they are the cotangent complexes of smooth Artin stacks. Therefore L_ψ is concentrated in degrees $[-1,1]$. Furthermore, if we examine the long exact cohomology sequence near $\mathrm{h}^1(L_\psi)$ we find

$$\mathrm{h}^1(\psi^* L_{\mathfrak{M}^{\mathrm{wt}}_{g,n,\beta}}) \to \mathrm{h}^1(L_{\mathfrak{M}^{\mathrm{wt}}_{A,B}}) \to \mathrm{h}^1(L_\psi) \to 0$$

and hence we must show that the first map is surjective. But this is dual to the map which takes an infinitesimal automorphism of the disconnected curve to an infinitesimal automorphism of the corresponding connected curve (obtained by glueing together the "nodal" marked points). The requirement of preserving the markings translates into that of fixing the node after the gluing operation, so the (infinitesimal) automorphism groups coincide. Hence $\mathrm{h}^1(L_\psi) = 0$ as claimed. (This also descends from the fact that the fibres of ψ are Deligne–Mumford.)

Lemma B.8. *$(h^* E_{\mathcal{E}^Q(A,B,X)}, \phi^* E_{\rho_Q}, \mathrm{ev}_q^* E_{\Delta_{X^r}})$ is a compatible triple for the triangle (8), hence*

$$\psi^! [\overline{Q}_{g,n}(X,\beta)] = \Delta^!_{X^r} [\mathcal{E}^Q(A,B,X)] \in A_*(\mathcal{D}^Q(A,B,X)).$$

Proof. We need to construct a morphism of triangles

$$\begin{tikzcd}
h^* E_{\mathcal{E}^Q(A,B,X)} \arrow[r] \arrow[d] & \phi^* E_{\rho_Q} \arrow[r] \arrow[d] & \mathrm{ev}_q^* E_{\Delta_{X^r}} \arrow[r, "{[1]}"] \arrow[d] & {} \\
h^* L_{\rho_E} \arrow[r] & L_{\rho_D} \arrow[r] & L_h \arrow[r, "{[1]}"] & {}
\end{tikzcd}$$

Consider the following diagram:

$$\begin{tikzcd}
h^* \tilde{C} \arrow[r, "\nu"] \arrow[dr, "\eta"'] & \varphi^* C \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & C \arrow[d, "\pi"] \\
& \mathcal{D}^Q(X,A,B) \arrow[r, "\varphi"] & \overline{Q}_{0,n}(X,\beta)
\end{tikzcd}$$

Here $\tilde{C}$ is the universal (disconnected) curve over $\mathcal{E}^Q(X,A,B)$, which we have pulled back to $\mathcal{D}^Q(X,A,B)$, while $\varphi^* C$ is the universal curve over

$\mathcal{D}^{Q}(X, A, B)$. Therefore the map $\nu : h^{*}\tilde{C} \to \varphi^{*}C$ is (fiberwise) a partial normalisation map given by detaching the nodes which connect the "trunk" of the centipede to the "legs."

There are natural sheaves $\mathcal{F}$ and $\tilde{\mathcal{F}}$ on C and $h^{*}\tilde{C}$ respectively, such that

$$\varphi^{*}E^{\vee}_{\rho_Q} = \mathrm{R}^{\bullet}\,\pi_{*}\mathcal{F}$$
$$h^{*}E^{\vee}_{\rho_E} = \mathrm{R}^{\bullet}\,\eta_{*}\tilde{\mathcal{F}}$$

Furthermore $\nu^{*}\mathcal{F} \simeq \tilde{\mathcal{F}}$, hence by tensoring the partial normalisation short exact sequence

$$0 \to \mathcal{O}_{\varphi^{*}C} \to \nu_{*}\mathcal{O}_{h^{*}\tilde{C}} \to \mathcal{O}_q \to 0$$

with $\mathcal{F}$ and applying the projection formula, we obtain

$$0 \to \mathcal{F} \to \nu_{*}\tilde{\mathcal{F}} \to \mathcal{F}_q \to 0$$

on $\varphi^{*}C$, where q is the locus of nodes connecting the trunk to the spine. (The fact that the morphism on the left is injective follows by applying the Snake Lemma to the short exact sequence defining $\mathcal{F}$.) To this we can apply $\mathrm{R}^{\bullet}\,\pi_{*}$ to obtain an exact triangle

$$\mathrm{R}^{\bullet}\,\pi_{*}\mathcal{F} \to \mathrm{R}^{\bullet}\,\eta_{*}\tilde{\mathcal{F}} \to \mathrm{R}^{\bullet}\,\pi_{*}\mathcal{F}_q \xrightarrow{[1]} \tag{10}$$

Finally, notice that, since quasimaps are required not to have base-points at the nodes, the fibre of the sheaf $\mathcal{F}$ at each of the nodes q can actually be identified with the tangent to the toric variety X at the image of the node itself, i.e. $\mathrm{R}^{\bullet}\,\pi_{*}\mathcal{F}_q \simeq \mathrm{ev}_q^{*}\,T_{X^r} = T_{\Delta_{X^r}}[-1]$.

The statement now follows from functoriality of virtual pull-backs. □

B.5. **The quasimap string equation for $\mathbb{P}^r$.** The string equation for the Gromov–Witten invariants of a smooth projective variety X is given by

$$\langle \mathbb{1}, \gamma_1\psi^{a_1}, \ldots, \gamma_n\psi^{a_n}\rangle^{X}_{g,n+1,\beta} = \sum_{i=1}^{n} \langle \gamma_1\psi^{a_1}, \ldots, \gamma_{i-1}\psi^{a_{i-1}}, \gamma_i\psi^{a_i-1}, \gamma_{i+1}\psi^{a_{i+1}} \ldots, \gamma_n\psi^{a_n}\rangle^{X}_{g,n,\beta}$$

where $\mathbb{1} \in H^{*}(X)$ is the unit class (by convention any term involving a negative power of ψ is set to zero). Since Gromov–Witten invariants and quasimap invariants coincide for $X = \mathbb{P}^r$ ([Man12b, Section 5.4]) we know that the same equation holds for quasimap invariants to $\mathbb{P}^r$.

Nevertheless, it would be illuminating to have a direct proof of this statement, without relying on the equivalence with Gromov–Witten theory. Amongst other things, such a proof would necessarily involve some non-trivial intersection computations in the cohomology ring of the quasimap space, which would be of independent interest.

The proof of the classical string equation (for Gromov–Witten invariants) relies on three key lemmas involving certain codimension–1 classes on the moduli space of stable maps. Let

$$\pi : \overline{\mathcal{M}}_{g,n+1}(X,\beta) \to \overline{\mathcal{M}}_{g,n}(X,\beta)$$

denote the contraction map given by forgetting the last marked point and stabilising. Then we have:

(1) $\psi_i = \pi^* \psi_i + D_{i,n+1}$
(2) $\psi_i \cdot D_{i,n+1} = 0$
(3) $D_{i,n+1} \cdot D_{j,n+1} = 0$ for $i \neq j$

Here $D_{i,n+1}$ is the locus of stable maps $(C, x_1, \ldots, x_{n+1}, f)$ such that we can split up C into two pieces, $C = C' \cup C''$ (intersecting in a single node) such that C'' has degree 0 and contains only the markings x_i and x_{n+1}.

[FIGURE]

We would like to have some analogue of these results in the quasimap setting. In fact, equations (2) and (3) carry over without difficulty. Equation (1), on the other hand, is rather more delicate.

In the stable map setting, equation (1) is proved by considering the following diagram

$$\begin{tikzcd}
\mathcal{C}_{g,n+1} \arrow[r, "\rho"] \arrow[dr, "\psi"'] & \pi^*\mathcal{C}_{g,n} \arrow[r, "\alpha"] \arrow[d, "\eta"] & \mathcal{C}_{g,n} \arrow[d, "\varphi"] \\
& \overline{\mathcal{M}}_{g,n+1}(X,\beta) \arrow[r, "\pi"] & \overline{\mathcal{M}}_{g,n}(X,\beta)
\end{tikzcd}$$

where the square on the right is cartesian. On fibres, the map ρ contracts rational components of $\mathcal{C}_{g,n+1}$ on which f is constant and which contain exactly three special points, one of which is x_{n+1}. Thus, we see that

$$\rho^*(x_i) = x_i + R_{i,n+1}$$

where $R_{i,n+1} \subseteq \mathcal{C}_{g,n+1}$ consists fibrewise of the rational tails containing only x_i and x_{n+1}; it is a closed substack of $\psi^{-1}(D_{i,n+1})$ of codimension 0.

On the other hand, we have (REFERENCE):

$$\rho^* \omega_\eta(\Sigma_{i=1}^n x_i) = \omega_\psi(\Sigma_{i=1}^n x_i)$$

Taking Chern classes and combining this with the above result we obtain:

$$\mathrm{c}_1(\rho^*\omega_\eta) = \mathrm{c}_1(\omega_\psi) - \Sigma_{i=1}^n R_{i,n+1}$$

We can now pull back along the section x_i and use the fact that $x_i^* R_{j,n+1} = \delta_{i,j} D_{i,n+1}$ to obtain:

$$\mathrm{c}_1(x_i^* \rho^* \omega_\eta) = \mathrm{c}_1(x_i^*\omega_\psi) - D_{i,n+1}$$

Now, $\rho^*\omega_\eta = \rho^*\alpha^*\omega_\varphi$, and so:

$$x_i^*\rho^*\omega_\eta = \pi^* x_i^* \omega_\varphi$$

Thus we end up with

$$\pi^* \mathrm{c}_1(x_i^*\omega_\varphi) = \mathrm{c}_1(x_i^*\omega_\psi) - D_{i,n+1}$$

which is equation (1) above.

What is different in the case of quasimaps? We have a similar-looking diagram

$$\begin{tikzcd} \mathcal{C}_{g,n+1} \arrow[r, "\rho"] \arrow[dr, "\psi"'] & \pi^*\mathcal{C}_{g,n} \arrow[r, "\alpha"] \arrow[d, "\eta"] & \mathcal{C}_{g,n} \arrow[d, "\varphi"] \\ & \overline{\mathcal{Q}}_{g,n+1}(X,\beta) \arrow[r, "\pi"] & \overline{\mathcal{Q}}_{g,n}(X,\beta) \end{tikzcd}$$

but now, because of the stronger stability condition, ρ also contracts the locus T_{n+1} consisting of rational tails (of any degree) with a single marking x_{n+1}. We claim that:

Conjecture B.9. $\rho^*\omega_\eta(\Sigma_{i=1}^n x_i) = \omega_\psi(\Sigma_{i=1}^n x_i - T_{n+1})$

Once we have this, the string equation follows as in the stable maps case by pulling back along the section x_i (and using the obvious fact that $x_i^* T_{n+1} = 0$).

Appendix C. Some intersection-theoretic lemmas

Consider a morphism of DM stacks $f\colon Y \to X$ over a smooth base $\mathfrak{M}$, such that X is *smooth* over $\mathfrak{M}$ and Y carries a virtual class given by a perfect obstruction theory $E^\bullet_{Y/\mathfrak{M}}$. Then, for every Cartesian diagram

$$\begin{tikzcd} G \arrow[r, "g"] \arrow[d, "q"] \arrow[dr, phantom, "\square"] & F \arrow[d, "p"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

and every class $\alpha \in A_*(F)$, we may define

$$f^!_\Delta(\alpha) = \Delta^!_X([Y]^{\mathrm{vir}} \times \alpha) \in A_*(G)$$

which we call a *diagonal* virtual pull-back. We first show that it coincides with the usual virtual pull-back along f in the presence of a compatible perfect obstruction theory relative to f.

Lemma C.1. *Assume that there exists a relative obstruction theory $E^\bullet_f$ compatible with $E^\bullet_{Y/\mathfrak{M}}$ and the standard (unobstructed) obstruction theory for X, i.e.*

$$\begin{tikzcd} f^*L^\bullet_{X/\mathfrak{M}} \arrow[r] \arrow[d, equal] & E^\bullet_{Y/\mathfrak{M}} \arrow[r] \arrow[d] & E^\bullet_f \arrow[r, "{[1]}"] \arrow[d] & {} \\ f^*L^\bullet_{X/\mathfrak{M}} \arrow[r] & L^\bullet_{Y/\mathfrak{M}} \arrow[r] & L^\bullet_f \arrow[r, "{[1]}"] & {} \end{tikzcd}$$

Then for every Cartesian diagram and every class $\alpha \in A_*(F)$ as above,

$$f^!_E(\alpha) = f^!_\Delta(\alpha).$$

Proof. Consider the following cartesian diagram:

$$\begin{tikzcd}
G \arrow[r, "q\times g"] \arrow[d, "g"] \arrow[dr, phantom, "\square"] & Y \times_{\mathfrak{M}} F \arrow[r, "\mathrm{pr}_1"] \arrow[d, "f\times \mathrm{Id}"] \arrow[dr, phantom, "\square"] & Y \arrow[d, "f"] \\
F \arrow[r, "p\times \mathrm{Id}"] \arrow[d, "p"] \arrow[dr, phantom, "\square"] & X \times_{\mathfrak{M}} F \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{Id}\times p"] & X \\
X \arrow[r, "\Delta_X"] & X \times_{\mathfrak{M}} X &
\end{tikzcd}$$

Then, by commutativity of virtual pull-backs, we have

$$\begin{aligned}
\Delta_X^!([Y]^{\mathrm{vir}} \times \alpha) &= \Delta^!((f_E^![X]) \times \alpha) \\
&= \Delta_X^!(f_E^!([X] \times \alpha)) \\
&= f_E^!(\Delta_X^!([X] \times \alpha)) \\
&= f_E^!(\alpha)
\end{aligned}$$

as required. □

Secondly, we show that the *diagonal* virtual pull-back behaves similarly to an ordinary virtual pull-back (e.g. commutes with other virtual pull-backs) even in the absence of a compatible perfect obstruction theory.

Lemma C.2. *The* diagonal *virtual pull-back morphism as defined above commutes with ordinary Gysin maps and with virtual pull-backs.*

Proof. First consider the case of ordinary Gysin maps. We must consider a cartesian diagram:

$$\begin{tikzcd}
Y'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & S \arrow[d, "k"] \\
Y' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X' \arrow[r] \arrow[d] & T \\
Y \arrow[r, "f"] & X &
\end{tikzcd}$$

with k a regular embedding and $f\colon Y \to X$ as before. We need to show that for all $\alpha \in A_*(X')$:

$$k^! f_\Delta^!(\alpha) = f_\Delta^! k^!(\alpha)$$

We form the cartesian diagram:

$$\begin{tikzcd}
Y'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y \times X'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & S \arrow[d, "k"] \\
Y' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y \times X' \arrow[r] \arrow[d] & T \\
X \arrow[r, "\Delta_X"] & X \times X &
\end{tikzcd}$$

And apply commutativity of usual Gysin morphisms. In the case where k is not a regular embedding but rather is equipped with a relative perfect obstruction theory, the same argument works with $k^!$ replaced by $k^!_{\mathrm{v}}$. □

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