GENUS 0 RELATIVE TORIC QUASIMAPS À LA GATHMANN

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ABSTRACT. Relative quasimaps are good.

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We first deal with genus 0 quasimaps to projective space, relative to a hyperplane. We give a Gathmann-like description of the space of relative quasimaps as a closed substack of the moduli space of (absolute) quasimaps to $\mathbb{P}^r$; it turns out to be irreducible of the expected dimension. Finally, we retrieve a Gathmann-type formula by pushforward along the comparison morphism $\chi\colon \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d) \to \overline{Q}_{0,n}(\mathbb{P}^r,d)$.

Fix coordinates on $\mathbb{P}^r$ such that the hyperplane H is $\{x_0 = 0\}$. Let $\alpha = (\alpha_1,\ldots,\alpha_n)$ be an n-tuple of nonnegative integers. Consider the following locus $\widetilde{Q}_{0,\alpha}(\mathbb{P}^r|H,d)$ inside $\overline{Q}_{0,n}(\mathbb{P}^r,d)$: the quasimaps $(C,x_1,\ldots,x_n,L,u_0,\ldots,u_r)$ such that, if Z is a connected component of the vanishing locus of u_0 in C, then one of the following holds:

(1) Z is a point, either unmarked, or one of the x_i's, and in this case u_0 vanishes at Z with multiplicity at least α_i.
(2) Z is a curve (*internal*); letting $C^{(1)}, \ldots, C^{(k)}$ be the (*external*) irreducible components adjacent to Z, with nodes $q_i = Z \cap C^{(i)}$, and $m^{(i)}$ the order of vanishing of $u_{0|C^{(i)}}$ at q_i, we must have

$$\deg(L_{|Z}) + \sum_{i=1}^{k} m^{(i)} \geq \sum_{x_j \in Z} \alpha_j$$

On the other hand, denote by $Q_{0,\alpha}(\mathbb{P}^r|H, d)$ the *nice locus*, consisting of actual maps from an irreducible curve (i.e. $\mathbb{P}^1$) and with specified tangency condition α at the markings $\mathbf{x}$. Notice that this is an irreducible, locally closed substack of $\overline{Q}_{0,n}(\mathbb{P}^r, d)$, by pretty much the same argument as in [**?**, Lemma 1.8]; it has codimension $\sum \alpha$. In fact it is isomorphic to the nice locus inside stable maps, that Gathmann denotes by $\mathcal{M}_{0,\alpha}(\mathbb{P}^r|H, d)$ [**?**, Def. 1.6] (the stricter stability condition has no effect when the source curve is irreducible, of course provided $n \geq 2$); hence:

Lemma 3.1. *The comparison morphism restricts to a birational morphism* $\overline{\mathcal{M}}_{0,\alpha}(\mathbb{P}^r|H, d) \to \widetilde{Q}_{0,\alpha}(\mathbb{P}^r|H, d)$*.*

Proof. The contraction of a rational tail R always happens far away from the markings, hence the only care we need to take is when the one component touching R is internal (call it Z); in this case, observe that $m^{(R)} \leq \deg(f_{|R})$ and the quasimap resulting from the contraction of R has $\deg(L_{|Z}) = \deg(f_{|Z}) + \deg(f_{|R})$, so the corresponding term only moves around the LHS of the α-tangency condition nr. 2.

Birationality follows from the fact that the comparison morphism restricts to give an isomorphism between the nice loci. □

Lemma 3.2. *With notations as above (with* $\sum \alpha \leq d$*),* $\widetilde{Q}_{0,\alpha}(\mathbb{P}^r|H, d)$ *is the closure of the nice locus* $Q_{0,\alpha}(\mathbb{P}^r|H, d)$ *inside* $\overline{Q}_{0,n}(\mathbb{P}^r, d)$*.*

Proof. $\widetilde{Q}_{0,\alpha}(\mathbb{P}^r|H, d) \subseteq \overline{Q_{0,\alpha}(\mathbb{P}^r|H, d)}$: we show that, given any quasimap satisfying the α-tangency conditions spelled above, it can be (infinitesimally) deformed to a stable *map* satisfying Gathmann's conditions [**?**, Def. 1.1 and Rmk. 1.4], and then appeal to [**?**, Prop. 1.14].

We induct on the number of components containing at least one base-point. If this number is zero, we're done (because quasimap stability is stronger than map stability); otherwise, pick such a component C_0, with base-points $p_1, \ldots, p_h$ and adjacent rational trees $R_1, \ldots, R_k$, joined to C_0 at the nodes $q_1, \ldots, q_k$. Since there are base-points but the quasimap respects the nondegeneracy condition, $\deg(L_{|C_0}) > 0$, and since $C_0 \simeq \mathbb{P}^1$ we can find a section w of $L_{|C_0} \simeq \mathcal{O}_{\mathbb{P}^1}(d_0)$ not vanishing at any of the base-points p_i's; then it is enough to deform the section $u_{r|C_0}$ to $u_{r|C_0} + \epsilon w$ (and keep the other sections the same) in order to delete the base-points belonging to C_0.

Notice that $u_{0|C_0}$ is unchanged, so the deformation still respects α-tangency at the markings lying on C_0 (whether the latter is an internal or an external component). We need to check that such a deformation can be extended to the whole curve C without changing the vanishing conditions on u_0. Notice that the action of PGL_{r+1} on $\mathbb{P}^r$ extends to an action of the group on the space of quasimaps; we can apply the matrix

$$\begin{bmatrix} 1 & & \\ & \ddots & \\ & \epsilon\frac{w(q_i)}{u_j(q_i)} & 1 \end{bmatrix}$$

to the restriction of the original quasimap to R_i, where j is any index s.t. $u_j(q_i) \neq 0$ (one such must exist because the node is not allowed to be a base-point), and by doing this separately to every rational tree springing from C_0 we get a deformation of the original quasimap that still has α-tangency with the hyperplane H (u_0 hasn't been touched at all), but the base-points on C_0 have been eliminated.

$\overline{Q_{0,\alpha}(\mathbb{P}^r|H,d)} \subseteq \widetilde{Q}_{0,\alpha}(\mathbb{P}^r|H,d)$: consider a family of relative quasimaps over a smooth curve S, such that the generic fiber lies in the nice locus. Then we may blow-up the source curve (which is a fibered surface) in the base-points of the quasimap (that are finitely many smooth points of the central fiber) in order to get an actual morphism to $\mathbb{P}^r$; we may as well suppose that the central fiber of the new family is stable. Notice that the central fiber actually belongs to Gathmann's space $\overline{\mathcal{M}}_{0,\alpha}(\mathbb{P}^r|H,d)$: we have just introduced some rational tails away from the markings, hence the only thing we have to check is, when we blow-up a base-point on an internal component, the rational tail will again be internal ($u_0 \equiv 0$ in a neighborhood of the base-point), so it will contribute to the LHS of the α-tangency condition nr. 2 in the very same way. We may now invoke [**?**, Lemma 1.9] and the quasimap case follows from Lemma 3.1. □

From now on we shall denote this closed substack by $\overline{Q}_{0,\alpha}(\mathbb{P}^r|H,d)$.

Increasing the multiplicity can be naively performed in the very same way as Gathmann did:

$$\sigma_k^m := x_k^* d^m_{C/\overline{Q}}(u_0) \in H^0(\overline{Q}, x_k^*\mathcal{P}^m_{C/\overline{Q}}(\mathcal{L}))$$

with $m = \alpha_k + 1$ cuts $\overline{Q}_{0,\alpha+e_k}(\mathbb{P}^r|H,d)$ inside $\overline{Q}_{0,\alpha}(\mathbb{P}^r|H,d)$, together with a bunch of degenerate contributions from quasimaps where the component on which x_k lies is internal (call it Z) and (notice the equality sign!)

$$\deg(L_{|Z}) + \sum m^{(i)} = \sum_{x_j \in Z} \alpha_j.$$

Of course, quasimap stability forces these degenerate contributions not to have any rational tail; this is really the only difference with the case of stable maps, and indeed we can pushforward Gathmann's formula along the comparison morphism $\chi\colon \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d) \to \overline{Q}_{0,n}(\mathbb{P}^r, d)$ and the only terms that are going to change are the degenerate ones with rational tails (in fact they disappear, since the restriction of the comparison map has positive dimensional fibers there). With an eye to the future, we remark that these contributions do matter when computing GW invariants of a CY hypersurface in projective space, and may well account for the divergence between GW and quasimap invariants in the CY case [**?**, Rmk. 1.6].

Lemma 3.3. $\chi^*(\psi_k) = \psi_k$ *and* $\chi^*(x_k^*\mathcal{L}) = \mathrm{ev}_k^*(\mathcal{O}_{\mathbb{P}^r}(H))$.

Proof. Recall that $\psi_k = c_1(x_k^*\omega_{C/\mathcal{M}})$ and contemplate the following diagram

$$\begin{tikzcd}
 & & & \mathbb{P}^r & \\
C_{\overline{\mathcal{M}}} \arrow[urrr, "f"] \arrow[rr, "\sigma^{ss}"] \arrow[dr] & & \chi^*C_{\overline{Q}} \arrow[ur, dashed] \arrow[rr] \arrow[dl] & & C_{\overline{Q}} \arrow[ul, dashed] \arrow[d] \\
 & \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d) \arrow[ul, bend left, "x_k"] \arrow[ur, bend right, "x_k"'] \arrow[rrr, "\chi"] & & & \overline{Q}_{0,n}(\mathbb{P}^r, d) \arrow[u, bend right, "x_k"']
\end{tikzcd}$$

where σ^{ss} is the strong stabilisation map, i.e. contracting the rational tails, which is an isomorphism near the markings. □

Lemma 3.4. $\dim(\overline{\mathcal{M}}_{0,(m^{(i)})}(\mathbb{P}^r|H, d) \cap \mathrm{ev}_1^*(p)) > 0$ *everytime* $rd > 1$*, where* p *is a point of* H*, so the pushforward along* χ *of a degenerate locus with rational tails is 0.*

Proof. $\dim(\overline{\mathcal{M}}_{0,(m^{(i)})}(\mathbb{P}^r|H, d) \cap \mathrm{ev}_1^*(p)) = (r-3)+(1-m^{(i)})+d(r+1)-(r-1) = (rd-1)+(d-m^{(i)})$. □

Proposition 3.5. *Denote by* $[D^{Q}_{\alpha,k}(\mathbb{P}^r|H, d)]$ *the sum of the (product) fundamental classes of*

$$\overline{Q}_{0,\alpha^{(0)}\cup(0,\dots,0)}(H, d_0) \times_{(\mathbb{P}^r)^k} \prod_{i=1}^{k} \overline{Q}_{0,(m^{(i)})\cup\alpha^{(i)}}(\mathbb{P}^r|H, d_i)$$

with coefficient $\frac{m^{(1)}\dots m^{(k)}}{k!}$*, where the sum runs over all splittings* $d = \sum d_i$ *and* $\alpha = \bigcup \alpha^{(i)}$ *such that the above spaces are well-defined, in particular* $|\alpha^{(0)}| + k$ *and* $|\alpha^{(i)}| + 1$ *are all* ≥ 2*, and such that*

$$d_0 + \sum_{i=1}^{k} m^{(i)} = \sum \alpha^{(0)}$$

The following formula holds

$$(\alpha_k\psi_k + x_k^*\mathcal{L}) \cdot [\overline{Q}_{0,\alpha}(\mathbb{P}^r|H, d)] = [\overline{Q}_{0,\alpha+e_k}(\mathbb{P}^r|H, d)] + [D^{Q}_{\alpha,k}(\mathbb{P}^r|H, d)].$$

Proof. Follows from [?, Thm. 2.6] by pushforward along $\chi\colon \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)\to \overline{Q}_{0,n}(\mathbb{P}^r,d)$, using the projection fomula and Lemmas 3.1, 3.3 and 3.4. □

4. Comparison with the GIT Construction

Let X be a hypersurface of degree a in $\mathbb{P}^r$. In the preceding sections, we have put a virtual class on $\overline{Q}_{g,n}(X,d)$ by way of the following Cartesian diagram:

$$\begin{tikzcd} \overline{Q}_{g,n}(X,d) \arrow[r] \arrow[d] & \overline{Q}_{g,n}(\mathbb{P}^r,d) \arrow[d, "\nu_a"] \\ \overline{Q}_{g,n}(H,ad) \arrow[r] & \overline{Q}_{g,n}(\mathbb{P}^N,ad) \end{tikzcd}$$

where $N=\binom{r+a}{a}-1$ and ν_a is the Veronese embedding. In fact, $\overline{Q}_{g,n}(X,d)$ is thought of as representing stable quasimaps to $\mathbb{P}^r$ such that the corresponding sections satisfy the equation for X inside $\mathbb{P}^r$, that is a homogeneous polynomial Q of degree a, i.e. gives a section of $L^{\otimes a}$ on the source curve C.

We wish to compare this with the GIT approach of [?]. Here X is seen as the GIT quotient of the affine cone $C_X\subseteq \mathbb{A}^{r+1}$ with respect to the diagonal $\mathbb{G}_m$-action. Objects of $\overline{Q}_{g,n}(X,d)^{\mathrm{GIT}}$ are diagrams of the form

$$\begin{tikzcd} P \arrow[r] \arrow[d, "\mathbb{G}_m"] & C_X \\ C & \end{tikzcd} \quad \text{or, equivalently,} \quad \begin{tikzcd} P\times_{\mathbb{G}_m} C_X \arrow[d, shift right, "\rho"'] \\ C \arrow[u, shift right, bend right, "u"'] \end{tikzcd}$$

and the dual perfect obstruction theory with respect to $\mathfrak{Bun}_{\mathbb{G}_m}$ is given by $R^\bullet\pi_*(u^*\mathbb{T}_\rho^\bullet)$, where $\pi\colon \mathcal{C}_{\mathfrak{Bun}}\to\mathfrak{Bun}_{\mathbb{G}_m}$ is the universal curve.

Notice that $\mathfrak{Bun}_{\mathbb{G}_m}\simeq\mathfrak{Pic}$ by taking the line bundle $L=P\times_{\mathbb{G}_m}\mathbb{A}^1\to C$ associated to the $\mathbb{G}_m$-torsor $P\to C$. Furthermore, the $\mathbb{G}_m$-equivariant embedding in a smooth stack

$$\begin{tikzcd} P\times_{\mathbb{G}_m} C_X \arrow[r, hook] \arrow[d, shift right, "\rho"'] & P\times_{\mathbb{G}_m}\mathbb{A}^{r+1}\simeq L^{\oplus r+1} \arrow[dl] \\ C \arrow[u, shift right, bend right, "u"'] & \end{tikzcd}$$

gives us $u^*T_\rho^\bullet\simeq[L^{\oplus r+1}\to L^{\otimes a}]$, where the arrow is induced by Q, and shows that both the modular interpretation and the obstruction theory coincide.

5. Recursion Formula in the General Case

We now move on to the general case. Let X be an arbitrary toric variety (smooth and proper) and $Y \subseteq X$ a very ample hypersurface (not necessarily toric). The complete linear system associated to $\mathcal{O}(Y)$ defines an embedding $i : X \hookrightarrow \mathbb{P}^N$ such that $i^{-1}(H) = Y$. By the functoriality property of quasimap spaces (see Appendix A) we have a map:

$$k := i_* : \overline{Q}_{0,n}(X, \beta) \to \overline{Q}_{0,n}(\mathbb{P}^N, d)$$

where $d = i_*\beta$. Furthermore since i is a closed embedding it follows that i_* is as well. It is easy to show that i_* restricts to a morphism between the relative spaces, and thus we have a diagram of embeddings

$$\begin{tikzcd}
\overline{Q}_{0,\alpha}(X|Y, \beta) \arrow[r, hook, "g"] \arrow[d, hook, "f"] \arrow[dr, phantom, "\square"] & \overline{Q}_{0,\alpha}(\mathbb{P}^N|H, d) \arrow[d, hook, "j"] \\
\overline{Q}_{0,n}(X, \beta) \arrow[r, hook, "k"] & \overline{Q}_{0,n}(\mathbb{P}^N, d)
\end{tikzcd}$$

which one can show is cartesian. As such we can define a virtual class on $\overline{Q}_{0,\alpha}(X|Y, \beta)$ by pullback; that is, we consider the cartesian diagram

$$\begin{tikzcd}
\overline{Q}_{0,\alpha}(X|Y, \beta) \arrow[r, hook, "f \times g"] \arrow[d, hook] \arrow[dr, phantom, "\square"] & \overline{Q}_{0,n}(X, \beta) \times \overline{Q}_{0,\alpha}(\mathbb{P}^N|H, d) \arrow[d, "k \times j"] \\
\overline{Q}_{0,n}(\mathbb{P}^N, d) \arrow[r, "\Delta"] & \overline{Q}_{0,n}(\mathbb{P}^N, d) \times \overline{Q}_{0,n}(\mathbb{P}^N, d)
\end{tikzcd}$$

which is equivalent to the previous one. The morphism Δ is a regular embedding because $\overline{Q}_{0,n}(\mathbb{P}^N, d)$ is smooth, and thus we can define:

$$[\overline{Q}_{0,\alpha}(X|Y, \beta)]^{\text{virt}} := \Delta^! \left([\overline{Q}_{0,n}(X, \beta)]^{\text{virt}} \times [\overline{Q}_{0,\alpha}(\mathbb{P}^N|H, d)] \right)$$

The idea is to prove the recursion formula for (X, Y) by pulling back the formula for $(\mathbb{P}^N, H)$ along $k = i_*$. In order to do this, we need to understand how the various virtual classes involved in the formula pull back along this map. This is the technical heart of the proof.

5.1. **Relative Spaces Pull Back.** But before getting into any of this, we need to explain what we mean by "pulling back along k." Indeed, k is not necessarily a regular embedding, so the Gysin map in the sense of [**?**] does not necessarily exist.

In [**?**] a generalisation of the Gysin map (called the **virtual pull-back**) is defined for morphisms endowed with a relative perfect obstruction theory. Moreover, a sufficient condition is given (Corollary 4.9) for this map to respect the virtual classes.

Lemma 5.1. *There exists a relative perfect obstruction theory E_k for the morphism*

$$k : \overline{Q}_{0,n}(X, \beta) \to \overline{Q}_{0,n}(\mathbb{P}^N, d)$$

and hence there exists a virtual pull-back morphism $k_v^!$. Moreover, E_k fits into a compatible triple with the standard obstruction theories for the quasimap spaces over $\mathfrak{M}_{0,n}$, so that:

$$k_v^![\overline{Q}_{0,n}(\mathbb{P}^N,d)] = [\overline{Q}_{0,n}(X,\beta)]^{\mathrm{virt}}$$

Proof. Note first that since k does not change the source curve of a quasimap we indeed have a commuting triangle:

$$\begin{tikzcd} \overline{Q}_{0,n}(X,\beta) \arrow[rr, "k"] \arrow[dr] & & \overline{Q}_{0,n}(\mathbb{P}^N,d) \arrow[dl] \\ & \mathfrak{M}_{0,n} & \end{tikzcd}$$

We have perfect obstruction theories $E_{\overline{Q}(X)/\mathfrak{M}}$ and $E_{\overline{Q}(\mathbb{P}^N)/\mathfrak{M}}$ and we want to find a perfect obstruction theory E_k. Consider the diagram of universal curves

$$\begin{tikzcd} \mathcal{C}_X \arrow[r, "\alpha"] \arrow[d, "\pi"] \arrow[dr, phantom, "\square"] & \mathcal{C}_{\mathbb{P}^N} \arrow[d, "\rho"] \\ \overline{Q}_{0,n}(X,\beta) \arrow[r, "k"] & \overline{Q}_{0,n}(\mathbb{P}^N,d) \end{tikzcd}$$

which is cartesian because k does not alter the source curve of any quasimap. We have sheaves $\mathcal{F}_X$ and $\mathcal{F}_{\mathbb{P}^N}$ on $\mathcal{C}_X$ and $\mathcal{C}_{\mathbb{P}^N}$ respectively such that:

$$E^\vee_{\overline{Q}(X)/\mathfrak{M}} = \mathrm{R}^\bullet\,\pi_*\mathcal{F}_X$$
$$E^\vee_{\overline{Q}(\mathbb{P}^N)/\mathfrak{M}} = \mathrm{R}^\bullet\,\rho_*\mathcal{F}_{\mathbb{P}^N}$$

It follows that when we pull back the latter obstruction theory to $\overline{Q}(X)$ we obtain:

$$k^*E^\vee_{\overline{Q}(\mathbb{P}^N)/\mathfrak{M}} = \mathrm{R}^\bullet\,\pi_*\alpha^*\mathcal{F}_{\mathbb{P}^N}$$

To construct a compatible triple, we require a morphism $k^*E_{\overline{Q}(\mathbb{P}^N)/\mathfrak{M}} \to E_{\overline{Q}(X)/\mathfrak{M}}$. It is therefore enough to construct a morphism of sheaves on $\mathcal{C}_X$

$$\mathcal{F}_X \to \alpha^*\mathcal{F}_{\mathbb{P}^N}$$

and then apply $\mathrm{R}^\bullet\,\pi_*$. This is analogous to the morphism $f^*T_X \to f^*T_{\mathbb{P}^N}|_X$ which is used in the stable maps setting. However the construction for quasimaps requires a little more ingenuity, because we do not have acess to a universal map f.

The sheaf $\mathcal{F}_X$ is defined on $\mathcal{C}_X$ by the short exact sequence

$$0 \to \mathcal{O}_{\mathcal{C}_X}^{\oplus r} \to \oplus_\rho \mathcal{L}_\rho \to \mathcal{F}_X \to 0$$

where $r = \operatorname{rk}\operatorname{Pic} X$ (implicitly we have chosen a basis for this $\mathbb{Z}$-module). Similarly $\mathcal{F}_{\mathbb{P}^N}$ is defined on $\mathcal{C}_{\mathbb{P}^N}$ by:

$$0 \to \mathcal{O}_{\mathcal{C}_{\mathbb{P}^N}} \to \mathcal{L}^{\oplus(N+1)} \to \mathcal{F}_{\mathbb{P}^N} \to 0$$

We will construct our morphism by first constructing a morhism:

$$\oplus_\rho \mathcal{L}_\rho \to \alpha^* \mathcal{L}^{\oplus(N+1)}$$

Recall that $i : X \hookrightarrow \mathbb{P}^N$ is given by homogeneous polynomials

$$P_0, \ldots, P_N \in S^X_\gamma \subset S^X = k[z_\rho : \rho \in \Sigma_X(1)]$$

in the Cox ring of X, where $\gamma \in \operatorname{Pic} X$ is some divisor class. We can write this (non-uniquely) as:

$$\gamma = \sum_\rho c_\rho D_\rho$$

Then given $i \in \{0, \ldots, N\}$ and sections v_ρ of $\mathcal{L}_\rho$ for all $\rho \in \Sigma_X(1)$ we can use the same trick as in the proof of functoriality (see Appendix A) to view $P_i(v_\rho)$ as a section of:

$$\otimes_\rho \mathcal{L}_\rho^{\otimes c_\rho} = \alpha^* \mathcal{L}$$

Thus we obtain a morphism

$$(P_0, \ldots, P_N) : \oplus_\rho \mathcal{L}_\rho \to \alpha^* \mathcal{L}^{\oplus(N+1)}$$

as required. On the other hand if we let $M_1, \ldots, M_r$ denote the chosen generators for Pic X then there exist unique numbers $b_1, \ldots, b_r$ such that:

$$i^* \mathcal{O}_{\mathbb{P}^N}(1) = \otimes_{i=1}^r M_i^{\otimes b_i}$$

These define a map $\mathcal{O}_{C_X}^{\oplus r} \to \mathcal{O}_{C_X}$ and so we obtain a digram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{C_X}^{\oplus r} \arrow[r] \arrow[d] & \oplus_\rho \mathcal{L}_\rho \arrow[r] \arrow[d] & \mathcal{F}_X \arrow[r] & 0 \\
& \mathcal{O}_{C_X} \arrow[r] & \alpha^* \mathcal{L}^{\oplus(N+1)} \arrow[r] & \alpha^* \mathcal{F}_{\mathbb{P}^N} \arrow[r] & 0
\end{tikzcd}$$

which one can check is commutative. By exactness of the first row, our morphism descends to a morphism $\mathcal{F}_X \to \alpha^* \mathcal{F}_{\mathbb{P}^N}$ if the composition

$$\mathcal{O}_{C_X}^{\oplus r} \to \oplus_\rho \mathcal{L}_\rho \to \alpha^* \mathcal{L}^{\oplus(N+1)} \to \alpha^* \mathcal{F}_{\mathbb{P}^N}$$

is zero. But this follows from the commutativity of the left square and the fact that the bottom row is a complex. Hence we obtain a morphism:

$$\mathcal{F}_X \to \alpha^* \mathcal{F}_{\mathbb{P}^N}$$

Applying $\mathrm{R}^\bullet \pi_*$ and dualising we obtain a morphism between the obstruction theories for the quasimap spaces, and we can complete this to obtain an exact triangle

$$k^* E_{\overline{\mathcal{Q}}\mathbb{P}^N/\mathfrak{M}} \to E_{\overline{\mathcal{Q}}(X)/\mathfrak{M}} \to E_k \xrightarrow{[1]}$$

on $\overline{Q}(X)$. The complex E_k is perfect (locally isomorphic to a bounded complex of vector bundles) because the other two are. On the other hand if we look at the long exact sequence in cohomology we find

$$0 \to \mathrm{h}^{-2}(E_k) \to \mathrm{h}^{-1}(k^* E_{\overline{Q}(\mathbb{P}^N)/\mathfrak{M}}) = 0$$

where the last term is zero because $\overline{Q}(\mathbb{P}^N) = \overline{Q}_{0,n}(\mathbb{P}^N, d)$ is unobstructed. Hence $\mathrm{h}^{-2}(E_k) = 0$ and it is easy to show using similar arguments that E_k is of perfect amplitude contained in $[-1, 0]$.

Finally the axioms of a triangulated category give a morphism of exact triangles

$$\begin{tikzcd}
k^* E_{\overline{Q}\mathbb{P}^N/\mathfrak{M}} \arrow[r] \arrow[d] & E_{\overline{Q}(X)/\mathfrak{M}} \arrow[r] \arrow[d] & E_k \arrow[r, "{[1]}"] \arrow[d] & {} \\
k^* L_{\overline{Q}(\mathbb{P}^N)/\mathfrak{M}} \arrow[r] & L_{\overline{Q}(X)/\mathfrak{M}} \arrow[r] & L_k \arrow[r, "{[1]}"] & {}
\end{tikzcd}$$

and it follows from a simple diagram chase that $E_k \to L_k$ is a relative perfect obstruction theory. ◻

We have thus produced a virtual pull-back morphism

$$k^!_{\mathrm{v}} : A_*(\overline{Q}_{0,n}(\mathbb{P}^N, d)) \to A_*(\overline{Q}_{0,n}(X, \beta))$$

and more generally for any cartesian diagram

$$\begin{tikzcd}
F \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & G \arrow[d] \\
\overline{Q}_{0,n}(X, \beta) \arrow[r, "k"] & \overline{Q}_{0,n}(\mathbb{P}^N, d)
\end{tikzcd}$$

we get a virtual pull-back morphism:

$$k^!_{\mathrm{v}} : A_*(G) \to A_*(F)$$

Lemma 5.2. *For any α we have:*

$$k^!_v[\overline{Q}_{0,\alpha}(\mathbb{P}^N|H, d)] = [\overline{Q}_{0,\alpha}(X|Y, \beta)]^{\mathrm{virt}}$$

Proof. Consider the following cartesian diagram:

$$\begin{tikzcd}
\overline{Q}(X|Y) \arrow[r, "f \times g"] \arrow[d, "g"] \arrow[dr, phantom, "\square"] & \overline{Q}(X) \times \overline{Q}(\mathbb{P}^N|H) \arrow[r, "\pi_1"] \arrow[d, "k \times \mathrm{Id}"] \arrow[dr, phantom, "\square"] & \overline{Q}(X) \arrow[d, "k"] \\
\overline{Q}(\mathbb{P}^N|H) \arrow[r, "j \times \mathrm{Id}"] \arrow[d, "j"] \arrow[dr, phantom, "\square"] & \overline{Q}(\mathbb{P}^N) \times \overline{Q}(\mathbb{P}^N|H) \arrow[r, "\pi_1"] \arrow[d, "\mathrm{Id} \times j"] & \overline{Q}(\mathbb{P}^N) \\
\overline{Q}(\mathbb{P}^N) \arrow[r, "\Delta"] & \overline{Q}(\mathbb{P}^N) \times \overline{Q}(\mathbb{P}^N) &
\end{tikzcd}$$

Then by commutativity of virtual pull-backs and their compatibility with flat pull-backs we have

$$\begin{aligned}
[\overline{Q}(X|Y)]^{\mathrm{virt}} &= \Delta^! \left([\overline{Q}(X)]^{\mathrm{virt}} \times [\overline{Q}(\mathbb{P}^N|H)]\right) \\
&= \Delta^! \pi_1^* [\overline{Q}(X)]^{\mathrm{virt}} \\
&= \Delta^! \pi_1^* k_{\mathrm{v}}^! [\overline{Q}(\mathbb{P}^N)] \\
&= \Delta^! k_{\mathrm{v}}^! \pi_1^* [\overline{Q}(\mathbb{P}^N)] \\
&= \Delta^! k_{\mathrm{v}}^! \left([\overline{Q}(\mathbb{P}^N)] \times [\overline{Q}(\mathbb{P}^N|H)]\right) \\
&= k_{\mathrm{v}}^! \Delta^! \left([\overline{Q}(\mathbb{P}^N)] \times [\overline{Q}(\mathbb{P}^N|H)]\right) \\
&= k_{\mathrm{v}}^! [\overline{Q}(\mathbb{P}^N|H)]
\end{aligned}$$

as required. □

5.2. **Centipede Loci and Obstruction Theories.** Thus, the first two terms of the recursion formula pull back easily along k. It remains to consider the third term, namely the virtual class of the comb locus. We wish to prove (see Lemma 5.13) that:

$$k_{\mathrm{v}}^! [D^{Q}_{\alpha,k}(\mathbb{P}^N|H,d)] = [D^{Q}_{\alpha,k}(X|Y,\beta)]^{\mathrm{virt}}$$

We will require a number of preparatory results. We begin by considering a simpler moduli space than the comb locus, which we call the CENTIPEDE LOCUS. We fix a partition $A = (A_0, \ldots, A_r)$ of the marked points and a partition $B = (\beta_0, \ldots, \beta_r)$ of the curve class satisfying the usual stability conditions for the comb loci, and then consider the space:

$$\mathcal{K}(X,A,B) := \overline{Q}_{0,A_0 \cup \{q_1,\ldots,q_r\}}(X,\beta_0) \times_{X^r} \prod_{i=1}^{r} \overline{Q}_{0,A_i \cup \{q_i\}}(X,\beta_i)$$

Notice that the comb locus lives inside this space as a closed substack. We equip this space with a virtual class in the usual way by pulling back the product class along the diagonal. That is, we set

$$\mathcal{L}(X,A,B) := \overline{Q}_{0,A_0 \cup \{q_1,\ldots,q_r\}}(X,\beta_0) \times \prod_{i=1}^{r} \overline{Q}_{0,A_i \cup \{q_i\}}(X,\beta_i)$$

which we equip with the class:

$$[\mathcal{L}(X,A,B)]^{\mathrm{virt}} := [\overline{Q}_{0,A_0 \cup \{q_1,\ldots,q_r\}}(X,\beta_0)]^{\mathrm{virt}} \times \prod_{i=1}^{r} [\overline{Q}_{0,A_i \cup \{q_i\}}(X,\beta_i)]^{\mathrm{virt}}$$

We then consider the cartesian diagram

$$\begin{tikzcd} \mathcal{K}(X,A,B) \arrow[r, "h"] \arrow[d, "\mathrm{ev}_q"] \arrow[dr, phantom, "\square"] & \mathcal{L}(X,A,B) \arrow[d, "\mathrm{ev}_q"] \\ X^r \arrow[r, "\Delta_{X^r}"] & X^r \times X^r \end{tikzcd} \tag{1}$$

and define:

$$[\mathcal{K}(X,A,B)]^{\mathrm{virt}} := \Delta^!_{X^r}([\mathcal{L}(X,A,B)]^{\mathrm{virt}})$$

On the other hand, we have the following cartesian diagram

$$\begin{tikzcd} \mathcal{K}(X,A,B) \arrow[r, "\varphi"] \arrow[d, "\rho"] \arrow[dr, phantom, "\square"] & \overline{Q}_{0,n}(X,\beta) \arrow[d, "\pi"] \\ \mathfrak{M}^{\mathrm{wt}}_{0,A,B} \arrow[r, "\psi"] & \mathfrak{M}^{\mathrm{wt}}_{0,n,\beta} \end{tikzcd} \tag{2}$$

where the moduli spaces on the bottom row are moduli spaces of weighted nodal curves, and we have set:

$$\mathfrak{M}^{\mathrm{wt}}_{0,A,B} := \mathfrak{M}^{\mathrm{wt}}_{0,A_0\cup\{q_1,\dots,q_r\},\beta_0} \times \prod_{i=1}^{r} \mathfrak{M}^{\mathrm{wt}}_{0,A_i\cup\{q_i\},\beta_i}$$

Remark 5.3. The horizontal maps are not injective: due to the existence of degree–0 components, there may be many possible equally valid ways of breaking up a nodal curve. For instance, consider the following example of two elements which map to the same curve under ψ.
[FIGURE]

Lemma 5.4. *The virtual class of $\mathcal{K}(X,A,B)$ as defined previously is induced by a perfect obstruction theory relative to the morphism $\psi \circ \rho$ in diagram* (2)*. Furthermore, there exists a compatible triple $(E_{\psi\circ\rho}, E_\pi, E_\varphi)$ and therefore a virtual pull-back morphism $\varphi^!_v$ such that:*

$$[\mathcal{K}(X,A,B)]^{\mathrm{virt}} = \varphi^!_v[\overline{Q}_{0,n}(X,\beta)]^{\mathrm{virt}}$$

Proof. See Lemmas 5.8 and 5.10 below. □

The proof will proceed by constructing a number of relative perfect obstruction theories. We start with the obstruction theory relative to the morphism

$$\pi : \mathcal{L}(X,A,B) \to \mathfrak{M}^{\mathrm{wt}}_{0,A,B}$$

and construct in sequence obstruction theories relative to the following morphisms:

$$
\begin{aligned}
\operatorname{ev}\times\pi &: \mathcal{L}(X,A,B) \to X^r \times X^r \times \mathfrak{M}^{\mathrm{wt}}_{0,A,B}\\
\operatorname{ev}\times\rho &: \mathcal{K}(X,A,B) \to X^r \times \mathfrak{M}^{\mathrm{wt}}_{0,A,B}\\
\rho &: \mathcal{K}(X,A,B) \to \mathfrak{M}^{\mathrm{wt}}_{0,A,B}\\
\psi\circ\rho &: \mathcal{K}(X,A,B) \to \mathfrak{M}^{\mathrm{wt}}_{0,n,\beta}\\
\varphi &: \mathcal{K}(X,A,B) \to \overline{Q}_{0,n}(X,\beta)
\end{aligned}
$$

Each perfect obstruction theory will be constructed from the preceding item in the list.

Lemma 5.5. *There exists a perfect obstruction theory for the morphism* $\operatorname{ev}\times\pi$ *which induces* $[\mathcal{L}(X,A,B)]^{\mathrm{virt}}$.

Proof. Consider the following sequence of morphisms:

$$
\begin{tikzcd}
\mathcal{L}(X,A,B) \arrow[r, "\operatorname{ev}\times\pi"] \arrow[rr, bend right, "\pi"'] & X^r \times X^r \times \mathfrak{M}^{\mathrm{wt}}_{0,A,B} \arrow[r, "\pi_2"] & \mathfrak{M}^{\mathrm{wt}}_{0,A,B}
\end{tikzcd}
$$

This induces an exact triangle of contangent complexes:

$$(\operatorname{ev}\times\pi)^* L_{\pi_2} = \operatorname{ev}^* L_{X^r\times X^r} \to L_\pi \to L_{\operatorname{ev}\times\pi} \xrightarrow{[1]}$$

By definition the product virtual fundamental class on $\mathcal{L}(X,A,B)$ is induced by a perfect obstruction theory:

$$E_\pi \to L_\pi$$

On the other hand, there exists a natural map

$$\operatorname{ev}^* L_{X^r\times X^r} \to E_\pi$$

which we now describe. The (dual of the) complex E_π is concentrated in degrees 0 and 1 and is given fibrewise on $\mathcal{L}(X,A,B)$ by:

$$H^0(C,\mathcal{F}) \xrightarrow{0} H^1(C,\mathcal{F})$$

Here $\mathcal{F}$ is the sheaf on C defined by the following short exact sequence:

$$0 \to \mathcal{O}_C^{\oplus r} \to \bigoplus_\rho L_\rho \to \mathcal{F} \to 0 \tag{3}$$

On the other hand (the dual of) $\operatorname{ev}^* L_{X^r\times X^r}$ is concentrated in degree 0 and is given fibrewise as the direct sum

$$\bigoplus_q T_{u(q)} X$$

where the sum runs over all the "nodal" marked points q of the disconnected curve C. Since none of these are basepoints, we can restrict to a neighbourhood of each q where we obtain a map $u : C \to X$. Then the sequence (3) is nothing more than the pullback along u of the Euler sequence on X:

$$0 \to \mathcal{O}_X^{\oplus r} \to \bigoplus_\rho \mathcal{O}(D_\rho) \to T_X \to 0$$

From this it follows that $\mathcal{F} \cong u^* T_X$ in a neighbourhood of each q, and hence we can evaluate at q to obtain a map

$$H^0(C, \mathcal{F}) \to \bigoplus_q T_q X$$

as required. Dualising we obtain a morphism $\mathrm{ev}^* L_{X^r \times X^r} \to E_\pi$ which we can complete to an exact triangle, obtaining a diagram:

$$\begin{tikzcd}
\mathrm{ev}^* L_{X^r \times X^r} \arrow[r] \arrow[d, "\mathrm{Id}"] & E_\pi \arrow[r] \arrow[d] & E_{\mathrm{ev} \times \pi} \arrow[r, "{[1]}"] \arrow[d] & {} \\
\mathrm{ev}^* L_{X^r \times X^r} \arrow[r] & L_\pi \arrow[r] & L_{\mathrm{ev} \times \pi} \arrow[r, "{[1]}"] & {}
\end{tikzcd} \tag{4}$$

It is simple to check (using the fact that $\mathrm{ev}^* L_{X^r \times X^r}$ is concentrated in degree 0) that $E_{\mathrm{ev} \times \pi} \to L_{\mathrm{ev} \times \pi}$ is a perfect obstruction theory for the morphism $\mathrm{ev} \times \pi$. □

Lemma 5.6. *There exists a perfect obstruction theory for the morphism* $\mathrm{ev} \times \rho$ *which induces* $[\mathcal{K}(X, A, B)]^{\mathrm{virt}}$.

Proof. We can use the following cartesian diagram

$$\begin{tikzcd}
\mathcal{K}(X, A, B) \arrow[r, "h"] \arrow[d, "\mathrm{ev} \times \rho"] \arrow[dr, phantom, "\square"] & \mathcal{L}(X, A, B) \arrow[d, "\mathrm{ev} \times \pi"] \\
X^r \times \mathfrak{M}^{\mathrm{wt}}_{0,A,B} \arrow[r, "\Delta \times \mathrm{Id}"] & X^r \times X^r \times \mathfrak{M}^{\mathrm{wt}}_{0,A,B}
\end{tikzcd}$$

to pull back the obstruction theory $E_{\mathrm{ev} \times \pi} \to L_{\mathrm{ev} \times \pi}$ to an obstruction theory $E_{\mathrm{ev} \times \rho} \to L_{\mathrm{ev} \times \rho}$, which induces $[\mathcal{K}(X, A, B)]^{\mathrm{virt}}$ as defined earlier (see [**?**, Proposition 7.2]). □

Lemma 5.7. *There exists a perfect obstruction theory for the morphism* ρ *which induces* $[\mathcal{K}(X, A, B)]^{\mathrm{virt}}$.

Proof. As earlier we can consider the composition

$$\begin{tikzcd}
\mathcal{K}(X, A, B) \arrow[r, "\mathrm{ev} \times \rho"] \arrow[rr, bend right, "\rho"'] & X^r \times \mathfrak{M}^{\mathrm{wt}}_{0,A,B} \arrow[r, "\pi_2"] & \mathfrak{M}^{\mathrm{wt}}_{0,A,B}
\end{tikzcd}$$

and thus obtain an exact triangle:

$$\mathrm{ev}^* L_{X^r} \to L_\rho \to L_{\mathrm{ev} \times \rho} \xrightarrow{[1]}$$

We then compose this with $E_{\mathrm{ev}\times\rho} \to L_{\mathrm{ev}\times\rho}$ to obtain a morphism of exact triangles

$$\begin{tikzcd} \mathrm{ev}^* L_{X^r} \arrow[r] \arrow[d, "\mathrm{Id}"] & E_\rho \arrow[r] \arrow[d] & E_{\mathrm{ev}\times\rho} \arrow[r, "{[1]}"] \arrow[d] & {} \\ \mathrm{ev}^* L_{X^r} \arrow[r] & L_\rho \arrow[r] & L_{\mathrm{ev}\times\rho} \arrow[r, "{[1]}"] & {} \end{tikzcd} \tag{5}$$

and it is simple to check that $E_\rho \to L_\rho$ is a perfect obstruction theory for $\mathcal{K}(X, A, B)$ over $\mathfrak{M}^{\mathrm{wt}}_{0,A,B}$. □

Lemma 5.8 (Lemma 5.4, First Part)**.** *There exists a perfect obstruction theory for the morphism $\psi \circ \rho$ which induces $[\mathcal{K}(X, A, B)]^{\mathrm{virt}}$.*

Proof. Compose $E_\rho \to L_\rho$ with the connecting homomorphism $L_\rho \to \rho^* L_\psi[1]$ to obtain a morphism of exact triangles:

$$\begin{tikzcd} \rho^* L_\psi \arrow[r] \arrow[d, "\mathrm{Id}"] & E_{\psi\circ\rho} \arrow[r] \arrow[d] & E_\rho \arrow[r, "{[1]}"] \arrow[d] & {} \\ \rho^* L_\psi \arrow[r] & L_{\psi\circ\rho} \arrow[r] & L_\rho \arrow[r, "{[1]}"] & {} \end{tikzcd} \tag{6}$$

We wish to prove that $E_{\psi\circ\rho} \to L_{\psi\circ\rho}$ is a perfect obstruction theory. We claim that $\rho^* L_\psi$ is concentrated in degrees $[-1, 0]$. To see this, consider the exact triangle of cotangent complexes associated to the morphism ψ:

$$\psi^* L_{\mathfrak{M}^{\mathrm{wt}}_{0,n,\beta}} \to L_{\mathfrak{M}^{\mathrm{wt}}_{0,A,B}} \to L_\psi \xrightarrow{[1]}$$

The first two terms are concentrated in degrees $[0, 1]$ because they are the cotangent complexes of smooth Artin stacks. Therefore L_ψ is concentrated in degrees $[-1, 1]$. Furthermore, if we examine the long exact cohomology sequence near $\mathrm{h}^1(L_\psi)$ we find

$$\mathrm{h}^1(\psi^* L_{\mathfrak{M}^{\mathrm{wt}}_{0,n,\beta}}) \to \mathrm{h}^1(L_{\mathfrak{M}^{\mathrm{wt}}_{0,A,B}}) \to \mathrm{h}^1(L_\psi) \to 0$$

and hence we must show that the first map is surjective. But this is dual to the map which takes an infinitesimal automorphism of the disconnected curve to an infinitesimal automorphism of the corresponding connected curve (obtained by glueing together the "nodal" marked points). Since these two curves have the same components, this map is an isomorphism. Hence $\mathrm{h}^1(L_\psi) = 0$ as claimed.

Aside 5.9. Intuitively this makes sense, since the fibres of ψ are Deligne–Mumford and hence should have cotangent complex suppported in negative degrees.

It then follows formally that $E_{\psi\circ\rho} \to L_{\psi\circ\rho}$ is a perfect obstruction theory for $\mathcal{K}(X, A, B)$ over $\mathfrak{M}^{\mathrm{wt}}_{0,n,\beta}$. □

Lemma 5.10 (Lemma 5.4, Second Part)**.** *There exist a perfect obstruction theory for the morphism φ which fits into a compatible triple $(E_{\psi\circ\rho}, E_\pi, E_\varphi)$.*

Proof. To prove the second part, we must construct a map:

$$\varphi^* E_\pi \to E_{\psi\circ\rho}$$

By diagram (6) above, to construct such a morphism is the same thing as to construct a morphism $\varphi^* E_\pi \to E_\rho$ such that the composition

$$\varphi^* E_\pi \to E_\rho \to \rho^* L_\psi[1]$$

is zero. But again by diagram (5) a morphism $\varphi^* E_\pi \to E_\rho$ is the same thing as a morphism $\varphi^* E_\pi \to E_{\mathrm{ev}\times\rho}$ such that the composition

$$\varphi^* E_\pi \to E_{\mathrm{ev}\times\rho} \to \mathrm{ev}^* L_{X^r}[1]$$

is zero. Recall that $E_{\mathrm{ev}\times\rho} = h^* E_{\mathrm{ev}\times\pi}$ and hence by diagram (4) above we get a map $\varphi^* E_\pi \to h^* E_{\mathrm{ev}\times\pi}$ if we can produce a map $\varphi^* E_\pi \to h^* E_\pi$, or equivalently a map $h^* E_\pi^\vee \to \varphi^* E_\pi^\vee$.

We are now dealing with sheaves that we understand. Consider the following diagram:

$$\begin{tikzcd}
h^*\tilde{C} \arrow[r, "\nu"] \arrow[dr, "\eta"'] & \varphi^* C \arrow[r, "\alpha"] \arrow[d, "\rho"] \arrow[dr, phantom, "\square"] & C \arrow[d, "\pi"] \\
& \mathcal{K}(X,A,B) \arrow[r, "\varphi"] & \overline{Q}_{0,n}(X,\beta)
\end{tikzcd}$$

Here $\tilde{C}$ is the universal (disconnected) curve over $\mathcal{L}(X,A,B)$, which we have pulled back to $\mathcal{K}(X,A,B)$. On the other hand $\varphi^* C$ is isomorphic to the universal curve over $\mathcal{K}(X,A,B)$. Therefore the map $\nu : h^*\tilde{C} \to \varphi^* C$ is a partial normalisation map given by normalising the nodes which connect the "trunk" of the centipede to the "legs."

There are natural sheaves $\mathcal{F}$ and $\tilde{\mathcal{F}}$ on C and $h^*\tilde{C}$ respectively, such that

$$E_\pi^\vee = \mathrm{R}^\bullet\, \pi_* \mathcal{F}$$
$$h^* E_\pi^\vee = \mathrm{R}^\bullet\, \eta_* \tilde{\mathcal{F}}$$

from which we obtain:

$$\varphi^* E_\pi^\vee = \mathrm{R}^\bullet\, \rho_* \alpha^* \mathcal{F}$$

Now, since ν is a partial normalisation there is a short exact sequence of sheaves on $\varphi^* C$:

$$0 \to \mathcal{O}_{\varphi^* C} \to \nu_* \mathcal{O}_{h^*\tilde{C}} \to \mathcal{O}_q \to 0$$

where q is the locus of nodes connecting the trunk to the spine. On the other hand it is clear that

$$\nu_* \tilde{\mathcal{F}} = \nu_* \mathcal{O}_{h^*\tilde{C}} \otimes \alpha^* \mathcal{F}$$

so tensoring the above exact sequence with $\alpha^* \mathcal{F}$ we obtain:

$$0 \to \alpha^* \mathcal{F} \to \nu_* \tilde{\mathcal{F}} \to \alpha^* \mathcal{F}_q \to 0$$

(The fact that the morphism on the left is injective follows by applying the Snake Lemma to the short exact sequence defining $\mathcal{F}$.) To this we can apply $\mathrm{R}^\bullet\,\rho_*$ to obtain an exact triangle

$$\mathrm{R}^\bullet\,\rho_*\alpha^*\mathcal{F} \to \mathrm{R}^\bullet\,\eta_*\tilde{\mathcal{F}} \to \mathrm{R}^\bullet\,\rho_*\alpha^*\mathcal{F}_q \xrightarrow{[1]} \tag{7}$$

the first morphism of which is the map $h^*E_\pi^\vee \to \varphi^*E_\pi^\vee$ that was promised.

Now we must show that the composition

$$\varphi^*E_\pi \to E_{\mathrm{ev}\times\rho} \to \mathrm{ev}^*\,L_{X^r}[1]$$

is zero. This follows from the existence of the exact triangle (7) above. To be precise, this triangle can be rewritten as

$$h^*E_\pi^\vee \to \varphi^*E_\pi^\vee \to h^*\,\mathrm{ev}^*\,L^\vee_{X^r\times X^r} \xrightarrow{[1]}$$

and hence we obtain a diagram

$$\begin{tikzcd}
\varphi^*E_\pi \arrow[r] \arrow[d, "\mathrm{Id}"] & h^*E_\pi \arrow[r] \arrow[d] & h^*\,\mathrm{ev}^*\,L_{X^r\times X^r}[1] \arrow[r, "{[1]}"] \arrow[d] & {} \\
\varphi^*E_\pi \arrow[r] & E_{\mathrm{ev}\times\rho} = h^*E_{\mathrm{ev}\times\pi} \arrow[r] & \mathrm{ev}^*\,L_{X^r}[1] &
\end{tikzcd}$$

where the third vertical morphism is obtain by pulling back the natural map

$$\Delta^*_{X^r}L_{X^r\times X^r} \to L_{X^r}$$

along ev. Then the bottom row composes to zero because the top row does (being an exact triangle).

Hence we obtain a map $\varphi^*E_\pi \to E_\rho$. Finally we must show that the composition

$$\varphi^*E_\pi \to E_\rho \to \rho^*L_\psi[1]$$

is zero. To see this, consider the composition

$$\varphi^*E_\pi \to \varphi^*L_\pi \to L_{\psi\circ\rho}$$

where the second map is induced by the sequence of morphisms

$$\begin{tikzcd}
\mathcal{K}(X,A,B) \arrow[r, "\varphi"] \arrow[rr, bend right, "\psi\circ\rho"'] & \overline{Q}_{0,n}(X,\beta) \arrow[r, "\pi"] & \mathfrak{M}^{\mathrm{wt}}_{0,n,\beta}
\end{tikzcd}$$

and form the diagram:

$$\begin{tikzcd}
\varphi^*E_\pi \arrow[r] \arrow[d] & E_\rho \arrow[r] \arrow[d] & \rho^*L_\psi[1] \arrow[d, "\mathrm{Id}"] & \\
L_{\psi\circ\rho} \arrow[r] & L_\rho \arrow[r] & \rho^*L_\psi[1] \arrow[r, "{[1]}"] & {}
\end{tikzcd}$$

Then the top row composes to zero because the bottom row does.

Thus we finally obtain a map

$$\varphi^* E_\pi \to E_\rho$$

and a simple (but quite lengthy) diagram chase shows that the obstruction theory E_φ induced by this map is perfect. Hence we have a virtual pull-back morphism $\varphi^!_{\mathrm{v}}$ such that

$$\varphi^!_{\mathrm{v}}[\overline{Q}_{0,n}(X,\beta)]^{\mathrm{virt}} = [\mathcal{K}(X,A,B)]^{\mathrm{virt}}$$

as claimed. □

This completes the proof of Lemma 5.4 and provides us with all the obstruction-theoretic results we require.

There is one other way of obtaining $\mathcal{K}(X,A,B)$ as a pullback. Recall that we have an embedding $i : X \hookrightarrow \mathbb{P}^N$. Let $i_*B = (i_*\beta_0, \ldots, i_*\beta_r)$. Then we have a cartesian diagram:

$$\begin{tikzcd}
\mathcal{K}(X,A,B) \arrow[r, "\eta"] \arrow[d, "\varphi_X"] \arrow[dr, phantom, "\square"] & \mathcal{K}(\mathbb{P}^N, A, i_*B) \arrow[d, "\varphi_{\mathbb{P}^N}"] \\
\overline{Q}_{0,n}(X,\beta) \arrow[r, "k"] & \overline{Q}_{0,n}(\mathbb{P}^N, d)
\end{tikzcd}$$

Lemma 5.11. $k^!_v[\mathcal{K}(\mathbb{P}^N, A, i_*B)] = [\mathcal{K}(X,A,B)]^{\mathrm{virt}}$

Proof. It follows from the construction above that the obstruction theory $E_{\varphi_X} \to L_{\varphi_X}$ is equal to the pullback of $E_{\varphi^N_{\mathbb{P}}} \to L_{\varphi_{\mathbb{P}^N}}$ along η. So:

$$(\varphi_X)^!_{\mathrm{v}} = (\varphi_{\mathbb{P}^N})^!_{\mathrm{v}} : A_*(\overline{Q}(X)) \to A_*(\mathcal{K}(X))$$

Then commutativity of virtual pull-backs gives

$$\begin{aligned}
k^!_{\mathrm{v}}[\mathcal{K}(\mathbb{P}^N)] &= k^!_{\mathrm{v}}(\varphi_{\mathbb{P}^N})^!_{\mathrm{v}}[\overline{Q}(\mathbb{P}^N)] \\
&= (\varphi_{\mathbb{P}^N})^!_{\mathrm{v}} k^!_{\mathrm{v}}[\overline{Q}(\mathbb{P}^N)] \\
&= (\varphi_{\mathbb{P}^N})^!_{\mathrm{v}}[\overline{Q}(X)]^{\mathrm{virt}} \\
&= (\varphi_X)^!_{\mathrm{v}}[\overline{Q}(X)]^{\mathrm{virt}} \\
&= [\mathcal{K}(X)]^{\mathrm{virt}}
\end{aligned}$$

where we have used Lemma 5.4 twice. □

5.3. **An Intersection-Theoretic Lemma.** Shortly we will be dealing with morphisms which do not admit relative perfect obstruction theories, and as such we will require a slightly different version of the virtual pull-back morphism.

Suppose that we have a cartesian diagram

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

with Z smooth. Then the diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \times Y \arrow[d] \\ Z \arrow[r, "\Delta"] & Z \times Z \end{tikzcd}$$

defines a morphism:

$$\Delta^! : A_*(X \times Y) \to A_*(W)$$

In particular if we fix a class $[Y]^{\text{virt}} \in A_*(Y)$ we get a pullback morphism

$$\begin{aligned} f_{\text{v}}^! : A_*(X) &\longrightarrow A_*(W) \\ \gamma &\mapsto \Delta^!(\gamma \times [Y]^{\text{virt}}) \end{aligned}$$

which we call (by abuse of terminology) a **virtual pull-back** morphism.

Lemma 5.12. *The virtual pull-back morphism as defined above commutes with ordinary Gysin maps and with virtual pull-backs.*

Proof. First consider the case of ordinary Gysin maps. We must consider a cartesian diagram:

$$\begin{tikzcd} X'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & S \arrow[d, "f"] \\ X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y' \arrow[r] \arrow[d] & T \\ X \arrow[r, "k"] & Y & \end{tikzcd}$$

with k a regular embedding and T smooth. We assume we have fixed some class $[S]^{\text{virt}} \in A_*(S)$, so that the virtual pull-back $f_{\text{v}}^!$ is defined. We need to show that for all $\gamma \in A_*(Y')$:

$$k^! f_{\text{v}}^!(\gamma) = f_{\text{v}}^! k^!(\gamma)$$

We form the cartesian diagram:

$$\begin{tikzcd}
X'' \times Y'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & T \arrow[d, "\Delta"] \\
X' \times S \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y' \times S \arrow[r] \arrow[d] & T \times T \\
X \times S \arrow[r, "k \times Id"] \arrow[d] \arrow[dr, phantom, "\square"] & Y \times S \arrow[d] & \\
X \arrow[r, "k"] & Y &
\end{tikzcd}$$

Then we have

$$\begin{aligned}
k^! f_{\mathrm{v}}^!(\gamma) &= (k \times \mathrm{Id})^! \Delta^! (\gamma \times [S]^{\mathrm{virt}}) \\
&= \Delta^! (k \times \mathrm{Id})^! (\gamma \times [S]^{\mathrm{virt}}) \\
&= \Delta^! (k^!(\gamma) \times [S]^{\mathrm{virt}}) \\
&= f_{\mathrm{v}}^! k^!(\gamma)
\end{aligned}$$

which completes the proof. In the case where k is not a regular embedding but rather is equipped with a relative perfect obstruction theory, the same argument works with $k^!$ replaced by $k_{\mathrm{v}}^!$. $\square$

5.4. **Comb Loci Pull Back.** We are finally able to prove that the third term in the recursion formula pulls back along k.

Lemma 5.13. *For any α we have:*

$$k_v^! [D^Q_{\alpha,k}(\mathbb{P}^N | H, d)] = [D^Q_{\alpha,k}(X|Y, \beta)]^{\mathrm{virt}}$$

Proof. We have a cartesian diagram:

$$\begin{tikzcd}
D^Q_{\alpha,k}(X|Y,\beta) \arrow[r, "k"] \arrow[d] \arrow[dr, phantom, "\square"] & D^Q_{\alpha,k}(\mathbb{P}^N|H,d) \arrow[d] \\
\overline{Q}_{0,n}(X,\beta) \arrow[r, "k"] & \overline{Q}_{0,n}(\mathbb{P}^N, d)
\end{tikzcd}$$

We can write $D^Q_{\alpha,k}(X|Y,\beta)$ as the disjoint union of spaces

$$D^Q(X|Y, A, B, M) = \overline{Q}_{0, A_0 \cup \{q_1, \ldots, q_r\}}(Y, \beta_0) \times_{Y^r} \prod_{i=1}^{r} \overline{Q}_{0, \alpha^{(i)} \cup (m_i)}(X|Y, \beta_i)$$

where A and B are partitions of the marked points and curve class as before, and $M = (m_1, \ldots, m_r)$ records the intersection multiplicities at the nodes which connect the internal component to the external components (the spine of the comb to the teeth).

As usual this space has a virtual class induced by pulling back the virtual class from the total product:

$$E^{Q}(X|Y,A,B,M) = \overline{Q}_{0,A_0\cup\{q_1,\ldots,q_r\}}(Y,\beta_0)\times\prod_{i=1}^{r}\overline{Q}_{0,\alpha^{(i)}\cup(m_i)}(X|Y,\beta_i)$$

In order to make the following discussion readable, we will need to introduce some shorthand notation. We suppose that the data of A,B and M has been fixed, and set:

$$\begin{aligned}\mathcal{K}(X|Y) &:= D^{Q}(X|Y,A,B,M)\\ \mathcal{L}(X|Y) &:= E^{Q}(X|Y,A,B,M)\end{aligned}$$

We then have a cartesian diagram:

$$\begin{tikzcd}
\mathcal{K}(X|Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{L}(X|Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{L}(\mathbb{P}^N|H) \arrow[d, "\theta"] \\
\mathcal{K}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{L}(X) \arrow[r] \arrow[d] & \mathcal{L}(\mathbb{P}^N) \\
X^r \arrow[r, "\Delta"] & X^r\times X^r &
\end{tikzcd}$$

The morphism θ does not have a perfect relative obstruction theory. Nevertheless, we can define a virtual pull-back morphism as follows.

Returning to our diagram above, there is a natural class on $\mathcal{L}(\mathbb{P}^N|H)$ so we have a well-defined virtual pull-back $\theta^!_{\mathrm{v}}$, and by the very definition:

$$[\mathcal{L}(X|Y)]^{\mathrm{virt}} = \theta^!_{\mathrm{v}}[\mathcal{L}(X)]^{\mathrm{virt}}$$

Hence we obtain:

$$\begin{aligned}[\mathcal{K}(X|Y)]^{\mathrm{virt}} &= \Delta^![\mathcal{L}(X|Y)] = \Delta^!\theta^!_{\mathrm{v}}[\mathcal{L}(X)]^{\mathrm{virt}}\\ &= \theta^!_{\mathrm{v}}\Delta^![\mathcal{L}(X)]^{\mathrm{virt}} = \theta^!_{\mathrm{v}}[\mathcal{K}(X)]^{\mathrm{virt}}\end{aligned}$$

Finally, consider the diagram:

$$\begin{tikzcd}
\mathcal{K}(X|Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{K}(\mathbb{P}^N|H) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{L}(\mathbb{P}^N|H) \arrow[d, "\theta"] \\
\mathcal{K}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{K}(\mathbb{P}^N) \arrow[r] \arrow[d] & \mathcal{L}(\mathbb{P}^N) \\
\overline{Q}(X) \arrow[r, "k"] & \overline{Q}(\mathbb{P}^N) &
\end{tikzcd}$$

Then we have:

$$\begin{aligned}k^!_{\mathrm{v}}[\mathcal{K}(\mathbb{P}^N|H)] &= k^!_{\mathrm{v}}\theta^!_{\mathrm{v}}[\mathcal{K}(\mathbb{P}^N)] = \theta^!_{\mathrm{v}}k^!_{\mathrm{v}}[\mathcal{K}(\mathbb{P}^N)]\\ &= \theta^!_{\mathrm{v}}[\mathcal{K}(X)]^{\mathrm{virt}} = [\mathcal{K}(X|Y)]^{\mathrm{virt}}\end{aligned}$$

The lemma follows because $D^{Q}_{\alpha,k}(X|Y,\beta)$ and $D^{Q}_{\alpha,k}(\mathbb{P}^N|H,d)$ are the disjoint union of the $\mathcal{K}(X|Y)$ and $\mathcal{K}(\mathbb{P}^N|H)$. □

Theorem 5.14. *Let X be a smooth and proper toric variety and let $Y \subseteq X$ be a very ample hypersurface (not necessarily toric). Then, with the set-up as in the preceding discussion, we have an equality*

$$(\alpha_k\psi_k + ev_k^*[Y])[\overline{Q}_{0,\alpha}(X|Y,\beta)]^{\mathrm{virt}} = [\overline{Q}_{0,\alpha+e_k}(X|Y,\beta)]^{\mathrm{virt}} + [D^{Q}_{\alpha,k}(X|Y,\beta)]^{\mathrm{virt}}$$

in the Chow group of $\overline{Q}_{0,n}(X,\beta)$.

Proof. Pull-back the equation for $\mathbb{P}^N$ relative H using the map $k^!_{\mathrm{v}}$, applying Lemmas 5.2 and 5.13. □

6. The Quasimap Mirror Theorem

Assuming that quasimap invariants for $\mathbb{P}^r$ coincide with Gromov-Witten invariants on the nose, we get the following result.

Definition 6.1. For a complete intersection X in $\mathbb{P}^r$ and $d > 0$, let

$$I_d^X = (\mathrm{ev}_1)_* \left(\frac{1}{z - \psi_1} [\overline{Q}_{0,2}(X, d)]^{\mathrm{vir}} \right)$$

where ev_1 is always thought of as landing in $\mathbb{P}^r$.

Set also $I_0^X = \mathbb{1}_{\mathbb{P}^r}$ and $I^X = \sum_{d \geq 0} I_d^X q^d$.

Theorem 6.2. *Let $X \subseteq \mathbb{P}^4$ be a smooth quintic 3-fold. Then*

$$\sum_{d \geq 0} q^d \prod_{i=0}^{5d} (X + iz) I_d^{\mathbb{P}^4} = X P(q) I^X$$

where

$$P(q) = 1 + \sum_{d>0} d q^d \langle H^4, \mathbb{1}_{\mathbb{P}^4} \rangle_{\overline{Q}_{0,\{5d,0\}}(\mathbb{P}^4|X,d)} = 1 + \sum_{d>0} q^d \frac{(5d)!}{(d!)^5} \sum_{i=d+1}^{5d-1} \frac{1}{i}.$$

Proof. We'll write it for a general CY hypersurface in $i \colon X_a \hookrightarrow \mathbb{P}^r$, so the degree of X is $a = r + 1$. Notice that dual bases for $H^*(\mathbb{P}^r)$ are given by $T^i = H^i$ and $T_i = H^{r-i}$, while (induced) dual bases for $i^* H^*(\mathbb{P}^r)$ are $S^i = H^i$ and $S_i = \frac{1}{a} H^{r-i-1}$; the restriction of H^r is 0.

Define

$$I_{d,(m)}^{\mathbb{P}^r|X} = (\mathrm{ev}_1)_* \left(\frac{1}{z - \psi_1} [\overline{Q}_{0,\{m,0\}}(\mathbb{P}^r|X, d)]^{\mathrm{vir}} \right),$$

which coincides with the absolute I-function defined above for $m = 0$, and

$$J_{d,(m)}^{\mathbb{P}^r|X} = (\mathrm{ev}_1)_* \left(m [\overline{Q}_{0,\{m,0\}}(\mathbb{P}^r|X, d)]^{\mathrm{vir}} + \frac{1}{z - \psi_1} [D_m^Q(\mathbb{P}^r|X, d)]^{\mathrm{vir}} \right).$$

Then, by Gathmann's formula, we can prove that

$$(X + mz) I_{d,(m)}^{\mathbb{P}^r|X} = I_{d,(m+1)}^{\mathbb{P}^r|X} + J_{d,(m)}^{\mathbb{P}^r|X}, \tag{8}$$

from which it follows that

$$\prod_{i=0}^{ad} (X + iz) I_d^{\mathbb{P}^r} = \sum_{m=0}^{ad} \prod_{i=m+1}^{ad} (X + iz) J_{d,(m)}^{\mathbb{P}^r|X}.$$

It is now a matter of evaluating the RHS. Notice that $J_{d,(m)}^{\mathbb{P}^r|X}$ is made of two parts:

- the boundary terms: the strong stability condition for quasimaps and the choice of working with only two markings makes these boundary contributions particularly simple to compute. The shape of the source curve is that of a snake which the hypersurface cuts into two pieces, the internal one of degree $d^{(0)}$, and the external one of degree $d^{(1)}$ and multiplicity $m^{(1)}$ of contact with X, with the first marking point belonging to the internal component and the second to the external one.

The invariants which we need to consider will hence be of the form

$$\langle T^i\psi_1^j, S_i\rangle_{\overline{Q}_{0,2}(X,d^{(0)})}\langle S^i, \mathbb{1}_{\mathbb{P}^r}\rangle_{\overline{Q}_{0,\{m^{(1)},0\}}(\mathbb{P}^r|X,.)d^{(1)}}$$

A dimensional computation

$$\begin{aligned}
0 \le \operatorname{codim} S_i &= \dim X - \operatorname{codim} S^i \\
&= \dim X - \operatorname{vdim} \overline{Q}_{0,\{m^{(1)},0\}}(\mathbb{P}^r|X, d^{(1)}) \\
&= \dim X - (\dim \mathbb{P}^r - 3 + 2 - m^{(1)} - K_{\mathbb{P}^r}\cdot d^{(1)}\ell) \\
&= m^{(1)} - X\cdot d^{(1)}\ell + K_X\cdot d^{(1)}\ell \\
&= m^{(1)} - X\cdot d^{(1)}\ell \le 0
\end{aligned}$$

forces $S_1 = \mathbb{1}_X$ and $S^1 = \frac{1}{a}H^{r-1}$, $m^{(1)} = ad^{(1)}$ hence

$$m = \alpha_1 = X\cdot d^{(0)}\ell + m^{(1)} = ad,$$

so this doesn't show up but at the very end of the "increasing the multiplicity" process.

- The other term in $J^{\mathbb{P}^r|X}_{d,(m)}$ is $m(\mathrm{ev}_1)_*[\overline{Q}_{0,\{m,0\}}(\mathbb{P}^r|X,d)]^{\mathrm{vir}}$; notice that it only gets insertions from the cohomology of $\mathbb{P}^r$ (restricted to X). On the other hand

  $$\operatorname{vdim}\overline{Q}_{0,\{m,0\}}(\mathbb{P}^r|X,d) = r - 3 + 2 - m + d(r+1) \ge r-1$$

  because $m \le ad$; since the restriction of H^r to X vanishes, the only insertion that contributes is H^{r-1}, forcing the equality $m = ad$.

So, in the end, we see that equation 8 reduces to

$$\begin{aligned}
\prod_{i=0}^{ad}(X+iz)I_d^{\mathbb{P}^r} &= J^{\mathbb{P}^r|X}_{d,(ad)} \\
&= \sum_{i=0,\dots,r-1;j\ge 0}(da)\langle H^{r-1}, \mathbb{1}_{\mathbb{P}^r}\rangle_{\overline{Q}_{0,\{ad,0\}}(\mathbb{P}^r|X,d)}H \\
&+ \sum_{\substack{0<d^{(0)}<d\\ d^{(0)}+d^{(1)}=d}} z^{j+1}H^{r-i}\langle H^i\psi_1^j, \mathbb{1}_X\rangle_{\overline{Q}_{0,2}(X,d^{(0)})}(ad^{(1)})\langle\frac{1}{a}H^{r-1}, \mathbb{1}_{\mathbb{P}^r}\rangle_{\overline{Q}_{0,\{ad^{(1)},0\}}(\mathbb{P}^r|X,d^{(1)})} \\
&+ z^{j+1}H^{r-i}\langle H^i\psi_1^j, \mathbb{1}_X\rangle_{\overline{Q}_{0,2}(X,d)}
\end{aligned}$$

from which the first claim of the theorem is now evident (with a bit of rearranging, using $X = aH$ and $i^*(H^r) = 0$, so in the last line everything is divisible by H).

In order to evaluate $P(q)$, we use again Gathmann's algorithm, this time in the other direction, to go all the way back to $\mathbb{P}^r$; then we make use of our assumption that quasimap invariants and ordinary GW coincide for the projective space. So it starts:

$$[\overline{Q}_{0,\{ad,0\}}(\mathbb{P}^r|X,d)]^{\mathrm{vir}} = (X+(ad-1)\psi_1)[\overline{Q}_{0,\{ad-1,0\}}(\mathbb{P}^r|X,d)]^{\mathrm{vir}}-[D^Q_{ad}(\mathbb{P}^r|X,d)]^{\mathrm{vir}}$$

When looking at the boundary, the invariants that come into play are of the form

$$\langle H^{r-1}, S_i\rangle_{\overline{Q}_{0,2}(X,d^{(0)})}\langle S^i, \mathbb{1}_{\mathbb{P}^r}\rangle_{\overline{Q}_{0,\{a(d-d^{(0)})-1,0\}}(\mathbb{P}^r|X,d-d^{(0)})}$$

but notice that they must vanish by dimensional reasons, since

$$\operatorname{codim}(S_i) = \dim X - 3 + 2 - K_X \cdot d^{(0)}\ell - (r-1) = -1.$$

So

$$\begin{aligned}
&d\langle H^{r-1}, \mathbb{1}_{\mathbb{P}^r}\rangle_{\overline{Q}_{0,\{ad,0\}}(\mathbb{P}^r|X,d)} \\
&= d[\overline{Q}_{0,2}(\mathbb{P}^r,d)] \cap \mathrm{ev}_1^*(H^{r-1})\prod_{i=0}^{ad-1}(\mathrm{ev}_1^* X + i\psi_1) \\
&= d[\overline{Q}_{0,2}(\mathbb{P}^r,d)] \cap \left((da-1)!\psi_1^{ad}\,\mathrm{ev}_1^*(H^{r-1}) + a\left(\sum_{j=1}^{ad-1}\frac{(ad-1)!}{j}\right)\psi_1^{ad-1}\,\mathrm{ev}_1^*(H^r)\right) \\
&= d\left((da-1)!\langle\psi_1^{ad-1}H^{r-1}\rangle_{0,1,d} + a\left(\sum_{j=1}^{ad-1}\frac{(ad-1)!}{j}\right)\langle\psi_1^{ad-2}H^r\rangle_{0,1,d}\right)
\end{aligned}$$

using the equality of quasimap and GW invariants and the string equation for the latter. These numbers can be extracted from the J-function for $\mathbb{P}^r$

$$I_d^{\mathbb{P}^r} = \prod_{i=1}^{d}\frac{1}{H+i}$$

from which

$$\begin{aligned}
\langle\psi_1^{ad-2}\,\mathrm{ev}_1^*(H^r)\rangle_{0,1,d} &= \frac{1}{(d!)^{r+1}} \\
\langle\psi_1^{ad-1}\,\mathrm{ev}_1^*(H^{r-1})\rangle_{0,1,d} &= -(r+1)\frac{1}{(d!)^{r+1}}\sum_{j=1}^{d}\frac{1}{j}
\end{aligned}$$

and the second claim of the theorem follows. □

Appendix A. Functoriality of Quasimap Spaces

In the case of stable maps, a morphism $f : X \to Y$ induces a morphism between the corresponding moduli spaces

$$\overline{\mathcal{M}}_{g,n}(X,\beta) \to \overline{\mathcal{M}}_{g,n}(Y, f_*\beta)$$

given by composition with f (in general this induced morphism may involve stabilisation of the source curve). Because of this, the construction of the moduli space of stable maps is said to be **functorial**.

It is natural to ask whether the same holds for the moduli space of quasimaps. Since here the objects of the moduli space are not maps, we cannot simply compose with f, and indeed it is not immediately clear how we should proceed. In [**?**, Section 3.1] a definition is given when f is an embedding into a projective space; however, this uses the more general language of GIT quotients which we seek to avoid here. As such, we will provide an alternative (but entirely equivalent) construction in the setting of toric varieties, which also relaxes the conditions on the map f and the target Y.
[1]

Our approach uses the language of Σ–collections introduced by D. Cox. This approach is natural insofar as a quasimap is a generalisation of a Σ–collection. We will refer extensively to [**?**] and [**?**], which we recommend as an introduction for any readers unfamiliar with the theory.

Let X and Y be smooth and proper toric varieties with fans $\Sigma_X \subseteq N_X$ and $\Sigma_Y \subseteq N_Y$. Suppose we are given $f : Y \to X$ (which we do not assume to be a toric morphism). By [**?**, Theorem 1.1] the data of such a map is equivalent to a Σ_X–collection on Y:

$$((L_\rho, u_\rho)_{\rho \in \Sigma_X(1)}, (\varphi_{m_x})_{m_x \in M_X})$$

In addition, [**?**] allows us to describe line bundles on Y and their global sections in terms of the homogeneous coordinates $(z_\tau)_{\tau \in \Sigma_Y(1)}$. All of these observations are combined into the following theorem, which is so useful that we will state it here in its entirety:

Theorem A.1. [**?**, Theorem 2.2] *The data of a morphism $f : Y \to X$ is the same as the data of homogeneous polynomials*

$$P_\rho \in S^Y_{\beta_\rho}$$

for $\rho \in \Sigma_X(1)$, where $\beta_\rho \in \operatorname{Pic} Y$ and $S^Y_{\beta_\rho}$ is the corresponding graded piece of the Cox ring

$$S^Y = k[z_\tau : \tau \in \Sigma_Y(1)]$$

This data is required to satisfy the following two conditions:

(1) $\sum_{\rho \in \Sigma_X(1)} \beta_\rho \otimes n_\rho = 0$ *in* $\operatorname{Pic} Y \otimes N_X$.

(2) $(P_\rho(z_\tau)) \notin Z(\Sigma_X) \subseteq \mathbb{A}_k^{\Sigma_X(1)}$ *whenever* $(z_\tau) \notin Z(\Sigma_Y) \subseteq \mathbb{A}_k^{\Sigma_Y(1)}$.

Furthermore, two such sets of data (P_ρ) and (P'_ρ) correspond to the same morphism if and only if there exists a $\lambda \in \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Pic} X, \mathbb{G}_m)$ such that

$$\lambda(D_\rho) \cdot P_\rho = P'_\rho$$

[1]We should probably look a bit harder to see if the definition exists elsewhere.

for all $\rho \in \Sigma_X(1)$. Finally, if we define $\tilde{f}(z_\tau) = (P_\rho(z_\tau))$ then this defines a lift of f to the prequotients:

$$\begin{tikzcd}
\mathbb{A}_k^{\Sigma_Y(1)} \setminus Z(\Sigma_Y) \arrow[r, "\tilde{f}"] \arrow[d, "\pi"] & \mathbb{A}_k^{\Sigma_X(1)} \setminus Z(\Sigma_X) \arrow[d, "\pi"] \\
Y \arrow[r, "f"] & X
\end{tikzcd}$$

Aside A.2. Throughout this section we will stick to the notation established above; in particular we will use ρ to denote a ray in $\Sigma_X(1)$ and τ to denote a ray in $\Sigma_Y(1)$.

Recall our goal: given a map $f : Y \to X$ we wish to define a "push-forward" map:

$$f_* : \overline{Q}_{g,n}(Y, \beta) \to \overline{Q}_{g,n}(X, f_*\beta)$$

Consider therefore a quasimap $(C, (L_\tau, u_\tau)_{\tau \in \Sigma_Y(1)}, (\varphi_{m_Y})_{m_Y \in M_Y})$ with target Y. Pick data $(P_\rho)_{\rho \in \Sigma_X(1)}$ corresponding to the map f, as in the theorem above; we will later see that our construction does not depend on this choice.

The idea of the construction is as follows. Let us pretend for a moment that C is toric and that the quasimap is without basepoints, so that we have an actual morphism $C \to Y$. Then we can lift this morphism to the prequotient as in the following diagram

$$\begin{tikzcd}
\mathbb{A}_k^{\Sigma_C(1)} \setminus Z(\Sigma_C) \arrow[r, "(u_\tau)"] \arrow[d] & \mathbb{A}_k^{\Sigma_Y(1)} \setminus Z(\Sigma_Y) \arrow[r, "(P_\rho)"] \arrow[d] & \mathbb{A}_k^{\Sigma_X(1)} \setminus Z(\Sigma_X) \arrow[d] \\
C \arrow[r] & Y \arrow[r] & X
\end{tikzcd}$$

from which it follows that the composition $C \to Y \to X$ is given in homogeneous coordinates by:

$$(P_\rho((u_\tau)_{\tau \in \Sigma_Y(1)}))_{\rho \in \Sigma_X(1)}$$

In general of course C is not a toric variety and the quasimap is not basepoint-free. Nevertheless, as we will see, we can still make sense of the expression $P_\rho(u_\tau)$ as a section of a line bundle on C. This will allow us to define the pushforward of our quasimap.

Let us begin. For each ρ, P_ρ is a polynomial in the z_τ; we can write it as

$$P_\rho(z_\tau) = \sum_{\underline{a}} P_\rho^{\underline{a}}(z_\tau) = \sum_{\underline{a}} \mu_{\underline{a}} \prod_\tau z_\tau^{a_\tau} \tag{9}$$

where the sum is over a finite number of multindices $\underline{a} = (a_\tau) \in \mathbb{N}^{\Sigma_Y(1)}$ and the $\mu_{\underline{a}}$ are nonzero scalars. For each $\underline{a}$ consider the following line bundle on C:

$$\tilde{L}_\rho^{\underline{a}} = \bigotimes_\tau L_\tau^{\otimes a_\tau}$$

Then we may take the following section of $\tilde{L}^{\underline{a}}_{\rho}$:

$$\tilde{u}^{\underline{a}}_{\rho} = P^{\underline{a}}_{\rho}(u_\tau) = \mu_{\underline{a}} \prod_{\tau} u_\tau^{a_\tau}$$

Thus each of the terms $P^{\underline{a}}_{\rho}$ of P_ρ defines a section $\tilde{u}^{\underline{a}}_{\rho}$ of a line bundle $\tilde{L}^{\underline{a}}_{\rho}$. But what we want is a single section $\tilde{u}_\rho$ of a single line bundle $\tilde{L}_\rho$. This is where the isomorphisms φ_{m_Y} come in.

Recall that we have a short exact sequence:

$$0 \longrightarrow M_Y \xrightarrow{\theta} \mathbb{Z}^{\Sigma_Y(1)} \longrightarrow \operatorname{Pic} Y \longrightarrow 0 \tag{10}$$

Let $\underline{a}$ and $\underline{b}$ be multindices appearing in the sum (9) above. By the homogeneity of P_ρ we have

$$\sum_{\tau} a_\tau D_\tau = \beta_\rho = \sum_{\tau} b_\tau D_\tau$$

which is precisely the statement that in the above sequence $\underline{a}$ and $\underline{b}$ map to the same element of $\operatorname{Pic} Y$ (namely β_ρ). Hence there exists an $m_Y \in M_Y$ such that:

$$\theta(m_Y) = \underline{a} - \underline{b}$$

Now, the isomorphism φ_{m_Y} (contained in the data of our original quasimap) is a map:

$$\varphi_{m_Y} : \bigotimes_{\tau} L_\tau^{\otimes \langle m_Y, n_\tau \rangle} \cong \mathcal{O}_C$$

By definition, $\theta(m_Y) = (\langle m_Y, n_\tau \rangle)_{\tau \in \Sigma_Y(1)}$. But also $\theta(m_Y) = (a_\tau - b_\tau)_{\tau \in \Sigma_Y(1)}$. Hence we have:

$$\varphi_{m_Y} : \bigotimes_{\tau} L_\tau^{\otimes a_\tau} \cong \bigotimes_{\tau} L_\tau^{\otimes b_\tau}$$

In other words, we have well-defined canonical isomorphisms

$$\tilde{L}^{\underline{a}}_{\rho} \cong \tilde{L}^{\underline{b}}_{\rho}$$

for all $\underline{a}$ and $\underline{b}$. Let us choose one such $\underline{a}$ (it doesn't matter which); call it $\underline{a}^\rho$. We define:

$$\tilde{L}_\rho = \tilde{L}^{\underline{a}^\rho}_{\rho}$$

Then for all $\underline{b}$ we can use the above isomorphism to view $\tilde{u}^{\underline{b}}_{\rho}$ as a section of $\tilde{L}_\rho$. Summing all of these together we obtain a section $\tilde{u}_\rho$ of $\tilde{L}_\rho$, which we can write (with abuse of notation) as:

$$\tilde{u}_\rho = \sum_{\underline{a}} \mu_{\underline{a}} \prod_{\tau} u_\tau^{a_\tau}$$

Note that if we had made a different choice of $\underline{a}^\rho$ above the result would have been isomorphic.

Thus far we have constructed line bundles and sections $(\tilde{L}_\rho, \tilde{u}_\rho)_{\rho \in \Sigma_X(1)}$ on C. It remains to define the isomorphisms

$$\tilde{\varphi}_{m_X} : \otimes_\rho \tilde{L}_\rho^{\otimes \langle m_X, n_\rho \rangle} \cong \mathcal{O}_C$$

for all $m_X \in M_X$. The left hand side is:

$$\otimes_\rho \tilde{L}_\rho^{\otimes \langle m_X, n_\rho \rangle} = \otimes_\rho \left(\otimes_\tau L_\tau^{\otimes a_\tau^\rho} \right)^{\otimes \langle m_X, n_\rho \rangle} = \otimes_\tau L_\tau^{\otimes \left(\sum_\rho a_\tau^\rho \langle m_X, n_\rho \rangle \right)}$$

Now, for $m_Y \in M_Y$ we have isomorphisms $\varphi_{m_Y} : \otimes_\tau L_\tau^{\otimes \langle m_Y, n_\tau \rangle} \cong \mathcal{O}_C$. Hence, in order to construct $\tilde{\varphi}_{m_X}$ we need to find an m_Y such that

$$\langle m_Y, n_\tau \rangle = \sum_\rho a_\tau^\rho \langle m_X, n_\rho \rangle$$

for all $\tau \in \Sigma_Y(1)$ (we will then set $\tilde{\varphi}_{m_X} = \varphi_{m_Y}$). Consider therefore the short exact sequence (10). Recall that $\theta(m_Y) = (\langle m_Y, n_\tau \rangle)_{\tau \in \Sigma_Y(1)}$. Hence we need to show that

$$\left(\sum_\rho a_\tau^\rho \langle m_X, n_\rho \rangle \right)_{\tau \in \Sigma_Y(1)}$$

belongs to the image of θ, i.e. that it belongs to the kernel of the second map (notice that m_Y is then unique because θ is injective). This is equivalent to saying that

$$\sum_\tau \sum_\rho a_\tau^\rho \langle m_X, n_\rho \rangle D_\tau = 0 \in \operatorname{Pic} Y$$

Now, we have

$$\sum_\tau a_\tau^\rho D_\tau = \beta_\rho$$

so that the above sum becomes

$$\sum_\rho \langle m_X, n_\rho \rangle \beta_\rho = \left\langle m_X, \sum_\rho \beta_\rho \otimes n_\rho \right\rangle = \langle m_X, 0 \rangle = 0$$

where $\sum_\rho \beta_\rho \otimes n_\rho = 0$ by Condition (1) in Theorem A.1. So there does indeed exist a (unique) $m_Y \in M_Y$ such that $\langle m_Y, n_\tau \rangle = \sum_\rho a_\tau^\rho \langle m_X, n_\rho \rangle$, so that we can set:

$$\tilde{\varphi}_{m_X} = \varphi_{m_Y} : \bigotimes_\rho \tilde{L}_\rho^{\otimes \langle m_X, n_\rho \rangle} \cong \mathcal{O}_C$$

Thus, we have produced a quasimap with target X:

$$(C, (\tilde{L}_\rho, \tilde{u}_\rho)_{\rho \in \Sigma_X(1)}, (\tilde{\varphi}_{m_X})_{m_X \in M_X})$$

The proof that this construction does not depend on the choice of (P_ρ) is straightforward and is left to the reader.

It remains to demonstrate that the quasimap thus constructed is nondegenerate and stable. Nondegeneracy follows immediately from Condition

(2) in Theorem A.1. Put differently: the original quasimap defined a rational map $C \dashrightarrow Y$, whereas the new quasimap defines a rational map which is simply the composition $C \dashrightarrow Y \to X$. Therefore the set of basepoints is exactly the same.

Stability is a bit more tricky: it is here that we will end up having to put some extra conditions on the map f. First, notice that there are no rational tails because the source curve is unchanged.

Next let $C' \subseteq C$ be a component with exactly 2 special points. Then we need to show (see [**?**, Definition 3.1.1]) that the following line bundle has positive degree on C':

$$\tilde{\mathcal{L}} = \bigotimes_{\rho} \tilde{L}_{\rho}^{\otimes \tilde{\alpha}_{\rho}}$$

Here the $\tilde{\alpha}_{\rho}$ are defined by fixing a polarisation on X:

$$\mathcal{O}_X(1) = \bigotimes_{\rho} \mathcal{O}_X(\tilde{\alpha}_{\rho} D_{\rho})$$

The choice of polarisation makes no difference: a quasimap is stable with respect to one polarisation if and only if it is stable with respect to all others. In order to make use of the fact that the original quasimap to Y was stable, we will make the following assumption on f:

(1) there exists an ample line bundle $\mathcal{O}_X(1)$ on X such that $f^*\mathcal{O}_X(1)$ is ample on Y

This is satisfied if, for example, f is an embedding (which is the only case we will need in this paper). Given this assumption, we can set $\mathcal{O}_Y(1) = f^*\mathcal{O}_X(1)$. We then have:

$$\begin{aligned} \mathcal{O}_Y(1) &= \bigotimes_{\rho} f^*\mathcal{O}_X(D_{\rho})^{\otimes \tilde{\alpha}_{\rho}} = \bigotimes_{\rho} \mathcal{O}_Y\Big(\sum_{\tau} a_{\tau}^{\rho} D_{\tau}\Big)^{\otimes \tilde{\alpha}_{\rho}} \\ &= \bigotimes_{\rho} \bigotimes_{\tau} \mathcal{O}_Y(a_{\tau}^{\rho} \tilde{\alpha}_{\rho} D_{\tau}) = \bigotimes_{\tau} \mathcal{O}_Y(D_{\tau})^{\otimes \sum_{\rho} a_{\tau}^{\rho} \tilde{\alpha}_{\rho}} \end{aligned}$$

Thus for $\tau \in \Sigma_Y(1)$ we have $\alpha_{\tau} = \sum_{\rho} a_{\tau}^{\rho} \tilde{\alpha}_{\rho}$ and by stability of the original quasimap the line bundle $\mathcal{L} = \otimes_{\tau} L_{\tau}^{\otimes \alpha_{\tau}}$ has positive degree on C'. But:

$$\mathcal{L} = \bigotimes_{\tau} L_{\tau}^{\otimes \alpha_{\tau}} = \bigotimes_{\rho} \bigotimes_{\tau} \left(L_{\tau}^{\otimes a_{\tau}^{\rho}} \right)^{\otimes \tilde{\alpha}_{\rho}} = \bigotimes_{\rho} \tilde{L}_{\rho}^{\otimes \tilde{\alpha}_{\rho}} = \tilde{\mathcal{L}}$$

We have thus proven that $\tilde{\mathcal{L}}$ has positive degree on C', so the pushed-forward quasimap is stable. This completes the proof of the following.

Theorem A.3. *Let X and Y be smooth proper toric varieties and $f : Y \to X$ a morphism. Assume that f satisfies Condition (1) above. Then there exists a natural push-forward map*

$$f_* : \overline{\mathcal{Q}}_{g,n}(Y, \beta) \to \overline{\mathcal{Q}}_{g,n}(X, f_*\beta)$$

which does not modify the underlying prestable curves.

Aside A.4. We expect that such a map exists even if f does not satisfy Condition (1). However, in this case we will need to modify the underlying prestable curves by contracting unstable components. The same is true in the stable maps case.

Finally, let us describe how this push-forward morphism behaves when f is a nonconstant map $\mathbb{P}^r \to \mathbb{P}^N$, since we will make use of this later. Write f in homogeneous coordinates as:

$$f[z_0, \ldots, z_r] = [f_0(z_0, \ldots, z_r), \ldots, f_N(z_0, \ldots, z_r)]$$

where the f_i are all homogeneous of degree a. Then given a quasimap with target $\mathbb{P}^r$

$$(C, L, u_o, \ldots, u_r)$$

the pushed-forward quasimap with target $\mathbb{P}^N$ is:

$$(C, L^{\otimes a}, f_0(u_0, \ldots, u_r), \ldots, f_N(u_0, \ldots, u_r))$$

(This is stable as long as $a > 0$, which is precisely when f satsfies Condition (1) above.)

Appendix B. The Comparison Morphism

We summarise the existence of the comparison morphism for $\mathbb{P}^r$ and how it implies that GW and quasimap invariants of projective space coincide. This has been proven in [**?**, Theorem 3] and [**?**, Section 4.3] (but see also [**?**, Proposition 4.1] and [**?**, Theorem 7.1] for inspiration). We shall try to clarify as many details as possible, for our own benefit and, hopefully, that of the novice reader.

In order to give a morphism $\chi\colon \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \to \overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d)$ we need to be able to canonically associate a family of quasimaps on a base S to any family of stable maps on the same base.

The pointwise construction is the following: a stable map has no base points, so the only thing that might prevent it from being a stable quasimap is the presence of rational tails (of positive degree, by the stable maps stability condition). Let $C = C^{(0)} \sqcup_{q_i} R_i$ be the source curve; the rational tail R_i has degree d_i and is joined to the permanent curve $C^{(0)}$ at the node q_i, which is the only special point on R_i; hence all the markings belong to $C^{(0)}$. The map to $\mathbb{P}^r$ is equivalent to the data of a line bundle $L = f^*\mathcal{O}_{\mathbb{P}^r}(1)$ on C and $r+1$ sections $s_0, \ldots, s_r$ thereof. We associate to such a stable map the quasimap $(C^{(0)}, \mathbf{x}; L_{|C^{(0)}} \otimes \mathcal{O}_{C^{(0)}}(\sum_i d_i q_i); \hat{s}_0, \ldots, \hat{s}_r)$, where $\hat{s}_j$ is the restriction of s_j to $C^{(0)}$, seen as a section of $L_{|C^{(0)}} \otimes \mathcal{O}_{|C^{(0)}}(\sum_i d_i q_i)$ through the inclusion $L_{|C^{(0)}} \hookrightarrow L_{|C^{(0)}} \otimes \mathcal{O}_{C^{(0)}}(\sum_i d_i q_i)$. Notice that the resulting quasimap has a base-point of order d_i at q_i.

The construction in families requires us to find a line bundle on the universal curve that is trivial on the rational tails and relatively ample elsewhere. This can be performed at the level of Picard stacks: let $\mathfrak{Pic}^{d,\mathrm{st}}_{g,n}$ be the open substack of $\mathfrak{Pic}(\pi\colon \mathfrak{C}_{g,n} \to \mathfrak{M}_{g,n})$ obtained by requiring that the total degree of the line bundle is d, the multi-degree is nonnegative and $\mathcal{L} \otimes \omega_\pi^{\log}$ is ample relative to π, where $\mathcal{L}$ is the universal line bundle. Let T^δ be the locus in the universal curve over $\mathfrak{Pic}_{g,n}^{d,\mathrm{st}}$ spanned by rational tails on which $\mathcal{L}$ has degree δ; this is a Cartier divisor by deformation theory and smoothness of the stack $\mathfrak{C}_{\mathfrak{Pic}}$. Notice that T^{δ_0} and T^{δ_1} (say $\delta_0 < \delta_1$) do intersect in a stratum of codimension 1 in both of them, where the rational tail splits into two rational components, the furthest from $C^{(0)}$ having degree δ_0.

[FIGURE]

Claim: the line bundle $\mathcal{M} = \mathcal{L} \otimes \omega_\pi^{\log} \otimes \bigotimes_{0<\delta\leq d} \mathcal{O}_{\mathfrak{C}}((\delta-1)T^\delta)$ on $\mathfrak{C}_{\mathfrak{Pic}}$ has degree 0 on every component of every rational tail, and is π-relatively ample elsewhere.

Proof. Consider a curve $C^{(0)} \sqcup_q R$ with a rational tail of degree δ, such that R consists of n many components $R^{(1)}, \ldots, R^{(n)}$, each of degree $\delta^{(1)}, \ldots, \delta^{(n)}$ respectively, numbered from the closest to the farthest from $C^{(0)}$; set $T_i = \bigcup_{j=i}^n R_j$ and $\epsilon_i = \delta - 1 - \sum_{j=1}^{i-1} \delta_j$.

[FIGURE]

A general one-parameter family in $\mathfrak{Pic}_{g,n}^{d,\mathrm{st}}$ will give us a smoothing of such a curve; the universal curve over such a family is a normal surface S; we can compute the degree of the restriction of $\mathcal{M}$ to components of the central fiber of this family by first restricting $\mathcal{M}$ to S, and then using intersection theory on this normal surface.

Notice that restricting $\bigotimes_{0<\delta\leq d} \mathcal{O}_{\mathfrak{C}}((\delta-1)T^\delta)$ to this family gives $\mathcal{O}_S(\sum_{j=1}^n \epsilon_j T_j)$. Since $R^{(i)}$ is a (-2)-curve for $i = 1, \ldots, n-1$, and $R^{(n)}$ is a (-1)-curve, we get

$$R^{(i)}.T_j = \begin{cases} 0, & \text{for } j < i \\ -1, & \text{for } j = i \\ 1, & \text{for } j = i+1 \\ 0 & \text{for } j > i+1 \end{cases}$$

hence $\deg(\mathcal{M}_{|R^{(i)}}) = \delta^{(i)} - \epsilon_i + \epsilon_{i+1} = 0$ for $i = 1 \ldots, n-1$, while for $i = n$ it is $\delta^{(n)} - 1 - \epsilon_n = 0$, as $\omega^{\log}$ is trivial on the (-2) curves and has degree -1 on $R^{(n)}$. The last assertion of the claim follows from the stability condition and the fact that $\mathcal{O}_{\mathfrak{C}}(T^\delta)$ is effective when restricted to $C^{(0)}$. □

By taking the relative Proj construction we obtain another curve $\hat{\mathfrak{C}} = \underline{\mathrm{Proj}}_{\mathfrak{Pic}} \left(\bigoplus_{k\geq 0} \pi_* \mathcal{M}^{\otimes k} \right)$ over $\mathfrak{Pic}_{g,n}^{d,\mathrm{st}}$, with a map ρ that contracts the rational tails

$$\begin{tikzcd} \mathfrak{C}_{\mathfrak{Pic}} \arrow[r, "\rho"] \arrow[dr, "\pi"'] & \hat{\mathfrak{C}} \arrow[d, "\pi'"] \\ & \mathfrak{Pic}^{d,\mathrm{st}}_{g,n} \end{tikzcd}$$

It is flat because it is a family of genus g curves over a reduced base. Furthermore, it can be checked by cohomology and base-change [**?**, Theorem 12.11][**?**, Corollary 1.5] (notice that the fibers of ρ are either points or rational curves) that $\hat{\mathcal{L}} = \rho_*\left(\mathcal{L} \otimes \bigotimes_{0<\delta\leq d} \mathcal{O}_{\mathfrak{C}}(\delta T^\delta)\right)$ is a line bundle on $\hat{\mathfrak{C}}$ of degree d relative to π' (such that $\rho^*\hat{\mathcal{L}} \simeq \mathcal{L} \otimes \bigotimes_{0<\delta\leq d} \mathcal{O}_{\mathfrak{C}}(\delta T^\delta)$), hence the universal property gives us a commutative diagram (with Cartesian square)

$$\begin{tikzcd} \mathfrak{C}_{\mathfrak{Pic}} \arrow[r, "\rho"] \arrow[dr, "\pi"'] & \hat{\mathfrak{C}} \arrow[r] \arrow[d, "\pi'"] \arrow[dr, phantom, "\square"] & \mathfrak{C}_{\mathfrak{Pic}} \arrow[d, "\pi"] \\ & \mathfrak{Pic}^{d,\mathrm{st}}_{g,n} \arrow[r, "\chi'"] & \mathfrak{Pic}^{d,\mathrm{st}}_{g,n} \end{tikzcd}$$

The very same construction, with the line bundles pulled back from the Picard stack, and the sections of $\mathcal{L}$ seen as sections of $\mathcal{L} \otimes \bigotimes_{0<\delta\leq d} \mathcal{O}_{\mathfrak{C}}(\delta T^\delta)$ through the inclusion of line bundles ($\mathcal{O}_{\mathfrak{C}}(T^\delta)$ is effective), and descended to sections of $\hat{\mathcal{L}}$ on $\hat{\mathfrak{C}}$ gives us the comparison morphism $\chi\colon \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \to \overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d)$, fitting in a commutative diagram

$$\begin{tikzcd} \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \arrow[r, "\chi"] \arrow[d, "\nu_{\mathcal{M}}"] & \overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d) \arrow[d, "\nu_{\mathcal{Q}}"] \\ \mathfrak{Pic}^{d,\mathrm{st}}_{g,n} \arrow[r, "\chi'"] & \mathfrak{Pic}^{d,\mathrm{st}}_{g,n} \end{tikzcd}$$

and, as before,

$$\begin{tikzcd} \mathcal{C}_{\mathcal{M}} \arrow[r, "\rho"] \arrow[dr, "\pi_{\mathcal{M}}"'] & \hat{\mathcal{C}} = \chi^*\mathcal{C}_{\mathcal{Q}} \arrow[r] \arrow[d, "\hat{\pi}"] \arrow[dr, phantom, "\square"] & \mathcal{C}_{\mathcal{Q}} \arrow[d, "\pi_{\mathcal{Q}}"] \\ & \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \arrow[r, "\chi"] & \overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d) \end{tikzcd}$$

The comparison between virtual fundamental classes is best outlined in the arXiv version of [**?**, Remark 5.20]. Call $\nu'_{\mathcal{M}} = \chi' \circ \nu_{\mathcal{M}}$. We may endow it with an obstruction theory by means of

$$\begin{tikzcd} \nu_{\mathcal{M}}^*\mathbb{L}_{\chi'} \arrow[r] \arrow[d] & \mathbb{E}_{\nu'_{\mathcal{M}}} \arrow[r] \arrow[d] & \mathbb{E}_{\nu_{\mathcal{M}}} \arrow[r, "{[1]}"] \arrow[d] & {} \\ \nu_{\mathcal{M}}^*\mathbb{L}_{\chi'} \arrow[r] & \mathbb{L}_{\nu'_{\mathcal{M}}} \arrow[r] & \mathbb{L}_{\nu_{\mathcal{M}}} \arrow[r, "{[1]}"] & {} \end{tikzcd}$$

Notice that χ' is a morphism (not of DM type) between smooth Artin stacks, hence we can only deduce that $\mathbb{L}_{\chi'}$ is supported in $[-1,1]$. It is therefore easily seen that $\mathbb{E}_{\nu'_{\mathcal{M}}}$ is also supported in $[-1,1]$; in order to show that it is actually a perfect obstruction theory, consider the long exact sequence

$$\begin{aligned} 0 \to h^{-1}\nu_{\mathcal{M}}^*\mathbb{L}_{\chi'} &\to h^{-1}\mathbb{E}_{\nu'_{\mathcal{M}}} \to h^{-1}\mathbb{E}_{\nu_{\mathcal{M}}} \\ \to h^{0}\nu_{\mathcal{M}}^*\mathbb{L}_{\chi'} &\to h^{0}\mathbb{E}_{\nu'_{\mathcal{M}}} \to h^{0}\mathbb{E}_{\nu_{\mathcal{M}}} \\ \to h^{1}\nu_{\mathcal{M}}^*\mathbb{L}_{\chi'} &\to h^{1}\mathbb{E}_{\nu'_{\mathcal{M}}} \to 0 \end{aligned}$$

and observe that, dually, $h^{-1}\nu_{\mathcal{M}}^*\mathbb{T}_{\chi'}$ injects into $h^0\mathbb{E}^\vee_{\nu_{\mathcal{M}}} \simeq h^0\mathbb{T}_{\nu_{\mathcal{M}}}$, because every infinitesimal automorphism of the rational tail induces a nontrivial deformation of the stable map (since the degree of the latter is positive on every component of the rational tail); we conclude that $h^1\mathbb{E}_{\nu'_{\mathcal{M}}} = 0$.

Claim: there is a morphism of obstruction theories $\chi^*\mathbb{E}_{\nu_Q} \to \mathbb{E}_{\nu_{\mathcal{M}}}$ [**?**, Lemma 4.19].

Dually, $\mathbb{E}^\vee_{\nu_{\mathcal{M}}} = R^\bullet\pi_{\mathcal{M}*}\mathcal{L}^{\oplus r+1} = R^\bullet\hat{\pi}_*(\rho_*\mathcal{L}^{\oplus r+1})$, while, by cohomology and base-change, $\chi^*\mathbb{E}^\vee_{\nu_Q} = R^\bullet\hat{\pi}_*(\hat{\mathcal{L}}^{\oplus r+1})$, where $\hat{\mathcal{L}} = \rho_*\left(\mathcal{L}\otimes\bigotimes_{0<\delta\leq d}\mathcal{O}_{\mathfrak{C}}(\delta T^\delta)\right)$, so $\mathbb{E}^\vee_{\nu_{\mathcal{M}}} \to \chi^*\mathbb{E}^\vee_{\nu_Q}$ comes from the inclusion of line bundles on $\mathcal{C}_{\mathcal{M}}$

$$\mathcal{L} \hookrightarrow \mathcal{L}\otimes\bigotimes_{0<\delta\leq d}\mathcal{O}_{\mathfrak{C}}(\delta T^\delta).$$

Claim: this morphism factors through $\mathbb{E}_{\nu'_{\mathcal{M}}}$.

$$\begin{tikzcd} & \chi^*\mathbb{E}_{\nu_Q} \arrow[dl, dashed, "\exists?"'] \arrow[d] \arrow[dr, "\phi"] & \\ \mathbb{E}_{\nu'_{\mathcal{M}}} \arrow[r] & \mathbb{E}_{\nu_{\mathcal{M}}} \arrow[r] & \nu_{\mathcal{M}}^*\mathbb{L}_{\chi'}[1] \end{tikzcd}$$

In order to prove that the dashed arrow exists, we need to show that ϕ is the zero map. Dually, we look at $\nu_{\mathcal{M}}^*\mathbb{T}_{\chi'}[-1] \xrightarrow{\phi^\vee} R^\bullet\hat{\pi}_*(\hat{\mathcal{L}}^{\oplus r+1})$. Notation: call R the rational tail, joined at the rest of the curve (which we denote by $(C^{(0)},\mathbf{p})$ as a marked curve), at the node q, which we may occasionally think of as a (smooth) point on $C^{(0)}$. We claim that:

- $h^0(\phi^\vee)$ is zero because: the LHS involves automorphisms of the rational tail that leave $C^{(0)}$ fixed, while the RHS involves deformations of $C^{(0)}$, so there is no possible interference.
- $h^1(\phi^\vee)$ is zero because: this is slightly awkward. There are two types of possible contributions to the LHS. They correspond to either moving the node q along $C^{(0)}$, or smoothing it. The former appears in the relative tangent of χ' only if the marked curve $(C^{(0)},\mathbf{p})$ has no automorphisms that may "move q back", i.e. $(C^{(0)},\mathbf{p})$ is a stable pointed curve. The latter matters only if $(C^{(0)},q,\mathbf{p})$ has no moduli, i.e. $(C^{(0)},\mathbf{p})$ is a rational tail with less than 3 markings. I will try

to justify why the first type vanishes under $h^1(\phi^\vee)$, and leave the second type because I do not understand it as yet. Look at the long exact sequence

$$\begin{aligned} 0 \to \operatorname{Hom}(\Omega_{C^{(0)}}, \mathcal{O}_{C^{(0)}}(-q - \textstyle\sum p_i)) &\to \operatorname{Hom}(\Omega_{C^{(0)}}, \mathcal{O}_{C^{(0)}}(-\textstyle\sum p_i)) \to \\ T_{C^{(0)},q} \to \operatorname{Ext}^1(\Omega_{C^{(0)}}, \mathcal{O}_{C^{(0)}}(-q - \textstyle\sum p_i)) &\to \operatorname{Ext}^1(\Omega_{C^{(0)}}, \mathcal{O}_{C^{(0)}}(-\textstyle\sum p_i)) \to 0 \end{aligned}$$

We are interested in what happens to

$$\frac{T_{C^{(0)},q}}{\operatorname{Im}\left(\operatorname{Hom}(\Omega_{C^{(0)}}, \mathcal{O}_{C^{(0)}}(-\sum p_i))\right)}$$

under $h^1(\phi^\vee)$. If we can show that $h^1(\phi^\vee)$ factors through $\operatorname{Ext}^1(\Omega_{C^{(0)}}, \mathcal{O}_{C^{(0)}}(-\sum p_i))$ we are in business. Indeed the natural maps

$$\begin{tikzcd} \mathrm{Def}_L \arrow[r] \arrow[d] & \mathrm{Def}_{(C,L)} \arrow[r] \arrow[d] & \mathrm{Def}_C \arrow[d] \\ H^1(\mathcal{O}_C) \arrow[r] & H^1(L^{\oplus r+1}) \arrow[r] & H^1(f^*T_{\mathbb{P}^r}) \end{tikzcd}$$

show that $h^1(\phi^\vee)$ factors through

$$\operatorname{Ext}^1(\Omega_{C^{(0)}}, \mathcal{O}_{C^{(0)}}(-q-\textstyle\sum p_i)) \to \operatorname{Ext}^1(\Omega_{C^{(0)}}, \mathcal{O}_{C^{(0)}}) \to \operatorname{Ext}^1(f^*\Omega_{\mathbb{P}^r}, \mathcal{O}_{C^{(0)}}) \simeq H^1(f^*T_{\mathbb{P}^r}).$$

- $h^2(\phi^\vee)$ is zero because: $\mathbb{E}^\vee_{\nu'_{\mathcal{M}}}$ is supported in $[0,1]$.

Now the cone $C(\phi)$ gives an obstruction theory relative to χ. A priori, it is supported in $[-2,0]$. By the octahedral axiom

$$\begin{tikzcd} \chi^*\mathbb{E}_{\nu_Q} \arrow[d, "\phi"] \arrow[dr, "\phi'"] & & \\ \mathbb{E}_{\nu'_{\mathcal{M}}} \arrow[r] \arrow[dd] & \mathbb{E}_{\nu_{\mathcal{M}}} \arrow[r] \arrow[d] & \nu^*_{\mathcal{M}}\mathbb{L}_{\chi'}[1] \\ & C(\phi') \arrow[ur] & \\ C(\phi) \arrow[ur] & & \end{tikzcd}$$

it is enough to observe that $C(\phi')$ is supported in $[-1,0]$ [?, Lemma 4.20] and that $\nu^*_{\mathcal{M}}\mathbb{L}_{\chi'}[1]$ is supported in degrees $[-2,0]$, in order to conclude that $C(\phi) = \mathbb{E}_\chi$ is a perfect obstruction theory. The conclusion that

$$\chi_*[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)]^{\mathrm{vir}} = [\overline{Q}_{g,n}(\mathbb{P}^r,d)]^{\mathrm{vir}}$$

follows from the connectedness of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)$ [?] (hence of $\overline{Q}_{g,n}(\mathbb{P}^r,d)$) and an application of the virtual push-forward theorem [?, Proposition 4.21].

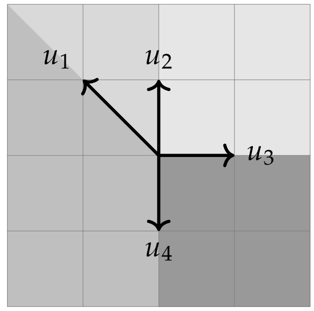

FIGURE 1. Toric fan for $\mathbb{F}_1$.

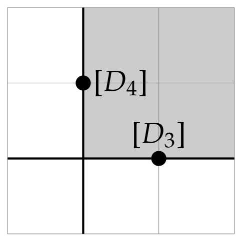

FIGURE 2. Nef cone $\mathrm{Nef}(\mathbb{F}_1)$.

We shall now explain with an example the reason why a naive attempt to extend the comparison morphism to a general toric variety fails. The problem in a nutshell is that not all toric divisors are nef: a rational tail contained in a divisor which is not nef may have negative degree $-d$ with respect to the corresponding line bundle; when contracting such a rational tail, we shall take the line bundle $L(-dq)$, but what to do with the sections? We would like to divide them by z^d, where z is a local coordinate around q, but no condition forces such a divisibility to happen. Otherwise said, there is now an inclusion $L_{|C^{(0)}}(-dq) \hookrightarrow L_{|C^{(0)}}$, but the (restriction of the) given sections of L do not necessarily live in the image of $H^0(C^{(0)}, L_{|C^{(0)}}(-dq)) \hookrightarrow H^0(C^{(0)}, L_{|C^{(0)}})$.

A concrete example is found when looking at the Hirzebruch surface $\mathbb{F}_1 = \mathrm{Bl}_p\, \mathbb{P}^1$.

$\mathrm{Pic}(\mathbb{F}_1)$ is generated by $[D_3]$ and $[D_4]$, with relations $[D_1] = [D_3]$ and $[D_2] = [D_4] - [D_3]$, and the intersection table is given by

$$\begin{cases} D_3^2 = 0 \\ D_3.D_4 = 0 \\ D_4^2 = 1 \end{cases}$$

When thinking of $\mathbb{F}_1$ as a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$, C_1 and C_3 represent the fibers of the bundle (over the toric points of $\mathbb{P}^1$), while C_4 (resp. C_2) is the zero/positive (resp. infinity/negative) section; when thinking of $\mathbb{F}_1$ as

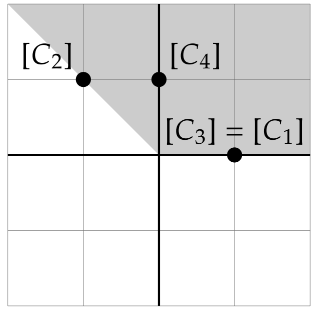

FIGURE 3. Mori cone $\overline{\mathrm{NE}}(\mathbb{F}_1)$.

$\mathrm{Bl}_p\,\mathbb{P}^1$, C_2 is the exceptional divisor, C_4 is the toric line not passing through p, and C_1, C_3 are the strict transforms of the toric lines through p.

Let us look at $\overline{\mathcal{M}}_{0,2}(\mathbb{F}_1, [C_4])$. Since $[C_4] = [C_2] + [C_3]$, there are going to be maps of the following sort: the source curve is reducible $R_1 \sqcup_q R_2$, R_1 is mapped isomorphically to a fiber (i.e. in class $[C_3]$) and R_2 is mapped isomorphically to C_2, all the markings belong to R_1. So R_2 is a rational tail and deserves to be contracted. Notice that the line bundle $\mathcal{O}(D_2)$ has degree -1 on R_2 (and 1 on R_1). In this case everything works well because the corresponding section $u_{2|R_1}$ must vanish at the node, so we can divide it by a chosen (once for all toric line bundles) section of $\mathcal{O}_{R_1}(q)$.

Consider now $\overline{\mathcal{M}}_{0,2}(\mathbb{F}_1, 2[C_2] + [C_3])$. Certainly there are going to be maps similar to the ones described above, with R_2 now covering C_2 2: 1. The point is that $\mathcal{O}(D_2)$ has degree -2 on R_2, but $u_{2|R_1}$ doesn't have to vanish at the node of order 2, so we are in trouble. Something is going on here: in this case there is a boundary component where the map is of the type that we have just described, and the requirement that $u_{2|R_1}$ vanishes of order 2 at the node defines precisely the intersection with the main component. Check this. Could we possibly exploit this phenomenon to define a smaller compactification, possibly even smaller than quasimaps?

APPENDIX C. THE QUASIMAP STRING EQUATION FOR $\mathbb{P}^r$

The string equation for the Gromov–Witten invariants of a smooth projective variety X is given by

$$\langle \mathbb{1}, \gamma_1\psi^{a_1}, \ldots, \gamma_n\psi^{a_n}\rangle^X_{g,n+1,\beta} = \sum_{i=1}^{n} \langle \gamma_1\psi^{a_1}, \ldots, \gamma_{i-1}\psi^{a_{i-1}}, \gamma_i\psi^{a_i-1}, \gamma_{i+1}\psi^{a_{i+1}} \ldots, \gamma_n\psi^{a_n}\rangle^X_{g,n,\beta}$$

where $\mathbb{1} \in H^*(X)$ is the unit class (by convention any term involving a negative power of ψ is set to zero). Since Gromov–Witten invariants and quasimap invariants coincide for $X = \mathbb{P}^r$ ([**?**, Section 5.4]) we know that the same equation holds for quasimap invariants to $\mathbb{P}^r$.

Nevertheless, it would be illuminating to have a direct proof of this statement, without relying on the equivalence with Gromov–Witten theory. Amongst other things, such a proof would necessarily involve some non-trivial intersection computations in the cohomology ring of the quasimap space, which would be of independent interest.

The proof of the classical string equation (for Gromov–Witten invariants) relies on three key lemmas involving certain codimension–1 classes on the moduli space of stable maps. Let

$$\pi : \overline{\mathcal{M}}_{g,n+1}(X,\beta) \to \overline{\mathcal{M}}_{g,n}(X,\beta)$$

denote the contraction map given by forgetting the last marked point and stabilising. Then we have:

(1) $\psi_i = \pi^*\psi_i + D_{i,n+1}$
(2) $\psi_i \cdot D_{i,n+1} = 0$
(3) $D_{i,n+1} \cdot D_{j,n+1} = 0$ for $i \neq j$

Here $D_{i,n+1}$ is the locus of stable maps $(C, x_1, \ldots, x_{n+1}, f)$ such that we can split up C into two pieces, $C = C' \cup C''$ (intersecting in a single node) such that C'' has degree 0 and contains only the markings x_i and x_{n+1}.

[FIGURE]

We would like to have some analogue of these results in the quasimap setting. In fact, equations (2) and (3) carry over without difficulty. Equation (1), on the other hand, is rather more delicate.

In the stable map setting, equation (1) is proved by considering the following diagram

$$\begin{tikzcd}
\mathcal{C}_{g,n+1} \arrow[r, "\rho"] \arrow[dr, "\psi"'] & \pi^*\mathcal{C}_{g,n} \arrow[r, "\alpha"] \arrow[d, "\eta"] & \mathcal{C}_{g,n} \arrow[d, "\varphi"] \\
& \overline{\mathcal{M}}_{g,n+1}(X,\beta) \arrow[r, "\pi"] & \overline{\mathcal{M}}_{g,n}(X,\beta)
\end{tikzcd}$$

where the square on the right is cartesian. On fibres, the map ρ contracts rational components of $\mathcal{C}_{g,n+1}$ on which f is constant and which contain exactly three special points, one of which is x_{n+1}. Thus, we see that

$$\rho^*(x_i) = x_i + R_{i,n+1}$$

where $R_{i,n+1} \subseteq \mathcal{C}_{g,n+1}$ consists fibrewise of the rational tails containing only x_i and x_{n+1}; it is a closed substack of $\psi^{-1}(D_{i,n+1})$ of codimension 0.

On the other hand, we have (REFERENCE):

$$\rho^*\omega_\eta(\Sigma_{i=1}^n x_i) = \omega_\psi(\Sigma_{i=1}^n x_i)$$

Taking Chern classes and combining this with the above result we obtain:

$$\mathrm{c}_1(\rho^*\omega_\eta) = \mathrm{c}_1(\omega_\psi) - \Sigma_{i=1}^n R_{i,n+1}$$

We can now pull back along the section x_i and use the fact that $x_i^* R_{j,n+1} = \delta_{i,j} D_{i,n+1}$ to obtain:

$$\mathrm{c}_1(x_i^* \rho^* \omega_\eta) = \mathrm{c}_1(x_i^* \omega_\psi) - D_{i,n+1}$$

Now, $\rho^* \omega_\eta = \rho^* \alpha^* \omega_\varphi$, and so:

$$x_i^* \rho^* \omega_\eta = \pi^* x_i^* \omega_\varphi$$

Thus we end up with

$$\pi^* \mathrm{c}_1(x_i^* \omega_\varphi) = \mathrm{c}_1(x_i^* \omega_\psi) - D_{i,n+1}$$

which is equation (1) above.

What is different in the case of quasimaps? We have a similar-looking diagram

$$\begin{tikzcd} \mathcal{C}_{g,n+1} \arrow[r, "\rho"] \arrow[dr, "\psi"'] & \pi^* \mathcal{C}_{g,n} \arrow[r, "\alpha"] \arrow[d, "\eta"] & \mathcal{C}_{g,n} \arrow[d, "\varphi"] \\ & \overline{\mathcal{Q}}_{g,n+1}(X,\beta) \arrow[r, "\pi"] & \overline{\mathcal{Q}}_{g,n}(X,\beta) \end{tikzcd}$$

but now, because of the stronger stability condition, ρ also contracts the locus T_{n+1} consisting of rational tails (of any degree) with a single marking x_{n+1}. We claim that:

Conjecture C.1. $\rho^* \omega_\eta (\Sigma_{i=1}^n x_i) = \omega_\psi (\Sigma_{i=1}^n x_i - T_{n+1})$

Once we have this, the string equation follows as in the stable maps case by pulling back along the section x_i (and using the obvious fact that $x_i^* T_{n+1} = 0$).