Report for: *Relative quasimaps and mirror formulae.*

1. Summary

After Givental [Giv96] proved genus zero mirror theorem for toric hyepersurfaces, there has been substantial progress in the study of mirror symmetry via the moduli spaces of quasimap by Ciocan-Fontanine and Kim ([CFK10], [CFKM14]). Classically, the invariants of mirror side was given by oscillating integral of superpotential on the mirror manifold which is called I-function. Ciocan-Fontanine and Kim alternatively defined I-function using the moduli space of quasimaps (in the similar way that one define J-function using the moduli space of stable maps) and provided new viewpoint of mirror symmetry via the wall-crossing formula between the moduli space of stable maps and the moduli space of quasimaps. This new version of mirror symmetry has several advantages over older one. One of them is that mirror theorem can be studied in great generalities in target space and also for the higher genus case.

On the other hand, relative Gromov-Witten invariants for the pair (X, D) was introduced by Li in the algebraic geometry setting (in symplectic geometry, by Ionel-Parker, Li-Ruan) for the study of degeneration formula of GW invariants ([Li01], [Li02]). One can ask the following problem; Is there also mirror symmetry in relative settings?

The paper under review tried to answer this question closely following the approach of Ciocan-Fontanine and Kim. They defined genus zero relative quasimap invariants for the pair (X, D) where X is toric variety and D is ample hypersurface. Using these relative quasimap invariants, they defined relative I-function and proved that this relative I-function coincide with the another relative I-function defined by Fan, Tseng and You ([FTY18]). (In [FTY18], they proved their version of relative I-function lies inside Givental's Lagrangian cone of relative GW invariants.) In conclusion, the contents of the paper under review combined with the result of [FTY18] make it possible to understand genus zero mirror symmetry in relative setting via wall-crossing formula between relative GW and relative quasimap invariants.

However, the authors of the paper under review took a similar approach of Gathemann (Who also defined relative Gromov-Witten invariants in his own way [Gat02].) for the definition of relative quasimap space. It is hard to expect that this approach will work for higher genus or more general target spaces. Nevertheless, I think the paper under review provided enough evidences to believe that there are mirror symmetry in relative settings via the wall crossing formula between relative stable maps and relative quasimaps in more general settings (where we can also possibly study higher genus case).

As far as I checked, there seemed to be no serious gap or mistakes and I am confident in the proofs of the theorems. In conclusion, I recommend the publication of the paper to IMRN.

2. SOME COMMENTS

(1) In Section 2.3 : *I recommend to give a precise statement about the toric settings of* X. *Also It will be better for the reader to state the definition of quasimap precisely or give a reference.*
(2) Page 8 line 14 : *I recommend to give a exact meaning of inequality.*
(3) Page 23 line -1 : *What is the meaning of subscript* 0 *in* S_0^X*? I recommend to change the notation* $S_0^X(x, \beta)$*, since usually S-operator notation is used for the series in q, not for the single invariant.*
(4) Page 24 line 6 : *Usually "generating" function refers to series in q (Novikov variable) not in series in z.*
(5) Page 35 line 3 : *It is common to define I-function as a series in q (Novikov variable).*