# A QUANTUM LEFSCHETZ THEOREM FOR QUASIMAP INVARIANTS VIA RELATIVE QUASIMAPS

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Abstract. We define moduli spaces of relative toric quasimaps in genus zero, in the spirit of A. Gathmann. When $X$ is a smooth toric variety and $Y$ is a very ample hypersurface in $X$ we construct a virtual class on the moduli space of relative quasimaps to $(X, Y)$ which can be used to define relative quasimap invariants of the pair. We obtain a recursion formula which expresses each relative invariant in terms of invariants of lower multiplicity. Finally we apply this formula to derive a quantum Lefschetz theorem expressing the restricted quasimap invariants of $Y$ in terms of those of $X$. We include several appendices collecting proofs of standard results in quasimap theory.

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## 1. Introduction

In this paper we construct moduli spaces of relative quasimaps as substacks of moduli spaces of (absolute) quasimaps. This provides a common generalisation of two different theories: stable quasimaps on the one hand, and relative stable maps (in the sense of A. Gathmann) on the other. In this introductory section we briefly recall these, providing the context for our work.

1.1. **Stable quasimaps.** The moduli space of *stable toric quasimaps* $\mathcal{Q}_{g,n}(X, \beta)$ was constructed by I. Ciocan-Fontanine and B. Kim [CFK10] as a compactification of the moduli space of smooth curves in a smooth and complete toric variety $X$. Roughly speaking, the objects are rational maps $C \dashrightarrow X$ where

$C$ is a nodal curve, subject to a stability condition; the precise definition depends on the description of $X$ as a GIT quotient. The space $\mathcal{Q}_{g,n}(X,\beta)$ is a proper Deligne–Mumford stack of finite type. It admits a virtual fundamental class, which is used to define curve-counting invariants for $X$ called *quasimap invariants*.

This theory agrees with that of stable quotients [MOP11] when both are defined, namely when $X$ is a projective space. There is a common generalisation given by the theory of stable quasimaps to GIT quotients [CFKM14]. For simplicity, however, we will work mostly in the toric setting[1]. Thus in this paper when we say "quasimaps" we are implicitly talking about toric quasimaps. Quasimap invariants provide an alternative system of curve counts to the more well-known Gromov–Witten invariants. These latter invariants are defined via moduli spaces of stable maps, and as such we will often refer to them as *stable map invariants*.

Genus zero quasimap invariants are expected to coincide with Gromov–Witten invariants when $X$ is a toric Fano variety [CM]; this has been proven (in all genera) for $X = \mathbb{P}^N$ [MOP11, Theorem 3] [Man12b, §5.4]. More generally, the case of a projective complete intersection of Fano index at least 2 can be obtained by combining A. Givental's mirror theorem [Giv98, Theorem 0.1] with the wall-crossing formulae of I. Ciocan-Fontanine and B. Kim [CFK10, Conjecture 7.2.10] [CFK14, §5.5 and Conjecture 6.3.1]. In general, however, the invariants differ, the difference being encoded by certain wall-crossing formulae [CFK14]. This can be interpreted using mirror symmetry: the idea is that quasimap invariants of $X$ should correspond to the $B$-side theory of $X^\vee$, the mirror to $X$, whereas Gromov–Witten invariants correspond to the $A$-side theory for $X$; see [CFK10, §7].

1.2. **Relative stable maps.** Let $Y$ be a smooth very ample hypersurface in a smooth projective variety $X$. In [Gat02] A. Gathmann constructs a moduli space of relative stable maps to the pair $(X,Y)$ as a closed substack of the moduli space of (absolute) stable maps to $X$

$$\mathcal{M}_{0,\alpha}(X|Y,\beta) \hookrightarrow \mathcal{M}_{0,n}(X,\beta).$$

The relative moduli space parametrises stable maps with prescribed tangencies to $Y$ at the marked points. Unfortunately this space does not admit a natural perfect obstruction theory. Nevertheless, because $Y$ is very ample it is still possible to construct a virtual fundamental class by intersection-theoretic methods, and hence one can define relative stable map invariants.

Gathmann establishes a recursion formula for these virtual classes which allows one to express any relative invariant of $(X,Y)$ in terms of absolute invariants of $Y$ and relative invariants with lower contact multiplicities. By successively increasing the contact multiplicites from zero to the maximum possible value, this gives an algorithm expressing the (restricted) invariants of $Y$ in terms of those of $X$: see [Gat02, Corollary 5.7]. In [Gat03b] this result

[1]This restriction is probably not essential for our arguments.

is applied to give an alternative proof of the mirror theorem for projective hypersurfaces [Giv96] [LLY97].

1.3. **Relative stable quasimaps.** In this paper we construct moduli spaces of relative quasimaps in genus zero. We prove a recursion relation similar to Gathmann's formula, and use this to derive a quantum Lefschetz formula for quasimap invariants. Our construction carries over to the setting of $\epsilon$-stable quasimaps [CFK10]; since for $\epsilon > 1$ these moduli spaces agree with the space of stable maps, one can view our construction as giving a common generalisation of the two stories outlined above.

The plan of the paper is as follows. In §§2.1 and 2.2 we provide a brief review of the theories of stable quasimaps and relative stable maps. Then in §2.3 we define the moduli space of relative quasimaps as a substack of the moduli space of (absolute) quasimaps:

$$\mathcal{Q}_{0,\alpha}(X|Y,\beta) \hookrightarrow \mathcal{Q}_{0,n}(X,\beta).$$

Here $X$ is a smooth toric variety, $Y$ is a smooth very ample hypersurface and $\alpha = (\alpha_1, \ldots, \alpha_n)$ encodes the orders of tangency of the marked points to $Y$. Note that we *do not* require $Y$ to be toric.

In §3 we examine the special case of a hyperplane $H \subseteq \mathbb{P}^N$. We find that the moduli space is irreducible of the expected dimension (in fact, more than this: it is the closure of the so-called "nice locus" consisting of maps from a $\mathbb{P}^1$ whose image is not contained in $H$). Thus it has an actual fundamental class, which we can use to define relative quasimap invariants. Another useful fact about this special case is that there exists a birational comparison morphism:

$$\chi : \mathcal{M}_{0,n}(\mathbb{P}^N, d) \to \mathcal{Q}_{0,n}(\mathbb{P}^N, d)$$

This restricts to a birational morphism between the relative spaces, which we use to push down Gathmann's formula to obtain a recursion formula for relative stable quasimaps. The stronger stability condition for quasimaps significantly simplifies the form of this recursion.

In §4 we turn to the case of an arbitrary pair $(X, Y)$ with $Y$ very ample. We use the embedding $X \hookrightarrow \mathbb{P}^N$ defined by $\mathcal{O}_X(Y)$ to construct a virtual class $[\mathcal{Q}_{0,\alpha}(X|Y,\beta)]^{\mathrm{virt}}$. We then prove the recursion formula for $(X, Y)$ by pulling back the formula for $(\mathbb{P}^N, H)$. This requires several comparison theorems for virtual classes, extending results in Gromov–Witten theory to the setting of quasimaps.

Finally in §5 we apply the recursion formula of §4 to obtain a quantum Lefschetz formula for quasimap invariants, that is, a formula expressing quasimap invariants of $Y$ in terms of those of $X$. This recovers Corollary 5.5.1 in [CFK14], and can be interpreted as a mirror theorem for $Y$. The argument is similar in spirit to the one given in [Gat03b], however the stronger stability condition considerably simplifies both the proof and the final result.

We also include several appendices, collecting together results which are well-known to experts but absent from the literature. Appendix A discusses the comparison morphism; Appendix B contains foundational results in quasimap theory, including functoriality and the splitting axiom; Appendix C contains a number of intersection-theoretic lemmas.

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1.4. **Table of notation.** We will use the following notation, most of which is introduced in the main body of the paper.

| | |
|---|---|
| $X$ | a smooth projective toric variety |
| $Y$ | a smooth very ample hypersurface in $X$ |
| $\Sigma$ | the fan of $X$ |
| $\Sigma(1)$ | the set of 1-dimensional cones of $\Sigma$ |
| $\rho$ | an element of $\Sigma(1)$ |
| $D_\rho$ | the toric divisor in $X$ corresponding to $\rho$ |
| $\mathcal{M}_{g,n}(X,\beta)$ | the moduli space of stable maps to $X$ |
| $\mathcal{M}_{0,\alpha}(X\|Y,\beta)$ | the moduli space of relative stable maps to $(X,Y)$; see §2.2 |
| $\mathcal{Q}_{g,n}(X,\beta)$ | the moduli space of toric quasimaps to $X$; see §2.1 |
| $\mathcal{Q}^{\circ}_{0,\alpha}(X\|Y,\beta)$ | the nice locus of relative quasimaps to $(X,Y)$; see §3.1 |
| $\mathcal{Q}_{0,\alpha}(X\|Y,\beta)$ | the moduli space of relative quasimaps to $(X,Y)$; see §2.3 |
| $\mathcal{D}^{\mathcal{Q}}_{\alpha,k}(X\|Y,\beta)$ | the quasimap comb locus; see §3.2 |
| $\mathcal{D}^{\mathcal{Q}}(X\|Y,A,B,M)$ | (a component of) the comb locus; see §3.2 |
| $\mathcal{E}^{\mathcal{Q}}(X\|Y,A,B,M)$ | the total product for the comb locus; see §4.3 |
| $\mathcal{D}^{\mathcal{Q}}(X,A,B)$ | the quasimap centipede locus; see Appendix B.3 |
| $\mathcal{E}^{\mathcal{Q}}(X,A,B)$ | the total product for the centipede locus; see Appendix B.3 |
| $\mathfrak{M}^{\mathrm{wt}}_{g,n}$ | the moduli stack of weighted prestable curves; see Appendix B.3 |
| $\mathfrak{Pic}^{d,\mathrm{st}}_{g,n}$ | an open substack of the relative Picard stack of the universal curve over $\mathfrak{M}_{g,n}$; see Appendix A |
| $\mathfrak{Bun}^{g,n}_{G}$ | the moduli stack of principal $G$-bundles on the universal curve over $\mathfrak{M}_{g,n}$; see Appendix B.4 |
| $\mathcal{Q}(f)$ | the push-forward morphism between quasimap spaces; see Appendix B.1 |
| $\chi$ | the comparison morphism from stable maps to quasimaps; see Appendix A |
| $f^!$ | Gysin morphism for $f$ a regular embedding |

| | |
|---|---|
| $f^!_{\mathrm{v}}$ | virtual pull-back for $f$ virtually smooth; see Appendix C |
| $f^!_{\Delta}$ | diagonal pull-back; see Appendix C |

## 2. Relative stable quasimaps

2.1. **Review of absolute stable quasimaps.** We briefly recall the definition and basic properties of the moduli space of toric quasimaps; see [CFK10] for more details.

**Definition 2.1** ([CFK10, Definition 3.1.1])**.** Let $N$ be a lattice, let $\Sigma \subseteq N_{\mathbb{Q}}$ be a fan, and let $X = X_\Sigma$ be the corresponding toric variety. Suppose that $X$ is smooth and projective. Let $M = N^\vee = \mathrm{Hom}(N, \mathbb{Z})$ and let $\mathcal{O}_{X_\Sigma}(1)$ be a fixed polarisation, which we can write (non-uniquely) in terms of the torus-invariant divisors as:

$$\mathcal{O}_{X_\Sigma}(1) = \otimes_{\rho \in \Sigma(1)} \mathcal{O}_{X_\Sigma}(D_\rho)^{\otimes \alpha_\rho}$$

for some $\alpha_\rho \in \mathbb{Z}$. We fix the following numerical invariants: a genus $g \geq 0$, number of marked points $n \geq 0$ and an effective curve class $\beta \in \mathrm{H}_2^+(X)$. A *stable (toric) quasimap* is given by the data

$$\Big( (C, x_1, \ldots, x_n), (L_\rho, u_\rho)_{\rho \in \Sigma(1)}, (\varphi_m)_{m \in M} \Big)$$

where:

1. $(C, x_1, \ldots, x_n)$ is a prestable curve of genus $g$ with $n$ marked points;
2. the $L_\rho$ are line bundles on $C$ of degree $d_\rho = D_\rho \cdot \beta$;
3. the $u_\rho$ are global sections of $L_\rho$;
4. $\varphi_m \colon \bigotimes_{\rho \in \Sigma(1)} L_\rho^{\otimes \langle \rho, m \rangle} \to \mathcal{O}_C$ are isomorphisms, such that $\varphi_m \otimes \varphi_{m'} = \varphi_{m+m'}$ for all $m, m' \in M$.

These are required to satisfy the following two conditions:

1. *nondegeneracy*: there is a finite (possibly empty) set of smooth and non-marked points $B \subseteq C$, called the *basepoints* of the quasimap, such that for all $x \in C \setminus B$ there exists a maximal cone $\sigma \in \Sigma_{\max}$ with $u_\rho(x) \neq 0$ for all $\rho \not\subset \sigma$;
2. *stability*: if we let $L = \otimes_\rho L_\rho^{\otimes \alpha_\rho}$ then the following $\mathbb{Q}$-divisor is ample
$$\omega_C(x_1 + \ldots + x_n) \otimes L^{\otimes \epsilon}$$
for every rational $\epsilon > 0$. Note that this does not depend on the choice of polarisation.

**Remark 2.2.** This definition is motivated by D. A. Cox's description of the functor of points of a toric variety in terms of $\Sigma$-collections [Cox95a]; see also Appendix B.1. A quasimap defines[2] a rational map $C \dashrightarrow X$ with base locus equal to $B$. In particular a quasimap without any basepoints defines a morphism $C \to X$. Thus maps with basepoints appear in the (virtual) boundary of the moduli space of quasimaps, in much the same way as maps with rational tails appear in the boundary of the moduli space of stable maps. This is something more than just a vague analogy; these loci

[2] This can be expressed in a more generalisable manner as follows: a quasimap is a map to the stack quotient $[\mathbb{A}^{\Sigma(1)}/(\mathbb{G}_{\mathrm{m}})^r]$ such that $B$ is the preimage of the unstable locus.

correspond to each other under the comparison morphism when $X \cong \mathbb{P}^N$; see Appendix A.

More generally, one can define the notion of a family of quasimaps over a base scheme $S$, and what it means for two such families to be isomorphic; one thus obtains a moduli stack

$$\mathcal{Q}_{g,n}(X, \beta)$$

of stable (toric) quasimaps to $X$, which is a proper Deligne–Mumford stack of finite type [CFK10, §3].

As with the case of stable maps, there is a combinatorial characterisation of stability which is easy to check in practice; a prestable quasimap is stable if and only if the following conditions hold:

(1) the line bundle $L = \otimes_\rho L_\rho^{\otimes \alpha_\rho}$ must have strictly positive degree on any rational component with fewer than three special points, and on any elliptic component with no special points;
(2) $C$ cannot have any rational components with fewer than two special points (that is, no *rational tails*).

Condition (1) is analogous to the ordinary stability condition for stable maps. Condition (2) is new, however, and gives quasimaps a distinctly different flavour to stable maps; we shall sometimes refer to it as the *strong stability condition*.

**Remark 2.3.** Unlike in Gromov–Witten theory, $\mathcal{Q}_{g,n+1}(X, \beta)$ is *not* the universal curve over $\mathcal{Q}_{g,n}(X, \beta)$ since markings cannot be basepoints. In fact there is not even a morphism between these spaces in general.

The moduli space $\mathcal{Q}_{g,n}(X, \beta)$ admits a perfect obstruction theory relative to the moduli space $\mathfrak{M}_{g,n}$ of source curves [CFK10, §5], and hence one can construct a virtual class

$$[\mathcal{Q}_{g,n}(X, \beta)]^{\text{virt}} \in \mathrm{A}_{\mathrm{vdim}\, \mathcal{Q}_{g,n}(X,\beta)} \left( \mathcal{Q}_{g,n}(X, \beta) \right)$$

where the virtual dimension is the same as for stable maps:

$$\mathrm{vdim}\, \mathcal{Q}_{g,n}(X, \beta) = (\dim X - 3)(1 - g) - (K_X \cdot \beta) + n$$

Since the markings are not basepoints there exist evaluation maps

$$\mathrm{ev}_i : \mathcal{Q}_{g,n}(X, \beta) \to X$$

and there are $\psi$-classes defined in the usual way by pulling back the relative dualising sheaf of the universal curve

$$\psi_i = \mathrm{c}_1(x_i^* \omega_{\mathcal{C}/\mathcal{Q}})$$

where $\mathcal{C} \to \mathcal{Q} = \mathcal{Q}_{g,n}(X, \beta)$ is the universal curve and $x_i : \mathcal{Q} \to \mathcal{C}$ is the section defining the $i$th marked point. Putting all these pieces together, we can define *quasimap invariants*:

$$\langle \gamma_1\psi_1^{k_1},\ldots,\gamma_n\psi_n^{k_n}\rangle^X_{g,n,\beta} = \int_{[Q_{g,n}(X,\beta)]^{\mathrm{virt}}} \prod_{i=1}^n \mathrm{ev}_i^\star(\gamma_i)\cdot\psi_i^{k_i}$$

We use the same correlator notation as in Gromov–Witten theory; this should not cause any confusion.

**Example 2.4.** Consider $Q_{0,2}(\mathbb{P}^2,1)$. What are its objects? By the strong stability condition (2) above, we see that the source curve must be irreducible. On the other hand since $\mathbb{P}^2$ has Picard rank 1 we may exploit the isomorphisms $\varphi_m$ to reduce ourselves to considering one line bundle equipped with three sections. Thus the data of the quasimap is $((C,x_1,x_2),L,u_0,u_1,u_2)$ where $(C,L)\cong(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(1))$.

Pick coordinates $[s:t]$ on $\mathbb{P}^1$ such that the marked points are $[1:0]$ and $[0:1]$. We can express the sections as $u_i = a_i s + b_i t$; the requirement that the markings are not basepoints then translates into the following stability condition:

$$(a_0,a_1,a_2)\neq(0,0,0) \quad \text{and} \quad (b_0,b_1,b_2)\neq(0,0,0).$$

The group $\mathrm{Aut}(C;x_1,x_2)\cong\mathbb{G}_\mathrm{m}$ acts by rotation $\lambda\colon[s:t]\mapsto[s:\lambda t]$, while $\mathrm{Aut}(L)\cong\mathbb{G}_\mathrm{m}$ acts by scalar multiplication on $\underline{a}$ and $\underline{b}$. Thus the $\mathbb{G}_\mathrm{m}^2$ action on $\mathbb{A}^6_{\underline{a},\underline{b}}$ is encoded by the following weight matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

It is now clear that the quotient is $\mathbb{P}^2\times\mathbb{P}^2$; in fact, we see that the evaluation map

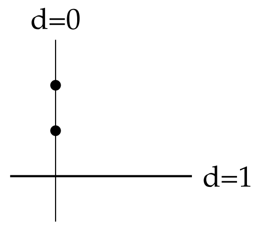
$$(\mathrm{ev}_1,\mathrm{ev}_2)\colon Q_{0,2}(\mathbb{P}^2,1)\to\mathbb{P}^2\times\mathbb{P}^2$$

is an isomorphism. It is given in the above notation by:

$$((\mathbb{P}^1;[1:0],[0:1]);\mathcal{O}_{\mathbb{P}^1}(1);u_0,u_1,u_2)\mapsto([a_0:a_1:a_2],[b_0:b_1:b_2])$$

Notice that the locus where $(a_0,a_1,a_2)=\mu(b_0,b_1,b_2)$, i.e. the diagonal in $\mathbb{P}^2\times\mathbb{P}^2$ is precisely the locus of quasimaps which have a basepoint. The point $[a_0:a_1:a_2]=[b_0:b_1:b_2]\in\mathbb{P}^2$ is the image of the underlying "residual map" of degree 0, obtained by dividing all the sections by a local equation of the basepoint (equivalently, by extending the rational map $C\dashrightarrow\mathbb{P}^2$ to a morphism $C\to\mathbb{P}^2$).

On the other hand, $(\mathrm{ev}_1,\mathrm{ev}_2)\colon\mathcal{M}_{0,2}(\mathbb{P}^2,1)\to\mathbb{P}^2\times\mathbb{P}^2$ is *not* an isomorphism. Off the diagonal, the images of the two marked points determine uniquely the image of the stable map, i.e. the line through them. On the diagonal however, the following maps with a rational tail appear:




The image of the degree 1 component under $f$ can be any line passing through the point of $\mathbb{P}^2$ to which the other component is contracted. Hence $\mathcal{M}_{0,2}(\mathbb{P}^2, 1) \simeq \mathrm{Bl}_\Delta(\mathbb{P}^2 \times \mathbb{P}^2)$. The comparison morphism $\chi$ (see Appendix A) can be interpreted as the blow-down map, and it induces an isomorphism of the rational tails-free locus with the basepoint-free locus.

**Remark 2.5.** There is a more general notion of *$\epsilon$-stable quasimap* [CFKM14, §7.1]. Here the stability condition, namely that the line bundle

$$\omega_C(x_1 + \ldots + x_n) \otimes L^{\otimes \epsilon}$$

is ample, is only required to hold for a fixed $\epsilon \in \mathbb{Q}_{>0}$ (instead of for arbitrary $\epsilon$, as was the case with ordinary quasimaps).

This has the effect of allowing some rational tails to appear, as long as their degree is high enough with respect to $\epsilon$. In order to keep the moduli space separated, one also has to bound the multiplicity of the basepoints that can occur.

By boundedness and the fact that the degree is an integer-valued function, there exist finitely many critical values of $\epsilon$ which divide $\mathbb{Q}_{>0}$ into chambers inside which the moduli spaces $\mathcal{Q}^{\epsilon}_{g,n}(X, \beta)$ do not change. For $\epsilon$ sufficiently small we recover the space of (ordinary) quasimaps, and for $\epsilon$ sufficiently large we obtain the moduli space of stable maps. Thus one can view the spaces of $\epsilon$-stable quasimaps as interpolating between these two extremes, and they have proven successful as a tool for comparing quasimap invariants to stable map invariants [CFK14].

Another variant of the theory, which will play a role in later sections, is that of *parametrised quasimaps* [CFK10, §7]. A parametrised quasimap comes with a preferred rational component, which is equipped with the extra data of an isomorphism with $\mathbb{P}^1$, and the stability condition is imposed *on all but the preferred component*. This mimics the construction of graph spaces in Gromov-Witten theory – for example, there is a $\mathbb{G}_{\mathrm{m}}$-action on $\mathcal{Q}G_{g,n}(X, \beta)$ by rotating the preferred component, which plays the role of the $\mathbb{G}_{\mathrm{m}}$-action that rotates the graph. The fixed loci and their equivariant normal bundles are well-understood, at least in the toric setting [CFK10, §7]. In the parameterised case we no longer require $2g - 2 + n \geq 0$, due to the modified stability condition. In particular it makes sense, and turns out to be very useful, to consider unmarked parametrised quasimaps $\mathcal{Q}G_{0,0}(X, \beta)$. In this case the source curve is necessarily irreducible.

**Example 2.6.** $QG_{0,0}(\mathbb{P}^N, d) = \mathbb{P}^k$ with $k = (N+1)(d+1) - 1$. Indeed, the curve and line bundle must be $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$ and we are left with choosing $N+1$ sections of $\mathcal{O}_{\mathbb{P}^1}(d)$ (not all zero) up to automorphisms of $\mathcal{O}_{\mathbb{P}^1}(d)$, i.e. up to scaling. For an early appearance of such spaces, see for instance [Ber00].

2.2. **Review of relative stable maps.** Given a smooth projective variety $X$ and a smooth very ample divisor $Y$, Gathmann's moduli space of relative stable maps parametrises stable maps to $X$ with specified tangencies to $Y$ at the marked points.

**Definition 2.7.** [Gat02, Definition 1.1] Let $X$ be a smooth projective variety and $Y \subseteq X$ a smooth very ample divisor. Fix a number $n \geq 0$ of marked points, an effective curve class $\beta \in \mathrm{H}_2^+(X)$ and an $n$-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non-negative integers such that $\Sigma_i \alpha_i \leq Y \cdot \beta$. The moduli space

$$\mathcal{M}_{0,\alpha}(X|Y, \beta)$$

of relative stable maps to $(X, Y)$ is defined to be the locus in $\mathcal{M}_{0,n}(X, \beta)$ of stable maps $(C \to S, (x_i : S \to C)_{i=1}^n, f : C \to X)$ satisfying the following two conditions:

(1) if $x_i$ is a marked point such that $\alpha_i > 0$ then $f(x_i) \in Y$;
(2) if we consider the class $f^*[Y] \in \mathrm{A}^0(f^{-1}Y \to S)$ then the difference $f^*[Y] - \sum_i \alpha_i x_i$ is an effective class.

These conditions define a closed substack of $\mathcal{M}_{0,n}(X, \beta)$. Condition (1) is required in order for the class $\sum_i \alpha_i x_i$ to make sense in $\mathrm{A}^0(f^{-1}Y \to S)$.

**Remark 2.8.** The notation in (2) comes from bivariant intersection theory: see [Ful98, §17]. Fibrewise, the condition is that the class $f^*[Y] - \sum_i \alpha_i x_i \in A_0(f^{-1}Y)$ is required to be effective.

**Remark 2.9.** When $\alpha = (0, \ldots, 0)$, condition (2) becomes $Y \cdot \beta \geq 0$, so $\mathcal{M}_{0,(0,\ldots,0)}(X|Y, \beta) = \mathcal{M}_{0,n}(X, \beta)$ as long as $Y$ is nef.

**Remark 2.10.** The definition given above works in families; however there is an equivalent, more combinatorial definition for individual maps which is more useful in practice (see [Gat02, Remark 1.4]): a stable map $(C, x_1, \ldots, x_n, f)$ is a relative stable map if and only if, for each connected component $Z$ of $f^{-1}(Y) \subseteq C$:

(1) if $Z$ is a point and is equal to a marked point $x_i$, then the multiplicity of $f$ to $Y$ at $x_i$ is greater than or equal to $\alpha_i$;
(2) if $Z$ is one-dimensional (hence a union of irreducible components of $C$) and if we let $C^{(i)}$ for $1 \leq i \leq r$ denote the irreducible components of $C$ adjacent to $Z$ and $m^{(i)}$ denote the multiplicity of $f|_{C^{(i)}}$ to $Y$ at the node $Z \cap C^{(i)}$, then:

$$Y \cdot f_*[Z] + \sum_{i=1}^{r} m^{(i)} \geq \sum_{x_i \in Z} \alpha_i \tag{1}$$

**Remark 2.11.** In case (2) above we call $Z$ an *internal* component and the $C^{(i)}$ *external* components. Note that $Z$ is not necessarily irreducible: the term "component" is justified by the fact that it is a *connected* component of $f^{-1}(Y)$.

**Remark 2.12.** In the case of maximal multiplicity $\Sigma_i \alpha_i = Y \cdot \beta$, all the inequalities in the above definition must actually be equalities.

In the case $X = \mathbb{P}^N$ and $Y = H$ a hyperplane, Gathmann showed [Gat02, Proposition 1.14] that $\mathcal{M}_{0,\alpha}(\mathbb{P}^N|H, d)$ is irreducible with dimension equal to the expected dimension:

$$\operatorname{vdim} \mathcal{M}_{0,\alpha}(X|Y,\beta) = \operatorname{vdim} \mathcal{M}_{0,n}(X,\beta) - \sum_{i=1}^{n} \alpha_i$$

Hence it has a fundamental class from which one can define relative Gromov–Witten invariants. More generally if $Y \subseteq X$ is very ample one can use the embedding $X \hookrightarrow \mathbb{P}^N$ given by $|\mathcal{O}_X(Y)|$ to obtain a cartesian diagram:

$$\begin{tikzcd}
\mathcal{M}_{0,\alpha}(X|Y,\beta) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{M}_{0,\alpha}(\mathbb{P}^N|H,d) \arrow[d] \\
\mathcal{M}_{0,n}(X,\beta) \arrow[r, "\varphi"] & \mathcal{M}_{0,n}(\mathbb{P}^N,d)
\end{tikzcd}$$

Then the fact that $\mathcal{M}_{0,n}(\mathbb{P}^N, d)$ is smooth allows one to define a virtual class on $\mathcal{M}_{0,\alpha}(X|Y,\beta)$ by diagonal pull-back (see Appendix C of the current paper):

$$[\mathcal{M}_{0,\alpha}(X|Y,\beta)]^{\mathrm{virt}} := \varphi^![\mathcal{M}_{0,\alpha}(\mathbb{P}^N|H,d)]$$

Thus one can define relative Gromov–Witten invariants in the usual way, by capping the virtual class with products of evaluation classes and psi classes.

In [Gat02, §§2-4] Gathmann establishes a recursion relation inside the Chow group of $\mathcal{M}_{0,\alpha}(X|Y,\beta)$. This describes what happens when we increase the multiplicity at one of the marked points by 1. Let us therefore fix a marked point $x_k \in \{x_1,\ldots,x_n\}$ and let $e_k = (0,\ldots,1,\ldots,0)$. Then

$$(\alpha_k\psi_k + \mathrm{ev}_k^*[Y]) \cap [\mathcal{M}_{0,\alpha}(X|Y,\beta)]^{\mathrm{virt}} = [\mathcal{M}_{0,\alpha+e_k}(X|Y,\beta)]^{\mathrm{virt}} + [\mathcal{D}_{\alpha,k}(X,\beta)]^{\mathrm{virt}}$$

where $\mathcal{D}_{\alpha,k}(X,\beta)$ is an appropriate *comb locus*. This parametrises relative stable maps where the component containing $x_k$ is mapped entirely into $Y$, and which satisfy inequality (1) for $\alpha$ but not for $\alpha + e_k$; these form a divisor in $\mathcal{M}_{0,\alpha}(X|Y,\beta)$.

Repeated application of this result shows that both the relative Gromov–Witten invariants of $(X,Y)$ and the (restricted) Gromov–Witten invariants of $Y$ are completely determined by the Gromov–Witten invariants of $X$ [Gat02, Corollary 5.7]. This result is then applied in [Gat03b] to obtain a new proof of the mirror theorem for projective hypersurfaces.

**Remark 2.13.** There are many other approaches to defining relative stable maps besides Gathmann's: the moduli space of maps to expanded degenerations of J. Li [Li01] [Li02], the twisted stable maps of D. Abramovich and B. Fantechi [AF16], the logarithmic stable maps with expansions of B. Kim [Kim10] and the logarithmic stable maps (without expansions) of M. Gross and B. Siebert [GS13] [GS16], Q. Chen [Che14] and D. Abramovich and Q. Chen [AC14]. However, the invariants defined via these theories are all known to coincide [AMW14] [Gat03a], so the choice of which moduli space to work with mainly depends on one's intended application.

2.3. **Definition of relative stable quasimaps.** We now give the main definition of the paper. From here on $X$ will denote a smooth projective toric variety and $Y \subseteq X$ a very ample hypersurface. We *do not* require that $Y$ is toric. Consider the line bundle $\mathcal{O}_X(Y)$ and the section $s_Y$ cutting out $Y$. By [Cox95b] we have a natural isomorphism of $\mathbb{C}$-vector spaces

$$\mathrm{H}^0(X, \mathcal{O}_X(Y)) = \left\langle \prod_\rho z_\rho^{a_\rho} : \Sigma_\rho a_\rho D_\rho = Y \right\rangle_{\mathbb{C}}$$

where the $z_\rho$ for $\rho \in \Sigma(1)$ are the generators of the Cox ring of $X$ and the $a_\rho$ are non-negative integers. We can therefore write $s_Y$ as

$$s_Y = \sum_{\underline{a}=(a_\rho)} \lambda_{\underline{a}} \prod_\rho z_\rho^{a_\rho}$$

for some scalars $\lambda_{\underline{a}} \in \mathbb{C}$. The idea is that a quasimap

$$((C, x_1, \ldots, x_n), (L_\rho, u_\rho)_{\rho\in\Sigma(1)}, (\varphi_m)_{m\in M})$$

should "map" a point $x \in C$ into $Y$ if and only if the section

$$u_Y := \sum_{\underline{a}} \lambda_{\underline{a}} \prod_\rho u_\rho^{a_\rho} \tag{2}$$

vanishes at $x$. We now explain how to make sense of expression (2). For each $\underline{a}$ we have a well-defined section

$$u_{\underline{a}} := \lambda_{\underline{a}} \prod_\rho u_\rho^{a_\rho} \in \mathrm{H}^0(C, \otimes_\rho L_\rho^{\otimes a_\rho})$$

and if we have $\underline{a}$ and $\underline{b}$ such that $\sum_\rho a_\rho D_\rho = Y = \sum_\rho b_\rho D_\rho$ then these differ by an element $m$ of $M$. Thus the isomorphism $\varphi_m$ allows us to view the sections $u_{\underline{a}}$ and $u_{\underline{b}}$ as sections of the same bundle, which we denote by $L_Y$. Then we can sum these together to obtain $u_Y$. There is a choice involved here, but up to isomorphism it does not matter; see the proof of functoriality in Appendix B.1 for more details.

The upshot is that we obtain a line bundle $L_Y$ on $C$, which plays the role of the "pull-back" of $\mathcal{O}_X(Y)$ along the "map" $C \to X$, and a global section

$$u_Y \in \mathrm{H}^0(C, L_Y)$$

which plays the role of the "pull-back" of $s_Y$.

**Definition 2.14.** With notation as above, let $n \geq 2$ be a number of marked points, $\beta \in \mathrm{H}_2^+(X)$ be an effective curve class and $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a collection of non-negative integers such that $\Sigma_i \alpha_i \leq Y \cdot \beta$. The *moduli space of relative stable quasimaps*

$$\mathcal{Q}_{0,\alpha}(X|Y, \beta) \subseteq \mathcal{Q}_{0,n}(X, \beta)$$

is defined to be the locus of quasimaps

$$((C \to S, (x_i : S \to C)_{i=1}^n), (L_\rho, u_\rho)_{\rho \in \Sigma(1)}, (\varphi_m)_{m \in M})$$

such that:

(1) if $x_i$ is a marking such that $\alpha_i > 0$, then $x_i^* u_Y = 0$;
(2) if we let $u_Y^*(0) \in \mathrm{A}_0(u_Y^{-1}(0))$ denote the class defined by the Gysin map for $L_Y$, then the difference $u_Y^*(0) - \Sigma_i \alpha_i x_i$ is an effective class.

The class $u_Y^*(0)$ is defined as follows. Consider the cartesian diagram

$$\begin{tikzcd} u_Y^{-1}(0) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & C \arrow[d, "u_Y"] \\ C \arrow[r, "0_Y"] & L_Y \end{tikzcd}$$

where $0_Y$ is the zero section. There is a Gysin map

$$0_Y^! : \mathrm{A}_*(C) \to \mathrm{A}_*(u_Y^{-1}(0))$$

and we define $u_Y^*(0) := 0_Y^!([C])$.

**Remark 2.15.** As in the case of relative stable maps (see §2.2) there is an equivalent definition which is more useful in practice: a quasimap is a relative quasimap if and only if for every connected component $Z$ of $u_Y^{-1}(0)$ we have that:

(1) if $Z$ is a point and is equal to a marked point $x_i$, then the order of vanishing of $u_Y$ at $x_i$ is greater than or equal to $\alpha_i$;
(2) if $Z$ is one-dimensional (hence a union of irreducible components) and if we let $C^{(i)}$ for $1 \leq i \leq r$ denote the irreducible components of $C$ adjacent to $Z$ and $m^{(i)}$ the order of vanishing of $u_Y$ at the node $Z \cap C^{(i)}$, then:

$$\deg L_Y|_Z + \sum_{i=1}^{r} m^{(i)} \geq \sum_{x_i \in Z} \alpha_i \tag{3}$$

## 3. Recursion formula for $\mathbb{P}^N$ relative a hyperplane

At this stage we do not know much about the moduli space of relative quasimaps. In this section we will examine the case $X = \mathbb{P}^N$ and $Y = H$ a hyperplane in detail.

3.1. **Basic properties of the moduli space.** We will now show that the moduli space

$$\mathcal{Q}_{0,\alpha}(\mathbb{P}^N|H,d)$$

is irreducible of the expected dimension, and thus admits a fundamental class. We then prove a recursion formula for these fundamental classes by pushing forward Gathmann's recursion formula along the comparison morphism:

$$\chi : \mathcal{M}_{0,n}(\mathbb{P}^N,d) \to \mathcal{Q}_{0,n}(\mathbb{P}^N,d)$$

Let us briefly recall what this morphism does. Every stable map defines a quasimap which is stable except for the fact that it may have rational tails. $\chi$ has the effect of contracting these rational tails and introducing a basepoint at the corresponding node, with multiplicity equal to the degree of the rational tail; see Appendix A for more details.

For the remainder of this section we set $X = \mathbb{P}^N$, denote the projective co-ordinates on $X$ by $[z_0 : \cdots : z_N]$ and set $Y = H = \{z_0 = 0\}$. Given a quasimap

$$((C, x_1, \ldots, x_n), L, u_0, \ldots, u_N) \in \mathcal{Q}_{0,n}(\mathbb{P}^N,d)$$

the line bundle $L_Y$ of the previous section is equal to $L$ and the section $u_Y$ is equal to $u_0$.

**Lemma 3.1.** The comparison morphism restricts to a morphism

$$\chi : \mathcal{M}_{0,\alpha}(\mathbb{P}^N|H,d) \to \mathcal{Q}_{0,\alpha}(\mathbb{P}^N|H,d)$$

*Proof.* We need to verify that a relative stable map is sent to a relative stable quasimap by $\chi$. Since the contraction of a rational tail $R$ always occurs away from the markings, we only need to examine the internal components $Z$ of the quasimap. To be more precise, we have to show that the inequality (3) is satisfied, using the fact that the inequality (1) is satisfied by the stable map that we started with. Let us describe this stable map around $Z$. For each basepoint $x$ on $Z$ there is a rational tail $R$ of the stable map attached to $Z$ at $x$. This is either internal (mapped into $H$) or external (not mapped entirely into $H$).

If $R$ is internal then both $R$ and $Z$ live inside the same connected component $Z'$ of $f^{-1}(H)$. Applying $\chi$ has the effect of contracting $R$ and increasing the degree of the line bundle on $Z$ by $H \cdot f_*[R]$. Thus the left hand side of inequality (1) is left unchanged, and since the right hand side is also unaltered we obtain inequality (3).

On the other hand if $R$ is external then the multiplicity $m^{(R)}$ of $R \cap Z$ satisfies:

$$m^{(R)} \leq H \cdot f_*[R]$$

Since applying $\chi$ has the effect of replacing $m^{(R)}$ by $H \cdot f_*[R]$ in the left hand side of (1), inequality (3) holds for the quasimap. Thus we obtain a morphism from the relative stable map space to the relative quasimap space, as claimed. □

Let us denote by

$$\mathcal{Q}^{\circ}_{0,\alpha}(\mathbb{P}^N|H,d) \subseteq \mathcal{Q}_{0,\alpha}(\mathbb{P}^N|H,d)$$

the *nice locus*, consisting of those quasimaps with irreducible source curve $C \cong \mathbb{P}^1$ and no basepoints (so that we have an actual map $u : C \to \mathbb{P}^N$) such that the curve is not mapped inside $H$ and $u$ has tangency at least $\alpha_i$ to $H$ at the marking $x_i$.

This is an irreducible, locally closed substack of $\mathcal{Q}_{0,n}(\mathbb{P}^N,d)$ of codimension $\Sigma_i\alpha_i$, by essentially the same argument as in [Gat02, Lemma 1.8]. In fact it is isomorphic to the nice locus inside the stable map space, denoted $\mathcal{M}_{0,\alpha}(\mathbb{P}^N|H,d)$ by Gathmann (see [Gat02, Def. 1.6]); the stricter stability condition has no effect when the source curve is irreducible.

**Lemma 3.2.** $\mathcal{Q}_{0,\alpha}(\mathbb{P}^N|H,d)$ is equal to the closure of the nice locus $\mathcal{Q}^{\circ}_{0,\alpha}(\mathbb{P}^N|H,d)$ inside $\mathcal{Q}_{0,n}(\mathbb{P}^N,d)$.

*Proof.* $\mathcal{Q}_{0,\alpha}(\mathbb{P}^N|H,d) \subseteq \overline{\mathcal{Q}^{\circ}_{0,\alpha}(\mathbb{P}^N|H,d)}$: we show that any relative stable quasimap can be infinitesimally deformed to a relative stable quasimap with no basepoints. This is in particular a relative stable map; we then appeal to [Gat02, Prop. 1.14] to deform this stable map and obtain a point in the nice locus. Since this deformation does not introduce any rational tails, this is also a deformation of quasimaps, and the statement follows.

We induct on the number of components containing at least one basepoint. Suppose this number is non-zero (otherwise there is nothing to prove) and pick such a component $C_0$, with base-points $y_1,\ldots,y_k$. Recall that this means that $u_i(y_j)=0$ for all $i$ and $j$. We will deform the section $u_N|_{C_0}$ to a new section $u'_N|_{C_0}$ in such a way that $u'_N|_{C_0}(y_j)\neq 0$ and in such a way that we do not introduce any new basepoints. Notice that since the relative condition only depends on $u_0$, the resulting deformed quasimap will still be a relative quasimap.

Now, by nondegeneracy and the fact that there exists a basepoint, we must have $\deg(L|_{C_0})>0$, and since $C_0\cong\mathbb{P}^1$ we can find a section $w_0$ of $L|_{C_0}\cong\mathcal{O}_{\mathbb{P}^1}(d_0)$ not vanishing at any of the base-points $y_i$. We then set

$$u'_N|_{C_0} := u_N|_{C_0} + \epsilon w_0$$

and $u'_i|_{C_0} = u_i|_{C_0}$ for all other $i$. Notice that $u'_N|_{C_0}(y_j)\neq 0$ for all $j$ as claimed. It is also clear that we do not introduce any new basepoints, since $u'_N|_{C_0}(y)=0$ implies $u_N|_{C_0}(y)=0$ (put differently: being a basepoint is a closed condition).

It remains to extend the section $u'_N|_{C_0}$ to a section $u'_N$ on the whole curve. Let $C_1,\ldots,C_r$ be the components of $C$ adjacent to $C_0$ and let $q_i=C_0\cap C_i$. We need to modify the sections $u_N|_{C_i}$ in such a way that $u'_N|_{C_i}(q_i)=u'_N|_{C_0}(q_i)$.

By nondegeneracy, we can choose a section $w_i$ of $L|_{C_i}$ such that $w_i(q_i)\neq 0$. Then set:

$$u'_N|_{C_i} := u_N|_{C_i} + \epsilon\left(\frac{w_0(q_i)}{w_i(q_i)}\right)\cdot w_i$$

Then indeed we have:

$$u'_N|_{C_i}(q_i) = u_N(q_i) + \epsilon \left(\frac{w_0(q_i)}{w_i(q_i)}\right) \cdot w_i(q_i) = u_N(q_i) + \epsilon w_0(q_i) = u'_N|_{C_0}(q_i)$$

We can continue this process, replacing $C_0$ by $C_i$; since the genus of the curve is zero there are no cycles in the dual intersection graph, and so we will never come to the same component twice. In this way we obtain a new quasimap

$$((C, x_1, \ldots, x_n), L, u_0, \ldots, u_{N-1}, u'_N)$$

over $\operatorname{Spec} \mathbb{C}[\epsilon]/(\epsilon^2)$ which has no basepoints on $C_0$. We can repeat this process for all the components of $C$ (using higher powers of $\epsilon$ each time in order to ensure that we never introduce additional basepoints) and thus we obtain an infinitesimal deformation of our original quasimap which has no basepoints, as required.

$\overline{\mathcal{Q}^\circ_{0,\alpha}(\mathbb{P}^N|H,d)} \subseteq \mathcal{Q}_{0,\alpha}(\mathbb{P}^N|H,d)$: consider a family of stable quasimaps over a smooth curve $S$, such that the generic fibre lies in the nice locus. We may blow-up the source curve (a fibered surface over $S$) in the locus of basepoints (which consists of finitely many smooth points of the central fiber) and repeat this process a finite number of times in order to obtain an actual morphism to $\mathbb{P}^N$. This has the effect of adding rational tails at the basepoints in the central fibre. If the morphism is constant on any of these rational tails we may contract them, and thus we obtain a family of stable maps which pushes down along $\chi$ to our original family of quasimaps.

The general fibre is not modified at all, and so is still in the nice locus. By [Gat02, Lemma 1.9] it follows that the central fibre is a relative stable map, and then by applying $\chi$ and appealing to Lemma 3.1 it follows that the same is true for the central fibre of the family of quasimaps. □

**Corollary 3.3.** The moduli space $\mathcal{Q}_{0,\alpha}(\mathbb{P}^N|H,d)$ is irreducible of the expected dimension. Hence it has a fundamental class.

*Proof.* This holds because the moduli space is equal to the closure of the nice locus, which is irreducible of the expected dimension. □

Since the moduli space of relative quasimaps has a fundamental class, we can define *relative quasimap invariants* for the pair $(\mathbb{P}^N, H)$:

$$\left\langle \gamma_1 \psi_1^{k_1}, \ldots, \gamma_n \psi_n^{k_n} \right\rangle_{0,\alpha,d}^{\mathbb{P}^N|H} := \int_{[\mathcal{Q}_{0,\alpha}(\mathbb{P}^N|H,\beta)]} \prod_{i=1}^n \operatorname{ev}_i^* \gamma_i \cdot \psi_i^{k_i}$$

We will now establish a number of properties of the fundamental class. These immediately imply corresponding properties of the relative invariants.

**Corollary 3.4.** The comparison morphism from relative stable maps to relative quasimaps is birational. In particular it sends the fundamental class to the fundamental class.

*Proof.* This follows because the comparison morphism restricts to an isomorphism on the nice locus, which by Lemma 3.2 is a dense open subset of both spaces. □

3.2. **Proof of the recursion formula.** We wish to obtain a recursion formula relating the quasimap invariants of multiplicity $\alpha$ with the quasimap invariants of multiplicity $\alpha + e_k$, as in [Gat02, Theorem 2.6]. For $m = \alpha_k + 1$ the following section (of the pull-back of the jet bundle of the universal line bundle)

$$\sigma_k^m := x_k^* d^m_{C/\mathcal{Q}}(u_0) \in \mathrm{H}^0(\mathcal{Q}, x_k^* \mathcal{P}^m_{C/\overline{\mathcal{Q}}}(\mathcal{L}))$$

vanishes along $\mathcal{Q}_{0,\alpha+e_k}(\mathbb{P}^N|H,d)$ inside $\mathcal{Q} = \mathcal{Q}_{0,\alpha}(\mathbb{P}^N|H,d)$, and also along a number of *comb loci*. The latter parametrise quasimaps for which $x_k$ belongs to an internal component $Z \subseteq C$ (a connected component of the vanishing locus of $u_0$), such that:

$$\deg(L|_Z) + \sum_{i=1}^{r} m^{(i)} = \sum_{x_i \in Z} \alpha_i$$

Quasimap stability means that quasimaps in the comb loci cannot contain any rational tails; this is really the only difference with the case of stable maps.

Indeed, we can push forward Gathmann's recursion formula for stable maps along the comparison morphism

$$\chi \colon \mathcal{M}_{0,\alpha}(\mathbb{P}^N|H,d) \to \mathcal{Q}_{0,\alpha}(\mathbb{P}^N|H,d)$$

and, due to Corollary 3.4 above, the only terms which change are the comb loci containing rational tails. In fact these disappear, since the restriction of the comparison map to these loci has positive-dimensional fibres:

**Lemma 3.5.** Consider a rational tail component in the comb locus of the moduli space of stable maps, i.e. a moduli space of the form:

$$\mathcal{M}_{0,(m^{(i)})}(\mathbb{P}^N|H,d)$$

and assume that $Nd > 1$. Then

$$\dim\left([\mathcal{M}_{0,(m^{(i)})}(\mathbb{P}^N|H,d)] \cap \mathrm{ev}_1^*(\mathrm{pt}_H)\right) > 0$$

where $\mathrm{pt}_H \in \mathrm{A}^{N-1}(H)$ is a point class. Thus the pushforward along $\chi$ of any comb locus with a rational tail is zero.

*Proof.* This is a simple dimension count. We have

$$\begin{aligned}\dim\left([\mathcal{M}_{0,(m^{(i)})}(\mathbb{P}^N|H,d)] \cap \mathrm{ev}_1^*(\mathrm{pt}_H)\right) &= (N-3) + d(N+1) + (1-m^{(i)}) - (N-1) \\ &= (Nd-1) + (d - m^{(i)})\end{aligned}$$

from which the lemma follows because $m^{(i)} \leq d$. □

**Remark 3.6.** With an eye to the future, we remark that these rational tail components contribute nontrivially to the Gromov–Witten invariants of a Calabi–Yau hypersurface in projective space, and so their absence from the quasimap recursion formula accounts for the divergence between Gromov–Witten and quasimap invariants in the Calabi–Yau case [Gat03b, Rmk. 1.6].

Since we wish to apply the projection formula to Gathmann's recursion relation, we should express the cohomological terms which appears as pullbacks:

**Lemma 3.7.** We have:

$$\chi^*(\psi_k) = \psi_k$$
$$\chi^*(\mathrm{ev}_k^* H) = \mathrm{ev}_k^* H$$

*Proof.* It suffices to show that:

$$\chi^* x_k^* \omega_{C/\mathcal{Q}} = x_k^* \omega_{C/\mathcal{M}}$$
$$\chi^* x_k^* \mathcal{L} = \mathrm{ev}_k^* \mathcal{O}_{\mathbb{P}^N}(H)$$

This follows by considering the following diagram:

$$\begin{tikzcd}
 & & \mathbb{P}^N & \\
C_{\overline{\mathcal{M}}} \arrow[urr, bend left, "f"] \arrow[r, "\sigma^{ss}"'] \arrow[dr] & \chi^* C_{\overline{\mathcal{Q}}} \arrow[ur, dashed] \arrow[rr] \arrow[d, shift right] \arrow[dr, phantom, "\square"] & & C_{\overline{\mathcal{Q}}} \arrow[ul, dashed] \arrow[d, shift right] \\
 & \mathcal{M}_{0,\alpha}(\mathbb{P}^N|H,d) \arrow[rr, "\chi"] \arrow[ul, bend left, "x_k"] \arrow[u, shift right, bend right, "x_k"'] & & \mathcal{Q}_{0,\alpha}(\mathbb{P}^N|H,d) \arrow[u, shift right, bend right, "x_k"']
\end{tikzcd}$$

where $\sigma^{ss}$ is the strong stabilisation map which contracts the rational tails. Note that $\sigma^{ss}$ is an isomorphism near the markings. ◻

**Proposition 3.8.** Define the *quasimap comb locus* $\mathcal{D}^{\mathcal{Q}}_{\alpha,k}(\mathbb{P}^N|H,d)$ as the union of the moduli spaces

$$\mathcal{D}^{\mathcal{Q}}(\mathbb{P}^N|H,A,B,M) := \mathcal{Q}_{0,A^{(0)}\cup\{q_1,\ldots,q_r\}}(H,d_0) \times_{H^r} \prod_{i=1}^{r} \mathcal{Q}_{0,(m^{(i)})\cup\alpha^{(i)}}(\mathbb{P}^N|H,d_i)$$

where the union runs over all splittings $A = (A^{(0)},\ldots,A^{(r)})$ of the markings (inducing a splitting $(\alpha^{(0)},\ldots,\alpha^{(r)})$ of the corresponding tangency conditions), $B = (d_0,\ldots,d_r)$ of the degree and all valid multiplicites $M = (m^{(1)},\ldots,m^{(r)})$ such that the above spaces are all well-defined (in particular we require that $|A^{(0)}| + r$ and $|A^{(i)}| + 1$ are all $\geq 2$) and such that

$$d_0 + \sum_{i=1}^{r} m^{(i)} = \sum \alpha^{(0)}$$

Write $[\mathcal{D}^{\mathcal{Q}}_{\alpha,k}(\mathbb{P}^N|H,d)]$ for the sum of the (product) fundamental classes, where each term is weighted by:

$$\frac{m^{(1)}\cdots m^{(r)}}{r!}$$

Then

$$(\alpha_k\psi_k + \mathrm{ev}_k^* H)\cdot[\mathcal{Q}_{0,\alpha}(\mathbb{P}^N|H,d)] = [\mathcal{Q}_{0,\alpha+e_k}(\mathbb{P}^N|H,d)] + [\mathcal{D}^{\mathcal{Q}}_{\alpha,k}(\mathbb{P}^N|H,d)].$$

*Proof.* This follows from [Gat02, Thm. 2.6] by pushing forward along $\chi$, using the projection fomula and Lemmas 3.4, 3.5 and 3.7 . □

**Remark 3.9.** In the discussion above we have implicitly used the fact that there exists a commuting diagram of comb loci:

$$\begin{array}{ccc}\mathcal{D}^{\mathcal{M}}(\mathbb{P}^N|H,A,B,M) & \longrightarrow & \mathcal{M}_{0,\alpha}(\mathbb{P}^N|H,d) \\ \downarrow & & \downarrow \\ \mathcal{D}^{\mathcal{Q}}(\mathbb{P}^N|H,A,B,M) & \longrightarrow & \mathcal{Q}_{0,\alpha}(\mathbb{P}^N|H,d)\end{array}$$

The vertical arrow on the left is a product of comparison morphisms (notice that $H \simeq \mathbb{P}^{r-1}$). The horizontal arrow at the top is the glueing morphism which glues together the various pieces of the comb to produce a single relative stable map. Here we explain how to define the corresponding glueing morphism for quasimaps, that is, the bottom horizontal arrow.

Suppose for simplicity that $r = 1$ and consider an element of the quasimap comb locus, consisting of two quasimaps:

$$((C^0, x_1^0,\ldots,x_{n_0}^0,q^0),L^0,u_0^0,\ldots,u_N^0)$$
$$((C^1, x_1^1,\ldots,x_{n_1}^1,q^1),L^1,u_0^1,\ldots,u_N^1)$$

such that $u^0(q^0) = u^1(q^1)$ in $\mathbb{P}^N$. We want to glue these quasimaps together at $q^0$, $q^1$. The definition of the curve is obvious; we simply take:

$$C = C^0\,{}_{q^0}\sqcup_{q^1} C^1$$

On the other hand, glueing together the line bundles $L^0$ and $L^1$ to obtain a line bundle $L$ over $C$ requires a choice of scalar $\lambda \in \mathbb{G}_{\mathrm{m}}$, in order to match up the fibres over $q^i$. Furthermore if the sections are to extend as well then this scalar must be chosen in such a way that it takes $(u_0^0(q^0),\ldots,u_N^0(q^0)) \in (L^0_{q^0})^{\oplus(N+1)}$ to $(u_0^1(q^1),\ldots,u_N^1(q^1)) \in (L^1_{q^1})^{\oplus(N+1)}$. Since neither $q^0$ nor $q^1$ are basepoints (because they are markings), these tuples are nonzero, and so $\lambda$ is unique if it exists. Furthermore it exists if and only if these tuples belong to the same $\mathbb{G}_{\mathrm{m}}$-orbit in $\mathbb{A}^{N+1}$. This is precisely the statement that $u^0(q^0) = u^1(q^1) \in \mathbb{P}^N$.

Similar arguments apply for $r > 1$, and for more general toric varieties.

## 4. Recursion formula in the general case

In this section we prove the main result of this paper: a recursion formula for relative quasimap invariants of a general pair $(X, Y)$.

**Theorem 4.1.** Let $X$ be a smooth and proper toric variety and let $Y \subseteq X$ be a very ample hypersurface (not necessarily toric). Then

$$(\alpha_k \psi_k + \mathrm{ev}_k^*[Y]) \cap [\mathcal{Q}_{0,\alpha}(X|Y,\beta)]^{\mathrm{virt}} = [\mathcal{Q}_{0,\alpha+e_k}(X|Y,\beta)]^{\mathrm{virt}} + [\mathcal{D}^{\mathcal{Q}}_{\alpha,k}(X|Y,\beta)]^{\mathrm{virt}}$$

in the Chow group of $\mathcal{Q}_{0,\alpha}(X|Y,\beta)$.

We begin by defining the terms that appear in the statement.

### 4.1. The virtual class on the space of relative quasimaps.

Let $X$ and $Y$ be as in the statement of Theorem 4.1. The complete linear system associated to $\mathcal{O}_X(Y)$ defines an embedding $i : X \hookrightarrow \mathbb{P}^N$ such that $i^{-1}(H) = Y$ for a certain hyperplane $H$. By the functoriality property of quasimap spaces (see Appendix B.1) we have a map:

$$k := \mathcal{Q}(i) : \mathcal{Q}_{0,n}(X,\beta) \to \mathcal{Q}_{0,n}(\mathbb{P}^N, d)$$

where $d = i_*\beta$. Since $i$ is a closed embedding it follows that $k$ is as well. Furthermore $k$ admits a compatible perfect obstruction theory (see Appendix B.2), so we have a notion of virtual pull-back along $k$.

It is easy to show that $k$ restricts to a morphism between moduli space of relative quasimaps, and thus we have a diagram of embeddings

$$\begin{tikzcd} \mathcal{Q}_{0,\alpha}(X|Y,\beta) \arrow[r, hook, "g"] \arrow[d, hook, "f"] \arrow[dr, phantom, "\square"] & \mathcal{Q}_{0,\alpha}(\mathbb{P}^N|H,d) \arrow[d, hook, "j"] \\ \mathcal{Q}_{0,n}(X,\beta) \arrow[r, hook, "k"] & \mathcal{Q}_{0,n}(\mathbb{P}^N,d) \end{tikzcd}$$

which one can show is cartesian. As such we can define a virtual class on $\mathcal{Q}_{0,\alpha}(X|Y,\beta)$ by pullback along $k$:

$$[\mathcal{Q}_{0,\alpha}(X|Y,\beta)]^{\mathrm{virt}} := k^![\mathcal{Q}_{0,\alpha}(\mathbb{P}^N|H,d)]$$

We use this class to define relative quasimap invariants in general:

$$\langle \gamma_1\psi_1^{k_1},\ldots,\gamma_n\psi_n^{k_n}\rangle^{X|Y}_{0,\alpha,\beta} := \int_{[\mathcal{Q}_{0,\alpha}(X|Y,\beta)]^{\mathrm{virt}}} \prod_{i=1}^n \mathrm{ev}_i^*(\gamma_i)\cdot\psi_i^{k_i}$$

These invariants will play a role in our proof of the quantum Lefschetz formula in §5.

### 4.2. Relative spaces pull back.

The idea is to prove the recursion formula for general $(X, Y)$ by pulling back the formula for $(\mathbb{P}^N, H)$ along $k$. In order to do this, we need to understand how the various virtual classes involved in the formula pull back along this map. The first two terms pull back by the very definition of the virtual class:

**Lemma 4.2.** $k^![\mathcal{Q}_{0,\alpha}(\mathbb{P}^N|H,d)] = [\mathcal{Q}_{0,\alpha}(X|Y,\beta)]^{\mathrm{virt}}$

It thus remains to consider the third term, namely the virtual class of the comb locus. This is the technical heart of the proof.

4.3. **Comb loci pull back.** As in the previous section, we define $\mathcal{D}^{Q}_{\alpha,k}(X|Y,\beta)$ to be the union of the moduli spaces

$$\mathcal{D}^{Q}(X|Y,A,B,M) := Q_{0,A^{(0)}\cup\{q_1,\ldots,q_r\}}(Y,\beta^{(0)}) \times_{Y^r} \prod_{i=1}^{r} Q_{0,\alpha^{(i)}\cup(m_i)}(X|Y,\beta^{(i)})$$

where the union runs over all splittings $A = (A^{(0)},\ldots,A^{(r)})$ of the markings (inducing a splitting $(\alpha^{(0)},\ldots,\alpha^{(r)})$ of the corresponding tangency requirements), $B = (\beta^{(0)},\ldots,\beta^{(r)})$ of the curve class $\beta$ and all valid multiplicities $M = (m^{(1)},\ldots,m^{(r)})$ such that the above spaces are non-empty and such that:

$$Y\cdot\beta^{(0)} + \sum_{i=1}^{r} m^{(i)} = \sum \alpha^{(0)}$$

We refer to the $\mathcal{D}^{Q}(X|Y,A,B,M)$ as *comb loci*.

**Remark 4.3.** Note that $Y$ is not in general toric, and so we should clarify what we mean by:

$$Q(Y) = Q_{0,A^{(0)}\cup\{q_1,\ldots,q_n\}}(Y,\beta^{(0)})$$

There are two possibilities here: one is to *define* this space as the cartesian product

$$\begin{tikzcd} Q(Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Q(H) \arrow[d] \\ Q(X) \arrow[r, "k"] & Q(\mathbb{P}^N) \end{tikzcd}$$

and equip it with the virtual class pulled back along $k$:

$$[Q(Y)]^{\mathrm{virt}} := k^{!}[Q(H)]$$

Using this definition, $Q(Y)$ consists of those quasimaps in $Q(X)$ for which $u_Y \equiv 0$. This has obvious advantages from the point of view of our computations, but is conceptually unsatisfying.

On the other hand, $X$ is a GIT quotient $\mathbb{A}^{\Sigma_X(1)} /\!\!/ \mathbb{G}_{\mathrm{m}}^r$, and $Y \subseteq X$ defines a $\mathbb{G}_{\mathrm{m}}^r$-invariant subvariety $C(Y)$ of $\mathbb{A}^{\Sigma_X(1)}$, which we call the *cone of* $Y$. Then $Y$ is equal to the GIT quotient

$$Y = C(Y) /\!\!/ \mathbb{G}_{\mathrm{m}}^r$$

and so we may use the more general theory of quasimaps to GIT quotients [CFKM14] to define $Q(Y)$ and its virtual class.

In fact these two definitions of $Q(Y)$ agree: there exists an isomorphism between these moduli spaces which preserves the virtual classes. We show this in Appendix B.4.

We now construct a virtual class on the comb locus $\mathcal{D}^{Q}(X|Y,A,B,M)$. Consider the product (*not* the fibre product over $Y^r$)

$$\mathcal{E}^{Q}(X|Y,A,B,M) := Q_{0,A^{(0)}\cup\{q_1,\ldots,q_r\}}(Y,\beta^{(0)}) \times \prod_{i=1}^{r} Q_{0,\alpha^{(i)}\cup(m_i)}(X|Y,\beta^{(i)})$$

which we may endow with the product virtual class (with weighting as before):

$$[\mathcal{E}^{Q}(X|Y,A,B,M)]^{\mathrm{virt}} := \left(\frac{m^{(1)}\cdots m^{(r)}}{r!}\right)\cdot\left([Q_{0,A^{(0)}\cup\{q_1,\ldots,q_r\}}(Y,\beta^{(0)})]^{\mathrm{virt}} \times \prod_{i=1}^{r}[Q_{0,\alpha^{(i)}\cup(m_i)}(X|Y,\beta^{(i)})]^{\mathrm{virt}}\right)$$

FIX ME fix the position of the square, also below

We have the following cartesian diagram

$$\begin{tikzcd} \mathcal{D}^{Q}(X|Y,A,B,M) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{E}^{Q}(X|Y,A,B,M) \arrow[d] \\ X^r \arrow[r, "\Delta_{X^r}"] & X^r \times X^r \end{tikzcd}$$

and we can use this to define a virtual class on the comb locus:

$$[\mathcal{D}^{Q}(X|Y,A,B,M)]^{\mathrm{virt}} := \Delta^{!}_{X^r}[\mathcal{E}^{Q}(X|Y,A,B,M)]^{\mathrm{virt}}$$

The virtual class on the union $\mathcal{D}^{Q}_{\alpha,k}(X|Y,\beta)$ of the comb loci is defined to be the sum of the virtual classes $[\mathcal{D}^{Q}(X|Y,A,B,M)]^{\mathrm{virt}}$.

**Remark 4.4.** This is the same definition of the virtual class of the comb locus that we gave in §3.2 in the case $(X,Y)=(\mathbb{P}^N,H)$.

On the other hand, there is another cartesian diagram:

$$\begin{tikzcd} \coprod_{B:i_*B=B'} \mathcal{D}^{Q}(X|Y,A,B,M) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{D}^{Q}(\mathbb{P}^N|H,A,B',M) \arrow[d] \\ Q_{0,n}(X,\beta) \arrow[r, "k"] & Q_{0,n}(\mathbb{P}^N,d) \end{tikzcd}$$

Recall that we are trying to show that the virtual class of the comb locus pulls back nicely along $k$. The result that we need is:

**Lemma 4.5.** $k^{!}[\mathcal{D}^{Q}(\mathbb{P}^N|H,A,B',M)]^{\mathrm{virt}} = \sum_{B:i_*B=B'}[\mathcal{D}^{Q}(X|Y,A,B,M)]^{\mathrm{virt}}$

For the proof of Lemma 4.5, let us introduce the following shorthand notation. We fix the the data of $A$, $B'$, $M$ and set:

$$
\begin{aligned}
\mathcal{D}(X|Y) &:= \textstyle\coprod_{B:i_*B=B'} \mathcal{D}^{Q}(X|Y,A,B,M) & \mathcal{D}(\mathbb{P}^N|H) &:= \mathcal{D}^{Q}(\mathbb{P}^N|H,A,B',M)\\
\mathcal{E}(X|Y) &:= \textstyle\coprod_{B:i_*B=B'} \mathcal{E}^{Q}(X|Y,A,B,M) & \mathcal{E}(\mathbb{P}^N|H) &:= \mathcal{E}^{Q}(\mathbb{P}^N|H,A,B',M)\\
\mathcal{D}(X) &:= \textstyle\coprod_{B:i_*B=B'} \mathcal{D}^{Q}(X,A,B) & \mathcal{D}(\mathbb{P}^N) &:= \mathcal{D}^{Q}(\mathbb{P}^N,A,B')\\
\mathcal{E}(X) &:= \textstyle\coprod_{B:i_*B=B'} \mathcal{E}^{Q}(X,A,B) & \mathcal{E}(\mathbb{P}^N) &:= \mathcal{E}^{Q}(\mathbb{P}^N,A,B')\\
\mathcal{Q}(X) &:= \mathcal{Q}_{0,n}(X,\beta) & \mathcal{Q}(\mathbb{P}^N) &:= \mathcal{Q}_{0,n}(\mathbb{P}^N,i_*\beta)
\end{aligned}
$$

Here $\mathcal{D}(X)$ and $\mathcal{E}(X)$ are the centipede loci introduced in Appendix B.3; they are defined in the same way as the comb loci, except that we replace both the quasimaps to $Y$ and the relative quasimaps to $(X,Y)$ by quasimaps to $X$. There is a cartesian diagram

$$
\begin{tikzcd}
\mathcal{E}(X|Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{E}(\mathbb{P}^N|H) \arrow[d, "\theta"] \\
\mathcal{E}(X) \arrow[r] & \mathcal{E}(\mathbb{P}^N)
\end{tikzcd}
$$

and, since $\mathcal{E}(\mathbb{P}^N)$ is smooth and there is a natural fundamental class on $\mathcal{E}(\mathbb{P}^N|H)$, we have a diagonal pull-back morphism $\theta^! = \theta^!_\Delta$ (see Appendix C).

**Lemma 4.6.** $[\mathcal{E}(X|Y)]^{\mathrm{virt}} = \theta^![\mathcal{E}(X)]^{\mathrm{virt}}$

*Proof.* It suffices to check that in the following cartesian diagram

$$
\begin{tikzcd}
\mathcal{Q}(Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{Q}(H) \arrow[d, "\theta"] \\
\mathcal{Q}(X) \arrow[r] & \mathcal{Q}(\mathbb{P}^N)
\end{tikzcd}
$$

we have $\theta^![\mathcal{Q}(X)]^{\mathrm{virt}} = [\mathcal{Q}(Y)]^{\mathrm{virt}}$. Depending on one's definition of $\mathcal{Q}(Y)$ (see Remark 4.3 above) this is either true by definition or is proved in Appendix B.4. ◻

Now consider the following cartesian diagram

$$
\begin{tikzcd}
\mathcal{D}(X) \arrow[r] \arrow[d, "\varphi_X"] \arrow[dr, phantom, "\square"] & \mathcal{D}(\mathbb{P}^N) \arrow[r] \arrow[d, "\varphi_{\mathbb{P}^N}"] \arrow[dr, phantom, "\square"] & \mathfrak{M}^{\mathrm{wt}}_{A,B} \arrow[d, "\psi"] \\
\mathcal{Q}(X) \arrow[r, "k"] & \mathcal{Q}(\mathbb{P}^N) \arrow[r] & \mathfrak{M}^{\mathrm{wt}}_{0,n,\beta}
\end{tikzcd}
$$

where $\mathfrak{M}^{\mathrm{wt}}_{0,n,\beta}$ is the moduli space of prestable curves weighted by the class $\beta$ [Cos06, §2] and:

$$
\mathfrak{M}^{\mathrm{wt}}_{A,B} := \mathfrak{M}^{\mathrm{wt}}_{0,A^{(0)}\cup\{q_1^0,\ldots,q_r^0\},\beta^{(0)}} \times \prod_{i=1}^{r} \mathfrak{M}^{\mathrm{wt}}_{0,A^{(i)}\cup\{q_i^1\},\beta^{(i)}}
$$

The maps $\mathcal{D}(X) \to \mathfrak{M}^{\mathrm{wt}}_{A,B}$ and $\mathcal{Q}(X) \to \mathfrak{M}^{\mathrm{wt}}_{0,n,\beta}$ admit relative perfect obstruction theories which are the same as the usual perfect obstruction theories relative to the moduli spaces of *unweighted* curves. Furthermore the morphism $\psi$ admits a perfect obstruction theory; see Appendix B.3 for details. Thus there are virtual pull-back morphisms $\psi^!$, and by the splitting axiom (see Lemma B.7) we have

$$[\mathcal{D}(X)]^{\mathrm{virt}} := \Delta^!_{X^r}[\mathcal{E}(X)]^{\mathrm{virt}} = \psi^![\mathcal{Q}(X)]^{\mathrm{virt}}$$

Commutativity of virtual pull-backs then implies that:

$$[\mathcal{D}(X)]^{\mathrm{virt}} = \psi^![\mathcal{Q}(X)]^{\mathrm{virt}} = \psi^!k^![\mathcal{Q}(\mathbb{P}^N)] = k^!\psi^![\mathcal{Q}(\mathbb{P}^N)] = k^![\mathcal{D}(\mathbb{P}^N)] \tag{4}$$

*Proof of Lemma 4.5.* Putting all the preceding results together, we consider the cartesian digram:

$$\begin{tikzcd}
\mathcal{D}(X|Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{E}(X|Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{E}(\mathbb{P}^N|H) \arrow[d, "\theta"] \\
\mathcal{D}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{E}(X) \arrow[r] \arrow[d] & \mathcal{E}(\mathbb{P}^N) \\
X^r \arrow[r, "\Delta_{X^r}"] & X^r \times X^r &
\end{tikzcd}$$

We then have:

$$\begin{aligned}
[\mathcal{D}(X|Y)]^{\mathrm{virt}} &= \Delta^!_{X^r}[\mathcal{E}(X|Y)]^{\mathrm{virt}} && \text{by definition} \\
&= \Delta^!_{X^r}\theta^![\mathcal{E}(X)]^{\mathrm{virt}} && \text{by Lemma 4.6} \\
&= \theta^!\Delta^!_{X^r}[\mathcal{E}(X)]^{\mathrm{virt}} && \text{by commutativity} \\
&= \theta^![\mathcal{D}(X)]^{\mathrm{virt}} && \text{by definition} \\
&= \theta^!k^![\mathcal{D}(\mathbb{P}^N)] && \text{by formula (4) above} \\
&= \theta^!k^!\Delta^!_{(\mathbb{P}^N)^r}[\mathcal{E}(\mathbb{P}^N)] && \text{by definition} \\
&= k^!\Delta^!_{(\mathbb{P}^N)^r}\theta^![\mathcal{E}(\mathbb{P}^N)] && \text{by commutativity} \\
&= k^!\Delta^!_{\mathbb{P}^{N^r}}[\mathcal{E}(\mathbb{P}^N|H)] && \text{by Lemma 4.6} \\
&= k^![\mathcal{D}(\mathbb{P}^N|H)] && \text{by definition}
\end{aligned}$$

Summing over all the components of $\mathcal{D}^{\mathcal{Q}}_{\alpha,k}(\mathbb{P}^N|H,d)$ we obtain the result. □

*Proof of Theorem 4.1.* Apply $k^!$ to Proposition 3.8, using Lemmas 4.2 and 4.5. □

## 5. Quantum Lefschetz for quasimaps

In [Gat03b] Gathmann applies his recursion formula for relative stable maps to obtain a new proof of the mirror theorem for hypersurfaces [Giv98]

[LLY97]. This can be viewed as a quantum Lefschetz formula, expressing the stable map invariants of $Y$ in terms of those of $X$.

In this section we carry out a similar computation in the quasimap setting, using the recursion found in Theorem 4.1 above. We work with generating functions for 2-pointed quasimap invariants (the minimal number of markings, due to the strong stability condition). The absence of rational tails in the quasimap moduli space makes the recursion much simpler than Gathmann's. We obtain a *quantum Lefschetz theorem for quasimap invariants* (Theorem 5.2); that is, a formula which expresses the quasimap invariants of $Y$ in terms of those of $X$.

Our formula can be viewed as a special case of [CFK14, Corollary 5.5.1], and so can be interpreted as a relation between certain residues of the $\mathbb{C}^*$-action on spaces of 0-pointed and 1-pointed parametrised quasimaps to $Y$. Some of the consequences of this formula are explored in [CFK14, Section 5.5]; for instance, it follows in the semipositive case that all primary $\epsilon$-quasimap invariants with a fundamental class insertion can be expressed in terms of 2-pointed invariants.

5.1. **Setup.** As before, we let $X = X_\Sigma$ be a smooth projective toric variety and $i\colon Y \hookrightarrow X$ a smooth very ample hypersurface. We also make the following two assumptions:

(1) $Y$ *semi-positive*: $-K_Y$ is nef;
(2) $Y$ *contains all curve classes*: the map $i_* : \mathrm{A}_1(Y) \to \mathrm{A}_1(X)$ is surjective.

By adjunction, $-K_X$ pairs strictly positively with every curve class coming from $Y$, hence with every curve class by Assumption (2). Thus $-K_X$ is ample; that is, $X$ is Fano[3]. Also note that if $\dim X \geq 3$ then Assumption (2) always holds, due to the Lefschetz hyperplane theorem.

We fix a homogeneous basis $\eta_0, \ldots, \eta_k$ for $\mathrm{H}^*(X) = \mathrm{H}^*(X, \mathbb{Q})$ and let $\eta^0, \ldots, \eta^k$ denote the dual basis. Without loss of generality we may suppose that $\eta^0 = \mathbb{1}_X$ and $\eta^1 = Y$. We get an induced basis $\rho_1 = i^*\eta_1, \ldots, \rho_k = i^*\eta_k$ for $i^* \mathrm{H}^*(X)$. Notice that $\rho_0 = i^*\eta_0 = i^* \mathrm{pt}_X = 0$ and $\rho_1 = i^*\eta_1 = \mathrm{pt}_Y$.

We can extend the $\rho_i$ to a basis $\rho_1, \ldots, \rho_l$ for $\mathrm{H}^*(Y)$ by adding $\rho_{k+1} \ldots, \rho_l$. Let $\rho^1, \ldots, \rho^l$ denote the dual basis; notice that $\rho^i$ is *not* equal to $i^*\eta^i$ (they don't even have the same dimension!).

5.2. **Generating functions for quasimap invariants.** As with many results in enumerative geometry, the quantum Lefchetz formula is most naturally stated in terms of generating functions. Here we define several such generating functions for the absolute quasimap invariants of $X$ and $Y$.

We work with two marked points since this is the minimum number required in order for the quasimap space to be nonempty. However since

[3]Kleiman's criterion says that a divisor $D$ is ample if and only if $D \cdot C > 0$ for every curve class $C$ in the closure of the effective cone. But since $X$ is a toric variety the effective cone is finitely generated in $\mathrm{A}_1(X)$, hence is closed in $\mathrm{A}_1(X)_\mathbb{R}$ as it is a finite intersection of half-spaces. So we only need to check $D \cdot C > 0$ for every effective curve class.

we only take insertions at the first marking we would like to think of these, morally speaking, as 1-pointed invariants (in Gromov–Witten theory the corresponding statement is literally true, due to the string equation).

For $X$ any smooth projective toric variety[4] we define:

$$S_0^X(z,\beta) = (\mathrm{ev}_1)_*\left(\frac{1}{z-\psi_1}[Q_{0,2}(X,\beta)]^{\mathrm{virt}}\right)$$

for every effective curve class $\beta \in \mathrm{H}_2^+(X)$. Similarly we define

$$S_0^X(z,q) = \sum_{\beta \geq 0} q^\beta S_0^X(z,\beta)$$

where by convention $S_0^X(z,0) = \mathbb{1}_X$. Here $q$ is a formal Novikov variable. These are generating functions for quasimap invariants of $X$ which take values in $\mathrm{H}^*(X)$.

The same definition applies to $Y$. However, sometimes we may wish to consider only insertions of cohomology classes coming from $X$. These are the so-called *restricted quasimap invariants*, and the corresponding generating function is defined as

$$\tilde{S}_0^Y(z,\beta) = (\mathrm{ev}_1)_*\left(\frac{1}{z-\psi_1}[Q_{0,2}(Y,\beta)]^{\mathrm{virt}}\right)$$

where crucially $\mathrm{ev}_1$ is viewed as *mapping to* $X$ instead of to $Y$. Thus $\tilde{S}_0^Y(z,\beta)$ takes values in $\mathrm{H}^*(X)$ and involves only quasimap invariants of $Y$ with insertions coming from $i^*\,\mathrm{H}^*(X)$; this is in contrast to $S_0^Y(z,\beta)$, which takes values in $\mathrm{H}^*(Y)$ and involves quasimap invariants of $Y$ with arbitrary insertions. As earlier, we can also define $\tilde{S}_0^Y(z,q)$.

Now, since $X$ and $Y$ are smooth we may use Poincaré duality to define a push-forward map on cohomology denoted $i_*\colon\ \mathrm{H}^k(Y) \to \mathrm{H}^{k+2}(X)$.

**Lemma 5.1.** $i_*S_0^Y(z,\beta) = \tilde{S}_0^Y(z,\beta)$.

*Proof.* This essentially follows by functoriality of cohomological push-forwards and the fact that we have a commuting triangle:

$$\begin{tikzcd} Q_{0,2}(Y,\beta) \arrow[r, "\mathrm{ev}_1"] \arrow[dr, "\mathrm{ev}_1"'] & Y \arrow[d, "i"] \\ & X \end{tikzcd}$$

However we will give a more concrete proof, in order to help familiarise the reader with the generating functions involved. First it is easy to see from

[4]or more generally any space for which the quasimap invariants are defined, for instance a smooth hypersurface in a toric variety.

the projection formula that:

$$i_*\rho^i = \begin{cases} \eta^i & \text{for } i = 1, \ldots, k \\ 0 & \text{for } i = k+1, \ldots, l \end{cases}$$

Now, we can write $S_0^Y(z,\beta)$ as:

$$S_0^Y(z,\beta) = \sum_{i=1}^{l} \left\langle \frac{\rho_i}{z-\psi_1}, \mathbb{1}_Y \right\rangle_{0,2,\beta}^{Y} \rho^i$$

Thus applying $i_*$ gives

$$i_* S_0^Y(z,\beta) = \sum_{i=1}^{l} \left\langle \frac{\rho_i}{z-\psi_1}, \mathbb{1}_Y \right\rangle_{0,2,\beta}^{Y} i_*\rho^i = \sum_{i=1}^{k} \left\langle \frac{\eta_i}{z-\psi_1}, \mathbb{1}_X \right\rangle_{0,2,\beta}^{Y} \eta^i = \tilde{S}_0^Y(z,\beta)$$

as claimed. □

5.3. **Quantum Lefschetz formula.** We now turn to our main result: a formula expressing the generating function $\tilde{S}_0^Y(z,q)$ for (restricted) quasimap invariants of $Y$ in terms of the quasimap invariants of $X$.

**Theorem 5.2.** Let $X$ and $Y$ be as above. Then

$$\frac{\sum_{\beta \geq 0} q^\beta \prod_{j=0}^{Y\cdot\beta}(Y+jz) S_0^X(z,\beta)}{P_0^X(q)} = \tilde{S}_0^Y(z,q) \tag{5}$$

where:

$$\begin{aligned} P_0^X(q) &= 1 + \sum_{\substack{\beta>0 \\ K_Y\cdot\beta=0}} q^\beta (Y\cdot\beta) \langle [\mathrm{pt}_Y], \mathbb{1}_X \rangle_{0,(Y\cdot\beta,0),\beta}^{X|Y} \\ &= 1 + \sum_{\substack{\beta>0 \\ K_Y\cdot\beta=0}} q^\beta (Y\cdot\beta)! \langle [\mathrm{pt}_X]\psi_1^{Y\cdot\beta-1}, \mathbb{1}_X \rangle_{0,2,\beta}^{X} \end{aligned}$$

Notice that $P_0^X(q)$ depends not only on $X$ but also on the divisor class of $Y$ in $X$; the superscript is supposed to indicate that the definition only involves quasimap invariants of $X$.

*Proof.* Define for $m = 0, \ldots, Y\cdot\beta$ the following generating function for 2-pointed relative quasimap invariants

$$S_{0,(m)}^{X|Y}(z,\beta) = (\mathrm{ev}_1)_* \left( \frac{1}{z-\psi_1} [\mathcal{Q}_{0,(m,0)}(X|Y,\beta)]^{\mathrm{virt}} \right)$$

where we view $\mathrm{ev}_1$ as mapping to $X$. This coincides with the absolute $S_0^X$-function defined above when $m=0$. Also define the following generating function for "comb loci invariants"

$$T_{0,(m)}^{X|Y}(z,\beta) = (\mathrm{ev}_1)_* \left( m[\mathcal{Q}_{0,(m,0)}(X|Y,\beta)]^{\mathrm{virt}} + \frac{1}{z-\psi_1} [\mathcal{D}_{(m,0),1}^{\mathcal{Q}}(X|Y,\beta)]^{\mathrm{virt}} \right)$$

where again we view $\mathrm{ev}_1$ as mapping to $X$. As in [Gat03b, Lemma 1.2], it follows from Theorem 4.1 that

$$(Y + mz)S^{X|Y}_{0,(m)}(z,\beta) = S^{X|Y}_{0,(m+1)}(z,\beta) + T^{X|Y}_{0,(m)}(z,\beta) \tag{6}$$

and we can apply this repeatedly to obtain:

$$\prod_{j=0}^{Y\cdot\beta}(Y + jz)S^{X}_{0}(z,\beta) = \sum_{m=0}^{Y\cdot\beta}\prod_{j=m+1}^{Y\cdot\beta}(Y + jz)T^{X|Y}_{0,(m)}(z,\beta) \tag{7}$$

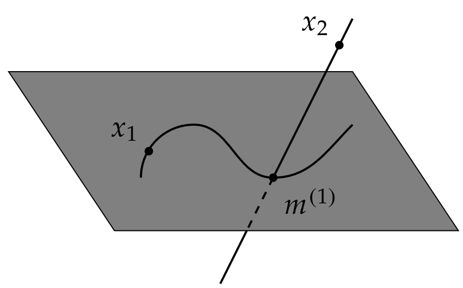
We now examine the right-hand side in detail. By definition, $T^{X|Y}_{0,(m)}(z,\beta)$ splits into two parts: those terms coming from the relative space and those terms coming from the comb loci.

Let us first consider the contribution of the comb loci. Since there are only two marked points and the first is required to lie on the internal component of the comb, it follows from the strong stability condition that there are only two options: a comb with 0 teeth or a comb with 1 tooth.

First consider the case of a comb with 0 teeth. The moduli space is then $\mathcal{Q}_{0,2}(Y,\beta)$ and we require that $Y \cdot \beta = m$. Thus this piece only contributes to $T^{X|Y}_{0,(Y\cdot\beta)}(z,\beta)$, and the contribution is:

$$\sum_{i=1}^{k}\left\langle \frac{\rho_i}{z-\psi_1}, \mathbb{1}_Y \right\rangle^{Y}_{0,2,\beta}\eta^i$$

Next consider the case of a comb with 1 tooth. Let $\beta^{(0)}$ and $\beta^{(1)}$ denote the curve classes of the internal and external components, respectively, and let $m^{(1)}$ be the contact order of the external component with $Y$. The picture is as follows




and the invariants which contribute take the form

$$\left\langle \frac{\rho_i}{z-\psi_1}, \rho^h \right\rangle^{Y}_{0,2,\beta^{(0)}}\left\langle \rho_h, \mathbb{1}_X \right\rangle^{X|Y}_{0,(m^{(1)},0),\beta^{(1)}}$$

for $i = 1, \ldots, k$ and $h = 1, \ldots, l$. By computing dimensions, we find

$$\begin{aligned} 0 \leq \operatorname{codim} \rho^h &= \dim Y - \operatorname{codim} \rho_h \\ &= \dim Y - \operatorname{vdim} Q_{0,(m^{(1)},0)}(X|Y, \beta^{(1)}) \\ &= \dim Y - (\dim X - 3 - K_X \cdot \beta^{(1)} + 2 - m^{(1)}) \\ &= K_Y \cdot \beta^{(1)} - Y \cdot \beta^{(1)} + m^{(1)} \\ &\leq 0 \end{aligned}$$

where the final equality follows from adjunction and the final inequality holds because $-K_Y$ is nef and $m^{(1)} \leq Y \cdot \beta_1$. This shows that the only non-trivial contributions come from curve classes $\beta^{(1)}$ such that $K_Y \cdot \beta^{(1)} = 0$, and that in this case the order of tangency must be maximal, i.e. $m^{(1)} = Y \cdot \beta^{(1)}$. Furthermore we must have $\operatorname{codim} \rho^h = 0$ and so $\rho^h = \rho^1 = \mathbb{1}_Y$ which implies $\rho_h = \rho_1 = \mathrm{pt}_Y$. Finally since $m^{(1)} = Y \cdot \beta^{(1)}$ we have

$$m = Y \cdot \beta^{(0)} + m^{(1)} = Y \cdot (\beta^{(0)} + \beta^{(1)}) = Y \cdot \beta$$

and so again this piece only contrbutes to $T^{X|Y}_{0,(Y\cdot\beta)}(z, \beta)$, and the contribution is:

$$\sum_{i=1}^{k} \left( \sum_{\substack{0<\beta^{(1)}<\beta \\ K_Y\cdot\beta^{(1)}=0}} (Y \cdot \beta^{(1)}) \left\langle \frac{\rho_i}{z - \psi_1}, \mathbb{1}_Y \right\rangle^{Y}_{0,2,\beta-\beta^{(1)}} \left\langle \mathrm{pt}_Y, \mathbb{1}_X \right\rangle^{X|Y}_{0,(Y\cdot\beta^{(1)},0),\beta^{(1)}} \right) \eta^i$$

where the $Y \cdot \beta^{(1)}$ factor comes from the weighting on the virtual class of the comb locus. Finally, we must examine the terms of $T^{X|Y}_{0,(m)}(z, \beta)$ coming from:

$$\mathrm{ev}_{1*}(m[Q_{0,(m,0)}(X|Y, \beta)]^{\mathrm{virt}})$$

Notice that we only have insertions from $i^* \mathrm{H}^*(X) \subseteq \mathrm{H}^*(Y)$, since $\mathrm{ev}_1$ is viewed as mapping to $X$. On the other hand

$$\begin{aligned} \operatorname{vdim} Q_{0,(m,0)}(X|Y, \beta) &= \dim X - 3 - K_X \cdot \beta + 2 - m \\ &= \dim X - 1 - K_Y \cdot \beta + Y \cdot \beta - m && \text{by adjunction} \\ &\geq \dim X - 1 + Y \cdot \beta - m && \text{since } -K_Y \text{ is nef} \\ &\geq \dim X - 1 && \text{since } m \leq Y \cdot \beta \end{aligned}$$

where in the second line we have applied the projection formula to $i$, and thus have implicity used Assumption (2), discussed in §5.1; namely that every curve class on $X$ comes from a class on $Y$.

Consequently the only insertions that can appear are those of dimension 0 and 1. However, the restriction of the 0-dimensional class $\eta_0 = \mathrm{pt}_X$ to $Y$ vanishes, as do the restrictions of all 1-dimensional classes except for $\eta_1$ (by the definition of the dual basis, since $\eta^1 = Y$). Thus the only insertion is $i^*\eta_1 = \rho_1 = \mathrm{pt}_Y$, and since $\eta^1$ has dimension 1 all the inequalities above must actually be equalities. Thus we only have a contribution if $-K_Y \cdot \beta = 0$ and $m = Y \cdot \beta$. The contribution to $T^{X|Y}_{0,(Y\cdot\beta)}(z,\beta)$ in this case is:

$$(Y \cdot \beta)\langle \mathrm{pt}_Y, \mathbb{1}_Y \rangle^{X|Y}_{0,(Y\cdot\beta,0),\beta}\eta^1$$

Thus we have calculated $T^{X|Y}_{0,(m)}(z,\beta)$ for all $m$; substituting into equation (7) we obtain

$$\begin{aligned}
\prod_{j=0}^{Y\cdot\beta}(Y+jz)S^X_0(z,\beta) &= T^{X|Y}_{0,(Y\cdot\beta)}(z,\beta) \\
&= \sum_{i=1}^{k}\left\langle \frac{\rho_i}{z-\psi_1}, \mathbb{1}_Y \right\rangle^Y_{0,2,\beta}\eta^i + \\
&\quad \sum_{i=1}^{k}\left(\sum_{\substack{0<\beta^{(1)}<\beta \\ K_Y\cdot\beta^{(1)}=0}}(Y\cdot\beta^{(1)})\left\langle \frac{\rho_i}{z-\psi_1}, \mathbb{1}_Y \right\rangle^Y_{0,2,\beta-\beta^{(1)}}\left\langle \mathrm{pt}_Y, \mathbb{1}_X \right\rangle^{X|Y}_{0,(Y\cdot\beta^{(1)},0),\beta^{(1)}}\right)\eta^i + \\
&\quad (Y\cdot\beta)\langle \mathrm{pt}_Y, \mathbb{1}_Y \rangle^{X|Y}_{0,(Y\cdot\beta,0),\beta}\eta^1
\end{aligned}$$

where the third term only appears if $K_Y \cdot \beta = 0$. We can rewrite this as:

$$\begin{aligned}
&\prod_{j=0}^{Y\cdot\beta}(Y+jz)S^X_0(z,\beta) \\
&\qquad = \tilde{S}^Y_0(z,\beta) + \sum_{\substack{0<\beta^{(1)}\leq\beta \\ K_Y\cdot\beta^{(1)}=0}}\left((Y\cdot\beta^{(1)})\left\langle \mathrm{pt}_Y, \mathbb{1}_X \right\rangle^{X|Y}_{0,(Y\cdot\beta^{(1)},0),\beta^{(1)}}\right)\tilde{S}^Y_0(z,\beta-\beta^{(1)})
\end{aligned}$$

It is now clear from the expression above that equation (5) in the statement of Theorem 5.2 holds, with:

$$P^X_0(q) = 1 + \sum_{\substack{\beta>0 \\ K_Y\cdot\beta=0}} q^\beta (Y\cdot\beta)\langle \mathrm{pt}_Y, \mathbb{1}_X \rangle^{X|Y}_{0,(Y\cdot\beta,0),\beta}$$

To complete the proof it thus remains to show that:

$$P^X_0(q) = 1 + \sum_{\substack{\beta>0 \\ K_Y\cdot\beta=0}} q^\beta (Y\cdot\beta)!\langle \psi_1^{Y\cdot\beta-1}\mathrm{pt}_X, \mathbb{1}_X \rangle^X_{0,2,\beta}$$

The aim therefore is to express the invariants

$$\langle [\mathrm{pt}_Y], \mathbb{1}_X \rangle^{X|Y}_{0,(Y\cdot\beta,0),\beta}$$

in terms of absolute invariants of $X$. It is thus not surprising that we have to apply the recursion from Theorem 4.1. We have:

$$[\mathcal{Q}_{0,(Y\cdot\beta,0)}(X|Y,\beta)]^{\mathrm{virt}} = ((Y\cdot\beta-1)\psi_1 + \mathrm{ev}_1^* Y)[\mathcal{Q}_{0,(Y\cdot\beta-1,0)}(X|Y,\beta)]^{\mathrm{virt}} - [\mathcal{D}^{\mathcal{Q}}_{(Y\cdot\beta-1,0),1}(X|Y,\beta)]^{\mathrm{virt}}$$

We begin by examining the contributions from the comb loci. As before, we have the contributions coming from combs with 0 teeth and combs with 1 tooth. The former contributions take the form

$$\langle [\mathrm{pt}_Y], \mathbb{1}_X\rangle^{Y}_{0,2,\beta}$$

which vanish because $\mathrm{vdim}\,\mathcal{Q}_{0,2}(Y,\beta) = \dim Y - 1 - K_Y\cdot\beta = \dim Y - 1$ whereas the insertion has codimension $\dim Y$. On the other hand, the latter contributions take the form

$$\langle [pt_Y], \rho^h\rangle^{Y}_{0,2,\beta^{(0)}} \langle \rho_h, \mathbb{1}_X\rangle^{X|Y}_{0,(Y\cdot(\beta-\beta^{(0)})-1,0),\beta-\beta^{(0)}}$$

and these must also vanish since:

$$\begin{aligned}
\mathrm{codim}\,\rho^h &= \dim Y - \mathrm{codim}\,\rho_h \\
&= \dim Y - \mathrm{vdim}\,\mathcal{Q}_{0,(Y\cdot(\beta-\beta^{(0)})-1,0)}(X|Y,\beta-\beta^{(0)}) \\
&= \dim Y - (\dim X - 3 - K_X\cdot(\beta-\beta^{(0)}) + 2 - Y\cdot(\beta-\beta^{(0)}) + 1) \\
&= -1 + K_X\cdot(\beta-\beta^{(0)}) + Y\cdot(\beta-\beta^{(0)}) \\
&= -1 + K_Y\cdot(\beta-\beta^{(0)}) \\
&\leq -1
\end{aligned}$$

Thus the comb loci do not contribute at all. Applying this recursively (the same argument as above shows that we never get comb loci contributions), we find that

$$\begin{aligned}
(Y\cdot\beta)\langle [\mathrm{pt}_Y], \mathbb{1}_X\rangle^{X|Y}_{0,(Y\cdot\beta,0),\beta} &= (Y\cdot\beta)\langle \rho_1, \mathbb{1}_X\rangle^{X|Y}_{0,(Y\cdot\beta,0),\beta} \\
&= (Y\cdot\beta)\langle \eta_1 \prod_{j=0}^{Y\cdot\beta-1}(Y+j\psi_1), \mathbb{1}_X\rangle^{X}_{0,2,\beta} \\
&= (Y\cdot\beta)!\langle [\mathrm{pt}_X]\psi_1^{Y\cdot\beta-1}, \mathbb{1}_X\rangle^{X}_{0,2,\beta}
\end{aligned}$$

where the second equality holds because $Y\cdot\eta_1 = \eta^1\cdot\eta_1 = [\mathrm{pt}_X]$ and $Y^2\cdot\eta_1 = 0$. This completes the proof of Theorem 5.2. □

**Corollary 5.3.** If $Y$ is Fano then there is no correction term:

$$\sum_{\beta\geq 0} q^\beta \prod_{j=0}^{Y\cdot\beta}(Y+jz)S_0^X(z,\beta) = \tilde{S}_0^Y(z,q)$$

**Corollary 5.4.** Let $Y = Y_5 \subseteq \mathbb{P}^4 = X$ be the quintic three-fold. Then

$$\tilde{S}_0^{Y_5}(z,q) = \frac{\mathbb{I}_{\mathrm{sm}}^{Y_5}(z,q)}{P^{Y_5}(q)}$$

where

$$\mathbb{I}_{\mathrm{sm}}^{Y_5}(z,q) = 5H + \sum_{d>0} \frac{\prod_{j=0}^{5d}(H+jz)}{\prod_{j=0}^{d}(H+jz)^5}\, q^d$$

and:

$$P^{Y_5}(q) = 1 + \sum_{d>0} \frac{(5d)!}{(d!)^5} q^d$$

*Proof.* Apply Theorem 5.2 and use the fact that the quasimap invariants of $\mathbb{P}^4$ coincide with the Gromov–Witten invariants, which are well-known from mirror symmetry. □

**Remark 5.5.** Theorem 5.2 agrees with [CZ14, Theorem 1] when both are applicable, namely when $X$ is a projective space.

5.4. **Comparison with the work of Ciocan-Fontanine and Kim.** Here we briefly explain how to compare our Theorem 5.2 to a formula obtained by Ciocan-Fontanine and Kim. We assume that the reader is familiar with the paper [CFK14], in particular §4 and §5. There they introduce (in the more general context of $\epsilon$-stable quasimaps) the following generating functions for quasimap invariants of $Y$:

(1) The $J^\epsilon$*-function*

$$J^\epsilon(\mathbf{t},z) = \sum_{m\geq 0,\beta\geq 0} \frac{q^\beta}{m!}(\mathrm{ev}_\bullet)_*\left(\prod_{i=1}^{m} \mathrm{ev}_i^*(\mathbf{t}) \cap \mathrm{Res}_{F_0}[QG_{0,m}^\epsilon(Y,\beta)]^{\mathrm{virt}}\right)$$

for $\mathbf{t} \in \mathrm{H}^*(X)$. Here $QG_{0,m}^\epsilon(Y,\beta)$ is the moduli space of $\epsilon$-stable quasimaps with a parametrised component, $F_0$ is a certain fixed locus of the natural $\mathbb{G}_{\mathrm{m}}$-action on this space, and $\mathrm{ev}_\bullet$ is the evaluation at the point $0 \in \mathbb{P}^1$ on the parametrised component. $\mathrm{Res}_{F_0}$ is the residue of the virtual class, i.e. the virtual class of the fixed locus divided by the Euler class of the virtual normal bundle (see [GP99] for details on virtual localisation). The variable $z$ is the $\mathbb{G}_{\mathrm{m}}$-equivariant parameter.

(2) The $S^\epsilon$*-operator*

$$S^\epsilon(\mathbf{t},z)(\gamma) = \sum_{m\geq 0,\beta\geq 0} \frac{q^\beta}{m!}(\mathrm{ev}_1)_*\left(\frac{\mathrm{ev}_2^*(\gamma)\cdot\prod_{j=3}^{2+m}\mathrm{ev}_j^*(\mathbf{t})}{z-\psi_1} \cap [Q_{0,2+m}^\epsilon(Y,\beta)]^{\mathrm{virt}}\right)$$

where $\mathbf{t},\gamma \in \mathrm{H}^*(X)$ and $z$ is a formal variable.

(3) The $P^\epsilon$*-series*

$$P^\epsilon(\mathbf{t},z) = \sum_{h=1}^{k} \rho^h \sum_{m\geq 0,\beta\geq 0} \frac{q^\beta}{m!}\left(\mathrm{ev}_1^*(\rho_h \boxtimes p_\infty) \cap [QG_{0,1+m}^\epsilon(Y,\beta)]^{\mathrm{virt}}\right)$$

where $\mathbf{t} \in \mathrm{H}^*(X)$ and $z$ is the $\mathbb{G}_\mathrm{m}$-equivariant parameter. Here we view $\mathrm{ev}_1$ as mapping to $Y \times \mathbb{P}^1$, and $p_\infty \in \mathrm{H}^*_{\mathbb{G}_\mathrm{m}}(\mathbb{P}^1)$ is the equivariant cohomology class defined by setting $p_\infty|_0 = 0$ and $p_\infty|_\infty = -z$ (where 0 and $\infty$ are the $\mathbb{G}_\mathrm{m}$-fixed points of $\mathbb{P}^1$). Thus $\mathrm{ev}_1^* p_\infty$ should be thought of as a Kronecker delta function: it is zero if the first marked point lies over 0, and $-z$ if it lies over $\infty$.

Given these definitions, Ciocan-Fontanine and Kim use localisation with respect to the $\mathbb{G}_\mathrm{m}$-action on the parametrised space to prove the following formula [CFK14, Theorem 5.4.1]:

$$J^\epsilon(\mathbf{t}, z) = S^\epsilon(\mathbf{t}, z)(P^\epsilon(\mathbf{t}, z))$$

They then observe that if we set $\mathbf{t} = 0$ and restrict to semi-positive targets, then the only class that matches non-trivially with $P^\epsilon|_{\mathbf{t}=0}$ is $[\mathrm{pt}_Y]$. The above formula then takes the simple form

$$\frac{J^\epsilon|_{\mathbf{t}=0}}{\langle [\mathrm{pt}_Y], P^\epsilon|_{\mathbf{t}=0}\rangle} = S^\epsilon(\mathbb{1}_Y)|_{\mathbf{t}=0} = \mathbb{1}_Y + \sum_{h=1}^{k} \rho^h \left( \sum_{\beta>0} q^\beta \left\langle \frac{\rho_h}{z-\psi}, \mathbb{1}_Y \right\rangle^{Y,\epsilon}_{0,2,\beta} \right) \tag{8}$$

see [CFK14, Corollary 5.5.1]. In our setting, $\epsilon = 0+$ and $Y$ embeds as a very ample hypersurface in a toric Fano variety $X$. We claim that in this case the formula above is equivalent to Theorem 5.2. More precisely:

**Lemma 5.6.** We have the following relations between our generating functions and the generating functions of Ciocan-Fontanine and Kim:

$$J^{0+}|_{\mathbf{t}=0} = \sum_{\beta \geq 0} q^\beta \prod_{j=0}^{Y\cdot\beta} (Y + jz) S_0^X(z, \beta) \tag{9}$$

$$\langle [\mathrm{pt}_Y], P^{0+}|_{\mathbf{t}=0}\rangle = P_0^X(q) \tag{10}$$

$$S^{0+}(\mathbb{1}_Y)|_{\mathbf{t}=0} = \tilde{S}_0^Y(z, q) \tag{11}$$

*Proof.* (11) is clear from the second equality of (8) and the definition of $\tilde{S}_0^Y(z,q)$. To show (9), let us look more closely at the left-hand side:

$$J^{0+}|_{\mathbf{t}=0} = \sum_{\beta\geq 0} q^\beta (\mathrm{ev}_\bullet)_* \left( \mathrm{Res}_{F_0} [\mathcal{Q}G_{0,0}(Y,\beta)]^{\mathrm{virt}} \right)$$

We have a diagram of fixed loci and evaluation maps

$$\begin{tikzcd} \mathcal{Q}G_{0,0}(Y,\beta) \arrow[d, hook, "i"] \arrow[dr, phantom, "\square"] & F_0^Y \arrow[l, hook'] \arrow[r, "\mathrm{ev}_\bullet"] \arrow[d, hook, "i"] & Y \arrow[d, hook, "i"] \\ \mathcal{Q}G_{0,0}(X,\beta) & F_0^X \arrow[l, hook'] \arrow[r, "\mathrm{ev}_\bullet"] & X \end{tikzcd}$$

and by a mild generalisation of [CFKM14, Propositions 6.2.2 and 6.2.3], we have an equality of $\mathbb{G}_\mathrm{m}$-equivariant classes

$$i_*[\mathcal{Q}G_{0,0}(Y,\beta)]^{\mathrm{virt}} = e(\pi_* E^Y_{0,0,\beta}) \cap [\mathcal{Q}G_{0,0}(X,\beta)]^{\mathrm{virt}}$$

where $\pi$ is the universal curve on $QG_{0,0}(X,\beta)$ and $E^Y_{0,0,\beta}$ is the equivariant line bundle on this curve associated to $\mathcal{O}_X(Y)$[5].

We would like to pull-back this equation to the fixed locus $F^X_0$ in order to obtain an equation involving the residues. Let us first briefly recall the definition of $F^X_0$. Since there are no markings, any quasimap in $QG_{0,0}(X,\beta)$ has irreducible source curve. For such a quasimap to be $\mathbb{G}_{\mathrm{m}}$-fixed we need that the degree of the quasimap is concentrated at the basepoints (i.e. the sum of the lengths of the basepoints should be equal to the degree); this is equivalent to requiring that the induced rational map is constant. Furthermore only the points 0 and $\infty$ of the parametrised component are allowed to be basepoints. The fixed loci are thus indexed by ordered partitions of the degree which record the length of the basepoints at 0 and $\infty$. $F^X_0$ is the locus on which all the degree is concentrated at $\infty$. This means that 0 is not a basepoint and we have an evaluation map $\mathrm{ev}_0$ (denoted $\mathrm{ev}_\bullet$ earlier). See [CFK14, §4] for more details: our $F^X_0$ is there denoted $F^{0,0,0}_{0,0,\beta}$.

Since the fibres of $\pi$ are irreducible, we have for $j \geq 0$ an exact sequence:

$$0 \to \pi_*(E^Y_{0,0,\beta}(-j\sigma_0)) \to \pi_* E^Y_{0,0,\beta} \to \sigma_0^* \mathcal{P}^{j-1}(E^Y_{0,0,\beta}) \to 0$$

where $\mathcal{P}^{j-1}$ denotes bundle of $(j-1)$-jets, and $\sigma_0$ is the section given by the point $0 \in \mathbb{P}^1$ of the parametrised component. The right-hand map is given by evaluating a section of $E^Y_{0,0,\beta}$ at the point 0 (as well as its derivatives up to order $j-1$). The left-hand term consists of sections of $E^Y_{0,0,\beta}$ which vanish at $\sigma_0$ to order $j$. If we set $j = Y \cdot \beta + 1$ then this term vanishes and we have:

$$\pi_* E^Y_{0,0,\beta} = \sigma_0^* \mathcal{P}^{Y\cdot\beta}(E^Y_{0,0,\beta})$$

On the other hand, we have

$$0 \to E^Y_{0,0,\beta} \otimes \omega_\pi^{\otimes j} \to \mathcal{P}^j(E^Y_{0,0,\beta}) \to \mathcal{P}^{j-1}(E^Y_{0,0,\beta}) \to 0$$

see [Gat02, §2]. Pulling back along $\sigma_0$ and taking Euler classes, we can compute recursively from $j = Y \cdot \beta$ to 0 and obtain a splitting

$$e(\pi_* E^Y_{0,0,\beta}) = \prod_{j=0}^{Y\cdot\beta} c_1(\sigma_0^* E^Y_{0,0,\beta} \otimes \omega_0^{\otimes j})$$

where $\omega_0 = \sigma_0^* \omega_\pi$ gives the cotangent space at the point 0. The bundle $\omega_0$ is (non-equivariantly) trivial since the source curves in $F^X_0$ are rigid; on the other hand the weight of the $\mathbb{G}_{\mathrm{m}}$-action on the cotangent space at 0 is $z$. We thus obtain:

$$i_*[F^Y_0]^{\mathrm{virt}} = \prod_{j=0}^{Y\cdot\beta} (\mathrm{ev}_0^* Y + jz) \cap [F^X_0]^{\mathrm{virt}}$$

[5]This is the parametrised analogue of the bundle $L_Y$ constructed in the definition of relative quasimaps; see §2.3.

Substituting this into $J^{0+}|_{\mathbf{t}=0}$ we find that:

$$\begin{aligned} J^{0+}|_{\mathbf{t}=0} &= \sum_{\beta \geq 0} q^\beta (\mathrm{ev}_\bullet)_* \left( \mathrm{Res}_{F_0^Y} [QG_{0,0}(Y,\beta)]^{\mathrm{virt}} \right) \\ &= \sum_{\beta \geq 0} q^\beta \prod_{j=0}^{Y\cdot\beta} (Y + jz) (\mathrm{ev}_\bullet)_* \left( \mathrm{Res}_{F_0^X} [QG_{0,0}(X,\beta)]^{\mathrm{virt}} \right) \end{aligned}$$

On the other hand, if we apply (8) with $X$ instead of $Y$, then the denominator on the left-hand side vanishes since $X$ is Fano. Comparing coefficients of $q^\beta$ we thus obtain

$$(\mathrm{ev}_\bullet)_* \, \mathrm{Res}_{F_0^X} [QG_{0,0}(X,\beta)]^{\mathrm{virt}} = S_0^X(z,\beta)$$

from which it follows that

$$J^{0+}|_{\mathbf{t}=0} = \sum_{\beta \geq 0} q^\beta \prod_{j=0}^{Y\cdot\beta} (Y + jz) S_0^X(z,\beta)$$

which proves (11).

It remains to show (10). Ciocan-Fontanine and Kim show that if we write the $1/z$-expansion of $J^\epsilon|_{\mathbf{t}=0}$ as

$$J^\epsilon|_{\mathbf{t}=0} = J_0^\epsilon(q) \mathbb{1}_Y + O(1/z)$$

then $\langle [\mathrm{pt}_Y], P^\epsilon|_{\mathbf{t}=0} \rangle = J_0^\epsilon(q)$. It thus remains to show that $J_0^{0+}(q) = P_0^X(q)$.

Since $X$ is a toric Fano variety, we have the following mirror symmetry calculation due to Givental [Giv96] (see also [CFK10, Definition 7.2.8]):

$$S_0^X(z,\beta) = \prod_{\rho \in \Sigma_X(1)} \frac{\prod_{j=-\infty}^{0} (D_\rho + jz)}{\prod_{j=-\infty}^{D_\rho\cdot\beta} (D_\rho + jz)} = \frac{\prod_{\rho \colon D_\rho\cdot\beta \leq 0} \prod_{j=D_\rho\cdot\beta}^{0} (D_\rho + jz)}{\prod_{\rho \colon D_\rho\cdot\beta > 0} \prod_{j=1}^{D_\rho\cdot\beta} (D_\rho + jz)}$$

We can then apply equation (11) to find $J^{0+}|_{\mathbf{t}=0}$, and hence also to find $J_0^{0+}(q)$. In the end we obtain:

$$J_0^{0+}(q) = \sum_{\beta \geq 0} q^\beta (Y\cdot\beta)! \frac{\prod_{\rho \colon D_\rho\cdot\beta < 0} (-1)^{-D_\rho\cdot\beta} (-D_\rho\cdot\beta)!}{\prod_{\rho \colon D_\rho\cdot\beta > 0} (D_\rho\cdot\beta)!}$$

On the other hand the coefficient

$$\langle [\mathrm{pt}_X] \psi_1^{Y\cdot\beta - 1}, \mathbb{1}_X \rangle_{0,2,\beta}^X$$

which appears in our $P_0^X(q)$-series also appears in $S_0^X(q,\beta)$. So again we can find it by appealing to Givental's calculation of $S_0^X(z,\beta)$.

$$\begin{aligned} \langle [\mathrm{pt}_X] \psi_1^{Y\cdot\beta - 1}, \mathbb{1}_X \rangle_{0,2,\beta}^X &= \mathrm{coeff}_{q^\beta z^{Y\cdot\beta}} \langle [\mathrm{pt}_X], S_0^X(z,\beta) \rangle \\ &= \frac{\prod_{\rho \colon D_\rho\cdot\beta < 0} (-1)^{-D_\rho\cdot\beta} (-D_\rho\cdot\beta)!}{\prod_{\rho \colon D_\rho\cdot\beta > 0} (D_\rho\cdot\beta)!} \end{aligned}$$

which proves (10). We may thus conclude that (8) coincides with our Theorem 5.2. □

## Appendix A. The comparison morphism

In this appendix we recall the construction of the comparison morphism for $\mathbb{P}^N$ and how it can be used to show that the stable map and the quasimap invariants of projective space coincide. This has been proven in [MOP11, Theorem 3] and [Man12b, Section 4.3] (but see also [Ber00, Proposition 4.1] and [PR03, Theorem 7.1] for inspiration). We will aim to provide as many details as possible, for our own benefit and, hopefully, that of the novice reader.

A.1. **Construction of the comparison morphism.** In order to give a morphism $\chi\colon \mathcal{M}_{g,n}(\mathbb{P}^N, d) \to \mathcal{Q}_{g,n}(\mathbb{P}^N, d)$ we need to be able to associate, to a family of stable maps on a base $T$, a family of quasimaps on the same base.

The pointwise construction is as follows: any stable map defines a quasimap with no basepoints. The only thing that might prevent this quasimap from being stable is the presence of rational tails (of positive degree, by the stability condition for stable maps). Let $C = C^{(0)} \sqcup_{q_i} R_i$ be the source curve; there is a "permanent" subcurve $C^{(0)}$ which is joined to a number of rational tails $R_i$ each of which has degree $d_i > 0$. We let $q_i$ be the node connecting $C^{(0)}$ and $R_i$; note that it is the only special point of $R_i$, and hence all the marked points belong to $C^{(0)}$. We obtain a stable quasimap by collapsing the rational tails and introducing basepoints of length $d_i$ at each of the (former) nodes $q_i$.

In other words, the comparison map is given by sending the quasimap $((C, x_1, \ldots, x_n), L, u_0, \ldots, u_N))$ to

$$((C^{(0)}, x_1, \ldots, x_n), L|_{C^{(0)}}(\Sigma_i d_i q_i), \hat{u_0}, \ldots, \hat{u_N})$$

where $\hat{u_i}$ is obtained from $u_i|_{C^{(0)}}$ via the natural inclusion:

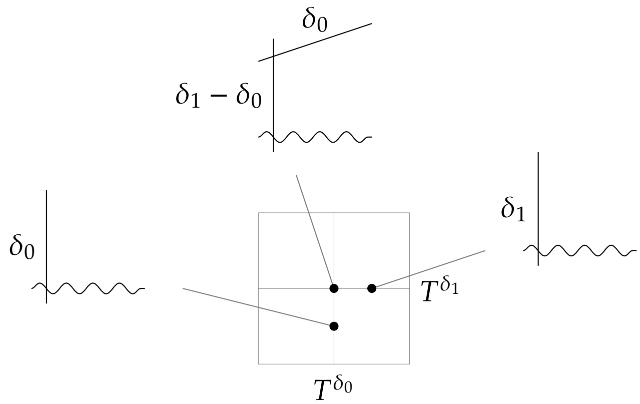
$$L|_{C^{(0)}} \to L|_{C^{(0)}}(\Sigma_i d_i q_i)$$

Locally around $q_j$ this has the effect of multiplying each $u_i$ by the $d_j$th power of the equation defining $q_j$, thus introducing a basepoint of length $d_j$.

The construction in families requires us to find a line bundle on the universal curve that is trivial on the rational tails (which we wish to contract) and relatively ample elsewhere. This can be performed at the level of Picard stacks: let $\mathfrak{Pic}^{d,\mathrm{st}}_{g,n}$ be the open substack of $\mathfrak{Pic}(\pi\colon \mathfrak{C}_{g,n} \to \mathfrak{M}_{g,n})$ obtained by requiring that the total degree of the line bundle is $d$, the degree on each component is nonnegative and $\mathcal{L} \otimes \omega_\pi(\Sigma_i x_i)$ is ample relative to $\pi$, where $\mathcal{L}$ is the universal line bundle.

Let $T^\delta$ be the locus in the universal curve $\mathfrak{C}_{\mathfrak{Pic}}$ over $\mathfrak{Pic}^{d,\mathrm{st}}_{g,n}$ spanned by rational tails on which $\mathcal{L}$ has degree $\delta$; this is a Cartier divisor by the smoothness of the stack $\mathfrak{C}_{\mathfrak{Pic}}$. Notice that $T^{\delta_0}$ and $T^{\delta_1}$ (say with $\delta_0 < \delta_1$)

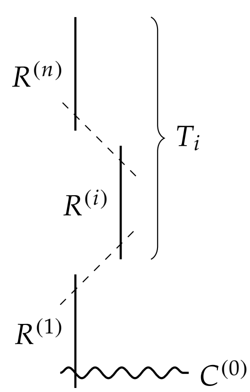
intersect in a stratum of codimension 1, where the rational tail splits into two rational components, the furthest from $C^{(0)}$ having degree $\delta_0$:




Define the following line bundle on $\mathfrak{C}_{\mathfrak{Pic}}$:

$$\mathcal{M} = \mathcal{L} \otimes \omega_\pi(\Sigma_i x_i) \otimes \bigotimes_{0<\delta\leq d} \mathcal{O}_{\mathfrak{C}_{\mathfrak{Pic}}}((\delta - 1)T^\delta)$$

We claim that $\mathcal{M}$ is trivial on every component of every rational tail, and $\pi$-relatively ample elsewhere. Consider a curve $C^{(0)} \sqcup_q R$ with a rational tail of degree $e$. We first consider the simple case where $R$ is isomorphic to a chain of $\mathbb{P}^1$s. We label the components $R^{(1)}, \ldots, R^{(n)}$, numbered from the closest to the farthest from $C^{(0)}$. We let $e_i$ denote the degree of $R^{(i)}$. We set $T_i = \bigcup_{j=i}^n R_j$ and $\epsilon_i = \deg \mathcal{L}|_{T_i} - 1 = e - 1 - \sum_{j=1}^{i-1} e_j$. The picture is as follows:




Because the stack $\mathfrak{Pic}_{g,n}^{d,\mathrm{st}}$ is smooth, a general one-parameter family inside this stack will give us a smoothing of our curve. The universal curve over such a family is a normal surface $S$; thus we can compute the degree of the restriction of $\mathcal{M}$ to the irreducible components of the central fiber of this family by first restricting $\mathcal{M}$ to $S$, and then using intersection theory on the normal surface $S$.

Notice that $T^\delta|_S = T_{i_\delta}$ where $i_\delta$ is the unique value of $i$ such that $\delta = \sum_{j=i_\delta}^n e_i$ (if no such $i$ exists then $T^\delta|_S = 0$). Thus the restriction of $\bigotimes_{0<\delta\leq d} \mathcal{O}_{\mathfrak{C}_{\mathfrak{Pic}}}((\delta-1)T^\delta)$ to $S$ gives:

$$\mathcal{O}_S(\Sigma_{j=1}^n \epsilon_j T_j)$$

Now, $R^{(i)}$ is a $(-2)$-curve for $i = 1, \ldots, n-1$, and so we have:

$$R^{(i)} \cdot T_j = \begin{cases} 0 & \text{for } j < i \\ -1 & \text{for } j = i \\ 1 & \text{for } j = i+1 \\ 0 & \text{for } j > i+1 \end{cases}$$

whereas for $R^{(n)}$ (which is a $(-1)$-curve) we have:

$$R^{(n)} \cdot T_j = \begin{cases} 0 & \text{for } j < n \\ -1 & \text{for } j = n \end{cases}$$

It follows that for $i = 1, \ldots, n-1$ we have

$$\deg \mathcal{M}|_{R^{(i)}} = \deg \mathcal{L}|_{R^{(i)}} - \epsilon_i + \epsilon_{i+1} = e_i - e_i = 0$$

whereas for $i = n$ we have:

$$\deg \mathcal{M}|_{R^{(n)}} = \deg \mathcal{L}|_{R^{(n)}} - 1 - \epsilon_n = e_n - 1 - (e_n - 1) = 0$$

Here we have used the fact that $\omega_\pi(\Sigma_i x_i)$ has degree 0 on $R^{(i)}$ for $i = 1, \ldots, n-1$ and degree $-1$ on $R^{(n)}$. Thus $\mathcal{M}$ is trivial on every component on every rational tail, as claimed. The fact that it is $\pi$-relatively ample on the rest of the curve follows from the stability condition and the fact that $\mathcal{O}_{\mathfrak{C}_{\mathfrak{Pic}}}(T^\delta)$ is effective when restricted to $C^{(0)}$.

[GENERAL CASE: UNDER CONSTRUCTION]

By taking the relative Proj construction we obtain another curve $\hat{\mathfrak{C}} = \underline{\mathrm{Proj}}_{\mathfrak{Pic}}\left(\bigoplus_{k\geq 0} \pi_* \mathcal{M}^{\otimes k}\right)$ over $\mathfrak{Pic}^{d,\mathrm{st}}_{g,n}$, with a map $\rho$ that contracts the rational tails

$$\begin{tikzcd} \mathfrak{C}_{\mathfrak{Pic}} \arrow[r, "\rho"] \arrow[dr, "\pi"'] & \hat{\mathfrak{C}} \arrow[d, "\pi'"] \\ & \mathfrak{Pic}^{d,\mathrm{st}}_{g,n} \end{tikzcd}$$

It is flat because it is a family of genus $g$ curves over a reduced base. Furthermore, it can be checked by cohomology and base-change [Har77, Theorem 12.11][Knu83, Corollary 1.5] (notice that the fibers of $\rho$ are either points or rational curves) that $\hat{\mathcal{L}} = \rho_*\left(\mathcal{L} \otimes \bigotimes_{0<\delta\leq d} \mathcal{O}_{\mathfrak{C}}(\delta T^\delta)\right)$ is a line bundle on $\hat{\mathfrak{C}}$ of degree $d$ relative to $\pi'$ (such that $\rho^*\hat{\mathcal{L}} \simeq \mathcal{L} \otimes \bigotimes_{0<\delta\leq d} \mathcal{O}_{\mathfrak{C}}(\delta T^\delta)$), hence the universal property gives us a commutative diagram (with Cartesian square)

$$\begin{tikzcd}
\mathfrak{C}_{\mathfrak{Pic}} \arrow[r, "\rho"] \arrow[dr, "\pi"'] & \hat{\mathfrak{C}} \arrow[r] \arrow[d, "\pi'"] \arrow[dr, phantom, "\square"] & \mathfrak{C}_{\mathfrak{Pic}} \arrow[d, "\pi"] \\
& \mathfrak{Pic}^{d,\mathrm{st}}_{g,n} \arrow[r, "\chi'"] & \mathfrak{Pic}^{d,\mathrm{st}}_{g,n}
\end{tikzcd}$$

The very same construction, with the line bundles pulled back from the Picard stack, and the sections of $\mathcal{L}$ seen as sections of $\mathcal{L} \otimes \bigotimes_{0<\delta\leq d} O_{\mathfrak{C}}(\delta T^{\delta})$ through the inclusion of line bundles ($O_{\mathfrak{C}}(T^{\delta})$ is effective), and descended to sections of $\hat{\mathcal{L}}$ on $\hat{\mathfrak{C}}$ gives us the comparison morphism $\chi\colon \mathcal{M}_{g,n}(\mathbb{P}^N, d) \to \mathcal{Q}_{g,n}(\mathbb{P}^N, d)$, fitting in a commutative diagram

$$\begin{tikzcd}
\mathcal{M}_{g,n}(\mathbb{P}^N, d) \arrow[r, "\chi"] \arrow[d, "\nu_{\mathcal{M}}"] & \mathcal{Q}_{g,n}(\mathbb{P}^N, d) \arrow[d, "\nu_{\mathcal{Q}}"] \\
\mathfrak{Pic}^{d,\mathrm{st}}_{g,n} \arrow[r, "\chi'"] & \mathfrak{Pic}^{d,\mathrm{st}}_{g,n}
\end{tikzcd}$$

and, as before,

$$\begin{tikzcd}
\mathcal{C}_{\mathcal{M}} \arrow[r, "\rho"] \arrow[dr, "\pi_{\mathcal{M}}"'] & \hat{\mathcal{C}} = \chi^*\mathcal{C}_{\mathcal{Q}} \arrow[r] \arrow[d, "\hat{\pi}"] \arrow[dr, phantom, "\square"] & \mathcal{C}_{\mathcal{Q}} \arrow[d, "\pi_{\mathcal{Q}}"] \\
& \mathcal{M}_{g,n}(\mathbb{P}^N, d) \arrow[r, "\chi"] & \mathcal{Q}_{g,n}(\mathbb{P}^N, d)
\end{tikzcd}$$

The comparison between virtual fundamental classes is best outlined in the arXiv version of [Man12b, Remark 5.20]. Call $\nu'_{\mathcal{M}} = \chi' \circ \nu_{\mathcal{M}}$. We may endow it with an obstruction theory by means of

$$\begin{tikzcd}
\nu^*_{\mathcal{M}}\mathbf{L}_{\chi'} \arrow[r] \arrow[d] & \mathbf{E}_{\nu'_{\mathcal{M}}} \arrow[r] \arrow[d] & \mathbf{E}_{\nu_{\mathcal{M}}} \arrow[r, "{[1]}"] \arrow[d] & {} \\
\nu^*_{\mathcal{M}}\mathbf{L}_{\chi'} \arrow[r] & \mathbf{L}_{\nu'_{\mathcal{M}}} \arrow[r] & \mathbf{L}_{\nu_{\mathcal{M}}} \arrow[r, "{[1]}"] & {}
\end{tikzcd}$$

Notice that $\chi'$ is a morphism (not of DM type) between smooth Artin stacks, hence we can only deduce that $\mathbf{L}_{\chi'}$ is supported in $[-1, 1]$. It is therefore easily seen that $\mathbf{E}_{\nu'_{\mathcal{M}}}$ is also supported in $[-1, 1]$; in order to show that it is actually a perfect obstruction theory, consider the long exact sequence

$$\begin{aligned}
0 &\to h^{-1}\nu^*_{\mathcal{M}}\mathbf{L}_{\chi'} \to h^{-1}\mathbf{E}_{\nu'_{\mathcal{M}}} \to h^{-1}\mathbf{E}_{\nu_{\mathcal{M}}} \\
&\to h^{0}\nu^*_{\mathcal{M}}\mathbf{L}_{\chi'} \to h^{0}\mathbf{E}_{\nu'_{\mathcal{M}}} \to h^{0}\mathbf{E}_{\nu_{\mathcal{M}}} \\
&\to h^{1}\nu^*_{\mathcal{M}}\mathbf{L}_{\chi'} \to h^{1}\mathbf{E}_{\nu'_{\mathcal{M}}} \to 0
\end{aligned}$$

and observe that, dually, $h^{-1}\nu^*_{\mathcal{M}}\mathbf{T}_{\chi'}$ injects into $h^0\mathbf{E}^{\vee}_{\nu_{\mathcal{M}}} \simeq h^0\mathbf{T}_{\nu_{\mathcal{M}}}$, because every infinitesimal automorphism of the rational tail induces a nontrivial deformation of the stable map (since the degree of the latter is positive on every component of the rational tail); we conclude that $h^1\mathbf{E}_{\nu'_{\mathcal{M}}} = 0$.

*Claim:* there is a morphism of obstruction theories $\chi^*\mathbf{E}_{\nu_Q} \to \mathbf{E}_{\nu_\mathcal{M}}$ [Man12b, Lemma 4.19].

Dually, $\mathbf{E}^\vee_{\nu_\mathcal{M}} = R^\bullet\pi_{\mathcal{M}*}\mathcal{L}^{\oplus r+1} = R^\bullet\hat{\pi}_*(\rho_*\mathcal{L}^{\oplus r+1})$, while, by cohomology and base-change, $\chi^*\mathbf{E}^\vee_{\nu_Q} = R^\bullet\hat{\pi}_*(\hat{\mathcal{L}}^{\oplus r+1})$, where $\hat{\mathcal{L}} = \rho_*\left(\mathcal{L} \otimes \bigotimes_{0<\delta\leq d} \mathcal{O}_\mathfrak{C}(\delta T^\delta)\right)$, so $\mathbf{E}^\vee_{\nu_\mathcal{M}} \to \chi^*\mathbf{E}^\vee_{\nu_Q}$ comes from the inclusion of line bundles on $\mathcal{C}_\mathcal{M}$

$$\mathcal{L} \hookrightarrow \mathcal{L} \otimes \bigotimes_{0<\delta\leq d} \mathcal{O}_\mathfrak{C}(\delta T^\delta).$$

*Claim:* this morphism factors through $\mathbf{E}_{\nu'_\mathcal{M}}$.

$$\begin{tikzcd}
 & \chi^*\mathbf{E}_{\nu_Q} \arrow[dl, dashed, "\exists?"'] \arrow[d] \arrow[dr, "\phi"] & \\
\mathbf{E}_{\nu'_\mathcal{M}} \arrow[r] & \mathbf{E}_{\nu_\mathcal{M}} \arrow[r] & \nu^*_\mathcal{M}\mathbf{L}_{\chi'}[1]
\end{tikzcd}$$

In order to prove that the dashed arrow exists, we need to show that $\phi$ is the zero map.

This follows formally from the following factorisation:

$$\begin{tikzcd}
 & & \mathbf{L}_\chi \arrow[dl, "{[1]}"'] & & \\
\chi^*\mathbf{E}_{\nu_Q} \arrow[r] \arrow[dd] & \chi^*\mathbf{L}_{\nu_Q} \arrow[rr] & & \mathbf{L}_{\nu'_\mathcal{M}} \arrow[ul] \arrow[dd] & \\
 & & & & \nu^*_\mathcal{M}\mathbf{L}_{\chi'}[1] \arrow[ul, "{[1]}"'] \\
\mathbf{E}_{\nu_\mathcal{M}} \arrow[rrr] & & & \mathbf{L}_{\nu_\mathcal{M}} \arrow[ur] &
\end{tikzcd}$$

Now the cone $C(\phi)$ gives an obstruction theory relative to $\chi$. A priori, it is supported in $[-2, 0]$. By the octahedral axiom

$$\begin{tikzcd}
\chi^*\mathbf{E}_{\nu_Q} \arrow[d, "\phi"] \arrow[dr, "\phi'"] & & & \\
\mathbf{E}_{\nu'_\mathcal{M}} \arrow[r] \arrow[dd] & \mathbf{E}_{\nu_\mathcal{M}} \arrow[rr] \arrow[dr] & & \nu^*_\mathcal{M}\mathbf{L}_{\chi'}[1] \\
 & & C(\phi') \arrow[ur] & \\
C(\phi) \arrow[urr] & & &
\end{tikzcd}$$

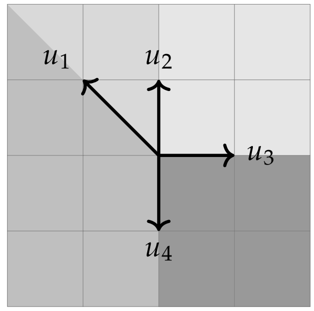



FIGURE 1. Toric fan for $\mathbb{F}_1$.

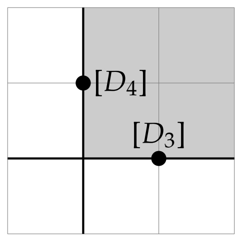



FIGURE 2. Nef cone $\mathrm{Nef}(\mathbb{F}_1)$.

it is enough to observe that $C(\phi')$ is supported in $[-1,0]$ [Man12b, Lemma 4.20] and that $\nu_{\mathcal{M}}^*\mathbf{L}_{\chi'}[1]$ is supported in degrees $[-2,0]$, in order to conclude that $C(\phi)=\mathbf{E}_\chi$ is a perfect obstruction theory. The conclusion that

$$\chi_*[\mathcal{M}_{g,n}(\mathbb{P}^N,d)]^{\mathrm{vir}}=[\mathcal{Q}_{g,n}(\mathbb{P}^N,d)]^{\mathrm{vir}}$$

follows from the connectedness of $\mathcal{M}_{g,n}(\mathbb{P}^N,d)$ [KP01] (hence of $\mathcal{Q}_{g,n}(\mathbb{P}^N,d)$) and an application of the virtual push-forward theorem [Man12b, Proposition 4.21].

We shall now explain with an example the reason why a naive attempt to extend the comparison morphism to a general toric variety fails. The problem in a nutshell is that not all toric divisors are nef: a rational tail contained in a divisor which is not nef may have negative degree $-d$ with respect to the corresponding line bundle; when contracting such a rational tail, we shall take the line bundle $L(-dq)$, but what to do with the sections? We would like to divide them by $z^d$, where $z$ is a local coordinate around $q$, but no condition forces such a divisibility to happen. Otherwise said, there is now an inclusion $L_{|C^{(0)}}(-dq)\hookrightarrow L_{|C^{(0)}}$, but the (restriction of the) given sections of $L$ do not necessarily live in the image of $H^0(C^{(0)},L_{|C^{(0)}}(-dq))\hookrightarrow H^0(C^{(0)},L_{|C^{(0)}})$.

A concrete example is found when looking at the Hirzebruch surface $\mathbb{F}_1=\mathrm{Bl}_p\,\mathbb{P}^1$.

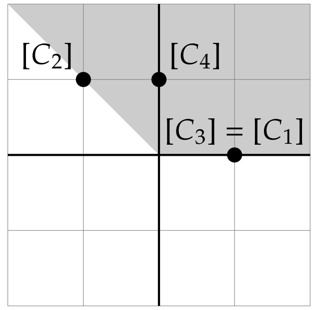



Figure 3. Mori cone $\overline{\mathrm{NE}}(\mathbb{F}_1)$.

$\mathrm{Pic}(\mathbb{F}_1)$ is generated by $[D_3]$ and $[D_4]$, with relations $[D_1] = [D_3]$ and $[D_2] = [D_4] - [D_3]$, and the intersection table is given by

$$\begin{cases} D_3^2 = 0 \\ D_3.D_4 = 0 \\ D_4^2 = 1 \end{cases}$$

When thinking of $\mathbb{F}_1$ as a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$, $C_1$ and $C_3$ represent the fibers of the bundle (over the toric points of $\mathbb{P}^1$), while $C_4$ (resp. $C_2$) is the zero/positive (resp. infinity/negative) section; when thinking of $\mathbb{F}_1$ as $\mathrm{Bl}_p\,\mathbb{P}^1$, $C_2$ is the exceptional divisor, $C_4$ is the toric line not passing through $p$, and $C_1, C_3$ are the strict transforms of the toric lines through $p$.

Let us look at $\mathcal{M}_{0,2}(\mathbb{F}_1, [C_4])$. Since $[C_4] = [C_2] + [C_3]$, there are going to be maps of the following sort: the source curve is reducible $R_1 \sqcup_q R_2$, $R_1$ is mapped isomorphically to a fiber (i.e. in class $[C_3]$) and $R_2$ is mapped isomorphically to $C_2$, all the markings belong to $R_1$. So $R_2$ is a rational tail and deserves to be contracted. Notice that the line bundle $\mathcal{O}(D_2)$ has degree $-1$ on $R_2$ (and 1 on $R_1$). In this case everything works well because the corresponding section $u_{2|R_1}$ must vanish at the node, so we can divide it by a chosen (once for all toric line bundles) section of $\mathcal{O}_{R_1}(q)$.

Consider now $\mathcal{M}_{0,2}(\mathbb{F}_1, 2[C_2] + [C_3])$. Certainly there are going to be maps similar to the ones described above, with $R_2$ now covering $C_2$ 2: 1. The point is that $\mathcal{O}(D_2)$ has degree $-2$ on $R_2$, but $u_{2|R_1}$ doesn't have to vanish at the node of order 2, so we are in trouble.

## Appendix B. Notes on quasimaps

In this appendix we collect several foundational results in quasimap theory, including:

(1) *Functoriality* (§B.1): given a morphism $f\colon Y \to X$ we describe the induced map:

$$\overline{Q}(f)\colon \mathcal{Q}_{g,n}(Y,\beta) \to \mathcal{Q}_{g,n}(X, f_*\beta)$$

We also discuss (§B.2) when $\overline{Q}(f)$ admits a compatible perfect obstruction theory.

(2) *Splitting axiom* (§B.3): this gives an equality between two natural virtual classes on boundary strata (i.e. loci where the underlying curve is reducible of a prescribed type).

(3) *Comparison with the GIT construction* (§B.4): we show that for a (not necessarily toric) hypersurface $Y \hookrightarrow X$, our definition of $Q(Y)$ as a substack of $Q(X)$ coincides with the definition of $Q(Y)$ given by the description of $Y$ as a GIT quotient (see [CFKM14]).

B.1. **Functoriality.** In the case of stable maps, a morphism $f : Y \to X$ induces a morphism between the corresponding moduli spaces

$$\mathcal{M}(f) : \mathcal{M}_{g,n}(Y, \beta) \to \mathcal{M}_{g,n}(X, f_*\beta)$$

given by (post)composition with $f$; in general this induced morphism may involve stabilisation of the source curve. Because of this, we may say that the construction of the moduli space of stable maps is *functorial*.

It is natural to ask whether the same holds for the moduli space of quasimaps, i.e. whether we have a morphism:

$$Q(f) : Q_{g,n}(Y, \beta) \to Q_{g,n}(X, f_*\beta)$$

Since here the objects of the moduli space are not maps, we cannot simply compose with $f$, but we should define an operation which agrees with composing with $f$ on the locus of quasimaps without basepoints. In [CFK14, Section 3.1] a definition (in the GIT context) is given when $f$ is an embedding into a projective space; we shall discuss the general situation of a morphism between toric varieties $f : Y \to X$.

Our approach uses the language of $\Sigma$–collections introduced by D. Cox. This approach is natural insofar as a quasimap is a generalisation of a $\Sigma$–collection. We will refer extensively to [Cox95b] and [Cox95a], which we recommend as an introduction for any readers unfamiliar with the theory.

Let $X$ and $Y$ be smooth and proper toric varieties with fans $\Sigma_X \subseteq N_X$ and $\Sigma_Y \subseteq N_Y$. Suppose we are given $f : Y \to X$ (which we do not assume to be a toric morphism). By [Cox95a, Theorem 1.1] the data of such a map is equivalent to a $\Sigma_X$–collection on $Y$:

$$((L_\rho, u_\rho)_{\rho \in \Sigma_X(1)}, (\varphi_{m_x})_{m_x \in M_X})$$

In addition, [Cox95b] allows us to describe line bundles on $Y$ and their global sections in terms of the homogeneous coordinates $(z_\tau)_{\tau \in \Sigma_Y(1)}$. All of these observations are combined into the following theorem, which is so useful that we will state it here in its entirety:

**Theorem B.1.** [Cox95a, Theorem 3.2] The data of a morphism $f : Y \to X$ is the same as the data of homogeneous polynomials

$$P_\rho \in S^Y_{\beta_\rho}$$

for $\rho \in \Sigma_X(1)$, where $\beta_\rho \in \operatorname{Pic} Y$ and $S^Y_{\beta_\rho}$ is the corresponding graded piece of the Cox ring

$$S^Y = k[z_\tau : \tau \in \Sigma_Y(1)]$$

This data is required to satisfy the following two conditions:

(1) $\sum_{\rho \in \Sigma_X(1)} \beta_\rho \otimes n_\rho = 0$ in $\operatorname{Pic} Y \otimes N_X$.

(2) $(P_\rho(z_\tau)) \notin Z(\Sigma_X) \subseteq \mathbb{A}^{\Sigma_X(1)}$ whenever $(z_\tau) \notin Z(\Sigma_Y) \subseteq \mathbb{A}^{\Sigma_Y(1)}$.

Furthermore, two such sets of data $(P_\rho)$ and $(P'_\rho)$ correspond to the same morphism if and only if there exists a $\lambda \in \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Pic} X, \mathbb{G}_{\mathrm{m}})$ such that

$$\lambda(D_\rho) \cdot P_\rho = P'_\rho$$

for all $\rho \in \Sigma_X(1)$. Finally, if we define $\tilde{f}(z_\tau) = (P_\rho(z_\tau))$ then this defines a lift of $f$ to the prequotients:

$$\begin{tikzcd} \mathbb{A}^{\Sigma_Y(1)} \setminus Z(\Sigma_Y) \arrow[r, "\tilde{f}"] \arrow[d, "q_Y"] & \mathbb{A}^{\Sigma_X(1)} \setminus Z(\Sigma_X) \arrow[d, "q_X"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

**Aside B.2.** Throughout this section we will stick to the notation established above; in particular we will use $\rho$ to denote a ray in $\Sigma_X(1)$ and $\tau$ to denote a ray in $\Sigma_Y(1)$.

Recall our goal: given a map $f \colon Y \to X$ we wish to define a "push-forward" map:

$$\mathcal{Q}(f) : \mathcal{Q}_{g,n}(Y, \beta) \to \mathcal{Q}_{g,n}(X, f_*\beta)$$

Consider therefore a quasimap $((C, x_1, \ldots, x_n), (L_\tau, u_\tau)_{\tau \in \Sigma_Y(1)}, (\varphi_{m_Y})_{m_Y \in M_Y})$ with target $Y$. Pick data $(P_\rho)_{\rho \in \Sigma_X(1)}$ corresponding to the map $f$, as in the theorem above; we will later see that our construction does not depend on this choice.

The idea of the construction is as follows. Assume for a moment that $C \simeq \mathbb{P}^1$ and that the quasimap is without basepoints, so that we have an actual morphism $C \to Y$. Then, by the theorem of Cox that we have just recalled, we can lift this morphism to the prequotient as in the following diagram

$$\begin{tikzcd} \mathbb{A}^2 \setminus \{0\} \arrow[r, "(u_\tau)"] \arrow[d] & \mathbb{A}^{|\Sigma_Y(1)|} \setminus Z(\Sigma_Y) \arrow[r, "(P_\rho)"] \arrow[d] & \mathbb{A}^{|\Sigma_X(1)|} \setminus Z(\Sigma_X) \arrow[d] \\ C \arrow[r] & Y \arrow[r] & X \end{tikzcd}$$

from which it follows that the composition $C \to Y \to X$ is given in homogeneous coordinates by:

$$(P_\rho((u_\tau)_{\tau \in \Sigma_Y(1)}))_{\rho \in \Sigma_X(1)}.$$

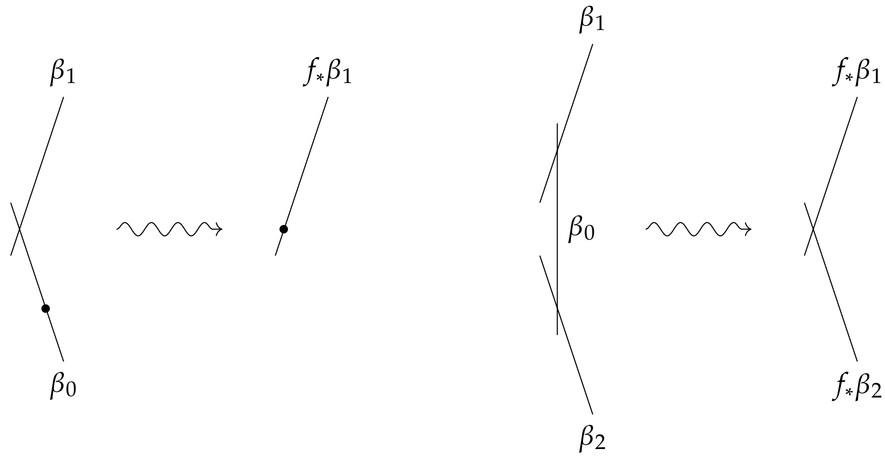
In general the quasimap to $Y$ will still be given by a bunch of line bundles and their sections on $C$, and we can still make sense of the expressions $P_\rho(u_\tau)$

as sections of line bundles on $C$ obtained from the given ones by means of the recipe encoded by $f$; morally, we can trivialise the given line bundles near a point of $C$ and treat the above polynomial expression as a function on this neighbourhood. This will allow us to define the pushforward along $f$ on the quasimap space.

Let us make this precise now. We shall first address the issue of stabilising the curve: in case $f_*\beta = 0$, observe that

$$\mathcal{Q}_{g,n}(X, 0) \simeq \mathcal{M}_{g,n} \times X$$

hence the stabilisation morphism is the usual $\mathcal{Q}_{g,n}(X, 0) \to \mathcal{M}_{g,n}$, while the map to $X$ is given by postcomposing with $f$ any of the evaluation morphisms. On the other hand it may well be that, even if $f_*\beta \neq 0$, the source curve is reducible and some of its components have a degree $\beta_0$ in $Y$ the pushforward of which is trivial in $X$.




Pick a polarisation $\mathcal{O}_X(1)$ and write $f^*\mathcal{O}_X(1) = \bigotimes_\tau L_\tau^{\otimes a_\tau}$. Notice that the line bundle $\omega_\pi(x_1 + \ldots + x_n) \otimes \bigotimes_\tau \mathcal{L}_\tau^{\otimes a_\tau}$ on the universal curve of $\mathcal{Q}_{g,n}(Y, \beta)$ provides the required contraction

$$\begin{tikzcd} \mathcal{C}_Y \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & \overline{\mathcal{C}} \arrow[dl, "\pi'"] \\ & \mathcal{Q}_{g,n}(Y, \beta) & \end{tikzcd}$$

Let the sections of $\overline{\mathcal{C}}$ be the ones of $\mathcal{C}_Y$ composed with $\phi$.

Resuming the notation above, for each $\rho \in \Sigma_X(1)$, $P_\rho$ is a polynomial in the $z_\tau$; we can write it as

$$P_\rho(z_\tau) = \sum_{\underline{a}} P_\rho^{\underline{a}}(z_\tau) = \sum_{\underline{a}} \mu_{\underline{a}} \prod_\tau z_\tau^{a_\tau} \tag{12}$$

where the sum is over a finite number of multindices $\underline{a} = (a_\tau) \in \mathbb{N}^{|\Sigma_Y(1)|}$ and the $\mu_{\underline{a}}$ are nonzero scalars. Observe that, for each $\underline{a}$, the line bundle $\bigotimes_\tau \mathcal{L}_\tau^{\otimes a_\tau}$ on $\mathcal{C}_Y$ is trivial on the $\rho$-contracted components, which are always rational curves; hence, by cohomology and base-change, the following gives a line bundle on $\overline{C}$:

$$\tilde{\mathcal{L}}_\rho^{\underline{a}} = \phi_* \bigotimes_\tau \mathcal{L}_\tau^{\otimes a_\tau}$$

Then we may take the following section of $\tilde{\mathcal{L}}_\rho^{\underline{a}}$:

$$\tilde{u}_\rho^{\underline{a}} = P_\rho^{\underline{a}}(u_\tau) = \mu_{\underline{a}} \prod_\tau u_\tau^{a_\tau};$$

notice that such sections must be constant on $\phi$-contracted components of $C_Y$ and so descend to $\overline{C}$. Thus each of the terms $P_\rho^{\underline{a}}$ of $P_\rho$ defines a section $\tilde{u}_\rho^{\underline{a}}$ of a line bundle $\tilde{\mathcal{L}}_\rho^{\underline{a}}$. But what we want is a single section $\tilde{u}_\rho$ of a single line bundle $\tilde{\mathcal{L}}_\rho$. This is where the isomorphisms $\varphi_{m_Y}$ come in.

Recall that we have a short exact sequence:

$$0 \longrightarrow M_Y \xrightarrow{\theta} \mathbb{Z}^{\Sigma_Y(1)} \longrightarrow \operatorname{Pic} Y \longrightarrow 0 \tag{13}$$

Let $\underline{a}$ and $\underline{b}$ be multindices appearing in the sum (12) above. By the homogeneity of $P_\rho$ we have

$$\sum_\tau a_\tau D_\tau = \beta_\rho = \sum_\tau b_\tau D_\tau$$

which is precisely the statement that in the above sequence $\underline{a}$ and $\underline{b}$ map to the same element of $\operatorname{Pic} Y$ (namely $\beta_\rho$). Hence there exists a unique $m_Y \in M_Y$ such that:

$$\theta(m_Y) = \underline{a} - \underline{b}$$

Now, the isomorphism $\varphi_{m_Y}$ (contained in the data of our original quasimap) is a map:

$$\varphi_{m_Y} : \bigotimes_\tau \mathcal{L}_\tau^{\otimes \langle m_Y, n_\tau \rangle} \cong \mathcal{O}_{C_Y}$$

By definition, $\theta(m_Y) = (\langle m_Y, n_\tau \rangle)_{\tau \in \Sigma_Y(1)}$. But also $\theta(m_Y) = (a_\tau - b_\tau)_{\tau \in \Sigma_Y(1)}$. Hence we have:

$$\varphi_{m_Y} : \bigotimes_\tau \mathcal{L}_\tau^{\otimes a_\tau} \cong \bigotimes_\tau \mathcal{L}_\tau^{\otimes b_\tau}$$

In other words, we have well-defined canonical isomorphisms

$$\tilde{\mathcal{L}}_\rho^{\underline{a}} \cong \tilde{\mathcal{L}}_\rho^{\underline{b}}$$

for all $\underline{a}$ and $\underline{b}$. Let us choose one such $\underline{a}$ (it doesn't matter which); call it $\underline{a}^\rho$. We define:

$$\tilde{\mathcal{L}}_\rho := \tilde{\mathcal{L}}_\rho^{\underline{a}^\rho}$$

Then for all $\underline{b}$ we can use the above isomorphism to view $\tilde{u}_\rho^{\underline{b}}$ as a section of $\tilde{\mathcal{L}}_\rho$. Summing all of these together with the appropriate coefficients we thus obtain a section $\tilde{u}_\rho$ of $\tilde{\mathcal{L}}_\rho$, which we can write (with abuse of notation) as:

$$\tilde{u}_\rho = \sum_{\underline{a}} \mu_{\underline{a}} \prod_\tau u_\tau^{a_\tau}$$

Note that if we had made a different choice of $\underline{a}^\rho$ above the result would have been isomorphic.

Thus far we have constructed line bundles and sections $(\tilde{\mathcal{L}}_\rho, \tilde{u}_\rho)_{\rho \in \Sigma_X(1)}$ on $\overline{C}$. It remains to define the isomorphisms

$$\tilde{\varphi}_{m_X} : \otimes_\rho \tilde{\mathcal{L}}_\rho^{\otimes \langle m_X, n_\rho \rangle} \cong \mathcal{O}_{\overline{C}}$$

for all $m_X \in M_X$. The left hand side is:

$$\otimes_\rho \tilde{\mathcal{L}}_\rho^{\otimes \langle m_X, n_\rho \rangle} = \otimes_\rho \left( \phi_* \otimes_\tau \mathcal{L}_\tau^{\otimes a_\tau^\rho} \right)^{\otimes \langle m_X, n_\rho \rangle} = \phi_* \otimes_\tau \mathcal{L}_\tau^{\otimes \left( \sum_\rho a_\tau^\rho \langle m_X, n_\rho \rangle \right)}$$

Now, for $m_Y \in M_Y$ we have isomorphisms $\varphi_{m_Y} : \otimes_\tau \mathcal{L}_\tau^{\otimes \langle m_Y, n_\tau \rangle} \cong \mathcal{O}_{C_Y}$. Hence, in order to construct $\tilde{\varphi}_{m_X}$ it is tempting to look for an $m_Y$ such that

$$\langle m_Y, n_\tau \rangle = \sum_\rho a_\tau^\rho \langle m_X, n_\rho \rangle$$

for all $\tau \in \Sigma_Y(1)$ (we will then set $\tilde{\varphi}_{m_X} = \varphi_{m_Y}$). Consider therefore the short exact sequence (13). Recall that $\theta(m_Y) = (\langle m_Y, n_\tau \rangle)_{\tau \in \Sigma_Y(1)}$. Hence we need to show that

$$\left( \sum_\rho a_\tau^\rho \langle m_X, n_\rho \rangle \right)_{\tau \in \Sigma_Y(1)}$$

belongs to the image of $\theta$, i.e. that it belongs to the kernel of the second map (notice that $m_Y$ is then unique because $\theta$ is injective). This is equivalent to saying that

$$\sum_\tau \sum_\rho a_\tau^\rho \langle m_X, n_\rho \rangle D_\tau = 0 \in \operatorname{Pic} Y$$

Now, we have

$$\sum_\tau a_\tau^\rho D_\tau = \beta_\rho$$

so that the above sum becomes

$$\sum_\rho \langle m_X, n_\rho \rangle \beta_\rho = \left\langle m_X, \sum_\rho \beta_\rho \otimes n_\rho \right\rangle = \langle m_X, 0 \rangle = 0$$

where $\sum_\rho \beta_\rho \otimes n_\rho = 0$ by Condition (1) in Theorem B.1. So there does indeed exist a (unique) $m_Y \in M_Y$ such that $\langle m_Y, n_\tau \rangle = \sum_\rho a_\tau^\rho \langle m_X, n_\rho \rangle$, so that we can set:

$$\tilde{\varphi}_{m_X} = \varphi_{m_Y} : \bigotimes_\rho \tilde{\mathcal{L}}_\rho^{\otimes \langle m_X, n_\rho \rangle} \cong \mathcal{O}_{\overline{C}}$$

Thus, we have produced a quasimap with target $X$ and class $f_*\beta$ on the base $Q_{g,n}(Y,\beta)$:

$$(\overline{C}, (\tilde{\mathcal{L}}_\rho, \tilde{u}_\rho)_{\rho\in\Sigma_X(1)}, (\tilde{\varphi}_{m_X})_{m_X\in M_X})$$

The proof that this construction does not depend on the choice of $(P_\rho)$ is straightforward and is left to the reader.

It remains to demonstrate that the quasimap thus constructed is nondegenerate and stable. Nondegeneracy follows immediately from Condition (2) in Theorem B.1. Put differently: the original quasimap defined a rational map $C \dashrightarrow Y$, whereas the new quasimap defines a rational map which is simply the composition $C \dashrightarrow Y \to X$. Therefore the set of basepoints is exactly the same.

Stability follows precisely from the construction of $\phi$: if $\mathcal{O}_X(1) = \bigotimes_\rho L_\rho^{\otimes b_\rho}$, then $\omega_{\pi'}(\bar{x}_1 + \ldots \bar{x}_n) \otimes \bigotimes_\rho \tilde{\mathcal{L}}_\rho^{\otimes b_\rho}$ will be $\pi'$-ample on $\overline{C}$, since we have contracted all the components on which $f^*\mathcal{O}_X(1)$ was trivial without introducing any rational tail.

Finally, by the universal property of $Q_{g,n}(X,\beta)$ there is an induced diagram

$$\begin{tikzcd} C_Y \arrow[r, "\phi"] \arrow[dr, "\pi"'] & \overline{C} \arrow[r] \arrow[d, "\pi'"] \arrow[dr, phantom, "\square"] & C_X \arrow[d, "\pi_X"] \\ & Q_{g,n}(Y,\beta) \arrow[r, "Q(f)"'] & Q_{g,n}(X, f_*\beta) \end{tikzcd}$$

This completes the proof of the following.

**Theorem B.3.** Let $X$ and $Y$ be smooth proper toric varieties and $f : Y \to X$ a morphism. Assume that $f$ satisfies Condition (1) above. Then there exists a natural push-forward map

$$Q(f)\colon Q_{g,n}(Y,\beta) \to Q_{g,n}(X, f_*\beta)$$

which does not modify the underlying prestable curves.

**Remark B.4.** Theorem B.1 tells us that we can lift any morphism between toric varieties to an equivariant morphism between the prequotients

$$\begin{tikzcd} \mathbb{A}^{\Sigma_Y(1)} \setminus Z(\Sigma_Y) \arrow[r, "\tilde{f}"] \arrow[d, "q_Y"] & \mathbb{A}^{\Sigma_X(1)} \setminus Z(\Sigma_X) \arrow[d, "q_X"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

where the torus homomorphism $G_Y = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}(Y), \mathbb{C}^*) \to G_X = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}(Y), \mathbb{C}^*)$ is induced in the obvious way by $f\colon Y \to X$. Now, thinking of quasimaps as maps to the quotient stack, functoriality is again clear by postcomposition with $\tilde{f}$ (notice that the preimage of the unstable locus of $X$ is the unstable locus of $Y$).

Finally, let us describe how this push-forward morphism behaves when $f$ is a nonconstant map $\mathbb{P}^r \to \mathbb{P}^N$, since we will make use of this later. Write $f$ in homogeneous coordinates as:

$$f[z_0, \ldots, z_r] = [f_0(z_0, \ldots, z_r), \ldots, f_N(z_0, \ldots, z_r)]$$

where the $f_i$ are all homogeneous of degree $a > 0$. Then given a quasimap with target $\mathbb{P}^r$

$$(C, L, u_o, \ldots, u_r)$$

the pushed-forward quasimap with target $\mathbb{P}^N$ is:

$$(C, L^{\otimes a}, f_0(u_0, \ldots, u_r), \ldots, f_N(u_0, \ldots, u_r))$$

B.2. **Relative obstruction theories for $\mathcal{Q}(Y) \to \mathcal{Q}(X)$.** Assume now that $f : Y \to X$ is a morphism (between projective varieties) satisfying one of the following equivalent conditions:

(1) $f$ is finite;
(2) for an(y) ample line bundle $\mathcal{O}_X(1)$ on $X$, $f^*\mathcal{O}_X(1)$ is ample on $Y$;
(3) for every effective curve class $\beta \in H_2^+(Y)$, $f_*\beta \neq 0$.

Observe then that the induced morphism

$$k = \mathcal{Q}(f) : \mathcal{Q}_{g,n}(Y, \beta) \to \mathcal{Q}_{g,n}(X, f_*\beta)$$

commutes with the projections to $\mathfrak{M}_{g,n}$, i.e. there is no need to stabilise the underlying curve. We are going to describe a situation in which it is possible to define an operation of pullback along $k$ on Chow groups.

Even in the easiest possible case when $Y \hookrightarrow X$ is a regular embedding, $k$ itself is not necessarily a regular embedding, and so the Gysin map in the sense of [Ful98] is not guaranteed to exist.

However, when $\mathcal{Q}_{g,n}(X, f_*\beta)$ is a smooth stack (or rather, when it is unobstructed, for instance when $X = \mathbb{P}^N$ and $g = 0$ or $(g, n) = (1, 0)$) there is a way around this. In [Man12a] a generalisation of the Gysin map called the *virtual pull-back* is defined for morphisms endowed with a relative perfect obstruction theory. Moreover, a sufficient condition is given [Man12a, Corollary 4.9] for this map to respect the virtual classes.

**Lemma B.5.** For a *finite* morphism of smooth toric varieties $f : Y \to X$, there exists a relative obstruction theory $\mathbf{E}_k$ for the morphism

$$k : \mathcal{Q}_{g,n}(Y, \beta) \to \mathcal{Q}_{g,n}(X, f_*\beta)$$

which fits into a compatible triple with the standard obstruction theories for the quasimap spaces over $\mathfrak{M}_{g,n}$. Furthermore, $\mathbf{E}_k$ is perfect *if $\mathcal{Q}_{g,n}(X, f_*\beta)$ is unobstructed*, so that:

$$k_{\mathrm{v}}^![\mathcal{Q}_{g,n}(X, f_*\beta)] = [\mathcal{Q}_{g,n}(Y, \beta)]^{\mathrm{virt}}$$

*Proof.* Note first that, since $k$ does not change the source curve of a quasimap, we indeed have a commuting triangle:

$$\begin{tikzcd}
Q_{g,n}(Y,\beta) \arrow[rr, "k"] \arrow[dr] & & Q_{g,n}(X, f_*\beta) \arrow[dl] \\
& \mathfrak{M}_{g,n} &
\end{tikzcd}$$

We have perfect obstruction theories $\mathbf{E}_{\overline{Q}(Y)/\mathfrak{M}}$ and $\mathbf{E}_{\overline{Q}(X)/\mathfrak{M}}$ and we want to find a perfect obstruction theory $\mathbf{E}_k$. Consider the diagram of universal curves

$$\begin{tikzcd}
C_Y \arrow[r, "\alpha"] \arrow[d, "\pi"] \arrow[dr, phantom, "\square"] & C_X \arrow[d, "\rho"] \\
Q_{g,n}(Y,\beta) \arrow[r, "k"] & Q_{g,n}(X, f_*\beta)
\end{tikzcd}$$

which is cartesian because $k$ does not alter the source curve of any quasimap. We have sheaves $\mathcal{F}_Y$ and $\mathcal{F}_X$ on $C_Y$ and $C_X$ respectively such that:

$$\mathbf{E}^\vee_{\overline{Q}(Y)/\mathfrak{M}} = \mathrm{R}^\bullet\, \pi_* \mathcal{F}_Y$$
$$\mathbf{E}^\vee_{\overline{Q}(X)/\mathfrak{M}} = \mathrm{R}^\bullet\, \rho_* \mathcal{F}_X$$

It follows (by flatness of $\rho$) that when we pull back the latter obstruction theory to $Q(Y)$ we obtain:

$$k^*\mathbf{E}^\vee_{\overline{Q}(X)/\mathfrak{M}} = \mathrm{R}^\bullet\, \pi_* \alpha^* \mathcal{F}_X$$

To construct a compatible triple, we require a morphism $k^*\mathbf{E}_{Q(X)/\mathfrak{M}} \to \mathbf{E}_{Q(Y)/\mathfrak{M}}$. Dually, it is therefore enough to construct a morphism of sheaves on $C_Y$

$$\mathcal{F}_Y \to \alpha^* \mathcal{F}_X$$

and then apply $\mathrm{R}^\bullet\, \pi_*$. This is analogous to the morphism $f^*T_Y \to f^*T_X|_Y$ which is used in the stable maps setting. However the construction for quasimaps requires a little more ingenuity, because we do not have access to a universal map $f$.

The sheaf $\mathcal{F}_Y$ is defined on $C_Y$ by the short exact sequence

$$0 \to \mathcal{O}_{C_Y}^{\oplus r_Y} \to \oplus_\tau \mathcal{L}_\tau \to \mathcal{F}_Y \to 0$$

where $r_Y = \operatorname{rk} \operatorname{Pic} Y$ (implicitly we have chosen a basis for this $\mathbb{Z}$-module). Similarly $\mathcal{F}_X$ is defined on $C_X$ by:

$$0 \to \mathcal{O}_{C_X}^{\oplus r_X} \to \oplus_\rho \mathcal{L}_\rho \to \mathcal{F}_X \to 0$$

We will construct our morphism by first constructing a morhism:

$$\oplus_\tau \mathcal{L}_\tau \to \alpha^*(\oplus_\rho \mathcal{L}_\rho)$$

Recall that $f\colon Y \to X$ is given by homogeneous polynomials

$$P_\rho \in S^Y_{\beta_\rho} \subset S^Y = k[z_\tau : \tau \in \Sigma_Y(1)]$$

in the Cox ring of $Y$, where $\beta_\rho = f^*[D_\rho] \in \operatorname{Pic} Y$. For all monomials appearing in $P_\rho$, if we look at their exponents $(a_\tau)_{\tau\in\Sigma_Y(1)}$, we have $\sum_{\tau\in\Sigma_Y(1)} a_\tau[D_\tau] = \beta_\rho$ by homogeneity, hence we can use the isomorphisms parametrised by $M_Y$ as in the proof of Proposition B.3 above in order to interpret the $(P_\rho)$ as a morphism

$$(P_\rho)_{\rho\in\Sigma_X(1)} \colon \bigoplus_\tau \mathcal{L}_\tau \to \bigoplus_\rho \bigotimes_\tau \mathcal{L}_\tau^{\otimes a_\tau^\rho} = \bigoplus_\rho \tilde{\mathcal{L}}_\rho = \alpha^*\left(\bigoplus_\rho \mathcal{L}_\rho\right)$$

where the notation is as in §B.1. Thus we have constructed a morphism $\oplus_\tau \mathcal{L}_\tau \to \alpha^*(\oplus_\rho \mathcal{L}_\rho)$.

On the other hand, $f\colon Y \to X$ induces a pullback map on line bundles $\operatorname{Pic}(X) \to \operatorname{Pic}(Y)$. Since we have implicitly chosen bases for these $\mathbb{Z}$-modules, this gives rise to a matrix, whose transpose we denote by:

$$Q \in \operatorname{Mat}_{r_X\times r_Y}(\mathbb{Z})$$

It is now clear by the functoriality construction that the square in the following diagram is commutative; hence it induces the (dashed) map of sheaves that we were hoping for

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_{C_Y}^{\oplus r_Y} \arrow[r] \arrow[d, "Q"] & \oplus_\tau \mathcal{L}_\tau \arrow[r] \arrow[d, "(P_\rho)"] & \mathcal{F}_Y \arrow[r] \arrow[d, dashed] & 0 \\ & \mathcal{O}_{C_Y}^{\oplus r_X} \arrow[r] & \alpha^*\left(\oplus_\rho \mathcal{L}_\rho\right) \arrow[r] & \alpha^*\mathcal{F}_X \arrow[r] & 0 \end{tikzcd} \tag{14}$$

Applying $\mathrm{R}^\bullet \pi_*$ and dualising we obtain a morphism between the obstruction theories for the quasimap spaces, and we can complete this to obtain an exact triangle

$$k^*\mathbf{E}_{\mathcal{Q}(X)/\mathfrak{M}} \to \mathbf{E}_{\mathcal{Q}(Y)/\mathfrak{M}} \to \mathbf{E}_k \xrightarrow{[1]}$$

on $\mathcal{Q}(Y)$. The axioms of a triangulated category then give a morphism of exact triangles:

$$\begin{tikzcd} k^*\mathbf{E}_{\mathcal{Q}(X)/\mathfrak{M}} \arrow[r] \arrow[d] & \mathbf{E}_{\mathcal{Q}(Y)/\mathfrak{M}} \arrow[r] \arrow[d] & \mathbf{E}_k \arrow[r, "{[1]}"] \arrow[d] & {} \\ k^*\mathbf{L}_{\mathcal{Q}(X)/\mathfrak{M}} \arrow[r] & \mathbf{L}_{\mathcal{Q}(Y)/\mathfrak{M}} \arrow[r] & \mathbf{L}_k \arrow[r, "{[1]}"] & {} \end{tikzcd}$$

It follows from a simple diagram chase that $\mathbf{E}_k \to \mathbf{L}_k$ is a relative obstruction theory. On the other hand, assuming that $\mathcal{Q}_{g,n}(X, f_*\beta)$ is unobstructed, we may look at the long exact sequence in cohomology and find

$$0 \to \mathrm{h}^{-2}(\mathbf{E}_k) \to \mathrm{h}^{-1}(k^*\mathbf{E}_{\mathcal{Q}(X)/\mathfrak{M}}) = 0$$

Hence $\mathrm{h}^{-2}(\mathbf{E}_k) = 0$ and it is easy to show using similar arguments that $\mathbf{E}_k$ is of perfect amplitude contained in $[-1, 0]$.

□

**Remark B.6.** The short exact sequence defining $\mathcal{F}_X$ should be thought of as the pull-back of the Euler sequence

$$0 \to \mathcal{O}_X^{\oplus r_X} \to \bigoplus_{\rho \in \Sigma_X(1)} \mathcal{O}_X(D_\rho) \to \mathrm{T}_X \to 0$$

along the map $C \to X$ (if such a map existed). In particular, if we work away from the locus of basepoints then $\mathcal{F}_X = u^* \mathrm{T}_X$.

In particular, for every smooth projective variety $i\colon X \hookrightarrow \mathbb{P}^N$, we have a virtual pull-back morphism

$$k_{\mathrm{v}}^! : A_*(\mathcal{Q}_{0,n}(\mathbb{P}^N, d)) \to A_*(\mathcal{Q}_{0,n}(X, \beta))$$

where $d = i_*\beta$, and more generally for any cartesian diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & G \arrow[d] \\ \mathcal{Q}_{0,n}(X,\beta) \arrow[r, "k"] & \mathcal{Q}_{0,n}(\mathbb{P}^N, d) \end{tikzcd}$$

we get an associated virtual pull-back morphism:

$$k_{\mathrm{v}}^! : A_*(G) \to A_*(F)$$

This is used in §4 to pull-back the recursion formula for the pair $(\mathbb{P}^N, H)$ and obtain a recursion formula in the general case.

B.3. **Splitting axiom.** In this section we consider certain boundary strata of the moduli space of quasimaps, called *centipede loci*. These are the analogues in the absolute setting of the comb loci which appear in the relative setting (§3.2). The general element of such a locus has a source curve with $r+1$ irreducible components, one "trunk" of the centipede and $r$ "legs." Each of these components has a prescribed genus, curve class and set of marked points.

Given such a locus, there are two natural virtual classes with which it can be equipped. One is the product virtual class induced by the absolute product of the $r+1$ quasimap spaces, and the other is the class pulled back from the ambient moduli space. In this section we show that these classes coincide. This is the quasimap version of the *splitting axiom* from Gromov–Witten theory, called the *cutting edges axiom* in [Beh97].

We first establish notation. Fix a smooth projective toric variety $X$ and numerical invariants $g, n, \beta$ such that the corresponding quasimap space is defined. Now fix partitions $G = (g_0, \ldots, g_r)$ of the genus, $A = (A_0, \ldots, A_r)$ of the marked points and $B = (\beta_0, \ldots, \beta_r)$ of the curve class and consider the following space (which we call the *centipede locus*):

$$\mathcal{D}^{\mathcal{Q}}(X, G, A, B) := \mathcal{Q}_{g_0, A_0 \cup \{q_1, \ldots, q_r\}}(X, \beta_0) \times_{X^r} \prod_{i=1}^{r} \mathcal{Q}_{g_i, A_i \cup \{q_i\}}(X, \beta_i)$$

Of course we assume that every element of the partition is in the stable range, so that every factor in the above product makes sense. See Remark 3.9 for a justification of why these are the correct boundary strata to consider. We can equip the centipede locus with the product virtual class in the following way. Set

$$\mathcal{E}^{Q}(X,G,A,B) := Q_{g_0,A_0\cup\{q_1,\ldots,q_r\}}(X,\beta_0)\times\prod_{i=1}^{r}Q_{g_i,A_i\cup\{q_i\}}(X,\beta_i)$$

which we endow with the product class:

$$[\mathcal{E}^{Q}(X,G,A,B)]^{\mathrm{virt}} := [Q_{g_0,A_0\cup\{q_1,\ldots,q_r\}}(X,\beta_0)]^{\mathrm{virt}}\times\prod_{i=1}^{r}[Q_{g_i,A_i\cup\{q_i\}}(X,\beta_i)]^{\mathrm{virt}}$$

We then consider the cartesian diagram:

$$\begin{tikzcd} \mathcal{D}^{Q}(X,G,A,B) \arrow[r, "h"] \arrow[d, "\mathrm{ev}_q"] \arrow[dr, phantom, "\square"] & \mathcal{E}^{Q}(X,G,A,B) \arrow[d, "\mathrm{ev}_q"] \\ X^r \arrow[r, "\Delta_{X^r}"] & X^r\times X^r \end{tikzcd} \tag{15}$$

Since $X$ is smooth $\Delta_{X^r}$ is a regular embedding, so we have a Gysin map which we use to define:

$$[\mathcal{D}^{Q}(X,G,A,B)]^{\mathrm{virt}} := \Delta_{X^r}^{!}[\mathcal{E}^{Q}(X,G,A,B)]^{\mathrm{virt}}$$

Notice that if we set

$$\mathfrak{M}^{\mathrm{wt}}_{G,A,B} := \mathfrak{M}^{\mathrm{wt}}_{g_0,A_0\cup\{q_1,\ldots,q_r\},\beta_0}\times\prod_{i=1}^{r}\mathfrak{M}^{\mathrm{wt}}_{g_i,A_i\cup\{q_i\},\beta_i}$$

then there is a morphism given by forgetting everything except the source curves and their classes

$$\rho_E : \mathcal{E}^{Q}(X,G,A,B)\to\mathfrak{M}^{\mathrm{wt}}_{G,A,B}$$

and the virtual class on $\mathcal{E}^{Q}(X,G,A,B)$ is induced by a perfect obstruction theory $\mathbf{E}_{\rho_E}\to\mathbf{L}_{\rho_E}$ given by the product of the standard obstruction theories for each factor:

$$Q_{g_i,A_i\cup\{q_i\}}(X,\beta_i)\to\mathfrak{M}^{\mathrm{wt}}_{g_i,A_i,\beta_i}$$

On the other hand, we have the following cartesian diagram

$$\begin{tikzcd} \mathcal{D}^{Q}(X,G,A,B) \arrow[r, "\varphi"] \arrow[d, "\rho_D"] \arrow[dr, phantom, "\square"] & Q_{0,n}(X,\beta) \arrow[d, "\rho_Q"] \\ \mathfrak{M}^{\mathrm{wt}}_{G,A,B} \arrow[r, "\psi"] & \mathfrak{M}^{\mathrm{wt}}_{g,n,\beta} \end{tikzcd} \tag{16}$$

The bottom horizontal map is not a closed immersion: due to the existence of degree–0 rational components, there may be many possible equally valid ways of breaking up a nodal curve. For instance, consider the following example of two elements which map to the same curve under $\psi$. [FIGURE]

Nevertheless $\psi$ has a natural perfect obstruction theory, given by $\mathbf{L}_\psi$: we only need to show that it is supported in $[-1,0]$. Consider the exact triangle:

$$\psi^*\mathbf{L}_{\mathfrak{M}^{\mathrm{wt}}_{g,n,\beta}} \to \mathbf{L}_{\mathfrak{M}^{\mathrm{wt}}_{G,A,B}} \to \mathbf{L}_\psi \xrightarrow{[1]}$$

The first two terms are concentrated in degrees $[0,1]$, because they are the cotangent complexes of smooth Artin stacks. Therefore $\mathbf{L}_\psi$ is concentrated in degrees $[-1,1]$. Furthermore, if we examine the long exact cohomology sequence near $\mathrm{h}^1(\mathbf{L}_\psi)$ we find

$$\mathrm{h}^1(\psi^*\mathbf{L}_{\mathfrak{M}^{\mathrm{wt}}_{g,n,\beta}}) \to \mathrm{h}^1(\mathbf{L}_{\mathfrak{M}^{\mathrm{wt}}_{G,A,B}}) \to \mathrm{h}^1(\mathbf{L}_\psi) \to 0$$

and hence we must show that the first map is surjective. But this is dual to the map which takes an infinitesimal automorphism of the disconnected curve to an infinitesimal automorphism of the corresponding connected curve (obtained by glueing together the "nodal" marked points). The space of infinitesimal automorphisms of a nodal curve splits into a direct sum of infinitesimal automorphisms of each component; since the glueing does not affect the components, we see that this map is an isomorphism. Hence $\mathrm{h}^1(\mathbf{L}_\psi) = 0$ as claimed; morally this follows from the fact that the fibres of $\psi$ are Deligne–Mumford.

Hence there is a virtual pull-back map $\psi^!$ which defines a class

$$\psi^![\mathcal{Q}_{g,n}(X,\beta)]^{\mathrm{virt}}$$

on $\mathcal{D}^{\mathcal{Q}}(X,G,A,B)$. By functoriality of virtual pull-backs, this is induced by the perfect obstruction theory:

$$\varphi^*\mathbf{E}_{\rho_Q} \to \mathbf{L}_{\rho_D}$$

Finally if we look at (15) we see that $\mathrm{ev}_q^* \mathbf{L}_{\Delta_{X^r}} \to \mathbf{L}_h$ is a perfect obstruction theory for the map $h$. To summarise, we have a triangle

$$\begin{tikzcd}
\mathcal{D}^{\mathcal{Q}}(X,G,A,B) \arrow[rr, "h"] \arrow[dr, "\rho_D"'] & & \mathcal{E}^{\mathcal{Q}}(X,G,A,B) \arrow[dl, "\rho_E"] \\
& \mathfrak{M}^{\mathrm{wt}}_{G,A,B} &
\end{tikzcd} \tag{17}$$

where all three morphisms are equipped with perfect obstruction theories. We simply need to check that these fit together in a compatible triple

**Lemma B.7.** There is a compatible triple

$$(h^*\mathbf{E}_{\rho_E}, \varphi^*\mathbf{E}_{\rho_Q}, \mathrm{ev}_q^* \mathbf{L}_{\Delta_{X^r}})$$

for the triangle (17). Hence by functoriality of virtual pull-backs we have:

$$\psi^![\mathcal{Q}_{g,n}(X,\beta)]^{\mathrm{virt}} = \Delta^!_{X^r}[\mathcal{E}^{\mathcal{Q}}(X,G,A,B)]^{\mathrm{virt}} = [\mathcal{D}^{\mathcal{Q}}(X,G,A,B)]^{\mathrm{virt}}$$

*Proof.* We need to construct a morphism of triangles

$$\begin{tikzcd}
h^*\mathbf{E}_{\rho_E} \arrow[r] \arrow[d] & \varphi^*\mathbf{E}_{\rho_Q} \arrow[r] \arrow[d] & \mathrm{ev}_q^* \mathbf{L}_{\Delta_{X^r}} \arrow[r, "{[1]}"] \arrow[d] & {} \\
h^*\mathbf{L}_{\rho_E} \arrow[r] & \mathbf{L}_{\rho_D} \arrow[r] & \mathbf{L}_h \arrow[r, "{[1]}"] & {}
\end{tikzcd}$$

Consider the following diagram:

$$\begin{tikzcd}
h^*\tilde{C} \arrow[r, "\nu"] \arrow[dr, "\eta"'] & \varphi^*C \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & C \arrow[d, "\pi"] \\
& \mathcal{D}^Q(X,G,A,B) \arrow[r, "\varphi"] & Q_{g,n}(X,\beta)
\end{tikzcd}$$

Here $\tilde{C}$ is the universal (disconnected) curve over $\mathcal{E}^Q(X,G,A,B)$, which we have pulled back to $\mathcal{D}^Q(X,G,A,B)$, while $\varphi^*C$ is the universal curve over $\mathcal{D}^Q(X,G,A,B)$. Therefore the map $\nu : h^*\tilde{C} \to \varphi^*C$ is (fiberwise) a partial normalisation map given by normalising the nodes which connect the "trunk" of the centipede to the "legs."

There are natural sheaves $\mathcal{F}$ and $\tilde{\mathcal{F}}$ on $C$ and $h^*\tilde{C}$ respectively, such that

$$\varphi^*\mathbf{E}^\vee_{\rho_Q} = \mathrm{R}^\bullet\,\pi_*\mathcal{F}$$
$$h^*\mathbf{E}^\vee_{\rho_E} = \mathrm{R}^\bullet\,\eta_*\tilde{\mathcal{F}}$$

Furthermore $\nu^*\mathcal{F} \simeq \tilde{\mathcal{F}}$, hence by tensoring the partial normalisation short exact sequence

$$0 \to \mathcal{O}_{\varphi^*C} \to \nu_*\mathcal{O}_{h^*\tilde{C}} \to \mathcal{O}_q \to 0$$

with $\mathcal{F}$ and applying the projection formula, we obtain

$$0 \to \mathcal{F} \to \nu_*\tilde{\mathcal{F}} \to \mathcal{F}_q \to 0$$

on $\varphi^*C$, where $q$ is the locus of nodes connecting the trunk to the spine. (The fact that the morphism on the left is injective follows by applying the Snake Lemma to the short exact sequence defining $\mathcal{F}$.) To this we can apply $\mathrm{R}^\bullet\,\pi_*$ to obtain an exact triangle

$$\mathrm{R}^\bullet\,\pi_*\mathcal{F} \to \mathrm{R}^\bullet\,\eta_*\tilde{\mathcal{F}} \to \mathrm{R}^\bullet\,\pi_*\mathcal{F}_q \xrightarrow{[1]} \tag{18}$$

Finally, notice that, since quasimaps are required not to have base-points at the nodes, the fibre of the sheaf $\mathcal{F}$ at each of the nodes $q$ can actually be identified with the tangent to the toric variety $X$ at the image of the node itself, i.e. $\mathrm{R}^\bullet\,\pi_*\mathcal{F}_q \cong \mathrm{ev}_q^*\,\mathrm{T}_{X^r} = \mathrm{ev}_q^*\,\mathrm{T}_{\Delta_{X^r}}[-1]$. Dualising sequence (18) we obtain

$$h^*\mathbf{E}_{\rho_E} \to \varphi^*\mathbf{E}_{\rho_Q} \to \mathrm{ev}_q^*\,\mathbf{E}_{\Delta_{X^r}} \xrightarrow{[1]}$$

as required. $\square$

B.4. **Comparison with the GIT construction.** Let $X$ be a smooth projective toric variety and $Y \hookrightarrow X$ a smooth very ample hypersurface. The complete linear system $|\mathcal{O}_X(Y)|$ gives an embedding $i : X \hookrightarrow \mathbb{P}^N$ which expresses $Y$ as the intersection inside $\mathbb{P}^N$ of $X$ and a certain hyperplane $H$: $Y = i^{-1}(H)$. We can *define* the moduli space of quasimaps to $Y$ via the following cartesian diagram:

$$\begin{tikzcd} \mathcal{Q}_{g,n}(Y,\beta) \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\square"] & \mathcal{Q}_{g,n}(H,d) \arrow[d, hook] \\ \mathcal{Q}_{g,n}(X,\beta) \arrow[r, hook, "k"] & \mathcal{Q}_{g,n}(\mathbb{P}^N,d) \end{tikzcd}$$

where $d = i_*\beta$. This moduli space is easy to describe: let $s_Y$ denote the section of $\mathcal{O}_X(Y)$ cutting out $Y$ inside $X$. Recall from §2.3 that for any quasimap

$$((C,x_1,\ldots,x_n),(L_\rho,u_\rho)_{\rho\in\Sigma_X(1)},(\varphi_m)_{m\in M_X}) \in \mathcal{Q}_{g,n}(X,\beta)$$

we can construct a section $u_Y$ of a line bundle $L_Y$ on $C$, which plays the role of the pull-back of $s_Y$ to $C$. Then

$$\mathcal{Q}_{g,n}(Y,\beta) \subseteq \mathcal{Q}_{g,n}(X,\beta)$$

consists of those quasimaps such that $u_Y \equiv 0$.

The cartesian diagram above can also be used to endow $\mathcal{Q}_{g,n}(Y,\beta)$ with a virtual class via virtual (or diagonal) pull-back along $k$. Thus we can define quasimap invariants for $Y$.

On the other hand, $Y$ has the natural structure of a GIT quotient

$$Y = C(Y) /\!\!/ G$$

where $C(Y) \subseteq \mathbb{A}^{\Sigma_X(1)}$ is the affine cone over $Y$ and $G = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}(X),\mathbb{G}_{\mathrm{m}}) \cong \mathbb{G}_{\mathrm{m}}^{r_X}$ acts on $C(Y)$ via the natural inclusion

$$\mathbb{G}_{\mathrm{m}}^{r_X} \hookrightarrow \mathbb{G}_{\mathrm{m}}^{\Sigma_X(1)}$$

(here $C(Y) \subseteq \mathbb{A}^{\Sigma_X(1)}$ is preserved by $G$ because it is cut out by a homogeneous equation). In [CFKM14] moduli spaces of quasimaps are constructed for GIT quotient targets (satisfying a number of conditions, all of which hold for $Y$). There is thus a moduli space

$$\mathcal{Q}^{\mathrm{GIT}}_{g,n}(Y,\beta)$$

which admits a virtual class. Hence we have two moduli spaces of quasimaps to $Y$, each equipped with a virtual class, and we want to check that these definitions agree.

Objects of $\mathcal{Q}^{\mathrm{GIT}}_{g,n}(Y,\beta)$ are diagrams of the form

$$\begin{tikzcd} P \arrow[r] \arrow[d] & C(Y) \\ C & \end{tikzcd}$$

where $C$ is a prestable curve, $P$ is a principal $G$-bundle and the map $P \to C(Y)$ is $G$-equivariant. Equivalently, an object consists of a prestable curve $C$, a principal $G$-bundle $P$ and a section $u$ of the associated $C(Y)$-bundle:

$$\begin{tikzcd} P \times_G C(Y) \arrow[d, shift right, "p"'] \\ C \arrow[u, bend right, "u"'] \end{tikzcd}$$

The obstruction theory on this space is defined relative to the stack $\mathfrak{Bun}_G$ parametrising principal $G$-bundles on the universal curve:

$$\mathcal{C}_{\mathfrak{M}_{g,n}} \to \mathfrak{M}_{g,n}$$

It is given by

$$\mathbf{E}^\vee_{\mathcal{Q}/\mathfrak{Bun}_G} = \mathrm{R}^\bullet\,\pi_*(u^*\mathbf{L}_p)$$

where $\pi$ is the universal curve over $\mathcal{Q} = \mathcal{Q}^{\mathrm{GIT}}_{g,n}(Y,\beta)$. There is a natural isomorphism

$$\mathfrak{Bun}_G^{g,n} \cong \times^{r_X}_{\mathfrak{M}_{g,n}} \mathfrak{Pic}_{\mathfrak{g},\mathfrak{n}}$$

given by sending $P$ to the $r_X$ individual factors of the affine bundle $P \times_G \mathbb{A}^{r_X}$. Furthermore there is a $G$-equivariant embedding

$$\begin{tikzcd} P \times_G C(Y) \arrow[r, hook, "j"] \arrow[d, shift right, "p"'] & P \times_G \mathbb{A}^{\Sigma_X(1)} \cong \bigoplus_{\rho\in\Sigma_X(1)} L_\rho \arrow[dl] \\ C \arrow[u, bend right, "u"'] & \end{tikzcd}$$

which expresses $P \times_G C(Y)$ as the vanishing locus of $u_Y$ in $\oplus_{\rho\in\Sigma_X(1)} L_\rho$. This shows that the two definitions of the moduli space agree.

Finally we must compare the virtual classes. Using the normal sheaf sequence for the inclusion $j$ (relative to the base $C$) we obtain a short exact sequence on $C$:

$$0 \to u^*\,\mathrm{T}_p \to \bigoplus_{\rho\in\Sigma_X(1)} L_\rho \to u^*\,\mathrm{N}_{P\times_G C(Y)/\oplus_{\rho\in\Sigma_X(1)} L_\rho} \to 0$$

Since $P \times_G C(Y)$ is defined by the vanishing of $u_Y$, we see that the final term is isomorphic to the line bundle $L_Y$ discussed above. Thus as elements of the derived category

$$u^*\,\mathrm{T}_p = \left[\bigoplus_{\rho\in\Sigma_X(1)} L_\rho \to L_Y\right]$$

Applying $\mathrm{R}^\bullet\,\pi_*$ we obtain on the left hand side the obstruction theory for the GIT moduli space relative $\mathfrak{Bun}_G^{g,n}$. On the other hand, the first term on the right hand side is the obstruction theory for $\mathcal{Q}(X)$ relative the product of the Picard stacks (isomorphic to $\mathfrak{Bun}_G^{g,n}$ via the discussion above) whereas the second term is the relative obstruction theory for $Q(Y)$ inside $Q(X)$. Thus the virtual classes agree, as claimed.

## Appendix C. Some intersection-theoretic lemmas

In this appendix we explicitly define the *diagonal pull-back* along a morphism whose target is unobstructed (used in [Gat02]) and verify that this agrees with the virtual pull-back of [Man12a] when both are defined. We also check that it satisfies some expected compatibility properties.

Consider a morphism of DM stacks $f\colon Y \to X$ over a smooth base $\mathfrak{M}$, such that $X$ is smooth over $\mathfrak{M}$ and $Y$ carries a virtual class given by a perfect obstruction theory $\mathbf{E}_{Y/\mathfrak{M}}$. Then, for every Cartesian diagram

$$\begin{tikzcd} G \arrow[r, "g"] \arrow[d, "q"] \arrow[dr, phantom, "\square"] & F \arrow[d, "p"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

and every class $\alpha \in \mathrm{A}_*(F)$, we may define

$$f^!_\Delta(\alpha) = \Delta^!_X([Y]^{\mathrm{vir}} \times \alpha) \in \mathrm{A}_*(G)$$

which we call the *diagonal (virtual) pull-back*. We first show that it coincides with the usual virtual pull-back along $f$ in the presence of a compatible perfect obstruction theory for $f$.

**Lemma C.1.** Assume that there exists a relative obstruction theory $\mathbf{E}_f$ compatible with $\mathbf{E}_{Y/\mathfrak{M}}$ and the standard (unobstructed) obstruction theory for $X$, i.e:

$$\begin{tikzcd} f^*\mathbf{L}_{X/\mathfrak{M}} \arrow[r] \arrow[d, "\mathrm{Id}"] & \mathbf{E}_{Y/\mathfrak{M}} \arrow[r] \arrow[d] & \mathbf{E}_f \arrow[r, "{[1]}"] \arrow[d] & {} \\ f^*\mathbf{L}_{X/\mathfrak{M}} \arrow[r] & \mathbf{L}_{Y/\mathfrak{M}} \arrow[r] & \mathbf{L}_f \arrow[r, "{[1]}"] & {} \end{tikzcd}$$

Then for every Cartesian diagram and every class $\alpha \in \mathrm{A}_*(F)$ as above,

$$f^!_{\mathrm{v}}(\alpha) = f^!_\Delta(\alpha).$$

*Proof.* Consider the following cartesian diagram:

$$\begin{tikzcd} G \arrow[r, "q\times g"] \arrow[d, "g"] \arrow[dr, phantom, "\square"] & Y \times_{\mathfrak{M}} F \arrow[r, "\mathrm{pr}_1"] \arrow[d, "f\times\mathrm{Id}"] \arrow[dr, phantom, "\square"] & Y \arrow[d, "f"] \\ F \arrow[r, "p\times\mathrm{Id}"] \arrow[d, "p"] \arrow[dr, phantom, "\square"] & X \times_{\mathfrak{M}} F \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{Id}\times p"] & X \\ X \arrow[r, "\Delta_X"] & X \times_{\mathfrak{M}} X & \end{tikzcd}$$

Then, by commutativity of virtual pull-backs, we have

$$\begin{aligned}\Delta^!_X([Y]^{\mathrm{vir}} \times \alpha) &= \Delta^!((f^!_{\mathrm{v}}[X]) \times \alpha) \\ &= \Delta^!_X(f^!_{\mathrm{v}}([X] \times \alpha)) \\ &= f^!_{\mathrm{v}}(\Delta^!_X([X] \times \alpha)) \\ &= f^!_{\mathrm{v}}(\alpha)\end{aligned}$$

as required. □

Secondly, we show that the *diagonal* virtual pull-back behaves similarly to an ordinary virtual pull-back (e.g. commutes with other virtual pull-backs) even in the absence of a compatible perfect obstruction theory.

**Lemma C.2.** The *diagonal* virtual pull-back morphism as defined above commutes with ordinary Gysin maps and with virtual pull-backs.

*Proof.* First consider the case of ordinary Gysin maps. We must consider a cartesian diagram:

$$\begin{tikzcd} Y'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & S \arrow[d, "k"] \\ Y' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X' \arrow[r] \arrow[d] & T \\ Y \arrow[r, "f"] & X & \end{tikzcd}$$

with $k$ a regular embedding and $f: Y \to X$ as before. We need to show that for all $\alpha \in A_*(X')$:

$$k^! f^!_\Delta(\alpha) = f^!_\Delta k^!(\alpha)$$

We form the cartesian diagram:

$$\begin{tikzcd} Y'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y \times X'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & S \arrow[d, "k"] \\ Y' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y \times X' \arrow[r] \arrow[d] & T \\ X \arrow[r, "\Delta_X"] & X \times X & \end{tikzcd}$$

And apply commutativity of usual Gysin morphisms. In the case where $k$ is not a regular embedding but rather is equipped with a relative perfect obstruction theory, the same argument works with $k^!$ replaced by $k^!_{\mathrm{v}}$. □

## References

[AC14] Dan Abramovich and Qile Chen. Stable logarithmic maps to Deligne-Faltings pairs II. *Asian J. Math.*, 18(3):465–488, 2014.

[AF16] Dan Abramovich and Barbara Fantechi. Orbifold techniques in degeneration formulas. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 16(2):519–579, 2016.

[AMW14] Dan Abramovich, Steffen Marcus, and Jonathan Wise. Comparison theorems for Gromov-Witten invariants of smooth pairs and of degenerations. *Ann. Inst. Fourier (Grenoble)*, 64(4):1611–1667, 2014.

[Beh97] K. Behrend. Gromov-Witten invariants in algebraic geometry. *Invent. Math.*, 127(3):601–617, 1997.

[Ber00] Aaron Bertram. Another way to enumerate rational curves with torus actions. *Invent. Math.*, 142(3):487–512, 2000.

[CFK10] Ionuţ Ciocan-Fontanine and Bumsig Kim. Moduli stacks of stable toric quasimaps. *Adv. Math.*, 225(6):3022–3051, 2010.

[CFK14] Ionuţ Ciocan-Fontanine and Bumsig Kim. Wall-crossing in genus zero quasimap theory and mirror maps. *Algebr. Geom.*, 1(4):400–448, 2014.

[CFKM14] Ionuţ Ciocan-Fontanine, Bumsig Kim, and Davesh Maulik. Stable quasimaps to GIT quotients. *J. Geom. Phys.*, 75:17–47, 2014.

[Che14] Qile Chen. Stable logarithmic maps to Deligne-Faltings pairs I. *Ann. of Math. (2)*, 180(2):455–521, 2014.

[CM] Tom Coates and Cristina Manolache. In preparation.

[Cos06] Kevin Costello. Higher genus Gromov-Witten invariants as genus zero invariants of symmetric products. *Ann. of Math. (2)*, 164(2):561–601, 2006.

[Cox95a] David A. Cox. The functor of a smooth toric variety. *Tohoku Math. J. (2)*, 47(2):251–262, 1995.

[Cox95b] David A. Cox. The homogeneous coordinate ring of a toric variety. *J. Algebraic Geom.*, 4(1):17–50, 1995.

[CZ14] Yaim Cooper and Aleksey Zinger. Mirror symmetry for stable quotients invariants. *Michigan Math. J.*, 63(3):571–621, 2014.

[Ful98] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998.

[Gat02] Andreas Gathmann. Absolute and relative Gromov-Witten invariants of very ample hypersurfaces. *Duke Math. J.*, 115(2):171–203, 2002.

[Gat03a] Andreas Gathmann. Gromov-Witten invariants of hypersurfaces, 2003. Habilitation thesis.

[Gat03b] Andreas Gathmann. Relative Gromov-Witten invariants and the mirror formula. *Math. Ann.*, 325(2):393–412, 2003.

[Giv96] Alexander B. Givental. Equivariant Gromov-Witten invariants. *Internat. Math. Res. Notices*, (13):613–663, 1996.

[Giv98] Alexander Givental. *A Mirror Theorem for Toric Complete Intersections*, pages 141–175. Birkhäuser Boston, Boston, MA, 1998.

[GP99] T. Graber and R. Pandharipande. Localization of virtual classes. *Invent. Math.*, 135(2):487–518, 1999.

[GS13] Mark Gross and Bernd Siebert. Logarithmic Gromov-Witten invariants. *J. Amer. Math. Soc.*, 26(2):451–510, 2013.

[GS16] M. Gross and B. Siebert. Intrinsic mirror symmetry and punctured Gromov-Witten invariants. *ArXiv e-prints*, September 2016.

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[Kim10] Bumsig Kim. Logarithmic stable maps. In *New developments in algebraic geometry, integrable systems and mirror symmetry (RIMS, Kyoto, 2008)*, volume 59 of *Adv. Stud. Pure Math.*, pages 167–200. Math. Soc. Japan, Tokyo, 2010.

[Knu83] Finn F. Knudsen. The projectivity of the moduli space of stable curves. II. The stacks $M_{g,n}$. *Math. Scand.*, 52(2):161–199, 1983.

[KP01] B. Kim and R. Pandharipande. The connectedness of the moduli space of maps to homogeneous spaces. In *Symplectic geometry and mirror symmetry (Seoul, 2000)*, pages 187–201. World Sci. Publ., River Edge, NJ, 2001.

[Li01] Jun Li. Stable morphisms to singular schemes and relative stable morphisms. *J. Differential Geom.*, 57(3):509–578, 2001.

[Li02] Jun Li. A degeneration formula of GW-invariants. *J. Differential Geom.*, 60(2):199–293, 2002.

[LLY97] Bong H. Lian, Kefeng Liu, and Shing-Tung Yau. Mirror principle. I. *Asian J. Math.*, 1(4):729–763, 1997.

[Man12a] Cristina Manolache. Virtual pull-backs. *J. Algebraic Geom.*, 21(2):201–245, 2012.

[Man12b] Cristina Manolache. Virtual push-forwards. *Geom. Topol.*, 16(4):2003–2036, 2012.

[MOP11] Alina Marian, Dragos Oprea, and Rahul Pandharipande. The moduli space of stable quotients. *Geom. Topol.*, 15(3):1651–1706, 2011.

[PR03] Mihnea Popa and Mike Roth. Stable maps and Quot schemes. *Invent. Math.*, 152(3):625–663, 2003.