RELATIVE QUASIMAPS À LA GATHMANN

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Abstract. abstract

1. Functoriality of Quasimap Spaces

In the case of stable maps, a morphism $f : X \to Y$ induces a morphism between the corresponding moduli spaces

$$\overline{\mathcal{M}}_{g,n}(X, \beta) \to \overline{\mathcal{M}}_{g,n}(Y, f_*\beta)$$

given by composition with f (in general this induced morphism may involve stabilisation of the source curve). Because of this, the construction of the moduli space of stable maps is said to be **functorial**.

It is natural to ask whether the same holds for the moduli space of quasimaps. Since here the objects of the moduli space are not maps, we cannot simply compose with f, and indeed it is not immediately clear how we should proceed. In [**?**, Section 3.1] a definition is given when f is an embedding into a projective space; however, this uses the more general language of GIT quotients which we seek to avoid here. As such, we will provide an alternative (but entirely equivalent) construction in the setting of toric varieties, while simultaneously relaxing the conditions on the map f and the target Y.
[1]

Our approach uses the language of Σ–collections introduced by D. Cox. This approach is natural insofar as a quasimap is a generalisation of a Σ–collection. We will refer extensively to [Cox95b] and [Cox95a], which we recommend as an introduction for any readers unfamiliar with the theory.

So let X and Y be smooth and proper toric varieties with fans $\Sigma_X \subseteq N_X$ and $\Sigma_Y \subseteq N_Y$. Suppose we are given $f : Y \to X$ (which we do not assume to be a toric morphism). By [Cox95a, Theorem 1.1] the data of such a map is equivalent to a Σ_X–collection on Y:

$$((L_\rho, u_\rho)_{\rho \in \Sigma_X(1)}, (\varphi_{m_x})_{m_x \in M_X})$$

In addition, [Cox95b] allows us to describe line bundles on Y and their global sections in terms of the homogeneous coordinates $(z_\tau)_{\tau \in \Sigma_Y(1)}$. All of these observations are combined into the following theorem, which is so useful that we will state it here in its entirety:

Theorem 1.1. [Cox95a, Theorem 2.2] *The data of a morphism $f : Y \to X$ is the same as the data of homogeneous polynomials*

$$P_\rho \in S^Y_{\beta_\rho}$$

for $\rho \in \Sigma_X(1)$, where $\beta_\rho \in \operatorname{Pic} Y$ and $S^Y_{\beta_\rho}$ is the corresponding graded piece of the Cox ring

$$S^Y = k[z_\tau : \tau \in \Sigma_Y(1)]$$

This data is required to satisfy the following two conditions:

(1) $\sum_{\rho \in \Sigma_X(1)} \beta_\rho \otimes n_\rho = 0$ *in* $\operatorname{Pic} Y \otimes N_X$.

(2) $(P_\rho(z_\tau)) \notin Z(\Sigma_X) \subseteq \mathbb{A}_k^{\Sigma_X(1)}$ *whenever* $(z_\tau) \notin Z(\Sigma_Y) \subseteq \mathbb{A}_k^{\Sigma_Y(1)}$.

[1]We should probably look a bit harder to see if the definition exists elsewhere.

Furthermore, two such sets of data (P_ρ) *and* (P'_ρ) *correspond to the same morphism if and only if there exists a* $\lambda \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}\, X, \mathbb{G}_m)$ *such that*

$$\lambda(D_\rho) \cdot P_\rho = P'_\rho$$

for all $\rho \in \Sigma_X(1)$. *Finally, if we define* $\tilde{f}(z_\tau) = (P_\rho(z_\tau))$ *then this defines a lift of* f *to the prequotients:*

$$\begin{tikzcd} \mathbb{A}_k^{\Sigma_Y(1)} - Z(\Sigma_Y) \arrow[r, "\tilde{f}"] \arrow[d, "\pi"] & \mathbb{A}_k^{\Sigma_X(1)} - Z(\Sigma_X) \arrow[d, "\pi"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

Aside 1.2. Throughout this section we will stick to the notation established above; in particular we will use ρ to denote a ray in $\Sigma_X(1)$ and τ to denote a ray in $\Sigma_Y(1)$.

Recall our goal: given a map $f : Y \to X$ we wish to define a "push-forward" map:

$$f_* : \overline{Q}_{g,n}(Y, \beta) \to \overline{Q}_{g,n}(X, f_*\beta)$$

Consider therefore a quasimap $(C, (L_\tau, u_\tau)_{\tau \in \Sigma_Y(1)}, (\varphi_{m_Y})_{m_Y \in M_Y})$ with target Y. Pick data $(P_\rho)_{\rho \in \Sigma_X(1)}$ corresponding to the map f, as in the theorem above; we will later see that our construction does not depend on this choice.

For each ρ, P_ρ is a polynomial in the z_τ; we can write it as

$$P_\rho(z_\tau) = \sum_{\underline{a}} P_\rho^{\underline{a}}(z_\tau) = \sum_{\underline{a}} \mu_{\underline{a}} \prod_\tau z_\tau^{a_\tau} \tag{1}$$

where the sum is over certain multindices $\underline{a} = (a_\tau) \in \mathbb{N}^{\Sigma_Y(1)}$ and the $\mu_{\underline{a}}$ are nonzero scalars. The homogeneity of P_ρ implies that for all $\underline{a}$ we have:

$$\sum_\tau a_\tau D_\tau = \beta_\rho \in \mathrm{Pic}\, Y$$

For each $\underline{a}$ consider the following line bundle on C:

$$\tilde{L}_\rho^{\underline{a}} = \bigotimes_\tau L_\tau^{\otimes a_\tau}$$

Then we may take the following section of $\tilde{L}_\rho^{\underline{a}}$:

$$\mu_{\underline{a}} \prod_\tau u_\tau^{a_\tau}$$

Thus each of the terms $P_\rho^{\underline{a}}$ of P_ρ defines a section $\tilde{u}_\rho^{\underline{a}}$ of a line bundle $\tilde{L}_\rho^{\underline{a}}$. But what we want is a single section $\tilde{u}_\rho$ of a single line bundle $\tilde{L}_\rho$. This is where the isomorphisms φ_{m_Y} come in.

Recall that we have a short exact sequence:

$$0 \longrightarrow M_Y \xrightarrow{\theta} \mathbb{Z}^{\Sigma_Y(1)} \longrightarrow \mathrm{Pic}\, Y \longrightarrow 0$$

Let $\underline{a}$ and $\underline{b}$ be multindices appearing in the sum (1) above. By the homogeneity of P_ρ we have

$$\sum_\tau a_\tau D_\tau = \beta_\rho = \sum_\tau b_\tau D_\tau$$

which is precisely the statement that in the above sequence $\underline{a}$ and $\underline{b}$ map to the same element of $\operatorname{Pic} Y$. Hence there exists an $m_Y \in M_Y$ such that:

$$\theta(m_Y) = \underline{a} - \underline{b}$$

Now, the isomorphism φ_{m_Y} (contained in the data of our original quasimap) is a map:

$$\varphi_{m_Y} : \bigotimes_\tau L_\tau^{\otimes \langle m_Y, u_\tau \rangle} \cong \mathcal{O}_C$$

By definition, $\theta(m_Y) = (\langle m_Y, u_\tau \rangle_{\tau \in \Sigma_Y(1)}$. But also $\theta(m_Y) = (a_\tau - b_\tau)_{\tau \in \Sigma_Y(1)}$. Hence we have:

$$\varphi_{m_Y} : \bigotimes_\tau L_\tau^{\otimes a_\tau} \cong \bigotimes_\tau L_\tau^{\otimes b_\tau}$$

In other words, we have well-defined canonical isomorphisms

$$\tilde{L}_\rho^{\underline{a}} \cong \tilde{L}_\rho^{\underline{b}}$$

for all $\underline{a}$ and $\underline{b}$. Let us choose one such $\underline{a}$ (it doesn't matter which) and define:

$$\tilde{L}_\rho = \tilde{L}_\rho^{\underline{a}}$$

Then for all $\underline{b}$ we can use the above isomorphism to view $\tilde{u}_\rho^{\underline{b}}$ as a section of $\tilde{L}_\rho$. Summing all of these together we obtain a section $\tilde{u}_\rho$ of $\tilde{L}_\rho$, which we can write (with abuse of notation) as:

$$\tilde{u}_\rho = \sum_{\underline{a}} \mu_{\underline{a}} \prod_\tau u_\tau^{a_\tau}$$

(Note that if we had made a different choice of $\underline{a}$ above the result would have been isomorphic.)

Thus, we have $(\tilde{L}_\rho, \tilde{u}_\rho)_{\rho \in \Sigma_X(1)}$ on C. It remains to define the isomorphisms

$$\tilde{\varphi}_{m_X} : \bigotimes_\rho \tilde{L}_\rho^{\otimes \langle m_X, u_\rho \rangle} \cong \mathcal{O}_C$$

for all $m_X \in M_X$.

2. Quasimaps to $\mathbb{P}^r$ relative to a hyperplane

We first deal with genus 0 quasimaps to projective space, relative to a hyperplane. We give a Gathmann-like description of the space of relative quasimaps as a closed substack of the moduli space of (absolute) quasimaps to $\mathbb{P}^r$; it turns out to be irreducible of the expected dimension. Finally, we retrieve a Gathmann-type formula by pushforward along the comparison morphism $\chi\colon \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d) \to \overline{Q}_{0,n}(\mathbb{P}^r, d)$.

Fix coordinates on $\mathbb{P}^r$ such that the hyperplane H is $\{x_0 = 0\}$. Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be an n-tuple of nonnegative integers. Consider the following locus $\widetilde{Q}_{0,\alpha}(\mathbb{P}^r|H, d)$ inside $\overline{Q}_{0,n}(\mathbb{P}^r, d)$: the quasimaps $(C, x_1, \dots, x_n, L, u_0, \dots, u_r)$ such that, if Z is a connected component of the vanishing locus of u_0 in C, then one of the following holds:

1. Z is a point, either unmarked, or one of the x_i's, and in this case u_0 vanishes at Z with multiplicity at least α_i.
2. Z is a curve (*internal*); letting $C^{(1)}, \dots, C^{(k)}$ be the (*external*) irreducible components adjacent to Z, with nodes $q_i = Z \cap C^{(i)}$, and $m^{(i)}$ the order of vanishing of $u_{0|C^{(i)}}$ at q_i, we must have

$$\deg(L_{|Z}) + \sum_{i=1}^{k} m^{(i)} \geq \sum_{x_j \in Z} \alpha_j$$

On the other hand, denote by $Q_{0,\alpha}(\mathbb{P}^r|H, d)$ the *nice locus*, consisting of actual maps from an irreducible curve (i.e. $\mathbb{P}^1$) and with specified tangency condition α at the markings $\mathbf{x}$. Notice that this is an irreducible, locally closed substack of $\overline{Q}_{0,n}(\mathbb{P}^r, d)$, by pretty much the same argument as in [Gat02, Lemma 1.8]; it has codimension $\sum \alpha$. In fact it is isomorphic to the nice locus inside stable maps, that Gathmann denotes by $\mathcal{M}_{0,\alpha}(\mathbb{P}^r|H, d)$ [Gat02, Def. 1.6] (the stricter stability condition has no effect when the source curve is irreducible, of course provided $n \geq 2$); hence:

Lemma 2.1. *The comparison morphism restricts to a birational morphism $\overline{\mathcal{M}}_{0,\alpha}(\mathbb{P}^r|H, d) \to \widetilde{Q}_{0,\alpha}(\mathbb{P}^r|H, d)$.*

Proof. The contraction of a rational tail R always happens far away from the markings, hence the only care we need to take is when the one component touching R is internal (call it Z); in this case, observe that $m^{(R)} \leq \deg(f_{|R})$ and the quasimap resulting from the contraction of R has $\deg(L_{|Z}) = \deg(f_{|Z}) + \deg(f_{|R})$, so the corresponding term only moves around the LHS of the α-tangency condition nr. 2.

Birationality follows from the fact that the comparison morphism restricts to give an isomorphism between the nice loci. □

Lemma 2.2. *With notations as above (with $\sum \alpha \leq d$), $\widetilde{Q}_{0,\alpha}(\mathbb{P}^r|H, d)$ is the closure of the nice locus $Q_{0,\alpha}(\mathbb{P}^r|H, d)$ inside $\overline{Q}_{0,n}(\mathbb{P}^r, d)$.*

Proof. $\widetilde{Q}_{0,\alpha}(\mathbb{P}^r|H,d) \subseteq \overline{Q_{0,\alpha}(\mathbb{P}^r|H,d)}$: we show that, given any quasimap satisfying the α-tangency conditions spelled above, it can be (infinitesimally) deformed to a stable *map* satisfying Gathmann's conditions [Gat02, Def. 1.1 and Rmk. 1.4], and then appeal to [Gat02, Prop. 1.14].

We induct on the number of components containing at least one base-point. If this number is zero, we're done (because quasimap stability is stronger than map stability); otherwise, pick such a component C_0, with base-points $p_1, \ldots, p_h$ and adjacent rational trees $R_1, \ldots, R_k$, joined to C_0 at the nodes $q_1, \ldots, q_k$. Since there are base-points but the quasimap respects the nondegeneracy condition, $\deg(L_{|C_0}) > 0$, and since $C_0 \simeq \mathbb{P}^1$ we can find a section w of $L_{|C_0} \simeq \mathcal{O}_{\mathbb{P}^1}(d_0)$ not vanishing at any of the base-points p_i's; then it is enough to deform the section $u_{r|C_0}$ to $u_{r|C_0} + \epsilon w$ (and keep the other sections the same) in order to delete the base-points belonging to C_0. Notice that $u_{0|C_0}$ is unchanged, so the deformation still respects α-tangency at the markings lying on C_0 (whether the latter is an internal or an external component). We need to check that such a deformation can be extended to the whole curve C without changing the vanishing conditions on u_0. Notice that the action of PGL_{r+1} on $\mathbb{P}^r$ extends to an action of the group on the space of quasimaps; we can apply the matrix

$$\begin{bmatrix} 1 & & \\ & \ddots & \\ & \epsilon\frac{w(q_i)}{u_j(q_i)} & 1 \end{bmatrix}$$

to the restriction of the original quasimap to R_i, where j is any index s.t. $u_j(q_i) \neq 0$ (one such must exist because the node is not allowed to be a base-point), and by doing this separately to every rational tree springing from C_0 we get a deformation of the original quasimap that still has α-tangency with the hyperplane H (u_0 hasn't been touched at all), but the base-points on C_0 have been eliminated.

$\overline{Q_{0,\alpha}(\mathbb{P}^r|H,d)} \subseteq \widetilde{Q}_{0,\alpha}(\mathbb{P}^r|H,d)$: consider a family of relative quasimaps over a smooth curve S, such that the generic fiber lies in the nice locus. Then we may blow-up the source curve (which is a fibered surface) in the base-points of the quasimap (that are finitely many smooth points of the central fiber) in order to get an actual morphism to $\mathbb{P}^r$; we may as well suppose that the central fiber of the new family is stable. Notice that the central fiber actually belongs to Gathmann's space $\overline{\mathcal{M}}_{0,\alpha}(\mathbb{P}^r|H,d)$: we have just introduced some rational tails away from the markings, hence the only thing we have to check is, when we blow-up a base-point on an internal component, the rational tail will again be internal ($u_0 \equiv 0$ in a neighborhood of the base-point), so it will contribute to the LHS of the α-tangency condition nr. 2 in the very same way. We may now invoke [Gat02, Lemma 1.9] and the quasimap case follows from Lemma 2.1. □

From now on we shall denote this closed substack by $\overline{Q}_{0,\alpha}(\mathbb{P}^r|H,d)$.

Increasing the multiplicity can be naively performed in the very same way as Gathmann did:

$$\sigma_k^m := x_k^* d^m_{C/\overline{Q}}(u_0) \in H^0(\overline{Q}, x_k^* \mathcal{P}^m_{C/\overline{Q}}(\mathcal{L}))$$

with $m = \alpha_k + 1$ cuts $\overline{Q}_{0,\alpha+e_k}(\mathbb{P}^r|H,d)$ inside $\overline{Q}_{0,\alpha}(\mathbb{P}^r|H,d)$, together with a bunch of degenerate contributions from quasimaps where the component on which x_k lies is internal (call it Z) and (notice the equality sign!)

$$\deg(L_{|Z}) + \sum m^{(i)} = \sum_{x_j \in Z} \alpha_j.$$

Of course, quasimap stability forces these degenerate contributions not to have any rational tail; this is really the only difference with the case of stable maps, and indeed we can pushforward Gathmann's formula along the comparison morphism $\chi\colon \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d) \to \overline{Q}_{0,n}(\mathbb{P}^r,d)$ and the only terms that are going to change are the degenerate ones with rational tails (in fact they disappear, since the restriction of the comparison map has positive dimensional fibers there). With an eye to the future, we remark that these contributions do matter when computing GW invariants of a CY hypersurface in projective space, and may well account for the divergence between GW and quasimap invariants in the CY case [Gat03, Rmk. 1.6].

Lemma 2.3. *$\chi^*(\psi_k) = \psi_k$ and $\chi^*(x_k^*\mathcal{L}) = \mathrm{ev}_k^*(\mathcal{O}_{\mathbb{P}^r}(H))$.*

Proof. Recall that $\psi_k = c_1(x_k^*\omega_{C/\mathcal{M}})$ and contemplate the following diagram

$$\begin{tikzcd}
 & & & \mathbb{P}^r & \\
C_{\overline{\mathcal{M}}} \arrow[urrr, "f"] \arrow[rr, "\sigma^{ss}"] \arrow[dr] & & \chi^* C_{\overline{Q}} \arrow[ur, dashed] \arrow[rr] \arrow[dl] & & C_{\overline{Q}} \arrow[ul, dashed] \arrow[d] \\
 & \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d) \arrow[ul, bend left, "x_k"] \arrow[ur, bend right, "x_k"'] \arrow[rrr, "\chi"] & & & \overline{Q}_{0,n}(\mathbb{P}^r,d) \arrow[u, bend right, "x_k"']
\end{tikzcd}$$

where σ^{ss} is the strong stabilisation map, i.e. contracting the rational tails, which is an isomorphism near the markings. □

Lemma 2.4. $\dim(\overline{\mathcal{M}}_{0,(m^{(i)})}(\mathbb{P}^r|H,d) \cap \mathrm{ev}_1^*(p)) > 0$ *everytime $rd > 1$, where p is a point of H, so the pushforward along χ of a degenerate locus with rational tails is 0.*

Proof. $\dim(\overline{\mathcal{M}}_{0,(m^{(i)})}(\mathbb{P}^r|H,d) \cap \mathrm{ev}_1^*(p)) = (r-3)+(1-m^{(i)})+d(r+1)-(r-1) = (rd-1)+(d-m^{(i)})$. □

Proposition 2.5. *Denote by $[D^{Q}_{\alpha,k}(\mathbb{P}^r|H,d)]$ the sum of the (product) fundamental classes of*

$$\overline{Q}_{0,\alpha^{(0)}\cup(0,\dots,0)}(H,d_0)\times_{(\mathbb{P}^r)^k}\prod_{i=1}^{k}\overline{Q}_{0,(m^{(i)})\cup\alpha^{(i)}}(\mathbb{P}^r|H,d_i)$$

with coefficient $\frac{m^{(1)}\dots m^{(k)}}{k!}$, where the sum runs over all splittings $d=\sum d_i$ and $\alpha=\bigcup\alpha^{(i)}$ such that the above spaces are well-defined, in particular $|\alpha^{(0)}|+k$ and $|\alpha^{(i)}|+1$ are all ≥ 2, and such that

$$d_0+\sum_{i=1}^{k}m^{(i)}=\sum\alpha^{(0)}$$

The following formula holds

$$(\alpha_k\psi_k+x_k^*\mathcal{L})\cdot[\overline{Q}_{0,\alpha}(\mathbb{P}^r|H,d)]=[\overline{Q}_{0,\alpha+e_k}(\mathbb{P}^r|H,d)]+[D^{Q}_{\alpha,k}(\mathbb{P}^r|H,d)].$$

Proof. Follows from [Gat02, Thm. 2.6] by pushforward along $\chi\colon\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)\to\overline{Q}_{0,n}(\mathbb{P}^r,d)$, using the projection fomula and Lemmas 2.1, 2.3 and 2.4. □

3. The quasimap mirror theorem

Assuming that quasimap invariants for $\mathbb{P}^r$ coincide with Gromov-Witten invariants on the nose, we get the following result.

Definition 3.1. For a complete intersection X in $\mathbb{P}^r$ and $d>0$, let

$$I_d^X=(\mathrm{ev}_1)_*\left(\frac{1}{z-\psi_1}[\overline{Q}_{0,2}(X,d)]^{\mathrm{vir}}\right)$$

where ev_1 is always thought of as landing in $\mathbb{P}^r$.

Set also $I_0^X=\mathbb{1}_{\mathbb{P}^r}$ and $I^X=\sum_{d\geq0}I_d^Xq^d$.

Theorem 3.2. *Let $X\subseteq\mathbb{P}^4$ be a smooth quintic 3-fold. Then*

$$\sum_{d\geq0}q^d\prod_{i=0}^{5d}(X+iz)I_d^{\mathbb{P}^4}=XP(q)I^X$$

where

$$P(q)=1+\sum_{d>0}dq^d\langle H^4,\mathbb{1}_{\mathbb{P}^4}\rangle_{\overline{Q}_{0,\{5d,0\}}(\mathbb{P}^4|X,d)}=1+\sum_{d>0}q^d\frac{(5d)!}{(d!)^5}\sum_{i=d+1}^{5d-1}\frac{1}{i}.$$

Proof. We'll write it for a general CY hypersurface in $i\colon X_a\hookrightarrow\mathbb{P}^r$, so the degree of X is $a=r+1$. Notice that dual bases for $H^*(\mathbb{P}^r)$ are given by $T^i=H^i$ and $T_i=H^{r-i}$, while (induced) dual bases for $i^*H^*(\mathbb{P}^r)$ are $S^i=H^i$ and $S_i=\frac{1}{a}H^{r-i-1}$; the restriction of H^r is 0.

Define

$$I_{d,(m)}^{\mathbb{P}^r|X}=(\mathrm{ev}_1)_*\left(\frac{1}{z-\psi_1}[\overline{Q}_{0,\{m,0\}}(\mathbb{P}^r|X,d)]^{\mathrm{vir}}\right),$$

which coincides with the absolute I-function defined above for $m = 0$, and

$$J^{\mathbb{P}^r|X}_{d,(m)} = (\mathrm{ev}_1)_* \left(m[\overline{Q}_{0,\{m,0\}}(\mathbb{P}^r|X, d)]^{\mathrm{vir}} + \frac{1}{z - \psi_1}[D^Q_m(\mathbb{P}^r|X, d)]^{\mathrm{vir}} \right).$$

Then, by Gathmann's formula, we can prove that

$$(X + mz) I^{\mathbb{P}^r|X}_{d,(m)} = I^{\mathbb{P}^r|X}_{d,(m+1)} + J^{\mathbb{P}^r|X}_{d,(m)}, \tag{2}$$

from which it follows that

$$\prod_{i=0}^{ad} (X + iz) I^{\mathbb{P}^r}_d = \sum_{m=0}^{ad} \prod_{i=m+1}^{ad} (X + iz) J^{\mathbb{P}^r|X}_{d,(m)}.$$

It is now a matter of evaluating the RHS. Notice that $J^{\mathbb{P}^r|X}_{d,(m)}$ is made of two parts:

- the boundary terms: the strong stability condition for quasimaps and the choice of working with only two markings makes these boundary contributions particularly simple to compute. The shape of the source curve is that of a snake which the hypersurface cuts into two pieces, the internal one of degree $d^{(0)}$, and the external one of degree $d^{(1)}$ and multiplicity $m^{(1)}$ of contact with X, with the first marking point belonging to the internal component and the second to the external one.

  The invariants which we need to consider will hence be of the form

  $$\langle T^i \psi_1^j, S_i \rangle_{\overline{Q}_{0,2}(X,d^{(0)})} \langle S^i, \mathbb{1}_{\mathbb{P}^r} \rangle_{\overline{Q}_{0,\{m^{(1)},0\}}(\mathbb{P}^r|X,.) d^{(1)}}$$

  A dimensional computation

  $$\begin{aligned} 0 \leq \operatorname{codim} S_i &= \dim X - \operatorname{codim} S^i \\ &= \dim X - \operatorname{vdim} \overline{Q}_{0,\{m^{(1)},0\}}(\mathbb{P}^r|X, d^{(1)}) \\ &= \dim X - (\dim \mathbb{P}^r - 3 + 2 - m^{(1)} - K_{\mathbb{P}^r} \cdot d^{(1)} \ell) \\ &= m^{(1)} - X \cdot d^{(1)} \ell + K_X \cdot d^{(1)} \ell \\ &= m^{(1)} - X \cdot d^{(1)} \ell \leq 0 \end{aligned}$$

  forces $S_1 = \mathbb{1}_X$ and $S^1 = \frac{1}{a} H^{r-1}$, $m^{(1)} = ad^{(1)}$ hence

  $$m = \alpha_1 = X \cdot d^{(0)} \ell + m^{(1)} = ad,$$

  so this doesn't show up but at the very end of the "increasing the multiplicity" process.
- The other term in $J^{\mathbb{P}^r|X}_{d,(m)}$ is $m(\mathrm{ev}_1)_*[\overline{Q}_{0,\{m,0\}}(\mathbb{P}^r|X, d)]^{\mathrm{vir}}$; notice that it only gets insertions from the cohomology of $\mathbb{P}^r$ (restricted to X). On the other hand

  $$\operatorname{vdim} \overline{Q}_{0,\{m,0\}}(\mathbb{P}^r|X, d) = r - 3 + 2 - m + d(r + 1) \geq r - 1$$

because $m \le ad$; since the restriction of H^r to X vanishes, the only insertion that contributes is H^{r-1}, forcing the equality $m = ad$.

So, in the end, we see that equation 2 reduces to

$$\begin{aligned}\prod_{i=0}^{ad}(X+iz)I_d^{\mathbb{P}^r} &= J_{d,(ad)}^{\mathbb{P}^r|X}\\ &= \sum_{i=0,\dots,r-1;j\ge 0}(da)\langle H^{r-1},\mathbb{1}_{\mathbb{P}^r}\rangle_{\overline{Q}_{0,\{ad,0\}}(\mathbb{P}^r|X,d)}H\\ &+\sum_{\substack{0<d^{(0)}<d\\ d^{(0)}+d^{(1)}=d}} z^{j+1}H^{r-i}\langle H^i\psi_1^j,\mathbb{1}_X\rangle_{\overline{Q}_{0,2}(X,d^{(0)})}(ad^{(1)})\langle\frac{1}{a}H^{r-1},\mathbb{1}_{\mathbb{P}^r}\rangle_{\overline{Q}_{0,\{ad^{(1)},0\}}(\mathbb{P}^r|X,d^{(1)})}\\ &+z^{j+1}H^{r-i}\langle H^i\psi_1^j,\mathbb{1}_X\rangle_{\overline{Q}_{0,2}(X,d)}\end{aligned}$$

from which the first claim of the theorem is now evident (with a bit of rearranging, using $X = aH$ and $i^*(H^r) = 0$, so in the last line everything is divisible by H).

In order to evaluate $P(q)$, we use again Gathmann's algorithm, this time in the other direction, to go all the way back to $\mathbb{P}^r$; then we make use of our assumption that quasimap invariants and ordinary GW coincide for the projective space. So it starts:

$$[\overline{Q}_{0,\{ad,0\}}(\mathbb{P}^r|X,d)]^{\mathrm{vir}} = (X+(ad-1)\psi_1)[\overline{Q}_{0,\{ad-1,0\}}(\mathbb{P}^r|X,d)]^{\mathrm{vir}}-[D_{ad}^{Q}(\mathbb{P}^r|X,d)]^{\mathrm{vir}}$$

When looking at the boundary, the invariants that come into play are of the form

$$\langle H^{r-1},S_i\rangle_{\overline{Q}_{0,2}(X,d^{(0)})}\langle S^i,\mathbb{1}_{\mathbb{P}^r}\rangle_{\overline{Q}_{0,\{a(d-d^{(0)})-1,0\}}(\mathbb{P}^r|X,d-d^{(0)})}$$

but notice that they must vanish by dimensional reasons, since

$$\operatorname{codim}(S_i) = \dim X - 3 + 2 - K_X\cdot d^{(0)}\ell - (r-1) = -1.$$

So

$$\begin{aligned} &d\langle H^{r-1},\mathbb{1}_{\mathbb{P}^r}\rangle_{\overline{Q}_{0,\{ad,0\}}(\mathbb{P}^r|X,d)}\\ &= d[\overline{Q}_{0,2}(\mathbb{P}^r,d)]\cap \mathrm{ev}_1^*(H^{r-1})\prod_{i=0}^{ad-1}(\mathrm{ev}_1^* X+i\psi_1)\\ &= d[\overline{Q}_{0,2}(\mathbb{P}^r,d)]\cap\left((da-1)!\psi_1^{ad}\,\mathrm{ev}_1^*(H^{r-1})+a\left(\sum_{j=1}^{ad-1}\frac{(ad-1)!}{j}\right)\psi_1^{ad-1}\,\mathrm{ev}_1^*(H^r)\right)\\ &= d\left((da-1)!\langle\psi_1^{ad-1}H^{r-1}\rangle_{0,1,d}+a\left(\sum_{j=1}^{ad-1}\frac{(ad-1)!}{j}\right)\langle\psi_1^{ad-2}H^r\rangle_{0,1,d}\right)\end{aligned}$$

using the equality of quasimap and GW invariants and the string equation for the latter. These numbers can be extracted from the J-function for $\mathbb{P}^r$

$$I_d^{\mathbb{P}^r} = \prod_{i=1}^{d} \frac{1}{H+i}$$

from which

$$\langle \psi_1^{ad-2} \operatorname{ev}_1^*(H^r) \rangle_{0,1,d} = \frac{1}{(d!)^{r+1}}$$

$$\langle \psi_1^{ad-1} \operatorname{ev}_1^*(H^{r-1} \rangle_{0,1,d} = -(r+1)\frac{1}{(d!)^{r+1}} \sum_{j=1}^{d} \frac{1}{j}$$

and the second claim of the theorem follows. □

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