ALTERNATIVE COMPACTIFICATIONS IN LOW GENUS GROMOV-WITTEN THEORY

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Contents

1. Review of toric varieties and quasimaps

1.1. **Definitions.** Let N be a lattice with dual $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$, $\Sigma \subseteq N_{\mathbb{R}}$ a rational polyhedral fan with associated toric variety X_Σ, with primitive generators v_ρ of the rays $\rho \in \Sigma(1)$. I assume throughout this text that $X = X_\Sigma$ is smooth and projective.

In [Cox95a] D.A. Cox gave a beautiful description of the functor of points of a smooth toric variety, generalising the well-known equivalence between maps to the projective space $\mathbb{P}^N$ and the data of a line bundle with $N+1$ sections that generate it.

Recall that the rays $\rho \in \Sigma(1)$ correspond to toric divisors D_ρ on X - which generate the Picard group -, while M can be thought of as the lattice of characters of the torus $T \subseteq X$, hence giving rational functions on X. In the smooth case the following sequence is exact:

$$0 \to M \to \mathbb{Z}^{\Sigma(1)} \to \mathrm{Pic}(X) \to 0 \tag{1}$$

Theorem 1.1 (Cox)**.** *Let X be a smooth toric variety with notation as above. A morphism $f : C \to X$ is equivalent to the data:*

$$\left((L_\rho, u_\rho)_{\rho \in \Sigma(1)}, (\varphi_m)_{m \in M}\right),$$

where $L_\rho \in \mathrm{Pic}(C)$, $u_\rho \in H^0(C, L_\rho)$, and $\varphi_m : \bigotimes_{\rho \in \Sigma(1)} L_\rho^{\otimes \langle v_\rho, m\rangle} \simeq \mathcal{O}_C$, satisfying the following conditions:

1. *nondegeneracy: for all $x \in C$ there exists a maximal cone $\sigma \in \Sigma_{\max}$ with $u_\rho(x) \neq 0$ for all rays $\rho \not\subset \sigma$;*
2. *compatibility: $\varphi_m \circ \varphi_{m'} = \varphi_{m+m'}$, $\forall m, m' \in M$.*

Remark 1.2. The isomorphisms φ_m can be used to reduce the number of line bundles down to the Picard rank of X.

Based on this and on previous work of A. Marian, D. Oprea and R. Pandharipande [MOP11] (similar ideas had in fact already appeared in the work of Drinfeld, Morrison-Plesser, and Givental), I. Ciocan-Fontanine and B. Kim introduced the following notion, which gives rise to an alternative compactification of the space of maps from smooth curves to a smooth projective toric variety, by strengthening the stability requirement and weakening the concept of map with respect to the usual Kontsevich's space of stable maps. For this let me fix a polarisation $\mathcal{O}_X(1)$ and an expression thereof as a combination of toric line bundles $\mathcal{O}_X(1) = \bigotimes_{\rho\in\Sigma(1)} \mathcal{O}(D_\rho)^{\otimes\alpha_\rho}$; it turns out that the resulting stability condition is independent of the choices made.

Definition 1.3. [CFK10, Definition 3.1.1] Let X be as above. We fix the following numerical invariants: a genus $g \geq 0$, a number of marked points $n \geq 0$, an effective curve class $\beta \in \mathrm{H}_2^+(X)$, and a positive rational number ϵ. A *quasimap* is given by the data

$$\left((C, x_1, \ldots, x_n), (L_\rho, u_\rho)_{\rho\in\Sigma(1)}, (\varphi_m)_{m\in M}\right)$$

where:

(1) $(C, x_1, \ldots, x_n)$ is a prestable curve of genus g with n marked points;
(2) $L_\rho \in \mathrm{Pic}^{d_\rho}(C)$ where $d_\rho = D_\rho \cdot \beta$ and $u_\rho \in H^0(C, L_\rho)$;
(3) $\varphi_m \colon \bigotimes_{\rho\in\Sigma(1)} L_\rho^{\otimes\langle v_\rho, m\rangle} \simeq \mathcal{O}_C$ are isomorphisms satisfying *compatibility* under the group structure;
(4) there is an at most finite set of smooth and non-marked points $B \subseteq C$, called the *basepoints* of the quasimap, such that *nondegeneracy* is satisfied for all $x \in C \setminus B$.

Furthermore a quasimap is said to be *ϵ-stable* if:

(5) the line bundle $\omega_C(x_1 + \ldots + x_n) \otimes L^{\otimes\epsilon}$ is ample, where $L = \otimes_\rho L_\rho^{\otimes\alpha_\rho}$ is determined by the polarisation,
(6) and $\epsilon\ell(x) \leq 1$, where

$$\ell(x) = \min\left\{\mathrm{ord}_x\left(\prod_{\rho\in\Sigma(1)\setminus\sigma(1)} u_\rho\right) \mid \sigma \in \Sigma_{\max}\right\}.$$

An isomorphism between two quasimaps is given by an isomorphism $\psi\colon (C, \mathbf{x}) \simeq (C', \mathbf{x}')$ of the underlying prestable curves, together with isomorphisms of the line bundles $\lambda_\rho \colon L_\rho \simeq \psi^* L'_\rho$ preserving the sections.

The above definitions make sense in families over arbitrary base schemes, and therefore determine the moduli space of ϵ-stable quasimaps as a cfg over (Sch), denoted by $\mathcal{Q}^\epsilon_{g,n}(X, \beta)$, which comes with universal structures, such as a curve, markings, line bundles and sections; note in particular that, since the basepoints cannot coincide with the markings, it comes with well-defined evaluation maps $\mathrm{ev}\colon \mathcal{Q}^\epsilon_{g,n}(X, \beta) \to X^n$.

Remark 1.4. There is a more general notion of quasimaps when the target belongs to a certain class of GIT quotients $W /\!\!/ G$ [CFKM14]; smooth toric varieties fit nicely in this framework, as they can always be described as a quotient $\mathbb{A}^{\Sigma(1)} /\!\!/ \mathbb{G}_{\mathrm{m}}^r$, where r is the Picard rank of X. In this light, quasimaps can be thought of as maps to the stack quotient $[W/G]$, such that the preimage of the unstable locus is finite and disjoint from the special points.

Remark 1.5. The ϵ-stability condition is introduced in order to interpolate between Kontsevich's stable maps (for $\epsilon \gg 0$, hence often denoted by $\epsilon = \infty$) and what in the sequel I will refer to just as quasimaps (for $\epsilon \ll 1$, denoted by $\epsilon = 0+$). Note that in the latter case necessarily $2g - 2 + n \geq 0$, and there cannot be *rational tails* by stability. In between these two extrema one should imagine that rational tails of higher and higher degree are exchanged for basepoints of order lower and lower; see the discussion following Example 1.8 below.

Example 1.6. Quasimaps to a point recover Hassett's moduli space of weighted pointed curves [Has03]. In particular, to set notation, $\mathcal{Q}_{g,n}(\{*\}, d) \simeq \overline{\mathcal{M}}_{g,n|d}/S_d$ where the first n-markings have weight 1, and the last d - corresponding to the basepoints - have weight 0+ and are unordered.

1.2. **Basic properties.**

Theorem 1.7. [CFK10, Theorems 3.2.1 and 4.0.1] $\mathcal{Q}_{g,n}(X, \beta)$ is a proper and virtually smooth DM stack of finite type over $\mathrm{Spec}(\mathbb{k})$.

Here is a brief sketch of the argument in [CFK10]. Let us examine properness first: assume we have a stable quasimap

$$((C_K, \mathbf{x}_K), (L_{i,K})_{i=1,\dots,\mathrm{rk}\,\mathrm{Pic}(X)}, (u_{\rho,K})_{\rho\in\Sigma(1)}))$$

over a discretely valued field K, such that C_K is smooth; we would like to extend it over the trait Δ, possibly after base-change. By semistable reduction for curves and Castelnuovo's criterion, we may find a regular model C over Δ with no -1-curves. Note that we may not appeal to the properness of the relative Picard functor in order to extend the line bundles (unless we restrict to genus 0). We exploit properness of the relative Quot functor instead: after twisting with a sufficiently ample polarisation $\mathcal{O}_{C_K}(m)$ on C_K, we find an exact sequence

$$0 \to L_{i,K} \to \mathcal{O}_{C_K}(m)^{\oplus N} \to Q_{i,K} \to 0$$

which may be extended to the whole of C; up to taking double duals, L_i is a line bundle. Now the u_ρ are only rational sections of the relevant line bundles. But observe that the given quasimap induces a bona fide map outside the base locus, i.e. from $C_K \setminus B_K$ to X_K. By normality of C and properness of X this can be extended to a map from $C \setminus (\overline{B_K} \cup B_0)$, where B_0 is a finite set of points of the central fiber; we thus get that the pullback of the toric line bundles are defined everywhere and so are the corresponding sections (by Hartogs). By twisting back with $\mathcal{O}_C(\overline{B_K})$ we obtain an extension of the quasimap we started with. There may be two issues at this point: unstable components are -2-curves on which the line bundles and sections are constant, hence they may be contracted by Artin's theory; on the other hand basepoints may be nodes in the central fiber, which can be resolved by blowing-up (the process terminates because the newly created -2-curves have nontrivial quasimap degree).

The morphism to the universal Picard stack that forgets the sections (u_ρ),

$$\mathcal{Q}_{g,n}(X,\beta) \to \bigtimes_{i=1}^{\operatorname{rk}\operatorname{Pic}(X)}{}_{\mathfrak{M}_{g,n}} \left(\mathfrak{Pic}^{d_{\rho_i}}_{g,n}\right)$$

admits a perfect obstruction theory given by $\bigoplus_{i=1}^{\operatorname{rk}\operatorname{Pic}(X)} \mathrm{R}^\bullet \pi_*\mathcal{L}_i$, where $\pi\colon \mathcal{C} \to \mathcal{Q}_{g,n}(X,\beta)$ is the universal curve and $\mathcal{L}_i$ the universal line bundles (see also [Wan12, CL12]). On the other hand the Euler sequence for X induces an exact sequence

$$0 \to \mathcal{O}_{\mathcal{C}}^{\oplus \operatorname{rk}\operatorname{Pic}(X)} \to \bigoplus_{\rho\in\Sigma(1)} \mathcal{L}_\rho \to \mathcal{F}_X \to 0$$

on the universal curve, which shows that a compatible perfect obstruction theory relative to $\mathfrak{M}_{g,n}$ is given by $\mathrm{R}^\bullet \pi_*\mathcal{F}_X$. Notice that the latter agrees with $\mathrm{R}^\bullet \pi_* f^*T_X$ on the locus of quasimaps with no basepoints (i.e. maps).

In particular $\mathcal{Q}^\epsilon_{g,n}(X,\beta)$ is endowed with a virtual class of the same dimension as $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{virt}}$. For cohomology classes $\delta_i \in H^*(X,\mathbb{Q})$ and natural numbers a_i, $i = 1,\ldots,n$, we may therefore define descendant ϵ-quasimap invariants by:

$$\langle \tau_{a_1}(\delta_1),\ldots,\tau_{a_n}(\delta_n)\rangle^\epsilon_{g,n,\beta} = \int_{[\mathcal{Q}^\epsilon_{g,n}(X,\beta)]^{\mathrm{virt}}} \prod_{i=1}^n \mathrm{ev}_i^*(\delta_i)\psi_i^{a_i}.$$

1.3. **The collapsing morphism.** I shall start this section with an example.

Example 1.8. Consider the evaluation map $\mathrm{ev}\colon \overline{\mathcal{M}}_{0,2}(\mathbb{P}^2,1) \to \mathbb{P}^2\times\mathbb{P}^2$: when the two image points are distinct, the map must necessarily paramerise the line between them, while when they coincide we are left with choosing a line through that point. This shows that $\overline{\mathcal{M}}_{0,2}(\mathbb{P}^2,1) \simeq \mathrm{Bl}_\Delta\, \mathbb{P}^2\times\mathbb{P}^2$, with ev the blow-down map. On the other hand $\mathrm{ev}\colon \mathcal{Q}_{0,2}(\mathbb{P}^2,1) \simeq \mathbb{P}^2\times\mathbb{P}^2$. The modular interpretation of the blow-down map is that it collapses rational tails to basepoints. This is in fact a general feature of quasimaps to projective space.

Lemma 1.9. There is a birational *collapsing* morphism $\chi\colon \overline{\mathcal{M}}_{0,n}(\mathbb{P}^N,d) \to \mathcal{Q}_{0,n}(\mathbb{P}^N,d)$.

The idea is to find a line bundle on the universal curve that will give us the contraction $\sigma_{\mathrm{ss}}\colon \mathcal{C} \to \hat{\mathcal{C}}$: if we work over $\mathfrak{Pic}^d_{0,n} = \mathfrak{M}^{\mathrm{wt=d}}_{0,n}$, then the locus

$\mathfrak{T}^\delta$ spanned by rational tails (trees) of (total) degree δ is a Cartier divisor in the universal curve $\mathfrak{C} \to \mathfrak{Pic}^d_{0,n}$. By pulling back to the universal curve over $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^N, d)$, we find that the line bundle

$$\omega_C\left(\sum_{i=1}^n x_i\right) \otimes f^*\mathcal{O}_{\mathbb{P}^N}(1) \otimes \bigotimes_{0<\delta\leq d} \mathcal{O}_C\left((\delta-1)\mathfrak{T}^\delta\right)$$

is semiample and trivial precisely on the rational tails. On the other hand the universal sections $u_0, \dots, u_N$ of $f^*\mathcal{O}_{\mathbb{P}^N}(1)$ induce sections $\tilde{u}_0, \dots, \tilde{u}_N$ of $\tilde{\mathcal{L}} := f^*\mathcal{O}_{\mathbb{P}^N}(1) \otimes \bigotimes_{0<\delta\leq d} \mathcal{O}_C\left(\delta\mathfrak{T}^\delta\right)$, that are constant along the rational tails by degree reasons, hence descend to sections $\bar{u}_0, \dots, \bar{u}_N$ of $\hat{\mathcal{L}} = \sigma_{\mathrm{ss},*}(\tilde{\mathcal{L}})$ (with basepoints where there used to be a rational tail attached).

Remark 1.10. A proof attributed to Jun Li can be found in [LLY97, Lemma 2.6] for the parametrised/graph space case; applying the same argument more carefully shows that the collapsing can be performed step-by-step, something that was exploited for example in [MtaMta07], as noted in [CFK14]. The statement generalises well to higher genus - where it is only a virtually birational morphism, see [MOP11, Theorem 3] and [Man12, Proposition 4.21]; instead it does not extend to the case that the target is any toric variety. The problem is that there could be some toric line bundle of negative degree along a rational tail; examples may be found already when looking at the Hirzebruch surfaces. A lengthier discussion takes place in [BN17, Appendix A].

1.4. **Generating functions and ϵ-wallcrossing.** Fixed X and the numerical invariants g, n, β, the quasimap spaces determine a finite wall-and-chamber structure on $\mathbb{Q}_{>0}$, such that $Q^\epsilon_{g,n}(X, \beta)$ is constant for ϵ in a fixed chamber, and changes across finitely many values of ϵ. I will briefly report on the beautiful results of Ciocan-Fontanine and Kim [CFK14, CFK17, CFK16] entailing relationships between the virtual classes and the resulting invariants when crossing an ϵ-wall.

Let me introduce some more notation. Fix a homogeneous basis $\gamma_0, \dots, \gamma_l$ of $H^*(X, \mathbb{Q})$, with dual basis $\gamma^0, \dots, \gamma^l$ with respect to the intersection product $\langle \cdot, \cdot \rangle$, and let $\mathbf{t} = \sum_{i=0}^l t_i\gamma_i$ be a general element of $H^*(X, \mathbb{Q})$. Adopting double bracket notation:

$$\langle\langle \delta_1\psi_1^{a_1}, \dots, \delta_n\psi_n^{a_n} \rangle\rangle^\epsilon_{g,n}(\mathbf{t}) = \sum_{\substack{\beta\in \mathrm{Eff}(X)\\ m\geq 0}} \frac{q^\beta}{m!} \langle \delta_1\psi_1^{a_1}, \dots, \delta_n\psi_n^{a_n}, \mathbf{t}, \dots, \mathbf{t}\rangle^\epsilon_{g,n+m,\beta}$$

we may consider the *big J^ϵ-function* as a generating series for genus 0 ϵ-quasimap invariants assuming the following form:

(2)
$$J^\epsilon(q, t, z) = \mathbb{1} + \frac{\mathbf{t}}{z} + \sum_{i=0}^l \gamma_i \sum_{\beta>0, \beta\cdot\mathcal{O}_X(1)\leq 1/\epsilon} q^\beta J^\epsilon_{i,\beta}(z) + \sum_{i=1}^l \gamma_i \langle\langle \frac{\gamma^i}{z(z-\psi_1)} \rangle\rangle^\epsilon_{0,1}(\mathbf{t}).$$

Among the different variations on the notion of quasimaps, that of *parametrized* quasimaps is particularly relevant: $QG^{\epsilon}_{g,n}(X,\beta)$ involves the extra data of a map of degree 1 to $\mathbb{P}^1$, singling out a specific rational component C_0 of C, which is exempt from the duty of satisfying the ϵ-stability condition; the $\epsilon=\infty$ case recovers the well-known construction of the graph space $\overline{\mathcal{M}}_{g,n}(X\times\mathbb{P}^1,(\beta,1))$ in Gromov-Witten theory. The $\mathbb{G}_{\mathrm{m}}$-action on $\mathbb{P}^1$ fixing two points 0 and ∞ lifts to an action on $QG^{\epsilon}_{g,n}(X,\beta)$, and the J^{ϵ}-function is in fact defined as a sum of residue integrals along some special $\mathbb{G}_{\mathrm{m}}$-fixed loci F_0, namely those where all the curve class is supported over $0\in\mathbb{P}^1$:

$$J^{\epsilon}(q,t,z)=\sum_{m\geq 0,\beta\geq 0} q^{\beta}\,\mathrm{ev}_{0,*}\left(\frac{\prod_{i=1}^{m}\mathrm{ev}_i(\mathbf{t})}{m!}\cap \mathrm{Res}_{F_0}[QG^{\epsilon}_{0,m}(X,\beta)]^{\mathrm{virt}}\right);$$

it is understood that the first three terms in the expression (2) correspond to the terms with unstable F_0 in the above sum, i.e. $(m,\beta)=(0,0),(1,0)$, and $m=0,\beta\neq 0$ but $\beta\cdot\mathcal{O}_X(1)\leq 1/\epsilon$ respectively. $J^{\infty}(q,t,z)$ coincides with Givental's big J-function; on the other hand Ciocan-Fontanine and Kim denote $J^{0+}(q,t,z)$ by $I(q,t,z)$.

The *small J^{ϵ}-function* is obtained by restricting to $\mathbf{t}=0$; the small I-function plays a central role in the theory and it turns out that all small J^{ϵ}-functions are polynomial q-truncations of the small I-function. The latter has been computed by Givental for toric targets and it is given by the following expression:

$$I(q,0,z)=\sum_{\beta\geq 0} q^{\beta}\frac{\prod_{\rho\in\Sigma(1)}\prod_{i=0}^{D_\rho\cdot\beta}(D_\rho+iz)}{\prod_{\rho\in\Sigma(1)}\prod_{i=-\infty}^{D_\rho\cdot\beta}(D_\rho+iz)} \tag{3}$$

Define $I_0(q)=1+O(q)\in\Lambda$ and $I_1(q)\in qH^{\leq 2}(X,\Lambda)$ (where Λ is the Novikov ring obtained by completing $\mathrm{Eff}(X)_{\mathbb{Q}}$ along the maximal ideal of non-zero classes) as the following coefficients in the $\frac{1}{z}$-expansion

$$I(q,0,z)=I_0(q)\mathbb{1}+I_1(q)\frac{1}{z}+O(\frac{1}{z^2}).$$

In the *toric semipositive* case ($-K_X$ nef), it can be computed from (3) that $I_0(q)=1$ and $I_1(q)$ is a sum over curve classes $\beta\in\mathrm{Ker}(K_X\cdot)$ such that there is exactly one ray $\rho_\beta\in\Sigma(1)$ with $D_{\rho_\beta}\cdot\beta<0$:

$$I_1(q)=\sum_{\substack{K_X\cdot\beta=0\\ \exists!\rho_\beta:D_{\rho_\beta}\cdot\beta<0}} q^{\beta}D_{\rho_\beta}\frac{(-1)^{D_{\rho_\beta}\cdot\beta+1}(-D_{\rho_\beta}\cdot\beta-1)!}{\prod_{\rho'\neq\rho_\beta}(D_{\rho'}\cdot\beta)!}.$$

Let me introduce one further generating function: let the S^{ϵ}-operator be defined by

$$S^{\epsilon}(q,t,z)(\gamma)=\sum_{i=0}^{l}\gamma_i\langle\langle\frac{\gamma^i}{z-\psi_1},\gamma\rangle\rangle^{\epsilon}_{0,2}(\mathbf{t}).$$

Lemma 1.11 (Birkhoff factorisation)**.** [CFK14, Theorem 1.3.1] In the toric semipositive case $J^\epsilon(q,t,z) = S^\epsilon(q,t,z)(\mathbb{1})$.

$S^\infty(q,t,z)(\mathbb{1})$ recovers Givental's fundamental solution matrix in Gromov-Witten theory.

Theorem 1.12. In the toric semipositive case [CFK14, Theorem 1.2.2]

$$J^\epsilon(q,t,z) = J^\infty(q, t + J_1^\epsilon(q), z)$$

and in particular

$$\sum_{\beta \in \mathrm{Eff}(X)} q^\beta \langle \tau_{a_1}(\delta_1), \delta_2, \dots, \delta_n \rangle^\epsilon_{0,n,\beta} = \sum_{\beta' \in \mathrm{Eff}(X)} q^{\beta'} \sum_{m \geq 0} \frac{1}{m!} \langle \tau_{a_1}(\delta_1), \delta_2, \dots, \delta_n, J_1^\epsilon(q), \dots, J_1^\epsilon(q) \rangle^\epsilon_{0,n+m,\beta'}$$

More generally [CFK17, Theorem 1.3.2]

$$\sum_{\beta \in \mathrm{Eff}(X)} q^\beta \langle \tau_{a_1}(\delta_1), \dots, \tau_{a_n}(\delta_n) \rangle^\epsilon_{g,n,\beta} = \sum_{\beta' \in \mathrm{Eff}(X)} q^{\beta'} \sum_{m \geq 0} \frac{1}{m!} \langle \tau_{a_1}(\delta_1), \dots, \tau_{a_n}(\delta_n), J_1^\epsilon(q), \dots, J_1^\epsilon(q) \rangle^\epsilon_{g,n+m,\beta'}$$

A few remarks: $0+ \leq \epsilon_1 < \epsilon_2 \leq \infty$ wall-crossing follows from the formulae above and the invertibility of $\mathbf{t} \mapsto \mathbf{t} + J_1^\epsilon$. Ciocan-Fontanine and Kim proved the genus 0 formulae in greater generality for semipositive GIT quotients admitting a torus action with finitely many isolated fixed points and 1-dimensional orbits - in which case though the formula I have written above must be corrected by introducing some $J_0^\epsilon(q)$ factors. The proof of the genus 0 result follows from a careful analysis of the fixed loci of the T-action on $Q^\epsilon_{0,2+m}(X,\beta)$ and how they change under embedding of X in projective space postcomposed with the collapsing morphism. The more general statement follows from an extension of Dubrovin's reconstruction that allows them to reduce any invariant to one-pointed descendant invariants. The higher genus result for semipositive toric varieties follows from an application of the localisation formula, that allows to express the residue integrals along fixed loci in terms of vertex, edge, and flag contributions, and the genus-invariance of certain universal polynommials appearing at the vertices, inspired by [MOP11].

I will now review a number of basic properties of quasimap spaces and comment on those that differ from ordinary Gromov-Witten theory.

1.5. **Functoriality.** Let $f\colon X \to Y$ be a toric morphism between smooth projective toric varieties, then there is an induced morphism at the level of quasimap spaces $Q(f)\colon Q_{g,n}(X,\beta) \to Q_{g,n}(Y, f_*\beta)$ (i.e. toric quasimaps are functorial). This is slightly more complicated than post-composing with f, but it has to coincide with this operation on the locus of quasimaps with no basepoints. Cox's description of the functor of points of a smooth toric variety becomes particularly useful when the source X is also toric, since [Cox95b] allows us to describe line bundles on X and their global sections in terms of the homogeneous coordinate ring $S^X = \mathbb{C}[z_\rho : \rho \in \Sigma_X(1)]$. Putting these observations together we get:

Theorem 1.13. [Cox95a, Theorem 3.2] A toric morphism $f : X \to Y$ as above is equivalent to the following data: homogeneous polynomials $P_\tau \in S^X_{\delta_\tau}$ where $\delta_\tau = f^*\mathcal{O}_X(D_\tau) \in \operatorname{Pic}(X), \forall \tau \in \Sigma_Y(1)$, and $S^X_{\delta_\tau}$ is the corresponding graded piece of the Cox ring of X. These are required to satisfy the following two conditions:

(1) $\sum_{\tau\in\Sigma_Y(1)} \delta_\tau \otimes v_\tau = 0$ in $\operatorname{Pic} X \otimes N_Y$, where v_τ is the primitive generator of the ray τ.
(2) $(P_\tau(z_\rho)) \notin Z(\Sigma_Y) \subseteq \mathbb{A}^{\Sigma_Y(1)}$ whenever $(z_\rho) \notin Z(\Sigma_X) \subseteq \mathbb{A}^{\Sigma_X(1)}$, where $Z(\Sigma) \subseteq \mathbb{A}^{\Sigma(1)}$ is the unstable locus determined by the Stanley-Reisner ideal of Σ.

Furthermore, two such sets of data (P_τ) and (P'_τ) correspond to the same morphism if and only if there exists a $\lambda \in \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Pic} Y, \mathbb{G}_{\mathrm{m}})$ such that

$$\lambda(D_\tau) \cdot P_\tau = P'_\tau$$

for all $\tau \in \Sigma_Y(1)$.

Namely, if we set $\tilde{f}(z_\rho) = (P_\tau(z_\rho))$ then this defines a lift of f to the prequotients:

$$\begin{tikzcd}
\mathbb{A}^{\Sigma_X(1)} \setminus Z(\Sigma_X) \arrow[r, "\tilde{f}"] \arrow[d, "q_X"] & \mathbb{A}^{\Sigma_Y(1)} \setminus Z(\Sigma_Y) \arrow[d, "q_Y"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

The second property above states precisely that $\tilde{f}^{-1}(Z(\Sigma_Y)) \subseteq Z(\Sigma_X)$. The last statement of the theorem explains that the ambiguity in the choice of a lifting $\tilde{f}$ corresponds exactly with the torus action on $\mathbb{A}^{\Sigma_Y(1)}$. Functoriality for quasimaps essentially comes from composing with $\tilde{f}$ in local trivialisations for $(L_\rho, u_\rho)_\rho$ (or when thinking of quasimaps as maps to the stack quotient).

In terms of line bundles and sections, suppose we are given a quasimap to X:

$$\xi = \left((C, \mathbf{x}), (L_\rho, u_\rho)_{\rho\in\Sigma_X(1)}, (\varphi_m)_{m\in M_X}\right),$$

and we want to associate a quasimap to Y to it. First of all we should stabilise the curve: if C_0 is an irreducible component of C, isomorphic to $\mathbb{P}^1$ with two special points, and of class β_0 for the given quasimap, such that $f_*(\beta_0) = 0$, then we may choose a polarisation $\mathcal{O}_Y(1)$ on Y and write $f^*\mathcal{O}(1) = \bigotimes_{\rho\in\Sigma_X(1)} \mathcal{O}_X(D_\rho)^{\otimes c_\rho}$. It follows that the line bundle $\omega_C(\mathbf{x}) \otimes$

$\bigotimes_{\rho\in\Sigma_X(1)} L_\rho^{\otimes c_\rho}$ is semiample on C and contracts C_0 as above. Call $q\colon C\to\bar{C}$ the resulting contraction morphism.

Pick data $(P_\tau)_{\tau\in\Sigma_Y(1)}$ associated to f, and write $P_\tau(z_\rho)=\sum_{\mathbf{a}^\tau}\mu_{\mathbf{a}}\prod_{\rho\in\Sigma_X(1)} z_\rho^{a_\rho^\tau}$, where $\mathbf{a}^\tau=(a_\rho^\tau)\in\mathbb{N}^{\Sigma_X(1)}$ is such that $[\mathbf{a}^\tau]=\delta_\tau\in\mathrm{Pic}(X)$. Choose one such $\mathbf{a}^\tau$; notice that all other $\mathbf{b}^\tau$ with $[\mathbf{b}^\tau]=\delta_\tau$ differ from $\mathbf{a}^\tau$ by a uniquely determined element of M_X (by the exactness of (1)). Set $L_\tau=\bigotimes_{\rho\in\Sigma_X(1)} L_\rho^{\otimes a_\rho^\tau}$; then $\prod_{\rho\in\Sigma_X(1)} u_\rho^{a_\rho^\tau}$ is a section of L_τ, and so are $\varphi_{\mathbf{b}^\tau-\mathbf{a}^\tau}(\prod_{\rho\in\Sigma_X(1)} u_\rho^{b_\rho^\tau})$; we may then set

$$u_\tau := \mu_{\mathbf{a}^\tau}\prod_{\rho\in\Sigma_X(1)} u_\rho^{a_\rho^\tau}+\sum_{\mathbf{b}^\tau\neq\mathbf{a}^\tau}\varphi_{\mathbf{b}^\tau-\mathbf{a}^\tau}\mu_{\mathbf{b}^\tau}\prod_{\rho\in\Sigma_X(1)} u_\rho^{b_\rho^\tau}.$$

Notice that (L_τ,u_τ) are trivial/constant along the fibers of q, hence they descend to $(\bar{L}_\tau,\bar{u}_\tau)$ on $\bar{C}$. We only need to define $\varphi_{m_Y}\colon \bigotimes_{\tau\in\Sigma_Y(1)}\bar{L}_\tau^{\langle m_Y,v_\tau\rangle}\simeq\mathcal{O}_{\bar{C}}$ for m_Y in M_Y.

The left hand side can be expanded as $q_*\left(\bigotimes_{\rho\in\Sigma_X(1)} L_\rho^{\sum_{\tau\in\Sigma_Y(1)} a_\rho^\tau\langle m_Y,v_\tau\rangle}\right)$. I claim that there is a unique $m_X\in M_X$ such that $(\langle m_X,v_\rho\rangle)_\rho=(\sum_{\tau\in\Sigma_Y(1)} a_\rho^\tau\langle m_Y,v_\tau\rangle)_\rho$. Again by the exact sequence (1), this is equivalent to $\sum_{\rho\in\Sigma_X(1)}\sum_{\tau\in\Sigma_Y(1)} a_\rho^\tau\langle m_Y,v_\tau\rangle\mathcal{O}_X(D_\rho)=0\in\mathrm{Pic}(X)$. But since $\delta_tau=\sum_{\rho\in\Sigma_X(1)} a_\rho^\tau\mathcal{O}_X(D_\rho)\in\mathrm{Pic}(X)$, this is precisely condition (1) in 1.13. Hence we can set $\varphi_{m_Y}=q_*(\varphi_{m_X})$ for the unique m_X that we have just found.

Finally, notice that the basepoint set of $\xi':=\left((\bar{C},\mathbf{x}),(\bar{L}_\tau,\bar{u}_\tau)_{\tau\in\Sigma_Y(1)},(\varphi_{m_Y})_{m_Y\in M_Y}\right)$ is the image of that of ξ under q, hence it is a finite set of non-special points by construction; also, ξ' is stable by the very construction of q. We may therefore set $Q(f)(\xi)=\xi'$.

We have also proved in [BN17, Lemma B.5] that if f is finite onto its image (i.e. it does not contract any effective curve class in X) and $\mathcal{Q}_{g,n}(Y,f_*\beta)$ is unobstructed, then $Q(f)$ admits a perfect obstruction theory compatible with the given ones, hence $Q(f)^![\mathcal{Q}_{g,n}(Y,f_*\beta)]^{\mathrm{virt}}=[\mathcal{Q}_{g,n}(X,\beta)]^{\mathrm{virt}}$, i.e. the virtual classes behave well under virtual pullback.

1.6. **Splitting axiom and CohFT.** This is discussed e.g. in [CFK17, §2.3.3]. There is a morphism $\mathrm{fgt}\colon\mathcal{Q}^\epsilon_{g,n}(X,\beta)\to\overline{\mathcal{M}}_{g,n}$ given by stabilising the source curve and forgetting all the extra data. For every $\epsilon\in\mathbb{Q}_{>0}$ and $2g-2+n>0$ there is a system of Λ-linear maps

$$\Omega^\epsilon_{g,n}\colon H^*(X,\Lambda)^{\otimes n}\to H^*(\overline{\mathcal{M}}_{g,n},\Lambda)$$
$$\Omega^\epsilon_{g,n}(\otimes_{i=1}^n\delta_i)=\mathrm{fgt}_*\left([\mathcal{Q}^\epsilon_{g,n}(X,\beta)]^{\mathrm{virt}}\cap\prod_{i=1}^n\mathrm{ev}_i^*(\delta_i)\right)$$

The splitting axiom holds for ϵ-quasimaps by noticing that the proof in [Beh97] is ϵ-independent; the details have been explicitly worked out in [BN17, Appendix B.3]. Recall its content: given a stable splitting $\xi = ((g_1, S_1), (g_2, S_2))$ such that $g = g_1 + g_2$ and $[n] = S_1 \coprod S_2$, there is a divisor $\iota_\xi \colon D_\xi \hookrightarrow \overline{\mathcal{M}}_{g,n}$, whose pullback under fgt is the boundary (virtual) divisor D_ξ^ϵ of $Q^\epsilon_{g,n}(X,\beta)$ that is the image of the proper gluing morphism from $\tilde{D}_\xi^\epsilon$ defined by the cartesian diagram

$$\begin{tikzcd}
\tilde{D}_\xi^\epsilon \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \coprod_{\substack{\beta=\beta_1+\beta_2 \\ \beta_1,\beta_2\in \mathrm{Eff}(X)}} Q^\epsilon_{g_1,S_1\cup\bullet}(X,\beta_1)\times Q^\epsilon_{g_2,S_2\cup\bullet}(X,\beta_2) \arrow[d, "\mathrm{ev}_\bullet\times\mathrm{ev}_\bullet"] \\
X \arrow[r, "\Delta"] & X\times X
\end{tikzcd}$$

Lemma 1.14. The construction above is compatible with virtual classes: $\iota_\xi^![Q^\epsilon_{g,n}(X,\beta)]^{\mathrm{virt}} = \Delta^!\left(\sum_{\beta=\beta_1+\beta_2}[Q^\epsilon_{g_1,S_1\cup\bullet}(X,\beta_1)]^{\mathrm{virt}} \boxtimes [Q^\epsilon_{g_2,S_2\cup\bullet}(X,\beta_2)]^{\mathrm{virt}}\right)$

A consequence of this is that, if we denote by $\tilde\iota_\xi\colon \overline{\mathcal{M}}_{g_1,S_1}\times\overline{\mathcal{M}}_{g_2,S_2}\to\overline{\mathcal{M}}_{g,n}$, the maps Ω satisfy:

$$\tilde\iota_\xi^*(\Omega^\epsilon_{g,n}(\otimes_{i=1}^n\delta_i)) = \sum_{i=0}^l \Omega^\epsilon_{g_1,|S_1|+1}((\otimes_{j\in S_1}\delta_j)\otimes\gamma_i)\Omega^\epsilon_{g_2,|S_2|+1}((\otimes_{j\in S_2}\delta_j)\otimes\gamma^i)$$

and similarly for the irreducible divisor. Hence $\{\Omega^\epsilon_{g,n}\}_{2g-2+n>0}$ determine a cohomological field theory over Λ on $(H^*(X,\Lambda),\langle\cdot,\cdot\rangle)$.

Finally define the *ϵ-quasimap quantum product* on generators by:

$$\gamma_i\circ_\epsilon\gamma_j = \sum_{k=1}^l \gamma_k\langle\langle\gamma_i,\gamma_j,\gamma^k\rangle\rangle^\epsilon_{0,3}(\mathbf{t}),$$

and extend by linearity to all of $H^*(X,\Lambda)$. The *small* product is obtained by letting $\mathbf{t}=0$.

Remark 1.15. The quasimap quantum product is clearly commutative; its associativity (equivalent to the WDVV equation for the genus 0 ϵ-quasimap potential) follows from considering the morphism $\mathrm{fgt}_{5,\dots,n}\circ\mathrm{fgt}\colon Q_{0,n}(X,\beta)\to\overline{\mathcal{M}}_{0,4}\simeq\mathbb{P}^1$ for $n\geq 4$, pulling back the rational equivalence between any two boundary points of $\overline{\mathcal{M}}_{0,4}$, and exploiting the splitting axiom to obtain that the expression

$$\sum_{\substack{\beta_1+\beta_2=\beta\\ n_1+n_2=n-4}} \langle\langle A,B,\delta_{i_1}\psi_{i_1}^{a_{i_1}},\dots,\delta_{i_{n_1}}\psi_{i_{n_1}}^{a_{i_{n_1}}}\rangle\rangle_{0,2+n_1,\beta_1}\langle\langle C,D,\delta_{j_1}\psi_{j_1}^{a_{j_1}},\dots,\delta_{j_{n_2}}\psi_{j_{n_2}}^{a_{j_{n_2}}}\rangle\rangle_{0,2+n_2,\beta_2}$$

is totally symmetric in A,B,C,D.

1.7. **String and dilaton equations.** The relation between $Q_{g,n+1}(X,\beta)$ and $Q_{g,n}(X,\beta)$ is delicate in quasimap theory; even when there is a forget/stabilise morphism (e.g. for $X=\mathbb{P}^N$), this may not be the universal curve, and the virtual class may not be compatible under pullback. As a consequence, the usual proof of the string, dilaton, and divisor equation is not ϵ-independent. On the other hand Ciocan-Fontanine and Kim proved that, for semi-positive targets, the string equation holds (with any number of descendant insertions) for the class $J_0^\epsilon(q)\mathbb{1}$ [CFK17, Proposition 3.4.1]. The same result for at most one-pointed descendants is already contained in [CFK14, Corollary 5.5.4], and it is sufficient to show that $J_0^\epsilon(q)\mathbb{1}$ is the unit for the ϵ-quasimap quantum product [CFK17, Remark 3.1.4]. Recall that $I_0(q) = 1$ in the toric semipositive case, and $J_0^\epsilon(q)$ is a q-truncation of $I_0(q)$ for every ϵ. From these remarks the next lemma follows.

Lemma 1.16. Let X be a semipositive toric variety. The ϵ-quasimap quantum cohomology $(H^*(X,\Lambda), \circ_\epsilon)$ is an associative, commutative algebra with unit $\mathbb{1}_X$.

Similarly they prove that the dilaton equation is satisfied by the class $(J_0^\epsilon(q)\mathbb{1})\psi - J_1^\epsilon(q)$ [CFK17, Lemma 3.4.3]. We collect these results in the following lemma, restricting to the special case of semipositive toric varieties.

Lemma 1.17. In the semipositive toric case, the following equations hold.

$$\text{(string)} \quad \langle \tau_{a_1}(\delta_1), \dots, \tau_{a_n}(\delta_n), \mathbb{1}\rangle_{g,n+1,\beta} = \sum_{i=1}^{n} \langle \tau_{a_1}(\delta_1), \dots, \tau_{a_i-1}(\delta_i), \dots, \tau_{a_n}(\delta_n)\rangle_{g,n,\beta}$$

$$\text{(dilaton)} \quad \sum_\beta q^\beta \langle \tau_{a_1}(\delta_1), \dots, \tau_{a_n}(\delta_n), \psi - J_1^\epsilon(q)\rangle_{g,n+1,\beta} = (2g-2+n)\sum_\beta q^\beta \langle \tau_{a_1}(\delta_1), \dots, \tau_{a_i-1}(\delta_i), \dots, \tau_{a_n}(\delta_n), \rangle_{g,n,\beta}$$

where the sum is over the effective curve classes β such that $2g-2+n+\epsilon\mathcal{O}_X(1)\cdot\beta > 0$.

1.8. **Divisor equation and final remarks.** On the other hand the divisor equation does not hold in general in the toric semipositive case. This can be seen by looking at the wall-crossing formulae of Theorem 1.12 and observing that, applying the divisor equation on the Gromov-Witten side, the coefficient coming out is not the same for all addends, indeed it depends on the curve class, and there can be many different β' contributing to the fixed β on the quasimap side. Notice though that the divisor equation holds in the toric Fano case.

It is a consequence of the wall-crossing formulae that the big J and I functions coincide in the *toric Fano* case; in fact in this case the mirror map is trivial, i.e. $I_1(q) = 0$, so in particular the quasimap invariants are ϵ-independent in this case. It is well-known that small quantum cohomology coincides with the Batyrev's quantum ring in this case [CK99, Example 11.2.5.2]. On the other hand they may differ in general in the semipositive case. Since Batyrev's ring naively accounts for smooth rational curves in the toric variety, the difference roughly speaking resides in the count of nodal curves; since quasimaps make rational tails unstable, it might be tempting to wonder whether small quasimap cohomology is isomorphic to the Batyrev's ring in the semipositive, non-Fano case.

Unfortunately the answer is no: as a counterexample consider the Hirzebruch surface $\mathbb{F}_2$. Thinking of it as a projective bundle over $\mathbb{P}^1$, call F the the pullback of $\mathcal{O}_{\mathbb{P}^1}(1)$, D_∞ the section such that $D_\infty^2 = -2$ and $D_0 = D_\infty + 2F$; I will use square brackets when I think of them as curve classes instead of as divisor classes. Then the relation $D_0 * D_\infty = q^{[F]}$ holds in Batyrev's ring. On the other hand,

$$I_1(q) = -D_\infty F(q^{[D_\infty]}), \quad \text{where} \quad F(z) = \sum_{k>0} z^k \frac{(2k-1)!}{(k!)^2}$$

and it can be shown by applying the wall-crossing formulae that

$$D_0 \circ_{0+} D_\infty = q^{[F]} \left(-e^{F(q^{[D_\infty]})} + \frac{e^{-F(q^{[D_\infty]}} - 1}{q^{[D_\infty]}} \right).$$

In particular the two rings do not coincide on the nose, but rather only after a coordiante transformation related to the mirror map.

One more comment is in order: since Witten's genus 0 topological recursion relation holds for ϵ-quasimaps by [CFK17, Corollary 2.3.4], it follows from the same proof in [CK99, Theorem 10.3.1] that a quantum differential operator annihilating $I(q, t_{|H^{\leq 2}(X)}, z)$ determines a relation in quasimap quantum cohomology. It is instead not true that $I(q, t_{|H^{\leq 2}(X)}, z) = e^{t/z} I(q, 0, z)$ due to the failure of the divisor equation. Hence $I(q, t_{|H^{\leq 2}(X)}, z)$ does not necessarily satisfy the GKZ system as in [CK99, §5.5.3], the relations associated to which are precisely the relations in Batyrev's quantum ring.

2. The localisation formula for toric quasimaps

The localisation formula is discussed in [CFK17, §5]. I review it and make a further step towards explicitness by expressing the various contributions combinatorially in terms of weights, following [Spi00].

2.1. **Notation from toric geometry.** Let X_Σ be a smooth complete toric variety, for $\Sigma \subseteq N$ a rational polyhedral fan and $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. Let us denote by r the Picard rank of X, by n its dimension, and by $N = n + r$ the number of rays in Σ. Let v_ρ denote the primitive generator of the ray ρ, and assume that $\Sigma(1)$ is an ordered set.

The (non-effective) action of the big torus $T = \mathbb{G}_{\mathrm{m}}^N$ on X induces an action on $\mathcal{Q}_{g,n}(X, \beta)$, by scaling the sections. Let us denote by $\lambda_1, \ldots, \lambda_N$ the corresponding weights.

Let us denote by $\{\sigma_i\}_{i\in\Sigma^{\max}}$ the T-fixed points on X (corresponding to maximal cones of Σ) and by $\{\tau_{i,j}\}_{i,j\in\Sigma^{\max}}$ the 1-dimensional orbits (corresponding to facets of the maximal cones; $\tau_{i,j}$ -if it exists- connects σ_i and σ_j).

Let σ_i and σ_j be two adjacent maximal cones (we write $\sigma_i \diamond \sigma_j$ for this relation); since X is smooth, $\{v_\rho\}_{\rho\prec\tau_{i,j}} \cup \{v_n\}$ is a $\mathbb{Z}$-basis of N (where v_n is the only ray in σ_i, but not in $\tau_{i,j}$), so we can find the dual basis $\{m_1, \dots, m_n\}$ of M. Define:

$$\lambda^{\sigma_i}_{\sigma_j} = \sum_{\rho\in\Sigma(1)} \langle m_n, v_\rho\rangle \lambda_\rho \quad \text{and} \quad \lambda^{\sigma}_{\text{tot}} = \prod_{\gamma\in\Sigma^{\max}:\ \gamma\diamond\sigma} \lambda^{\sigma}_{\gamma}$$

Compare with [Spi00, §§6.4 and 7.3].

Lemma 2.1. Let σ_i be a T-fixed point on X and $\tau_{i,j}$ be a 1-dimensional orbit through it, furthermore let D_ρ be a toric divisor. Then the weight of the T-action on $\mathcal{O}(D_\rho)_{\sigma_i}$ is

$$\begin{cases} \lambda^{\sigma_i}_{\sigma_j} & \text{if } \rho \prec \sigma_i \text{ and } \tau_{i,j} \cup \{v_\rho\} = \sigma_i \\ 0 & \text{otherwise.} \end{cases}$$

The weight of the T-action on $T(\tau_{i,j})_{\sigma_i}$ is $\lambda^{\sigma_i}_{\sigma_j}$.

Proof. Let σ_i be spanned by $\{v_{i_1}, \dots, v_{i_n}\}$. If $[z_1 : \dots : z_N]$ are homogeneous coordinates on X, then local coordinates around σ_i are given by

$$\left(x_{i_1} = z_{i_1} \prod_{j\neq i_1} z_j^{\langle m_{i_1}, v_j\rangle}, \dots, x_{i_n} = z_{i_n} \prod_{j\neq i_n} z_j^{\langle m_{i_n}, v_j\rangle} \right),$$

where $\{m_{i_1}, \dots, m_{i_n}\}$ is the dual basis of $\{v_{i_1}, \dots, v_{i_n}\}$.

If $\rho \nprec \sigma_i$ then the weight is 0 because we can find a divisor representing $\mathcal{O}(D_\rho)$ that does not pass through σ_i. Otherwise $\rho = i_j$ for some $j \in \{1, \dots, n\}$, so D_ρ has local equation $x_{i_j} = 0$ near σ_i, which makes the first statement clear.

The second part follows from the exact sequence

$$0 \to T\tau_{i,j} \to TX_{|\tau_{i,j}} \to \bigoplus_{\rho\prec\tau_{i,j}} \mathcal{O}_{\tau_{i,j}}(D_\rho) \to 0$$

together with the Euler exact sequence for TX and the first part. □

2.2. ***T*-fixed loci.** The following discussion is inspired by [MOP11, §7.3]. T-fixed loci for $Q_{g,n}(X,\beta)$ are indexed by decorated graphs

$$(\Gamma, v, \gamma, b, \varepsilon, \delta, \mu)$$

where:

(1) $\Gamma = (V, E)$ is a graph with vertex set V and edge set E (no self-edges allowed); let us denote by F the set of flags $f = (v, e)$ such that v is adjacent to e;

(2) $\mathrm{v}\colon V \to \{\sigma_i\}_{i\in\Sigma^{\max}}$ assigns a fix point to each vertex;
(3) $\gamma\colon V \to \mathbb{Z}_{\geq 0}$ is a genus assignment;
(4) $b\colon V \to H_2^+(X,\mathbb{Z})$ assigns an effective curve class to each vertex;
(5) $\varepsilon\colon E \to \{\tau_{i,j}\}_{i,j\in\Sigma^{\max}}$ assigns a one-dimensional orbit to each edge;
(6) $\delta\colon E \to \mathbb{Z}_{\geq 1}$ specifies the degree of the covering map;
(7) $\mu\colon \{1,\dots,n\} \to V$ is a distribution of the markings to the vertices V.

These data are required to satisfy a number of compatibility conditions:

- Γ must be connected;
- if $e\colon v_1 \to v_2$ then $\varepsilon(e) = \tau_{i,j}$ with $\mathrm{v}(v_1) = \sigma_i$ and $\mathrm{v}(v_2) = \sigma_j$ (or viceversa);
- $h^1(\Gamma) + \sum_{v\in V}\gamma(v) = g$;
- b is compatible with v, namely[1] $b(v)\cdot D_\rho \geq 0$ for all $\rho \not< \mathrm{v}(v)$;

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- $\sum_{v\in V} b(v) + \sum_{e\in E}\delta(e)[\varepsilon(e)] = \beta$.

We are going to denote by $\mathrm{val}\colon V \to \mathbb{Z}_{\geq 1}$ the number of edges adjacent to a vertex, and by $\deg\colon V \to \mathbb{Z}_{\geq 2}$ the sum of val with the number of marked points associated to each vertex.

The corresponding T-fixed locus is isomorphic, up to a finite map, to:

$$\mathcal{M}_\Gamma := \prod_{v\in V} \overline{\mathcal{M}}_{\gamma(v),\deg(v)|\sum_{\rho\not<\mathrm{v}(v)} b(v)\cdot D_\rho}$$

The moduli spaces corresponding to degenerate vertices (where $\deg(v) = 2$, $\gamma(v) = 0$, and $b(v) = 0$) are treated as points in this product; let us denote the collection of such vertices by $V^{\deg}$, and in particular by $V^{\deg}_{\mathrm{val}=2}$ those with valence 2 (i.e. those corresponding to nodes between two non-contracted components). Notice that $\overline{\mathcal{M}}_{g,n|d}/S_d \simeq \mathcal{Q}_{g,n}(\mathbb{A}^1 /\!\!/ \mathbb{G}_{\mathrm{m}}, d)$. Hence the finite map has degree

$$a_\Gamma := |\mathbf{A}|\cdot\prod_{v\in V}\prod_{\rho\not<\mathrm{v}(v)}(b(v)\cdot D_\rho)!$$

where $|\mathbf{A}|$ can be extrapolated from

$$0 \to \prod_{e\in E}\mathbb{Z}/\delta(e)\mathbb{Z} \to \mathbf{A} \to \mathrm{Aut}(\Gamma) \to 0.$$

The corresponding quasimap can be described as follows: edges correspond to maps (without basepoints) from $\mathbb{P}^1$ to the corresponding 1-dimensional T-orbit $\varepsilon(e)$, of degree $\delta(e)$ and totally ramified at the two T-fixed points. Pick instead a vertex $v\in V$: notice that, for every maximal cone σ_i, the collection $\{D_\rho\}_{\rho\not<\sigma_i}$ constitutes a basis of $\mathrm{Pic}(X)$ (since every support function can be made into vanishing on every $\rho\not<\sigma_i$ by subtracting an appropriate $m\in M$). According to $\mathrm{v}(v) = \sigma_i$ then, we may write

[1] This condition implies that $b(v)$ is effective; in fact a curve is effective if and only if it matches non-negatively with any polarisation $\mathcal{O}_X(1)$; but $\{D_\rho\}_{\rho\not<\mathrm{v}(v)}$ is a basis of $\mathrm{Pic}(X)$ and the coefficients of $\mathcal{O}_X(1)$ in this basis must all be positive, since a power of it is very ample hence it cuts an effective divisor on X.

$\mathcal{O}(D_{i_j}) = \bigotimes_{\rho \not< \sigma_i} \mathcal{O}(D_\rho)^{\otimes a_{i_j,\rho}}$ for each $i_j,\ j = 1, \dots, n$ such that $v_{i_j} \in \sigma_i$. For a marked curve C_v in the mixed moduli space $\overline{\mathcal{M}}_{\gamma(v),\deg(v)|\sum_{\rho \not< \sigma_i} b(v)\cdot D_\rho}$ with markings

$$\{p_1, \dots, p_{\deg(v)}\} \cup \bigcup_{\rho \not< \sigma_i} \{q_{\rho,1}, \dots, q_{\rho, b(v)\cdot D_\rho}\}$$

the corresponding quasimap is given by:

$$\left((C_v, \{p_1, \dots, p_{\deg(v)}\}), (\mathcal{O}_{C_v} \hookrightarrow \mathcal{O}_{C_v}(\sum_{j=1}^{b(v)\cdot D_\rho} q_{\rho,j}) =: L_\rho)_{\rho \not< \sigma_i}, (\mathcal{O}_{C_v} \xrightarrow{0} \bigotimes_{\rho \not< \sigma_i} L_\rho^{\otimes a_{i_j,\rho}} =: L_{i_j})_{j=1,\dots,n} \right)$$

Gluing along flags $f \in F$ is made possible by the required compatibilities.

2.3. **The obstruction theory.** Recall that $Q_{g,n}(X, \beta)$ has a perfect obstruction theory relative to $\mathfrak{M}_{g,n}$ given by $\mathrm{R}^\bullet \pi_* \mathcal{F}_X$, where $\mathcal{F}_X$ is the sheaf on the universal curve defined by the exact sequence

$$0 \to \mathcal{O}_{C_Q} \otimes \mathfrak{t} \to \bigoplus_{\rho \in \Sigma(1)} \mathcal{L}_\rho \to \mathcal{F}_X \to 0. \tag{4}$$

where $\mathfrak{t}$ is the Lie algebra of the (small) torus $G := \mathrm{Hom}(\mathrm{Pic}(X), \mathbb{G}_\mathrm{m})$ and the first map is determined by the derivative of the action of G on $\mathbb{A}^N$ at the identity element of G. When restricting to a single quasimap $\big((C, \mathbf{p}), (L_\rho, u_\rho)\big)$, we get an exact sequence:

$$\begin{aligned} 0 \to \mathrm{Ext}^0(\Omega_C(\mathbf{p}), \mathcal{O}_C) \to H^0(C, \mathcal{F}_X) \to \mathcal{T}^0_{|C} \to \\ \to \mathrm{Ext}^1(\Omega_C(\mathbf{p}), \mathcal{O}_C) \to H^1(C, \mathcal{F}_X) \to \mathcal{T}^1_{|C} \to 0 \end{aligned}$$

where $\mathcal{T}^0 - \mathcal{T}^1$ is the tangent-obstruction bundle. Let us denote by $B_1, \dots, B_6$ the objects in the above exact sequence, by $(-)^f$ and $(-)^m$ the fixed and moving parts respectively. We shall study the fixed/moving decomposition of B_3 and B_6 by considering that of B_1, B_2, B_4, and B_5. It is understood that the analysis of each B_i can be conducted by looking at the partial normalisation of C at the nodes connecting a contracted to a (non-)contracted component, from which it follows that each term factors into a vertex, edge, and flag contribution (to which we are going to refer freely in the following).

Recall that an object is fixed if we can find an isomorphism between it and its image under the torus action (for every element of the torus). A preliminary observation is that the edge contribution is precisely the same as in the stable maps case, since a non-contracted component must be a $\mathbb{P}^1$ covering a 1-dimensional orbit, totally ramified at 0 and ∞, which must furthermore be nodes or markings (by stability), hence there cannot be any basepoint (by non-degeneracy). On the other hand, a vertex contribution may come from a totally basepoint quasimap; in this case, write as above $\sigma_i := \mathrm{v}(v) = \langle \rho_{i_1}, \dots, \rho_{i_n} \rangle$, and notice that the n (zero) sections $u_{i_1}, \dots, u_{i_n}$ are going to be unaffected by the torus action, while the non-trivial action on $\{u_\rho\}_{\rho \not< \sigma_i}$ can be adjusted by taking an appropriate automorphism of the corresponding line bundles (of which there are precisely $r = \operatorname{rk}\operatorname{Pic}(X_\Sigma)$); it is apparent from this analysis that the underlying curve C_v is not altered by the action, i.e. we can take id_{C_v}.

Focussing on the *fixed part* first, observe that B_1^f comprises a 1-dimensional contribution from both the edges and the genus 0, $\deg(v) = 2$ (but $b(v) \neq 0$) components, which cancels out with a corresponding term in B_4^f; this corresponds to the (rotation) automorphisms of a two-pointed rational curve. The remaining part of B_4^f corresponds to deformations of the contracted components (leaving the dual graph fixed). On the other hand the fixed part of B_2 and B_5 may be simulataneously studied from the normalisation exact sequence:

(5)

$$0 \to H^0(C, \mathcal{F}_X) \to \bigoplus_{v\in V} H^0(C_v, \mathcal{F}_{X|C_v}) \oplus \bigoplus_{e\in E} H^0(C_e, \mathcal{F}_{X|C_e}) \to \bigoplus_{f=(v,e)\in F} TX_{\mathrm{v}(v)} \to$$
$$\to H^1(C, \mathcal{F}_X) \to \bigoplus_{v\in V} H^1(C_v, \mathcal{F}_{X|C_v}) \oplus \bigoplus_{e\in E} H^1(C_e, \mathcal{F}_{X|C_e}) \to 0$$

notice that the third term is justified because nodes and markings are not basepoints; for degenerate vertices v it is intended that $H^0(C_v, \mathcal{F}_{X|C_v}) = TX_{\mathrm{v}(v)}$ cancels out with the corresponding flag term, and $H^1(C_v, \mathcal{F}_{X|C_v}) = 0$. The only contribution to the fixed part comes from $H^0(C_v, \mathcal{F}_{X|C_v})$; namely, let $\sigma_i = \mathrm{v}(v)$ be as above, and choose $\{\mathcal{O}_X(D_\rho)\}_{\rho\not<\sigma_i}$ as a basis of $\operatorname{Pic}(X_\Sigma)$: then the exact sequence (4) can be rewritten as

(6)
$$0 \to \mathcal{O}_C^{\oplus r} \xrightarrow{(\mathrm{can},0)} \underbrace{\bigoplus_{\rho\not<\sigma_i} L_\rho}_{} \oplus \bigoplus_{j=1}^{n} L_{i_j} \to \mathcal{F}_X \to 0$$
$$= \bigoplus_{\rho\not<\sigma_i} \mathcal{O}_C\Big(\sum_{j=1}^{b(v)\cdot D_\rho} q_j\Big)$$

The fixed part of $B_2 - B_5$ then results in $\oplus_{\rho\not<\sigma_i} H^0(C, \mathcal{O}_C(\mathbf{q})_{|\mathbf{q}})$, which coincides with the relative tangent of $\overline{\mathcal{M}}_{g(v),\mathrm{val}(v)|\sum_{\rho\not<\sigma_i} b(v)\cdot D_\rho}$ over $\mathfrak{M}_{g(v),\mathrm{val}(v)}$. This discussion proves that the fixed part of the restriction of the perfect obstruction theory to the fixed loci corresponds to their standard obstruction theory. Let us move on now to the *moving part*.

As compared to the case of stable maps, the stability condition implies that there are no rational tails, hence $B_1^m = 0$ (see [Spi00, Lemma 7.2]). Let us introduce a useful notation at this point: for a flag $f = (v, e)$, if $\mathrm{v}(v) = \sigma_i$ and $\varepsilon(e) = \tau_{i,j}$, denote by $\lambda_f := \frac{1}{\delta(e)}\lambda_{\sigma_j}^{\sigma_i}$. Since deformations of contracted components are T-fixed, B_4^m comes from smoothing nodes between a non-contracted component and:

- another non-contracted component: this is the case of a degenerate vertex v of valence two; if $f_1 = (v, e_1)$ and $f_2 = (v, e_2)$ are the two flags containing it, then the weight is given by $\lambda_{f_1} + \lambda_{f_2}$;
- a contracted component: such a node determines a marking on the corresponding contracted component, and let ψ_f denote the c_1 of the cotangent line bundle at that point; in this case the flag $f = (v, e)$ is such that v is non-degenerate, and the contribution is given by $\lambda_f - \psi_f$.

Summing up, we have the following (see [Spi00, Lemma 7.3]):

Lemma 2.2. The moving part from deformations of the underlying curve can be expressed as:

$$e^T(B_4^m) = \prod_{f\in F:\, v(f)\in V^{\text{non-deg}}} (\lambda_f - \psi_f) \prod_{v\in V^{\text{deg}}_{\text{val}=2}:\, v\in f_1, f_2} (\lambda_{f_1} + \lambda_{f_2}),$$

Finally, the moving part of $B_2 - B_5$ can be analysed from the normalisation exact sequence (5). As we have already remarked, around non-contracted components the situation is just the same as in the case of stable maps, so the edge contributions are unaltered.

$$\begin{aligned} H^0(C, \mathcal{F}_X) - H^1(C, \mathcal{F}_X) = &\bigoplus_{v\in V} H^0(C_v, \mathcal{F}_{X|C_v}) - \bigoplus_{v\in V} H^1(C_v, \mathcal{F}_{X|C_v}) \\ &+ \bigoplus_{e\in E} H^0(C_e, f_e^* TX) - \bigoplus_{e\in E} H^1(C_e, f_e^* TX) \\ &- \bigoplus_{f=(v,e)\in F} TX_{\mathrm{v}(v)} \end{aligned} \tag{7}$$

The computation for the edge contributions involves both the torus action on the line bundle, and a non-trivial action on C_e (induced by that on $\varepsilon(e)$ but weighted by $\frac{1}{\delta(e)}$). Notice that, by the Euler sequence, to understand the tangent bundle of X it is enough to consider the restriction of the toric divisors to $\varepsilon(e)$, or equivalently that of the universal line bundles to C_e; let us denote by σ_1 and σ_2 the image of the vertices of e, and for every maximal cone $\sigma_2 \neq \gamma \diamond \sigma_1$ denote by $d_e^\gamma = \mathcal{O}(D_{\bar\rho}) \cdot \varepsilon(e)$, where $\bar\rho$ is the only ray such that $\sigma_1 = \langle (\sigma_1 \cap \gamma) \cup \{v_{\bar\rho}\}\rangle$. Compare with [Spi00, Lemma 7.4 and Corollary 7.5].

Lemma 2.3. The edge contribution to the moving part is given by

$$e^T(H^0(C_e, f_e^* TX)^m) = (-1)^{\delta_e} \frac{(\delta_e!)^2}{\delta_e^{2\delta_e}} (\lambda_{\sigma_2}^{\sigma_1})^{2\delta_e} \prod_{\substack{\gamma\in\Sigma^{\max}: \\ \sigma_2\neq\gamma\diamond\sigma_1, d_e^\gamma\geq 0}} \prod_{k=0}^{d_e^\gamma} (\lambda_\gamma^{\sigma_1} - \frac{k}{\delta_e} \lambda_{\sigma_2}^{\sigma_1})$$

and by Serre duality

$$e^T(H^1(C_e, f_e^*TX)^m) = \prod_{\substack{\gamma\in\Sigma^{\max}:\\ \sigma_2\neq\gamma\diamond\sigma_1, d_e^\gamma\leq -2}} \prod_{k=d_e^\gamma}^{-2} (\lambda_\gamma^{\sigma_1} - \frac{k+1}{\delta_e}\lambda_{\sigma_2}^{\sigma_1})$$

with notation as above. Notice that these expressions are independent of the ordering of $\{\sigma_1, \sigma_2\}$.

The vertex contribution is easily computed from (6) if we restrict to the genus zero case; otherwise it seems that the first line of (7) is not a two-term comlex of vector bundles and we still need to find a better presentation.

Lemma 2.4. The vertex contribution to the moving part is given by

$$e^T(H^0(C_v, \mathcal{F}_{X|C_v})^m - H^1(C_v, \mathcal{F}_{X|C_v})^m) = \prod_{j=1}^n (\lambda_{\sigma_{i_j}}^{\sigma_i})^{b(v)\cdot D_{i_j}+1}$$

where $\sigma_i = \mathrm{v}(v)$, $\sigma_{i_j} \diamond \sigma_i$, and v_{i_j} is the only ray of σ_i complementary to σ_{i_j}. In particular when $b(v) = 0$ this reduces to

$$e^T(TX_{\sigma_i}) = \lambda_{\mathrm{tot}}^{\sigma_i}$$

2.4. **Genus 0 quasimap invariants.** Recall that the cohomology of a smooth complete toric variety $H^*(X, \mathbb{Z})$ is generated by $H^2(X, \mathbb{Z}) \simeq \mathrm{Pic}(X)$; choose a basis $D_1, \ldots, D_r$. For cohomological insertions $\xi_1, \ldots, \xi_m$, fix once and for all a writing $\xi_k = \prod_{i=1}^r D_i^{l_{i,k}}$.

Proposition 2.5. The genus-zero quasimap invariants of X are given by

$$\langle \xi_1, \ldots, \xi_m \rangle_{0,m,\beta}^{0+} = \sum \frac{1}{a_\Gamma} \int_{\mathcal{M}_\Gamma} \frac{\prod_{k=1}^m \prod_{j=1}^r (\lambda_j^k)^{l_{j,k}}}{e^T(N_\Gamma^{\mathrm{vir}})} \tag{8}$$

where: the sum runs over all fixed point loci $(\Gamma, v, \gamma, b, \varepsilon, \delta, \mu)$; setting $\sigma(k) = \mathrm{v}(\mu(k))$, λ_j^k is given by:

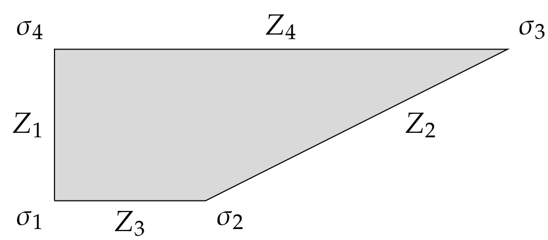
maybe I should just explain this convention earlier in the text, that given a max.l cone σ there's a bijection between rays of σ and neighbouring max.l cones

$$\begin{cases} 0 & \text{if } v_j \notin \sigma(k); \\ \lambda_{\sigma(j)}^{\sigma(k)} & \text{if } \sigma(j) \diamond \sigma(k) \text{ and } \sigma(k) = \langle (\sigma(j) \cap \sigma(k)) \cup \{v_j\} \rangle; \end{cases}$$

and the inverse of the T-equivariant Euler class of the virtual normal bundle is given by

(9)
$$\frac{1}{e^T(N_\Gamma^{\mathrm{vir}})} = \prod_{f\in F:\, v(f)\in V^{\text{non-deg}}} \frac{1}{\lambda_f - \psi_f} \cdot \prod_{\substack{v\in V^{\deg}_{\mathrm{val}=2} \\ f_i:=(v,e_i)\in F, i=1,2}} \frac{1}{\lambda_{f_1}+\lambda_{f_2}} \cdot$$
$$\prod_{\substack{e\in E,\delta_e:=\delta(e), \\ \{\sigma_1,\sigma_2\}:=\mathrm{v}\{\partial e\}}} \left(\frac{(-1)^{\delta_e}\delta_e^{2\delta_e}}{(\delta_e!)^2(\lambda^{\sigma_1}_{\sigma_2})^{2\delta_e}} \prod_{\substack{\gamma\in\Sigma^{\max}: \\ \sigma_2\neq\gamma\diamond\sigma_1, d_e^\gamma:=\dots}} \frac{\prod_{k=d_e^\gamma+1}^{-1}(\lambda^{\sigma_1}_\gamma - \frac{k}{\delta_e}\lambda^{\sigma_1}_{\sigma_2})}{\prod_{k=0}^{d_e^\gamma}(\lambda^{\sigma_1}_\gamma - \frac{k}{\delta_e}\lambda^{\sigma_1}_{\sigma_2})} \right) \cdot$$
$$\prod_{\substack{v\in V \\ \sigma:=\mathrm{v}(v)}} \prod_{\substack{\gamma\in\Sigma^{\max}:\,\gamma\diamond\sigma \\ \rho_\gamma:=\sigma\setminus\gamma}} \frac{1}{(\lambda^\sigma_\gamma)^{b(v)\cdot D_{\rho_\gamma}}} \cdot \prod_{v\in V} (\lambda^{\mathrm{v}(v)}_{\mathrm{tot}})^{\mathrm{val}(v)-1}$$

2.5. **Example.** We shall compute a quasimap invariant of the Hirzebruch surface $\mathbb{F}_2$ and compare it to the corresponding Gromov-Witten invariant. The moment polytope is:



The cohomology ring is given by:

$$H^*(\mathbb{F}_2,\mathbb{Z}) \simeq \mathbb{Z}[Z_1,\dots,Z_4]/\langle Z_1 - Z_2, Z_4 - Z_3 - 2Z_2, Z_1Z_2, Z_3Z_4\rangle$$

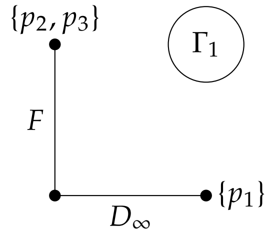
We work out the following weights for the $\mathbb{G}_{\mathrm{m}}^4$-action:

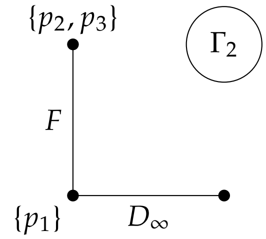
When thinking of them as curve classes, we'll write $F = [Z_1], D_0 = [Z_4], D_\infty = [Z_3]$.

$\lambda^{\sigma_1}_{\sigma_2} = -\lambda^{\sigma_2}_{\sigma_1}$	$\lambda_1 - \lambda_2$	$\lambda^{\sigma_3}_{\sigma_4} = -\lambda^{\sigma_4}_{\sigma_3}$	$\lambda_2 - \lambda_1$
$\lambda^{\sigma_2}_{\sigma_3} = -\lambda^{\sigma_3}_{\sigma_2}$	$2\lambda_1 + \lambda_3 - \lambda_4$	$\lambda^{\sigma_4}_{\sigma_1} = -\lambda^{\sigma_1}_{\sigma_4}$	$\lambda_4 - 2\lambda_2 - \lambda_3$

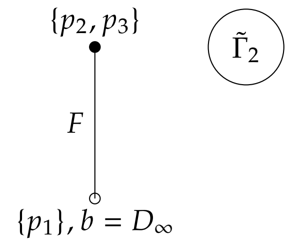
We are going to consider the space $Q_{0,3}(\mathbb{F}_2, D_\infty + F)$ (and the analogous stable maps space) with insertions $\xi_1 = Z_3$, $\xi_2 = Z_4$, $\xi_3 = Z_1Z_4$.

The relevant graphs in the stable maps case are:





In the case of quasimaps Γ_2 is unstable instead the following graph must be taken into account:



The white circle is supposed to indicate that there is a full basepoint component at that vertex.

3. Relative quasimaps

The relative problem in Gromov-Witten theory consists in studying maps with not only incidence, but also tangency conditions to a divisor. It has seen great advancements in the past twenty years, of which I will sample some most relevant ones in algebraic geometry (there is an almost parallel and similarly rich story in symplectic geometry):

- R. Vakil's pioneering work on rational and elliptic curves in projective space, satisfying tangency conditions to a hyperplane [Vak00];
- A. Gathmnn's work in genus 0, enhancing the previous one to the case of a smooth very ample divisor in any variety [Gat02];
- J. Li's moduli space of maps to expanded targets, removing the ampleness hypothesis on the divisor and developing the framework for the all important degeneration formula [Li01, Li02], which has been reviewed by B. Kim in a more explicitly logarithmic [Kim10], and by D. Abramovich and B. Fantechi in a more stacky [AF16] context;
- a host of recent works around D. Abramovich, Q. Chen, M. Gross, and B. Siebert's space of logarithmic stable maps [Che14, AC14, GS13], which allows for example the divisor to be snc.

It would be interesting to have a full relative theory for ϵ-quasimaps, including a degeneration formula and wall-crossing formulae for relative invariants as well; one application (suggested by I. Ciocan-Fontanine) would be to the study of the higher genus ϵ-quasimap invariants of the quintic threefold via Maulik-Pandharipande's degeneration scheme [MP06]. In [BN17] we have made a first step towards this program, namely we have introduced spaces of genus 0 relative quasimaps to a smooth very ample divisor, extending the work of Gathmann. Among the peculiarities of this approach are the fact that the tangency needs not be maximal (i.e. not all intersection points of the curve with the divisor must be marked with the respective tangency order requirement), and the fact that the relative spaces are nested (they become smaller as the tangency requirement gets higher), with a nice formula expressing their virtual classes in terms of one another, corrected by a boundary term exhibiting a clear recursive structure. This led Gathmann to the discovery of an algorithm for computing relative invariants, or, even further, restricted invariants of the hyperplane section, recursively, starting from the descendant theory of the ambient space; under positivity assumptions, he was able to realise this in a different proof of what goes under the name of Givental's mirror theorem, or the quantum Lefschetz principle [Gat03]. In this section I will report on my work with N. Nabijou.

3.1. **Review of Gathmann's work.** The starting point is the case of $\mathbb{P}^N$ relative to a hyperplane H. The most down-to-earth approach works in this case: fixed a tangency condition $\alpha \in \mathbb{N}^n$ with $\sum \alpha \leq d$, Gathmann defines the relative space $\overline{\mathcal{M}}_{0,\alpha}(\mathbb{P}^N|H,d) \subseteq \overline{\mathcal{M}}_{0,n}(\mathbb{P}^N,d)$ as the closure of the "nice" locus of maps $f\colon (\mathbb{P}^1,\mathbf{x}) \to \mathbb{P}^N$ such that $f(\mathbb{P}^1) \subsetneq H$ and $f^{-1}(H) - \sum \alpha_i x_i$ is an effective divisor. He then proves that the reducible maps that lie in the closure can be characterised combinatorially as follows: a stable map $(C, x_1, \ldots, x_n, f)$ is in $\overline{\mathcal{M}}_{0,\alpha}(\mathbb{P}^N|H,d)$ if and only if, for each connected component Z of $f^{-1}(H) \subseteq C$:

(1) if Z is a point and is equal to a marked point x_i, then f is tangent to H at x_i to order at least α_i;
(2) if Z is a curve, and if we let $C^{(i)}$ for $1 \leq i \leq r$ denote the irreducible components of C adjacent to Z and $m^{(i)}$ denote the multiplicity of $f|_{C^{(i)}}$ to H at the node $Z \cap C^{(i)}$, then:

$$(*) \qquad H \cdot f_*[Z] + \sum_{i=1}^{r} m^{(i)} \geq \sum_{x_i \in Z} \alpha_i$$

Now this definition works well for every pair $(X|Y)$ of a variety with a smooth divisor, by just replacing H with Y in the description above. On the other hand, the definition as the closure of the nice locus makes it clear that $\overline{\mathcal{M}}_{0,\alpha}(\mathbb{P}^N|,d)$ is irreducible of the expected dimension:

$$\dim(\overline{\mathcal{M}}_{0,\alpha}(\mathbb{P}^N|H,d)) = \dim(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^N,d)) - \sum \alpha.$$

If we then assume that Y is very ample, we may use the complete linear system assoiated to it in order to embed $\phi_{|Y|}\colon X \hookrightarrow \mathbb{P}^N$ and find a hyperplane $H \subseteq \mathbb{P}^N$ such that $H \cap X = Y$. Set $d = \phi_{|Y|,*}(\beta)$ for a curve class $\beta \in \mathrm{Eff}(X)$. The following cartesian diagram holds true:

$$\begin{tikzcd} \overline{\mathcal{M}}_{0,\alpha}(X|Y,\beta) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{\mathcal{M}}_{0,n}(X,\beta) \arrow[d, "\Phi_{|Y|}"] \\ \overline{\mathcal{M}}_{0,\alpha}(\mathbb{P}^N|H,d) \arrow[r] & \overline{\mathcal{M}}_{0,n}(\mathbb{P}^N,d) \end{tikzcd}$$

Furthermore $\mathrm{R}^\bullet\pi_*f^*N_{X/\mathbb{P}^N}$ provides a perfect obstruction theory for $\Phi_{|Y|}$, hence we may endow $\overline{\mathcal{M}}_{0,\alpha}(X|Y,\beta)$ with a virtual class

$$[\overline{\mathcal{M}}_{0,\alpha}(X|Y,\beta)]^{\mathrm{virt}} = \Phi^!_{|Y|}[\overline{\mathcal{M}}_{0,\alpha}(\mathbb{P}^N|H,d)]$$

which also has the expected codimension $\sum\alpha$ with respect to $[\overline{\mathcal{M}}_{0,n}(X,\beta)]^{\mathrm{virt}}$.

As anticipated, the combinatorial description shows that $\overline{\mathcal{M}}_{0,\alpha+e_k}(X|Y,\beta)\subseteq \overline{\mathcal{M}}_{0,\alpha}(X|Y,\beta)$, and it is a divisor. Gathmann finds a natural line bundle with section that cuts it out, together with a number of boundary correction terms. Let me start with some heuristics: assume $n=1$ for this. Let $Y=V(s)\subseteq X$. Then $\overline{\mathcal{M}}_{0,(1)}(X|Y,\beta)\subseteq\overline{\mathcal{M}}_{0,1}(X,\beta)$ is cut out by the section $\mathrm{ev}_1*(s)$ of $\mathrm{ev}_1^*\mathcal{O}_X(Y)$. Now the restriction of ev_1 to $\overline{\mathcal{M}}_{0,(1)}(X|Y,\beta)$ maps to Y; by composing $df_{x_1}\colon TC_{x_1}\to TX_{f(x_1)}$ with the projection to $N_{Y/X,x_1}$ we get a map that has to vanish if f is tangent to H at x_1. Rearranging we get a section $\mathrm{ev}_1^*(d^1s)$ of $x_1^*(T^*C)\otimes\mathrm{ev}_1^*\mathcal{O}_X(Y)$ that defines $\overline{\mathcal{M}}_{0,(2)}(X|Y,\beta)$ within $\overline{\mathcal{M}}_{0,(1)}(X|Y,\beta)$. More generally we may use the jet bundles (or bundles of principal parts) of $\mathcal{O}_X(Y)$: there is an exact sequence

$$0\to x_1^*\Omega_C^{\otimes m+1}\otimes\mathrm{ev}_1^*\mathcal{O}_X(Y)\to ev_1^*\mathcal{P}^{m+1}(\mathcal{O}_X(Y))\to ev_1^*\mathcal{P}^m(\mathcal{O}_X(Y))\to 0$$

and s induces a section $\mathrm{ev}_1^*(d^{m+1}s)$ of the middle term (which should be thought of as the Taylor expansion up to order $m+1$ of s at x_1), the image of which in the rightmost term (i.e. its truncation up to order m) vanishes on $\overline{\mathcal{M}}_{0,(m)}(X|Y,\beta)$, hence inducing a section of the line bundle $x_1^*\Omega_C^{\otimes m+1}\otimes\mathrm{ev}_1^*\mathcal{O}_X(Y)$ which vanishes along $\overline{\mathcal{M}}_{0,(m+1)}(X|Y,\beta)$. Notice though that it also vanishes (for every m, really) on those maps such that the irreducible component containing x_1 lies entirely inside Y. We have thus arrived to Gathmann's formula:

Theorem 3.1. [Gat02, Theorem 2.6] In the Chow group of $\overline{\mathcal{M}}_{0,\alpha}(X|Y,\beta)$

$$(\alpha_k\psi_k+\mathrm{ev}_k^*\mathcal{O}_X(Y))\cap[\overline{\mathcal{M}}_{0,\alpha}(X|Y,\beta)]^{\mathrm{virt}}=[\overline{\mathcal{M}}_{0,\alpha+e_k}(X|Y,\beta)]^{\mathrm{virt}}+[\mathcal{D}_{\alpha,k}(X,\beta)]^{\mathrm{virt}}$$

where $\mathcal{D}_{\alpha,k}(X,\beta)$ is a sum over all $r\geq 0$ determining the number of adjacent external components $C^{(1)},\dots,C^{(r)}$, $M=(m^{(1)},\dots,m^{(r)})\in\mathbb{N}^r_{>0}$ determining the order of contact of $C^{(i)}$ with Y at the intersection with the internal component $C^{(0)}$, $A=(\alpha^{(0)},\dots,\alpha^{(r)})$ and $B=(\beta_0,\dots,\beta_r)$ dictating the splitting of the markings/tangency requirements and curve class respectively, of the following "comb loci":

$$\mathcal{D}_{A,B,M}(X) = \overline{\mathcal{M}}_{0,|\alpha^{(0)}|+r}(Y,\beta_0) \times_{H^r} \prod_{i=1}^{r} \overline{\mathcal{M}}_{0,\alpha^{(i)}\cup m^{(i)}}(X|Y,\beta_i)$$

These are endowed with the product virtual fundamental class, weighted by a factor $\frac{m^{(1)}\cdot\ldots\cdot m^{(r)}}{r!}$; the denominator is just there to make the extra gluing markings unordered, while in a sense the numerator is the real content of Gathmann's formula. As noticed above the section $\mathrm{ev}_k^*(d^m s)$ vanishes whenever x_k lies on an internal component; the sum must therefore run over all those splitting types such that $\mathcal{D}_{A,B,M}(X)$ lies in $\overline{\mathcal{M}}_{0,\alpha}(X|Y,\beta)$ but not in $\overline{\mathcal{M}}_{0,\alpha+e_k}(X|Y,\beta)$, namely those satisfying $\alpha_k \in \alpha^{(0)}$ and

$$\mathcal{O}_X(Y)\cdot\beta_0 + \sum M = \sum \alpha^{(0)}.$$

The proof goes roughly as follows: the reduction to the case of $(\mathbb{P}^N|H)$ is just a matter of virtual intersection calculus; in the unobstructed case the shape of the formula follows simply from the above arguments and a(n actual) dimension computation, so the hard part is to get the coefficients right. Locally one may reduce further to $(\mathbb{P}^1,\{\infty\})$ by projecting from a generic $(N-2)$-plane inside H; finally the proof for $\mathbb{P}^1$ is based on Vakil's observation that, for maps from cuves to curves, the obstruction theory only sees the special loci (markings, nodes, ramification points, and contracted components), so that one can split the deformation space and reduce to a much simpler moduli problem.

Remark 3.2. Gathmann's recursion can be recovered in the framework of Jun Li's spaces of maps to expansions (or rather Kim's logarithmic stable maps) when the target is $(\mathbb{P}^N|H)$. This was suggested by J. Wise and D. Ranganathan. It is in a sense surprising because one gets the non-maximal from the maximal tangency case; but in fact this idea was already contained in the proof of Gathmann's formula. Here is how we can recover Theorem 3.1: add $h = d - \sum\alpha$ auxiliary markings of multiplicity 1, so consider $\overline{\mathcal{M}}^{\mathrm{Kim}}_{0,\alpha\cup(1,\ldots,1)}(\mathbb{P}^N|H,d)$; forgetting the auxiliary markings and the log structures, collapsing the target, and stabilising the source curve, we get an $(h! : 1)$ cover of $\overline{\mathcal{M}}_{0,\alpha}(\mathbb{P}^N|H,d)$ (generically we are just marking the h simple intersections of the image of C with H, and endowing C with the pullback of the divisorial log structure on $\mathbb{P}^N$; here we use that $g=0$ and the target is $\mathbb{P}^N$, but notice that the approach will work whenever the nice locus is dense in the relative space). Denote by $c\colon \overline{\mathcal{M}}^{\mathrm{Kim}}_{0,\alpha\cup(1,\ldots,1)}(\mathbb{P}^N|H,d) \to \overline{\mathcal{M}}_{0,\alpha\cup(1,\ldots,1)}(\mathbb{P}^N,d)$ the map that forgets the log structures, collapses the target, and stabilises the source curve; we may then pullback Gathmann's line bundle and section along c. The vanishing locus of the section thus obtained consists of maps such that the target is expanded, and x_k lies on a non-trivial (i.e. having either positive horizontal degree, or at least three markings) component at higher level. Notice that, as soon as there is more than one non-trivial component at higher level, collapsing the target has positive-dimensional fibers. Hence we are reduced to a sum over bipartite graphs with only one vertex at higher level; we recognise in these the shape of the comb loci appearing in Gathmann's formula. Also, I claim that the only case in which one of the auxiliary markings (say x_{n+1}) may lie on the component containing x_k is when the latter only contains x_k and x_{n+1} among the markings, and has zero horizontal degree - otherwise the map collapsing the target and forgetting the auxiliary markings will have positive dimensional fibers. The first case recovers exactly $\overline{\mathcal{M}}_{0,\alpha+e_k}(\mathbb{P}^N|H,d)$! The claim follows from the fact that the rubber space $\overline{\mathcal{M}}_{0,\alpha}(\mathbb{P}_H(\mathcal{O}\oplus\mathcal{O}(1)),\beta)^{\sim}$ has the same dimension as $\overline{\mathcal{M}}_{0,|\alpha|}(H,\beta)$, see [GV05, §2.4]. Now the coefficient with which each of these loci appears has been computed by several authors in the framework of the degeneration formula (see e.g. [Kim10, Remark 6.3.1.2] and [KLR18, Equation (1.7)]): the log structure is determined at the level of ghost sheaves by the underlying morphism, and the numerator counts the number of log liftings, while the denominator again clears out the ordering of the edges. There is already in Gathmann's proof a subtelty, that $\overline{\mathcal{M}}_{0,\alpha+e_k}(\mathbb{P}^N|H,d)$ on one side comes with a coefficient α_k+1, while on the other side it comes with an α_k from the comparison between ψ_k and $\mathrm{fgt}^*_{n+1,\ldots,n+h}\,\psi_k$. All the degeneracy loci that persist under pushforward cover $(h!:1)$ one of the comb loci in Gathmann's formula.

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3.2. **Definitions and propositions.** I come now to the definition of relative quasimaps [BN17, §2.3]. Let X be a smooth toric variety, $Y=V(s)$ a smooth very ample divisor (not necessarily toric). As in the discussion of functoriality abov, we may write $\mathcal{O}_X(Y)=\bigotimes_{\rho\in\Sigma(1)}\mathcal{O}_X(D_\rho)^{\otimes a_\rho}$ and

$$s=\sum_{\mathbf{a}:[\mathbf{a}]=\mathcal{O}_X(Y)}\mu_{\mathbf{a}}\prod_{\rho\in\Sigma(1)}z_\rho^{a_\rho}$$

in terms of Cox's homogeneous coordinates. For a quasimap

$$\Big((C,x_1,\ldots,x_n),(L_\rho,u_\rho)_{\rho\in\Sigma(1)},(\varphi_m)_{m\in M}\Big)$$

set $L_Y=\bigotimes_{\rho\in\Sigma(1)}L_\rho^{\otimes a_\rho}$ and

$$u_Y=\mu_{\mathbf{a}}\prod_{\rho\in\Sigma(1)}u_\rho^{a_\rho}+\sum_{\mathbf{a}\neq\mathbf{b}:[\mathbf{b}]=\mathcal{O}_X(Y)}\mu_{\mathbf{b}}\varphi_{\mathbf{b}-\mathbf{a}}\left(\prod_{\rho\in\Sigma(1)}u_\rho^{b_\rho}\right).$$

Definition 3.3. A quasimap belongs to the relative space $Q_{0,\alpha}(X|Y,\beta)\subseteq Q_{0,n}(X,\beta)$ if for every connected component Z of $s_Y^{-1}(0)$, the following holds:

(1) if Z is a point and is equal to a marked point x_i, then $s_Y^*(0)$ has order at least α_i at x_i;

(2) if Z is a curve, and if we let $C^{(i)}$ for $1 \leq i \leq r$ denote the irreducible components of C adjacent to Z and $m^{(i)}$ denote the multiplicity of $f|_{C^{(i)}}$ to H at the node $Z \cap C^{(i)}$, then:

$$\deg(L_{Y|Z}) + \sum_{i=1}^{r} m^{(i)} \geq \sum_{x_i \in Z} \alpha_i \tag{$*$}$$

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