ALTERNATIVE COMPACTIFICATIONS IN LOW GENUS GROMOV-WITTEN THEORY

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Contents

1. Reduced invariants and the Li-Zinger's formula

Contrary to the genus zero case, $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^N,d)$ is not a smooth stack; in fact it is not even equidimensional. A classical example is given by $\overline{\mathcal{M}}_{1,0}(\mathbb{P}^2,3)$:

- smooth planar cubics are elliptic curves by the degree-genus formula; viewing them as $E\hookrightarrow\mathbb{P}^2$ gives a component of dimension 9 of $\overline{\mathcal{M}}_{1,0}(\mathbb{P}^2,3)$, often referred to as the *main* component;
- a contracted elliptic curve attached to a rational tail that normalises a nodal cubic determines the generic point of a different component of dimension 10, which I am going to denote by $D^1(\mathbb{P}^2,3)$;
- finally a contracted genus one curve with two rational tails parametrising the union of a line and a quadric in $\mathbb{P}^2$ describes the generic element of yet another 9-dimensional component, that I shall denote by $D^2(\mathbb{P}^2,3)$.

Furthermore, the boundary of the main component admits a neat description: $\overline{\mathcal{M}}_{1,0}(\mathbb{P}^2,3)^{\mathrm{main}}\cap D^1(\mathbb{P}^2,3)$ consists of those maps where the rational tail normalises a cusp, and the elliptic curve is contracted exactly to the singular point (this has dimension 8, thus being a divisor in *main*); while $\overline{\mathcal{M}}_{1,0}(\mathbb{P}^2,3)^{\mathrm{main}}\cap D^2(\mathbb{P}^2,3)$ has the line tangent to the conic.

The description above generalises indeed to all moduli space of maps to $\mathbb{P}^N$ in genus one: besides the *main component*, which is the closure of the locus of maps from a smooth elliptic curve, for every positive integer k and partition $\lambda\vdash d$ into k positive parts, there is an irreducible *boundary component* $D^\lambda(\mathbb{P}^N,d)$ defined to be the closure of the locus where:

(i) the source curve is obtained by gluing a smooth k-pointed elliptic curve E with as many rational tails $R_i \cong \mathbb{P}^1,\ i = 1, \dots, k$,
(ii) the map contracts the elliptic curve E to a point,
(iii) and it has degree λ_i on the rational tail R_i.

Indeed $D^\lambda(\mathbb{P}^N, d)$ is the image of the gluing morphism from

$$\left(\overline{\mathcal{M}}_{1,k} \times \prod_{i=1}^{k} \overline{\mathcal{M}}_{0,1}(\mathbb{P}^N, \lambda_i)\right) \times_{(\mathbb{P}^N)^k} \mathbb{P}^N.$$

I will denote by D^k the union of all $D^\lambda(\mathbb{P}^N, d)$ where λ has k parts. Notice that an analogous description holds in the case of a positive number of markings, except that the combinatorial data should also include a partition $\mu \vdash n$ into $k+1$ parts (the 0-th of which telling how many points lie on E).

Proposition 1.1. Let the smallest connected arithmetic genus one subcurve go under the name of *core* or *circuit*.

(1) The ones above are all the irreducible components of $\overline{\mathcal{M}}_1(\mathbb{P}^N, d)$:

$$\overline{\mathcal{M}}_1(\mathbb{P}^N, d) = \overline{\mathcal{M}}_1(\mathbb{P}^N, d)^{\text{main}} \cup \bigcup_{\lambda} D_\lambda(\mathbb{P}^N, d).$$

(2) A map $[f]$ lies in *the boundary of the main component* if and only if:
- f is non-constant on at least one irreducible component of the core,
- or, if f contracts the core, writing $C = E\ {}_{\mathbf{p}}\sqcup_{\mathbf{q}} \bigsqcup_{i=1}^k R_i$ with E the *maximal* contracted subcurve genus one, then $\{\mathrm{d}f(T_{q_i}R_i)\}_{i=1}^k$ are *linearly dependent* in $T_{f(E)}\mathbb{P}^r$.

In this case we say that $[f]$ is smoothable.

This is essentially due to R. Vakil and A. Zinger, see [Vak00, Lemma 5.9] [VZ07] [Vis12, ???]. I shall later discuss a proof of the second fact based on local equations for the moduli space.

Let me carry the comparison to the genus zero situation one step further: assume we are interested in the Gromov-Witten theory of a complete intersection in $\mathbb{P}^N$, say a hypersurface X of degree l, cut out by a certain section $s \in H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(l))$. Letting $(\pi, f)\colon \mathcal{C}_{0,n}(\mathbb{P}^N, d) \to \overline{\mathcal{M}}_{0,n}(\mathbb{P}^N, d) \times \mathbb{P}^N$ be the universal curve and stable map, recall that there is an induced section $\tilde{s}$ of the sheaf $E = \pi_* f^* \mathcal{O}_{\mathbb{P}^N}(l)$ on $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^N, d)$, which vanishes along $\overline{\mathcal{M}}_{0,n}(X, d)$; in fact more is true: E is a vector bundle of rank $dl + 1$ (by cohomology and base-change, and a Riemann-Roch computation) and after shifting it provides us with a perfect obstruction theory for the inclusion $\iota\colon \overline{\mathcal{M}}_{0,n}(X, d) \hookrightarrow \overline{\mathcal{M}}_{0,n}(\mathbb{P}^N, d)$, so that the following result holds.

Proposition 1.2. [CKL01, KKP03] With notation as above,

$$\iota_*[\overline{\mathcal{M}}_{0,n}(X, d)]^{\text{vir}} = c_{dl+1}(E) \cap [\overline{\mathcal{M}}_{0,n}(\mathbb{P}^N, d)].$$

In particular, this result makes the restricted Gromov-Witten invariants of X into *twisted* Gromov-Witten invariants of projective space, which are computable e.g. by localisation [Kon95].

Again the situation in genus one is much more intricate: $\pi_* f^* \mathcal{O}_{\mathbb{P}^N}(l)$ generically has rank dl on the main component, but the rank jumps to $dl+1$ on the boundary, where the elliptic curve is contracted - as can be seen by constancy of the Euler characteristic and the fact that $\mathrm{R}^1 \pi_* f^* \mathcal{O}_{\mathbb{P}^N}(l)$, that always satisfies cohomology and base-change, is supported on such boundary loci. In other words, the natural obstruction theory for $\overline{\mathcal{M}}_{1,n}(X,d) \hookrightarrow \overline{\mathcal{M}}_{1,n}(\mathbb{P}^N,d)$ is not perfect.

A possible approach to this problem is the one taken by J. Li, R. Vakil and A.Zinger in a series of papers [**?**]: roughly, they produce a desingularisation $\mathcal{VZ}_{1,n}(\mathbb{P}^N,d)$ of the main component, on which the cone $\pi_* f^* \mathcal{O}_{\mathbb{P}^N}(l)$ is seen to contain a vector bundle E of rank dl, and for a hypersurface $X_l \subseteq \mathbb{P}^N$ as before (more generally for a projective complete intersection) they define *reduced invariants* by integrating against

$$[\iota_* \mathcal{VZ}_{1,n}(X,d)]^{\mathrm{red}} := c_{dl}(E) \cap [\mathcal{VZ}_{1,n}(\mathbb{P}^N,d)].$$

In particular reduced invariants may again be computed by torus localisation [Zin09, **?**]. Let me describe Vakil and Zinger's construction more precisely: it is an iterated blow-up that makes all the boundary components intersect main in a divisor (within the latter). Let $\mathfrak{M}_1^{\mathrm{wt}}$ denote the stack of prestable curves with a weight assignment, namely to every irreducible component in the fiber a non-negative integer is associated in a way compatible with specialisation morphisms: $\mathfrak{M}_1^{\mathrm{wt}}$ is étale, non-separated over $\mathfrak{M}_1$ and it was first considered in [**?**]; by an abuse of notation I will keep denoting by $\mathfrak{M}_1^{\mathrm{wt}}$ the open, bounded locus where the curve is weighted-stable, i.e. every rational component of weight zero has at least three special points. $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^N,d)$ admits a map to $\mathfrak{M}_1^{\mathrm{wt}}$ by retaining only the degree of the map. Let $\Theta_k \subseteq \mathfrak{M}_1^{\mathrm{wt}}$ denote the closure of the locus where the elliptic curve has weight zero, and k rational tails of positive weight attached to it. Define $\mathfrak{M}^{(0)} = \mathfrak{M}_1^{\mathrm{wt}}$ and $\mathfrak{M}^{(k)}$ iteratively as the blow-up of the strict transform of Θ_k in $\mathfrak{M}^{(k-1)}$. Notice that the blow-up loci are always smooth, hence so is $\mathfrak{M}^{(k)}$ for every k; also, by weighted-stability, for every fixed total degree d the procedure stops after finitely many steps: denote by $\widetilde{\mathfrak{M}}$ the end result. Form the cartesian diagram

$$\begin{tikzcd} \widetilde{\mathcal{VZ}}_{1,n}(\mathbb{P}^N,d) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{\mathcal{M}}_{1,n}(\mathbb{P}^N,d) \arrow[d] \\ \widetilde{\mathfrak{M}} \arrow[r] & \mathfrak{M}_1^{\mathrm{wt}} \end{tikzcd}$$

Now the pullback of any boundary component of $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^N,d)$ has the same dimension in $\widetilde{\mathcal{VZ}}_{1,n}(\mathbb{P}^N,d)$ and intersects its main component (which is denoted by $\mathcal{VZ}_{1,n}(\mathbb{P}^N, d)$ and is smooth [?]) in a Cartier divisor. Denoting $\mathcal{L} = \tilde{f}^*\mathcal{O}_{\mathbb{P}^N}(1)$ on the universal curve, notice that $\mathrm{R}^\bullet\,\pi_*\mathcal{L}$ may be resolved locally by picking a smooth section $\mathcal{A}$ of the universal curve passing through the core and writing:

$$0 \to \tilde{\pi}_*\mathcal{L} \to \tilde{\pi}_*(\mathcal{L} \otimes \mathcal{O}_C(\mathcal{A})) \xrightarrow{\mathrm{res}_{\mathcal{A}}} \tilde{\pi}_*(\mathcal{L}(\mathcal{A})_{|\mathcal{A}}) \to \mathrm{R}^1\,\tilde{\pi}_*\mathcal{L} \to 0$$

By restricting to $\mathcal{VZ}_{1,n}(\mathbb{P}^N, d)$ we see then that the image of the middle arrow is $\tilde{\pi}_*(\mathcal{L}(\mathcal{A})_{|\mathcal{A}})(-\Xi)$ where Ξ denotes the intersection of the main component of he Vakil-Zinger's desingularisation with the union of the boundary components, hence $\tilde{\pi}_*\mathcal{L}$ is a vector bundle (being expressed as the kernel of a vector bundle morphism). A similar argument works for all the tensor powers of $\mathcal{L}$, and in particular we let $E := \tilde{\pi}_*\mathcal{L}^{\otimes l}$. The previous discussion is not completely accurate, but can be made so by studying the arrow $\mathrm{res}_{\mathcal{A}}$ in local coordinates [HL10, ???].

It is clear from the construction above that reduced invariants shoud have a better enumerative meaning than ordinary Gromov-Witten invariants, in the sense that they discard most boundary contributions from nodal curves. In the realm of symplectic geometry, it has been proven by Li and Zinger [LZ07] that for every *primary* insertion $(\delta_1, \dots, \delta_n) \in H^*(X)^{\otimes n}$ and curve class $\beta \in H_2^+(X)$

$$\langle \delta_1, \dots, \delta_n \rangle^X_{1,n,\beta} - \langle \delta_1, \dots, \delta_n \rangle^{X,\mathrm{red}}_{1,n,\beta} = \begin{cases} 0 & \text{if } \dim(X) = 2, \\ \frac{2-K_X\cdot\beta}{24}\langle \delta_1, \dots, \delta_n \rangle^X_{0,n,\beta} & \text{if } \dim(X) = 3. \end{cases}$$

Li-Zinger's equation tells us that in the case of a threefold, which is the most natural one where to look for such a comparison result because the virtual dimension (hence the meaningful insertions) does not depend on the genus, the difference between ordinary and reduced genus one invariants is given by the corresponding genus zero invariant multiplied by a certain correction factor. The relation has been proved in algebraic geometry for the quintic threefold [CL15], and it is an adaptation of their work that I am going to discuss in the next few sections. Together with Cristina Manolache and Tom Coates I am also working towards a different proof of the formula: the key issue is to prove that the components of the intrinsic normal cone supported on the boundary components compare well (as in [Man12b]) to their genus zero relatives - or rather excluding those that do not contribute at all. I shall not attempt a detailed discussion here; I would rather work out the case of projective spaces of low dimension, which can be dealt with entirely by hand. Let me recall the following

Lemma 1.3. [Ful98, Proposition 1.8] Let $U \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} Z$ be complementary open and closed subvarieties. Then the following sequence is exact:

$$A_k(Z) \xrightarrow{i^*} A_k(X) \xrightarrow{j^*} A_k(U) \to 0$$

This means that, whenever we are looking for a virtual class of dimension d, we are welcome to discard any closed suscheme of dimension less than d. Furthermore cones and Gysin pullbacks behave well under restriction to open subsets [Ful98, Proposition 4.2(b) and Theorem 6.2(b)]. Also $\widetilde{\mathcal{VZ}}_{1,n}(\mathbb{P}^N,d) \to \overline{\mathcal{M}}_{1,n}(\mathbb{P}^N,d)$ is virtually birational, and all insertions come from downstairs, so we do not really need to work on the Vakil-Zinger desingularisation if we do not want to.

Let me start by loooking at $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^1,d)$. Except for $d=1$, where the main component is empty, there are $n+2$ components: the main one and a boundary component for every partition of n into two parts, which is the image of gluing

$$\overline{\mathcal{M}}_{1,k+1} \times \overline{\mathcal{M}}_{0,n-k+1}(\mathbb{P}^1,d).$$

All components have the same dimension, which is also equal to the virtual dimension; this in particular means that we can discard all the intersections, thus every component is smooth. Notice that from the exact triangle

$$E^{\bullet}_{\overline{\mathcal{M}}(\mathbb{P}^1)/\mathfrak{M}} \to E^{\bullet}_{\overline{\mathcal{M}}(\mathbb{P}^1)} \to \rho^* T^{\bullet}_{\mathfrak{M}} \xrightarrow{[1]}$$

the relative obstruction space $\mathbb{E}^\vee \boxtimes \mathrm{ev}_q^* T_{\mathbb{P}^1}$ - where $\mathbb{E}$ is the Hodge bundle and q is the gluing node - cancels out with the normal bundle of the boundary divisor in the moduli space of curves, which is $\mathbb{L}_{q,E}^\vee \boxtimes \mathbb{L}_{q,R}^\vee$: notice that the difference between λ_1 and ψ_1 (when there are markings on the elliptic tail) is supported precisely on the intersections between two boundary components, while $\mathrm{d}f_q$ gives an isomoprhism $\mathbb{L}_{q,R}^\vee \xrightarrow{\sim} \mathrm{ev}_q^* T_{\mathbb{P}^1}$, except when the gluing node is a ramification point for the map, which happens at the intersection with main by Proposition 1.1. This shows that the virtual class is the sum of the fundamental classes of all the components. Clearly no boundary component will contribute to the primary invariants. On the other hand it is also clear that any descendant invariant will determine (at most) one boundary component contributing to it.

The case of $\mathbb{P}^2$ is slightly more complicated: the boundary component D^1 has excess dimension 1, and two irreducible components $D^{\lambda=(d),\mu=(A_0,A_1)}$ and $D^{\lambda,\mu'}$ intersect if and only if $A_0 \subseteq A_0'$ and $A_1' \subseteq A_1$, or viceversa; in this case the dimension of the intersection is equal to the virtual dimension. Finally D^2 also has dimension equal to the virtual one. Notice that the restriction of the intrinsic normal cone to $D^1 \setminus \overline{\mathcal{M}}_{1,n}(\mathbb{P}^2,d)^{\mathrm{main}}$ is pulled back from the boundary divisor in $\mathfrak{M}_{1,n}^{\mathrm{wt}}$, hence it is a line bundle and its components are in bijection with those of D^1; we may therefore study it componentwise. It is an easy Chern class computation that

$$c_1\left(\frac{\mathbb{E}^\vee \boxtimes \mathrm{ev}_q^* T_{\mathbb{P}^2}}{\mathbb{L}_{q,E}^\vee \boxtimes \mathbb{L}_{q,R}^\vee}\right) = 3\cdot 1 \boxtimes \mathrm{ev}_q^* H + 2\cdot\lambda_1 \boxtimes 1 - \psi \boxtimes 1 - 1 \boxtimes \psi$$

It follows from arguments similar to the above ones that D^2 contributes with its fundamental class. We may wrap it all up in the following formula:

$$
\begin{aligned}
&\langle \tau_{h_1}(H^{k_1}),\dots,\tau_{h_n}(H^{k_n})\rangle^{\mathbb{P}^2}_{1,n,d} = \langle \tau_{h_1}(H^{k_1}),\dots,\tau_{h_n}(H^{k_n})\rangle^{\mathbb{P}^2,\mathrm{red}}_{1,n,d} + \\
&\sum_{A_0\coprod A_1=[n]}\Big(\langle \psi^{h^0_1},\dots,\psi^{h^0_{n_0}},1\rangle^{\{*\}}_{1,A_0\cup\{q_E\}}\langle 3H^{\sum_{i\in A_0}k_i+1}-\tau_1(H^{\sum_{i\in A_0}k_i}),\tau_{h^1_1}(H^{k^1_1}),\dots,\tau_{h^1_{n_1}}(H^{k^1_{n_1}})\rangle^{\mathbb{P}^2}_{0,\{q_R\}\cup A_1,d} \\
&\quad +\langle \psi^{h^0_1},\dots,\psi^{h^0_{n_0}},2\lambda_1-\psi\rangle^{\{*\}}_{1,A_0\cup\{q_E\}}\langle H^{\sum_{i\in A_0}k_i},\tau_{h^1_1}(H^{k^1_1}),\dots,\tau_{h^1_{n_1}}(H^{k^1_{n_1}})\rangle^{\mathbb{P}^2}_{0,\{q_R\}\cup A_1,d}\Big)+ \\
&\sum_{\substack{A_0\coprod A_1\coprod A_2=[n]\\ d_1+d_2=d:\ d_1,d_2>0}}\sum_{j=0}^{2}\langle \psi^{h^0_1},\dots,\psi^{h^0_{n_0}},1,1\rangle^{\{*\}}_{1,n_0+2}\langle H^{j+\sum_{i\in A_0}k_i},\tau_{h^1_1}(H^{k^1_1}),\dots,\tau_{h^1_{n_1}}(H^{k^1_{n_1}})\rangle^{\mathbb{P}^2}_{0,1+n_1,d_1}\langle H^{2-j},\tau_{h^2_1}(H^{k^2_1}),\dots,\tau_{h^2_{n_2}}(H^{k^2_{n_2}})\rangle^{\mathbb{P}^2}_{0,1+n_2,d_2}
\end{aligned}
$$

Notice in particular that ordinary and reduced invariants coincide for primary insertions (using the string equation).

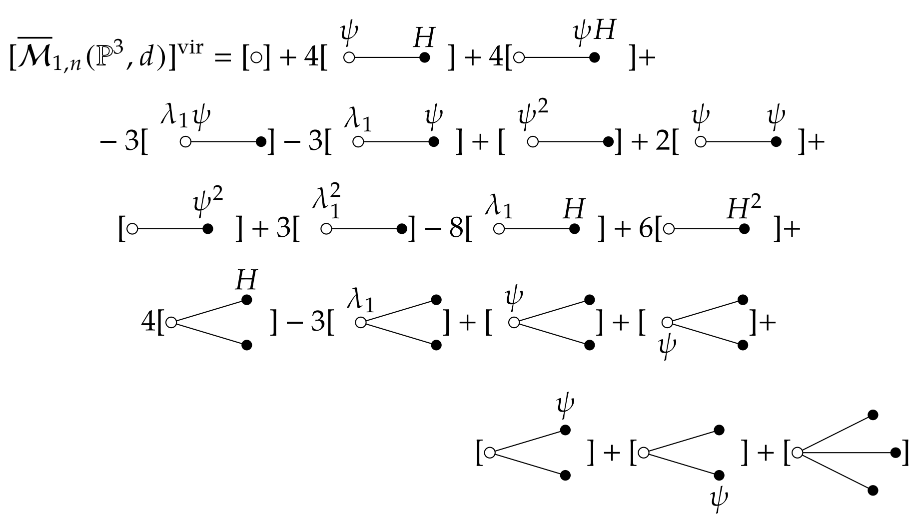
Analogous computations work for $\mathbb{P}^3$: D^1 has excess dimension two and the obstruction bundle is

$$\frac{\mathbb{E}^\vee \boxtimes \mathrm{ev}_q^* T_{\mathbb{P}^3}}{\mathbb{L}^\vee_{q,E}\boxtimes\mathbb{L}^\vee_{q,R}}$$

D^2 has excess dimension one and obstructions

$$\frac{\mathbb{E}^\vee \boxtimes \mathrm{ev}_q^* T_{\mathbb{P}^3}}{\mathbb{L}^\vee_{q_1,E}\boxtimes\mathbb{L}^\vee_{q,R_1}\oplus\mathbb{L}^\vee_{q_2,E}\boxtimes\mathbb{L}^\vee_{q,R_2}}$$

while D^3 has dimension equal to the virtual dimension. A Chern class computation implies (schematic notation suggested by N. Nabijou):



If we restrict our attention to primary invariants, the only surviving terms are:

$$[\circ] + 4[\overset{\psi}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{H}{\bullet}] - 3[\overset{\lambda_1\psi}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\bullet] - 3[\overset{\lambda_1}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{\psi}{\bullet}] +$$
$$[\overset{\psi^2}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\bullet] + 2[\overset{\psi}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{\psi}{\bullet}] + 3[\overset{\lambda_1^2}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\bullet] - 8[\overset{\lambda_1}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{H}{\bullet}]$$

We may then compute the following contributions:

(1) gives the reduced invariants,
(2) $4\langle\psi\rangle_{1,1}\langle H,-\rangle^{\mathbb{P}^3}_{0,n+1,d} = \frac{4d}{24}GW_0$ by divisor,
(3) $-3\langle\lambda_1\psi,1\rangle_{1,2}\langle-\rangle^{\mathbb{P}^3}_{0,n,d} = \frac{-3n}{24}GW_0$ since there are n choices for the marking on the genus one curve,
(4) $-3\langle\lambda_1\rangle_{1,1}\langle\psi,-\rangle^{\mathbb{P}^3}_{0,n+1,d} = \frac{-3(n-2)}{24}GW_0$ by dilaton,
(5) $\langle\psi^2,1\rangle_{1,2}\langle-\rangle^{\mathbb{P}^3}_{0,n,d} = \frac{n}{24}GW_0$,
(6) $2\langle\psi\rangle_{1,1}\langle\psi,-\rangle^{\mathbb{P}^3}_{0,n+1,d} = \frac{2(n-2)}{24}GW_0$ by dilaton,
(7) vanishes since $\langle\lambda_1^2,1\rangle_{1,2} = 0$,
(8) $-8\langle\lambda_1\rangle_{1,1}\langle H,-\rangle^{\mathbb{P}^3}_{0,n+1,d} = \frac{-8d}{24}GW_0$ by divisor.

Summing up we obtain the following:

Proposition 1.4 (Li-Zinger formula for $\mathbb{P}^3$)**.**

$$\langle\delta_1,\ldots,\delta_n\rangle^{\mathbb{P}^3}_{1,n,d} = \langle\delta_1,\ldots,\delta_n\rangle^{\mathbb{P}^3,\mathrm{red}}_{1,n,d} + \frac{2-4d-3n}{24}\langle\delta_1,\ldots,\delta_n\rangle^{\mathbb{P}^3}_{0,n,d}.$$

Notice that this is slightly different from Li-Zinger's original formula. An alternative approach would be comparing every boundary component to its genus zero relative by virtual pushforward: if we are only interested in primary insertions, notice that the components of D^1 such that more than one marking dwells on the elliptic curve will not contribute, since the excess bundle has rank two.

2. Genus one singularities

A different approach to reducing the complexity of $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^N,d)$ is the one followed by M. Viscardi in [Vis12], building on work of D.I. Smyth on the minimal model program for $\overline{\mathcal{M}}_{1,n}$ [Smy11a]. Rather than blowing up and desingularising the main component, the idea is to collapse a number of boundary components, only keeping their intersection with main and the boundary components of smaller dimension. Vakil's description of the smoothable elements of $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^N,d)$ (see Proposition 1.1(2)) suggests to do so by allowing maps from more singular (than nodal) curves, and simultaneously making their semistable models unstable, in order to preserve the separatedness of the moduli space.

The easiest example is the following: the cusp

$$\Bbbk[\![x, y]\!]/(y^2 - x^3)$$

is the only unibranch singularity of genus one (meaning that there exists a flat family of smooth elliptic curves degenerating to an irreducible curve of geometric genus 0 and with only one singular point, around which the curve is formally isomorphic to the spectrum of the ring above). It is a well-known computation [HM98, §3.C] that the semistable reduction of the cusp has a rational tail (normalising the singularity) attached to an elliptic curve at the preimage of the singular point; this indicates that we should make curves of genus one with only one special point unstable. On the other hand the only singularities appearing in the fibers of the miniversal deformation of the cusp are nodes, hence for a curve being *at worst cuspidal* (i.e. smooth, nodal or cuspidal) is an open condition. Consider then the following:

Definition 2.1. The moduli space of 1-stable maps $\overline{\mathcal{M}}_{1,n}^{(1)}(X, \beta)$ parametrises $f\colon (C, \mathbf{p}) \to X$ such that

(1) $(C, p_1, \ldots, p_n)$ is at worst cuspidal, of arithmetic genus one, and $\mathbf{p}$ is an n-tuple of smooth and disjoint sections;
(2) if C_0 is a minimal connected subcurve of C contracted by f, the number of markings on C_0 added to the number of intersections of C_0 with $\overline{C \setminus C_0}$ is at least 3 if $p_a(C) = 0$ and *at least* 2 if $p_a(C) = 1$;
(3) $f_*[C] = \beta \in H_2^+(X)$ and $\mathrm{Aut}(C, f)$ is finite.

Lemma 2.2. $\overline{\mathcal{M}}_{1,n}^{(1)}(X, \beta)$ is a proper DM stack of finite type over $\Bbbk$.

For example $\overline{\mathcal{M}}^{(1)}(\mathbb{P}^2, 3)$ has only two components, since D^1 is gone! On the other hand the intersection of D^1 with main and with D^2 has been filled up by maps from cuspidal curves. This is the main instance of Smyth-Viscardi's spaces that I am going to be concerned with, but there is a well-developed theory which I shall quickly review here.

Definition 2.3. Let (C, p) be the germ of a curve singularity, with normalisation $\nu\colon \tilde{C} \to C$. Let me denote by m the number of branches of C at p (i.e. irreducible components of $\tilde{C}$) and by $\delta = \dim_{\Bbbk}(\nu_*\mathcal{O}_{\tilde{C}}/\mathcal{O}_C)$. The *genus* of (C, p) is defined by

$$g = \delta - m + 1.$$

Proposition 2.4. [Smy11a, Proposition A.3] There is up to isomorphism only one germ of Gorenstein curve singularity (C, p) of genus one with m branches, namely

$$\hat{\mathcal{O}}_{C,p} \simeq \begin{cases} \Bbbk[\![x, y]\!]/(y^2 - x^3) & \text{(cusp) if } m = 1, \\ \Bbbk[\![x, y]\!]/y(y - x^2) & \text{(tacnode) if } m = 2, \end{cases}$$

and the germ of m generic lines through the origin in $\mathbb{A}^{m-1}$ for $m \geq 3$. It is called the *elliptic m-fold point*.

All these singularities are smoothable (see the proof of [Smy11a, Theorem 3.8]). Smyth studied their semistable models: pick a smoothing $\bar{\mathcal{C}}$ of the m-fold elliptic point and consider its semistable model with *regular* total space $\tilde{\mathcal{C}}$; let $(E, q_1, \ldots, q_m)$ be the exceptional locus of the contraction (the fiber over the m-fold elliptic point) marked with its intersection with the rest of the curve (m rational trees). Call a semistable genus one curve $(E, q_1, \ldots, q_m)$ a *semistable tail* if it arises in this way. Smyth shows [Smy11a, Proposition 2.12] that semistable tails can be characterised combinatorially as those curves such that the distance (on the dual graph) from q_i to the core of E is constant for all $i = 1, \ldots, m$ (*balancing condition*). The idea is that $\bar{\mathcal{C}}$ is a normal surface, and it is Gorenstein if and only if the central fiber is. In this case, denoting by $\phi \colon \tilde{\mathcal{C}} \to \bar{\mathcal{C}}$ the contraction, $\phi^*\omega_{\bar{\mathcal{C}}} = \omega_{\tilde{\mathcal{C}}}(D)$ for some divisor D supported on the exceptional locus E and such that $\omega_{\tilde{\mathcal{C}}}(D)_{|E} \simeq \mathcal{O}_E$; we can find appropriate weights on the components of E such that this holds precisely in the balanced case. On the other hand, if balancing holds, we may contract E by applying the Proj construction to $\omega_{\tilde{\mathcal{C}}}(D)$ (possibly twisted by some sections of $\tilde{\mathcal{C}}$ far away from E) as above, and then show that the resulting $\bar{\mathcal{C}}$ is Gorenstein by relating its dualising sheaf to the $\mathcal{O}_{\bar{\mathcal{C}}}(1)$ from the Proj construction. The Smyth singularity one gets when contracting a balanced curve is determined by the number of rational trees attached from it, i.e. the number of branches of the singularity itself. These singularities have no moduli (on the other hand the global configuration of the curve may have moduli, i.e. the positioning of special points). This discussion indicates what are the stable curves/maps that we should make unstable in order to keep the moduli space separated. Before passing to maps, let me make the following:

Remark 2.5. The operation of collapsing elliptic tails to cusps (1-stabilisation) works well in families (in fact, even when the elliptic tail is a component of a curve of higher genus [Sch91]). On the other hand, (m)-stabilisation is not well-defined. Here is an example from [BCM18, Remark 4.12]: the

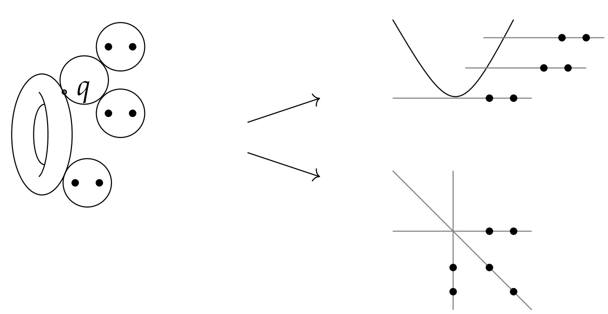


Figure 1. Two different plausible 3-stabilisations.

point is that the maximal unpointed genus one subcurve is not balanced; on the other hand, say we could consistently contract the minimal genus one subcurve: by smoothing the node q and doing so, we would get a family of 3-fold elliptic points specialising to the tacnode, which can be excluded by studying the miniversal deformation of the latter. This problem is studied in [Smy11b, §4.1]. I will discuss a specific variation [BCM18, Theorem 4.4] that I am going to need later on.

Proposition 2.6. There exists a morphism $\mathfrak{M}_1^{\mathrm{wt}=d,\mathrm{st}} \to \mathfrak{M}_1^{\mathrm{wt}=d,\mathrm{st}}(1)$ which extends the identity on the smooth locus.

Recall that $\mathfrak{M}_1^{\mathrm{wt}=d,\mathrm{st}}$ is the open and bounded substack within the Artin stack of prestable curves with a weight assignment, defined by the conditions that the total weight is d and the curve is weighted-stable; similarly $\mathfrak{M}_1^{\mathrm{wt}=d,\mathrm{st}}(1)$ is the stack of weighted 1-stable (at worst cuspidal) curves.

The proof is inspired by [HH09, §2] and [RSW17, §3.7]. We construct the contraction over $\mathfrak{M}_1^{\mathrm{div}}$ (the moduli space of nodal curves of genus one with a simple smooth divisor of total degree d that makes them weighted-stable) first, and then show that it descends to $\mathfrak{M}_1^{\mathrm{wt,st}}$. We do this in order to have a natural polarisation.

Let $\mathcal{E}$ be the locus inside the universal curve over $\mathfrak{M}_1^{\mathrm{wt,st}}$ spanned by elliptic tails of weight 0; this is a Cartier divisor, and abusing notation I will denote by $\mathcal{E}$ all its pullbacks. Moreover let $\mathfrak{D}^1$ be its image in $\mathfrak{M}_1^{\mathrm{wt,st}}$, which is a Cartier divisor as well.

Consider the following line bundle on the universal curve over $\mathfrak{M}_1^{\mathrm{div}}$:

$$\mathcal{N} := \omega_\pi(\mathcal{E}) \otimes \mathcal{O}_{\mathcal{C}}(2\mathcal{D}), \tag{1}$$

where $\mathcal{D}$ is the universal Cartier divisor over $\mathfrak{M}_1^{\mathrm{div}}$.

Proposition 2.7. Let $\overline{\mathcal{C}} = \underline{\mathrm{Proj}}_{\mathfrak{M}_1^{\mathrm{div}}}(\bigoplus_{n\geq 0} \pi_*\mathcal{N}^{\otimes n})$. Then $\overline{\mathcal{C}}$ is a family of weighted 1-stable curves and ϕ is a regular morphism:

$$\begin{tikzcd} (\mathcal{C},\mathcal{D}) \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & (\overline{\mathcal{C}},\phi(\mathcal{D})) \arrow[dl, "\bar{\pi}"] \\ & \mathfrak{M}_1^{\mathrm{div}} & \end{tikzcd}$$

This defines the 1-stabilisation morphism $\mathfrak{M}_1^{\mathrm{div}} \to \mathfrak{M}_1^{\mathrm{div}}(1)$.

Notice that $\mathcal{N}$ is trivial on the locus of elliptic tails, so this will be contracted by ϕ. We need to prove that $\mathcal{N}$ is π-semiample (regularity of ϕ) and that $\pi_*\mathcal{N}$ is locally free (flatness of $\bar{\pi}$). This is clear on the smooth locus, but to prove it along $\mathfrak{D}^1$ we use [RSW17, Lemmma 3.7.2.2].

Lemma 2.8 (Pullback with a boundary)**.** Let $\pi\colon \mathcal{C} \to S$ be a proper family of curves over a smooth basis, and let $\mathcal{N}$ be a line bundle on $\mathcal{C}$ such that $\mathrm{R}^1\,\pi_*\mathcal{N}$ is a line bundle supported on a Cartier divisor $\mathfrak{D} \subseteq S$. Then for every DVR scheme Δ with closed point 0 and generic point η, and for every morphism $f\colon \Delta \to S$ such that $f(0) \in \mathfrak{D}$ and $f(\eta) \in S \setminus \mathfrak{D}$ we have

$$f^*\pi_*\mathcal{N} \cong \pi_{\Delta,*}f_C^*\mathcal{N}.$$

Lemma 2.9. The line bundle $\mathcal{N}$ is π-semi-ample, i.e. the natural map

$$\pi^*\pi_*\mathcal{N}^{\otimes n} \to \mathcal{N}^{\otimes n}$$

is surjective for $n \gg 0$.

Proof. Outside the locus of elliptic tails $\mathcal{N}$ is π-ample. We are left with checking at points of an elliptic tail; thanks to the above Lemma we can reduce to the case that C is the central fiber of a one-parameter smoothing with regular total space. This is proved in Smyth's contraction lemma [Smy11a, Lemma 2.12]. □

Lemma 2.10. $\pi_*\mathcal{N}$ is locally free on $\mathfrak{M}_1^{\mathrm{div}}$.

Proof. Compare with [RSW17, Proposition 3.7.2.1]. We check that $\pi_*\mathcal{N}$ has constant rank. On $\mathfrak{M}_1^{\mathrm{div}} \setminus \mathfrak{D}^1$, $\mathrm{R}^1\,\pi_*\mathcal{N} = 0$, so $\pi_*\mathcal{N}$ satisfies Cohomology and Base Change [Har77, Theorem III.12.11] and its rank is determined by Riemann-Roch, hence constant. Given a point x on the boundary $\mathfrak{D}^1$, we can pick a one-parameter smoothing as above, and we can check the rank at x by looking at $\pi_*f^*\mathcal{N}$ over Δ. Now $f^*\mathcal{N}$ is flat over Δ, so $\pi_*f^*\mathcal{N}$ is as well, which implies torsion-free and thus constant rank. □

Proof. 2.7 Let $S \to \mathfrak{M}_1^{\mathrm{div}}$ be a smooth atlas, then we have:

$$\begin{tikzcd}
(\mathcal{C}_S, \mathcal{D}_S) \arrow[rr, "\phi_S"] \arrow[dr, "\pi_S"'] & & (\overline{\mathcal{C}}_S, \phi(\mathcal{D}_S)) \arrow[dl, "\bar{\pi}_S"] \\
& S &
\end{tikzcd}$$

where $\overline{\mathcal{C}}_S = \underline{\mathrm{Proj}}_S(\bigoplus_{n\geq 0}\pi_{S,*}\mathcal{N}^{\otimes n})$, ϕ_S is a proper and birational, and $\bar{\pi}_S$ is a flat family. To verify that this defines a morphism $S \to \mathfrak{M}_1^{\mathrm{div}}(1)$ we have to argue that $\overline{\mathcal{C}}_S$ has reduced fibers and only nodes and cusps as singularities. After pulling back to a generic Δ, this is again Smyth's contraction lemma [Smy11a, Lemma 2.13]. $\mathfrak{D}$ is pushed forward along ϕ and it satisfies weighted-stability, since ϕ is an isomorphism outside of the locus of elliptic tails. To conclude that this defines a morphism $\mathfrak{M}_1^{\mathrm{div}} \to \mathfrak{M}_1^{\mathrm{div}}(1)$ it is enough to verify that there is an isomorphism $\mathrm{pr}_1^*\overline{\mathcal{C}}_\mathrm{S} \cong \mathrm{pr}_2^*\overline{\mathcal{C}}_\mathrm{S}$ satisfying the cocycle condition, where $\mathrm{pr_i}\colon \mathrm{S}' = \mathrm{S} \times_{\mathfrak{M}_1^{\mathrm{div}}} \mathrm{S} \rightrightarrows \mathrm{S}$.

Now $\mathrm{pr}_i^*\overline{\mathcal{C}}_S$ is obtained by applying the Proj construction to $\mathrm{pr}_\mathrm{i}^*(\pi_{\mathrm{S},*}\mathcal{N}) \cong \pi_{\mathrm{S}',*}(\mathrm{pr}_\mathrm{i}^*\mathcal{N})$, which are isomorphic because $S' \to S$ is flat. On the other hand $\mathrm{pr}_1^*\mathcal{N} \cong \mathrm{pr}_2^*\mathcal{N}$ since $\mathcal{N}$ is the pullback of a line bundle on $\mathfrak{M}_1^{\mathrm{div}}$. The cocycle condition is derived similarly. □

Lemma 2.11. The 1-stabilisation for curves with a divisor induces an analogous morphism at the level of weighted curves:

$$\begin{tikzcd} \mathfrak{M}_1^{\mathrm{div}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathfrak{M}_1^{\mathrm{div}}(1) \arrow[d] \\ \mathfrak{M}_1^{\mathrm{wt,st}} \arrow[r, "\exists"] & \mathfrak{M}_1^{\mathrm{wt,st}}(1) \end{tikzcd}$$

Proof. Étale locally on $\mathfrak{M}_1^{\mathrm{wt,st}}$ we can choose smooth sections s_i of the universal curve so that the Cartier divisor $\mathcal{D} = \sum s_i$ has degree compatible with the weight function, so in particular it makes $\mathcal{N} = \omega_\pi(\mathcal{E}) \otimes \mathcal{O}_C(2\mathcal{D})$ trivial on the elliptic tails and π-ample elsewhere. For a smooth atlas $S \to \mathfrak{M}_1^{\mathrm{wt,st}}$, this observation allows us to define a lifting $S \to \mathfrak{M}_1^{\mathrm{div}}$, and thus a morphism $\xi\colon S \to \mathfrak{M}_1^{\mathrm{wt,st}}(1)$ through the construction of Proposition 2.7.

In order to show that this descends to a morphism $\mathfrak{M}_1^{\mathrm{wt,st}} \to \mathfrak{M}_1^{\mathrm{wt,st}}(1)$ we need to verify that there exists $\mathrm{pr}_1^*(\xi) \cong \mathrm{pr}_2^*(\xi)$ satisfying the cocycle condition, where $\mathrm{pr_i}\colon \mathrm{S}' = \mathrm{S} \times_{\mathfrak{M}_1^{\mathrm{wt,st}}} \mathrm{S} \rightrightarrows \mathrm{S}$.

This boils down to checking that for two different choices of a lifting $\mathcal{D}_1, \mathcal{D}_2\colon S \to \mathfrak{M}_1^{\mathrm{div}}$ there exists a unique isomorphism

$$\overline{C}_1 = \underline{\mathrm{Proj}}_S\left(\bigoplus_{n\geq 0} \pi_*(\mathcal{N}_1))^{\otimes n}\right) \cong \underline{\mathrm{Proj}}_S\left(\bigoplus_{n\geq 0} \pi_*(\mathcal{N}_2))^{\otimes n}\right) = \overline{C}_2.$$

By construction there is a birational map ψ:

$$\begin{tikzcd} & C_S \arrow[dl, "\phi_1"'] \arrow[dr, "\phi_2"] & \\ \overline{C}_1 \arrow[rr, dashed, "\psi"] & & \overline{C}_2. \end{tikzcd}$$

We want to show that ψ extends to a regular morphism. Notice that $\overline{C}_i$ is normal, $i = 1, 2$: indeed since S is smooth and the singularities of the fibers are in codimension 1, $\overline{C}_i$ is regular in codimension 1; since both S (smooth) and the fibers (Cohen-Macaulay) satisfy Serre's condition S_2, so does the total space of $\overline{C}_i$ by [**?**, Thorem 23.9]. By Zariski's connectedness theorem $\phi_{i,*}\mathcal{O}_{C_S} \cong \mathcal{O}_{\overline{C}_i}$. By construction $\mathrm{Exc}(\phi_1) = \mathrm{Exc}(\phi_2)$ is the locus of elliptic tails of weight 0, so in particular ϕ_2 contracts all the fibers of ϕ_1. Then [**?**, Lemma 1.15] implies that ϕ_2 factors through ϕ_1, and viceversa. This proves the regularity of ψ and its inverse. Notice that ψ is unique as it is the only extension of $\phi_2 \circ \phi_1^{-1}$. □

This concludes the proof of Proposition 2.6. I finally come to M. Viscardi's definition of alternative compactification of the space of maps [Vis12, Definition 2.15].

Definition 2.12. The moduli space of m-stable maps $\overline{\mathcal{M}}_{1,n}^{(m)}(X,\beta)$ parametrises $f\colon (C,\mathbf{p}) \to X$ of class β such that:

(1) C has only nodes and l-fold elliptic points as singularities, with $l \leq m$, $p_a(C) = 1$ and $p_1, \dots, p_n$ are smooth and disjoint sections,
(2) if $E \subseteq C$ is a connected subcurve contracted by f, and $p_a(E) = 1$, then the *level* is

$$|\{i \in [n] :\ p_i \in E\}| + |E \cap \overline{C \setminus E}| > m,$$

(3) $|\mathrm{Aut}(C, f)| < +\infty$.

Notice that the usual Behrend-Fantechi's construction [BF97, Proposition 6.2] endows $\overline{\mathcal{M}}_{1,n}^{(m)}(X,\beta)$ of the usual dimension. Setting $m = 0$ recovers the usual space of Kontsevich's stable maps.

Proposition 2.13. [Vis12, Theorem 3.6] $\overline{\mathcal{M}}_{1,n}^{(m)}(X,\beta)$ is a proper DM stack of finite type over $\Bbbk$.

Checking the valuative criterion for properness goes as follows: assume the generic fiber is smooth (in general we may work componentwise on the generic fiber); we may first complete the family over Δ as an ordinary stable map. If f contracts the core and it does not satisfy m-stability, we can make it into doing so by a sequence of operations, alternating between contracting the core to a Smyth's singularity and *sprouting*, i.e. blowing up at markings or nodes along the core. Notice that more sprouting may be required in order for the map to descend to the singularity (see [BCM18, Remark 2.6]). As anticipated, Viscardi's moduli space of m-stable maps to $\mathbb{P}^N$ "collapses" the boundary components $D_1^k(\mathbb{P}^N, d)$ for $k \leq m$ (the pointed case is subtler, as usual), hence the following result [Vis12, Corollary 5.10].

Proposition 2.14. For $m \geq m_0(d,n)$, $\overline{\mathcal{M}}_{1,n}^{(m)}(\mathbb{P}^N, d)$ is irreducible.

Finally, I would like to discuss how these two lines of thought (Li-Vakil-Zinger desingularisation and reduced invariants, compared to Smyth-Viscardi alternative compactifications) are not unrelated. A main issue with the Vakil-Zinger desingularisation is that the iterated blow-up procedure makes the modular interpretation of the resulting space not immediately clear. This has recently been fixed by D. Ranganathan, K. Santos-Parker and J. Wise with the introduction of the notion of an *aligned* log curve. Recall the description of log smooth curves, due to F. Kato[Kat00], and the parallel with pointed prestable curves. Let us work over a geometric point $S = \mathrm{Spec}(\Bbbk = \bar{\Bbbk})$. In the genus one case, we can modify the usual dual graph construction by collapsing all the vertices corresponding to components of the core (in case the latter is a circle of $\mathbb{P}^1$) to one vertex, called the *circuit* and denoted by $\circ$. Define an $\overline{\mathcal{M}}_S$-valued function on the dual graph by

$$\lambda(v) = \sum_{e \in [\circ, v]} \rho_e,$$

i.e. by associating to a vertex/component the sum of all the smoothing parameters of the edges/nodes separating it from the circuit.

Definition 2.15. Let S be a log scheme and $C \to S$ a log smooth curve of genus one on it. We say that C is *radially aligned* if for every geometric point $s \in S$ the values of $\lambda(v)$ are *comparable* in $\overline{\mathcal{M}}_{S,s}$ for every vertex v of $\Gamma(C_s)$.

There exist *minimal* radially aligned log structures, hence the moduli problem over $(LogSch)$ is the enhancement of a moduli stack over (Sch) endowed with a log structure, after work of D. Gillam [Gil12]. At this level

$$\mathfrak{M}_{1,n}^{\mathrm{rad}} \to \mathfrak{M}_{1,n}$$

is a log modification, due to the key observation that a log blow-up along a log ideal K determines on every one of its charts a minimal element among the generators of K, hence a sequence of log blow-ups will serve the goal of ordering a collection of sections of $\overline{\mathcal{M}}_S$ [Par17, Lemma 3.36]. Pictorially, it corresponds to a subdivision of the dual minimal monoid (see [RSW17, §3.3-3.4]). On the other hand, for every *stable* radially aligned curve over a geometric point S, and for every integer $m \geq 0$, we may find a section δ_m of $\overline{\mathcal{M}}_S$ such that $\delta_m = \lambda(v)$ for some vertex of $\Gamma(C)$ and the circle of radius δ_m has inner valence less than m and outer valence strictly larger than m. δ_m behaves well under specialisation/generisation, hence it can be defined over any base. The power of log structures is two-fold at this point:

(1) it produces a log-modification $\tilde{C} \to C$ by subdividing the edges where they meet the δ_m-radius circle (i.e. blowing up some nodes and markings on the components with $\lambda(v) < \delta_m$);
(2) by combining λ and δ_m, it produces a $\tilde{\pi}$-semiample line bundle on $\tilde{C}$, the morphism associated to which contracts the strict interior of the δ_m-radius circle, producing a diagram as follows, with $\bar{\pi}\colon \bar{C} \to S$ an m-stable Smyth's curve.

$$\begin{tikzcd} & \tilde{C} \arrow[dl] \arrow[dr] & \\ C \arrow[dr, "\pi"] & & \bar{C} \arrow[dl, "\bar{\pi}"] \\ & S & \end{tikzcd}$$

Remark 2.16. It may seem that this construction sometimes happens to contract unbalanced curves, but it is not the case; notice that the blow-ups in point (1) above may occur along non-reduced centers. In fact, the strict interior of a circle around the circuit is the correct generalisation of a balanced elliptic tail when $C \to S$ is not just a one-parameter smoothing with regular total space.

Finally notice that the above construction in the stable case provides a resolution of indeterminacy of the birational map between different Smyth's compactifications:

$$\begin{tikzcd} & \overline{\mathcal{M}}_{1,n}^{\mathrm{rad}} \arrow[dl] \arrow[dr] & \\ \overline{\mathcal{M}}_{1,n} \arrow[rr, dashed] & & \overline{\mathcal{M}}_{1,n}^{(m)} \end{tikzcd}$$

More to the point, there is an extension of this construction to the realm of maps.

Definition 2.17. A *centrally aligned map* $f\colon (C,\mathbf{p}) \to X$ over S is a log morphism (where X has the trivial log structure) with a section δ_0 of $\overline{\mathcal{M}}_S$, such that $\delta_0 = \lambda(v)$ for v of minimal distance to the circuit among the non-contracted components, $\lambda(w)$ is comparable with δ_0 for every w, and the $\lambda(w)$ are comparable with one another whenever they are less than δ_0.

The section δ_0 together with λ defines a modification $\tilde{C}$ and a contraction to $\bar{C}$ with a Smyth's singularity as aboce.

Definition 2.18. A centrally aligned map satisfies the *factorisation property* if its pullback to $\tilde{C}$ descends to $\bar{C}$:

$$\begin{tikzcd} & \tilde{C} \arrow[dl] \arrow[dr] & \\ C \arrow[dr, "f"] & & \bar{C} \arrow[dl, dashed, "\exists \bar{f}"] \\ & X & \end{tikzcd}$$

Theorem 2.19. [RSW17, Theorems 4.6.3.2 and 4.5.1] The moduli space of centrally aligned maps to projective space is isomorphic to the Vakil-Zinger blow-up. The factorisation property identifies the main component.

Notice that by construction $\bar{f}$ is non-constant on at least one branch of the core. For every Gorenstein curve of genus one with no separating nodes, the dualising sheaf is trivial [Smy11a, Lemma 3.3]. Therefore $H^1(\bar{C}, \bar{f}^*\mathcal{O}_{\mathbb{P}^N}(1)) = 0$ by Serre duality, and the projection to $\mathfrak{M}^{\mathrm{cen}}$ is unobstructed (a perfect obstruction theory is given by $\mathrm{R}^\bullet\,\bar{\pi}_*\bar{f}^*T_{\mathbb{P}^N}$), so every element satisfying factorisation is smoothable; on the other hand factorisation is a closed condition [RSW17, Theorem 4.3]. This proves the second claim above.

Remark 2.20. From the discussion above we see that the construction in [RSW17] might be suitable to extend the definition of reduced invariants to a larger class of varieties than projective complete intersections. It would be enough that $\mathcal{VZ}_{1,n}(X,\beta)$ is irreducible and $Y \subseteq X$ is such that $N_{Y/X}$ is an ample vector bundle, so that $\mathrm{R}^1\,\bar{\pi}_*\bar{f}^*N_{Y/X} = 0$ and the inclusion

$\mathcal{VZ}_{1,n}(Y,\beta) \hookrightarrow \mathcal{VZ}_{1,n}(X,\beta)$ admits a perfect obstruction theory. On the other hand if we want to prove that $\mathcal{VZ}_{1,n}(X,\beta)$ is smooth following in the steps above, we need T_X ample, which is quite a restrictive condition (the only smooth variety with ample tangent bundle is the projective space, by a theorem of Mori [Mor79], on the other hand I think it should be possible to extend e.g. to the orbifold theory of weighted projective spaces).

In the next section I am going to discuss a result that I have obtained with F. Carocci and C. Manolache, which illustrates the relation between the Li-Vakil-Zinger's and the Smyth-Viscardi's plans along a slightly different line: namely, the idea is that the reduced invariants may be recovered as m-stable invariants, when m is big enough that all the boundary components contributing non-trivially to the Li-Zinger's formula have been deleted. We demonstrate this principle in the case of the quintic threefold, for which only D^1 matters, so that it is enough to allow maps from cuspidal curves.

Theorem 2.21. [BCM18] For a smooth quintic threefold $X_5 = V(\mathbf{w}) \subseteq \mathbb{P}^4$,

$$GW_1^{\mathrm{red}}(X_5) = GW_1^{(1)}(X_5).$$

This is really a result about the virtual cycles, hence it holds for any (descendent) insertion. Here is an idea of how the proof could go: recall from Proposition 2.6 that there is a well-defined 1-stabilisation morphism at the level of prestable weighted curves; the following fiber product

$$\begin{tikzcd}
\mathcal{Z}_X \arrow[r, "\phi"] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{\mathcal{M}}^{(1)}(X_5, d) \arrow[d] \\
\mathfrak{M}_1^{\mathrm{wt}=d,\mathrm{st}} \arrow[r] & \mathfrak{M}_1^{\mathrm{wt}=d,\mathrm{st}}(1)
\end{tikzcd}$$

is endowed with a class $[\mathcal{Z}_X]^{\mathrm{vir}}$ by virtual pullback, such that $\phi_*[\mathcal{Z}_X]^{\mathrm{vir}} = [\overline{\mathcal{M}}^{(1)}(X_5, d)]^{\mathrm{vir}}$, because $\mathfrak{M}_1^{\mathrm{wt}=d,\mathrm{st}} \to \mathfrak{M}_1^{\mathrm{wt}=d,\mathrm{st}}(1)$ is proper [BCM18, Lemma 4.19] and birational and by commutativity of virtual pullback with pushforward [Man12a]. On the other hand there is a closed embedding of $\mathcal{Z}_X$ into ordinary stable maps $\overline{\mathcal{M}}_1(X, d)$ [BCM18, Lemma 4.13]; it is an isomorphism with the substack of maps satisfying factorisation through 1-stabilisation of the underlying weighted curve. In particular $\mathcal{Z}_X$ has a main component, and all the boundary components except $D^1(X, d)$. Unfortunately it is hard to study the intrinsic normal cone of $\overline{\mathcal{M}}_1(X, d)$ directly, hence we resort to an indirect approach, extending work of H.L. Chang, Y. Hu, Y.-H. Kiem and J. Li to the situation at hand.

3. p-fields, local equations and a splitting of the cone

The indirect approach is the following: we introduce the moduli space of 1-stable maps with p-fields, $\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4, d)^p$, the geometry of which is closely related to that of $\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4,d)$ (in fact, it is a line bundle over the boundary, while the main components are isomorphic); $\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4,d)^p$ is endowed with a cosection localised virtual class carrying the same information as $[\overline{\mathcal{M}}_1^{(1)}(X,d)]^{\mathrm{vir}}$. By pulling back along the 1-stabilisation as above, we may define $\mathcal{Z}_{\mathbb{P}^4}$ and $\mathcal{Z}^p$; unfortunately $\mathcal{Z}^p$ does not compare directly with $\overline{\mathcal{M}}_1(\mathbb{P}^4,d)^p$. On the other hand, after studying local equations for the moduli space and a Vakil-Zinger blow-up, we are able to split the normal cone of $\widetilde{\mathcal{Z}}^p$ directly, and show that the only non-trivial contribution to the invariants comes from the main component.

In [CL12], Chang and Li develop a general theory of moduli spaces of sections: let us work over a base B - an algebraic stack - and let $\pi\colon C\to B$ be a family of proper l.c.i. curves (they restrict to the nodal case, but all their arguments carry through unchanged, as we have checked in [BCM18, §3]). Let $\mathcal{V}$ be representable, quasi-projective, and smooth over C. The *space of sections* of $\mathcal{V}$ over C is a B-stack $\mathfrak{S}$ defined by:

$$\mathfrak{S}(S\to B)=\{\text{sections of } \mathcal{V}_S\to C_S\}\,;$$

it has a dual perfect obstruction theory relative to B [CL12, Proposition 2.5]:

$$\phi_{\mathfrak{S}/B}\colon \mathbb{T}_{\mathfrak{S}/B}\to\mathbb{E}_{\mathfrak{S}/B}:=\mathrm{R}^\bullet\,\pi_{\mathfrak{S},*}\mathfrak{e}^*T_{\mathcal{V}/C}.$$

This applies in particular when $\mathcal{V}$ is a vector bundle over C (or an open within one), and in this case we get a *cone* of sections $C(\mathcal{V})$; in fact by Serre duality

$$C(\mathcal{V})=\underline{\mathrm{Spec}}_B\,\mathrm{Sym}^\bullet(\mathrm{R}^1\,\pi_*(V^\vee\otimes\omega_{C/B})).$$

If we let $B=\mathfrak{P}$ be the universal Picard stack over $\mathfrak{M}$, we recover the moduli space of stable maps to $\mathbb{P}^N$ as an open (defined by stability and the fact that the section do not vanish simultaneously) inside $C(\pi_*(\mathcal{L})^{\oplus N+1})$; the relative obstruction theory $\mathrm{R}^\bullet\,\pi_*\mathcal{L}^{\oplus N+1}$ over $\mathfrak{P}$ is compatible with the usual one $\mathrm{R}^\bullet\,\pi_*f^*T_{\mathbb{P}^N}$ over $\mathfrak{M}$, by the Euler sequence and $\mathbb{T}_{\mathfrak{P}/\mathfrak{M}}\simeq\mathrm{R}^\bullet\,\pi_*\mathcal{O}_C[1]$ [CL12, Lemma 2.8]. This is familiar from the theory of quasimaps (it is just a change in the stability condition cutting a different open within the Artin stack of sections).

Definition 3.1. The moduli space of 1-stable maps with p-fields $\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4,d)^p$ parametrises 1-stable maps $\bar f\colon\bar C\to\mathbb{P}^4$ together with a *p-field*:

$$\psi\in H^0(\bar C,\bar f^*\mathcal{O}_{\mathbb{P}^4}(-5)\otimes\omega_{\bar\pi}).$$

It is the cone of sections of the line bundle $\bar{\mathcal{P}}=\bar f^*\mathcal{O}_{\mathbb{P}^4}(-5)\otimes\omega_{\bar\pi}$ over $\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4,d)$. Notice that it is isomorphic to $\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4,d)$ on the open locus of the main component, while it is a line bundle over the boundary components; in particular it is not a proper space. It has a perfect obstruction theory relative to $\bar{\mathfrak{P}}=\mathfrak{Pic}(\bar C\to\mathfrak{M}_1(1))$ given by $\mathrm{R}^\bullet\,\bar\pi_*(\bar{\mathcal{L}}^{\oplus 5}\oplus\bar{\mathcal{P}})$, where $\bar{\mathcal{P}}$ is defined as above replacing $\bar f^*\mathcal{O}_{\mathbb{P}^4}(1)$ by the universal line bundle $\bar{\mathcal{L}}$.

The quintic polynomial $\mathbf{w}$ determines a cosection of the obstruction bundle as follows: there is a morphism of vector bundles on the universal curve over $\overline{\mathfrak{P}}$: $h_1\colon \mathrm{Vb}(\overline{\mathcal{L}}^{\oplus 5} \oplus \overline{\mathcal{P}}) \to \mathrm{Vb}(\omega_{\bar\pi}), \quad \mathrm{h}_1(\mathbf{x}, \mathrm{p}) = \mathrm{p}\mathbf{w}(\mathrm{x}_0, \ldots, \mathrm{x}_4)$. By differentiating it and pulling it back along the universal evaluation

$$\begin{tikzcd}
 & \mathrm{Vb}(\overline{\mathcal{L}}^{\oplus 5}) \setminus \{0\} \oplus \mathrm{Vb}(\overline{\mathcal{P}}) \arrow[d] \\
\overline{\mathcal{C}} \arrow[ur, bend left, "\mathfrak{e}"] \arrow[r] \arrow[d, "\bar\pi"] & \overline{\mathcal{C}} \arrow[d, "\bar\pi"] \\
\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4, d)^p \arrow[r] & \overline{\mathfrak{P}}
\end{tikzcd}$$

we obtain a cosection of the relative obstruction sheaf

$$\sigma_1\colon \mathrm{Ob}_{\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4,\mathrm{d})^{\mathrm{p}}/\overline{\mathfrak{P}}} = \mathrm{R}^1\,\bar\pi_*(\overline{\mathcal{L}}^{\oplus 5} \oplus \overline{\mathcal{P}}) \to \mathrm{R}^1\,\bar\pi_*(\omega_{\bar\pi}) \simeq \mathcal{O}_{\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4,\mathrm{d})^{\mathrm{p}}} \tag{2}$$

$$\sigma_{1|(u,\psi)}(\mathring{\mathbf{x}}, \mathring{p}) = \mathring{p}\mathbf{w}(u) + \psi \sum_{i=0}^{4} \partial_i \mathbf{w}(u)\mathring{x}_i$$

The degeneracy locus of this cosection is precisely $\overline{\mathcal{M}}_1^{(1)}(X, d)$ by Serre duality and smoothness of X. We may therefore apply Kiem-Li's machinery of cosection-localised virtual cycles [KL13][CL12, §5], to obtain a class supported on the degeneracy locus of σ_1:

$$[\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4, d)^p]^{\mathrm{vir}}_{\mathrm{loc}} = 0^!_{\sigma_1,\mathrm{loc}}[\mathfrak{C}_{\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4,d)^p/\overline{\mathfrak{P}}}] \in A_0\left(\overline{\mathcal{M}}_1^{(1)}(X, d)\right).$$

Theorem 3.2.

$$\deg[\overline{\mathcal{M}}_{1,1}^{(1)}(\mathbb{P}^4, d)^p]^{\mathrm{vir}}_{\mathrm{loc}} = (-1)^{5d}\deg[\overline{\mathcal{M}}_1^{(1)}(X, d)]^{\mathrm{vir}}$$

This is proved by deforming to the normal cone and then working out explicitly the localised virtual pullback along $\overline{\mathcal{M}}_1^{(1)}(N_{X/\mathbb{P}^4}, d)^p \to \overline{\mathcal{M}}_1^{(1)}(X, d)$, which has a symmetric obstruction theory. All the details are spelled out in [CL12] and summarised in [BCM18, §3]. Consider now the fiber diagram:

$$\begin{tikzcd}
\mathcal{Z}^p \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4, d)^p \arrow[d] \\
\mathfrak{Z} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{\mathfrak{P}} \arrow[d] \\
\mathfrak{M}_1^{\mathrm{wt,st}} \arrow[r] & \mathfrak{M}_1^{\mathrm{wt,st}}(1)
\end{tikzcd}$$

Remark 3.3. The stack $\mathfrak{Z}$ parametrises $\mathcal{C} \to \overline{\mathcal{C}}$ with a line bundle $\overline{\mathcal{L}}$ on $\overline{\mathcal{C}}$. By pulling back $\overline{\mathcal{L}}$ to $\mathcal{C}$ we obtain a morphism $\mathfrak{Z} \to \mathfrak{P}$. This is generically an isomorphism, but has 1-dimensional fibers over the locus of elliptic tails, due to the fact that $\mathrm{Pic}(\overline{C}) \to \mathrm{Pic}(C)$ has kernel $\mathbb{G}_a$ when $\overline{C}$ has a cusp; on the other hand the line bundle thus obtained is always trivial on the elliptic tail, therefore $\mathfrak{Z} \to \mathfrak{P}$ is not surjective, as it misses the non-trivial line bundles of degree 0 on the elliptic tail.

Similarly the stack $\mathcal{Z}^p$ parametrises $C \xrightarrow{\phi} \bar{C} \xrightarrow{\bar{f}} \mathbb{P}^4$ with a p-field $\psi \in H^0(\overline{C}_S, f_S^*\mathcal{O}_{\mathbb{P}^4}(-5)\otimes\omega_{\hat{\pi}_S})$. We were not able to compare $\mathcal{Z}^p$ with $\overline{\mathcal{M}}_1(\mathbb{P}^4, d)^p$ directly, since denoting by $\overline{\mathcal{L}} = f^*\mathcal{O}_{\mathbb{P}^4}(1)$ and by $\mathcal{L} = \phi^*\overline{\mathcal{L}}$ we only have a map $\mathrm{R}^1\,\hat{\pi}_*\overline{\mathcal{L}} \to \mathrm{R}^1\,\pi_*\mathcal{L}$ on $\mathcal{Z}^p$ which is not an isomorphism.

$\mathcal{Z}^p$ is endowed with a localised virtual cycle that has the same degree of $[\overline{\mathcal{M}}^{(1)}_{1,1}(\mathbb{P}^4, d)^p]^{\mathrm{vir}}_{\mathrm{loc}}$, by commutativity of localised virtual pullback with pushforward [BCM18, Lemma 4.17].

In order to study the intrinsic cone of $\mathcal{Z}^p$ we need to embed it, at least locally, in a smooth space, and understand the equations of the embedding. Since locally $\mathcal{Z}^p$ sits inside $\mathcal{Z}_{\mathbb{P}^4}\times\mathbb{A}^1$, it is enough to find local equations for $\mathcal{Z}_{\mathbb{P}^4}$. Recall that $\mathcal{Z}_{\mathbb{P}^4}$ is an open inside the cone of sections $\bar{\pi}_*\overline{\mathcal{L}}^{\oplus 5}$ over $\overline{\mathfrak{P}}$. It is standard to resolve $\mathrm{R}^\bullet\,\bar{\pi}_*\overline{\mathcal{L}}$ twisting by a sufficiently $\bar{\pi}$-ample line bundle on $\overline{C}$, but in the genus one case this can be performed very efficiently by twisting with a local section $\mathcal{A}$ passing through the core (a line bundle of positive degree on a Gorenstein genus one curve has no h^1); $\bar{\pi}_*\overline{\mathcal{L}}$ is then the kernel of the restriction map $\bar{\pi}_*\overline{\mathcal{L}}(\mathcal{A}) \to \bar{\pi}_*\overline{\mathcal{L}}(\mathcal{A})_{|\mathcal{A}}$. Furthermore this map can be expressed very explicitly in local coordinates.

For technical reasons we work over $\mathfrak{Z}^{\mathrm{div}} := \mathfrak{Z}\times_{\mathfrak{M}_1(1)}\mathfrak{M}_1^{\mathrm{div}}(1)$: locally on $\mathcal{Z}_{\mathbb{P}^4}$ we can choose a hyperplane $H_0 \subseteq \mathbb{P}^4$ that pulls back to a simple divisor contained in the smooth locus of the curve, say $D = \sum_{i=1}^d \delta_i$, and choose coordinates on $\mathbb{P}^4$ such that $H_0 = \{x_0 = 0\}$.

4. On the relative problem

References

[BCM18] L. Battistella, F. Carocci, and C. Manolache. Reduced invariants from cuspidal maps. *arXiv:1801.07739*, January 2018.

[BF97] K. Behrend and B. Fantechi. The intrinsic normal cone. *Invent. Math.*, 128(1):45–88, 1997.

[CKL01] David A. Cox, Sheldon Katz, and Yuan-Pin Lee. Virtual fundamental classes of zero loci. In *Advances in algebraic geometry motivated by physics (Lowell, MA, 2000)*, volume 276 of *Contemp. Math.*, pages 157–166. Amer. Math. Soc., Providence, RI, 2001.

[CL12] Huai-Liang Chang and Jun Li. Gromov-Witten invariants of stable maps with fields. *Int. Math. Res. Not. IMRN*, (18):4163–4217, 2012.

[CL15] Huai-Liang Chang and Jun Li. An algebraic proof of the hyperplane property of the genus one GW-invariants of quintics. *J. Differential Geom.*, 100(2):251–299, 2015.

[Ful98] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998.

[Gil12] W. D. Gillam. Logarithmic stacks and minimality. *Internat. J. Math.*, 23(7):1250069, 38, 2012.

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[HH09] Brendan Hassett and Donghoon Hyeon. Log canonical models for the moduli space of curves: the first divisorial contraction. *Trans. Amer. Math. Soc.*, 361(8):4471–4489, 2009.

[HL10] Yi Hu and Jun Li. Genus-one stable maps, local equations, and Vakil-Zinger's desingularization. *Math. Ann.*, 348(4):929–963, 2010.

[HM98] Joe Harris and Ian Morrison. *Moduli of curves*, volume 187 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998.

[Kat00] Fumiharu Kato. Log smooth deformation and moduli of log smooth curves. *Internat. J. Math.*, 11(2):215–232, 2000.

[KKP03] Bumsig Kim, Andrew Kresch, and Tony Pantev. Functoriality in intersection theory and a conjecture of Cox, Katz, and Lee. *J. Pure Appl. Algebra*, 179(1-2):127–136, 2003.

[KL13] Young-Hoon Kiem and Jun Li. Localizing virtual cycles by cosections. *J. Amer. Math. Soc.*, 26(4):1025–1050, 2013.

[Kon95] Maxim Kontsevich. Enumeration of rational curves via torus actions. In *The moduli space of curves (Texel Island, 1994)*, volume 129 of *Progr. Math.*, pages 335–368. Birkhäuser Boston, Boston, MA, 1995.

[LZ07] Jun Li and Aleksey Zinger. Gromov-Witten invariants of a quintic threefold and a rigidity conjecture. *Pacific J. Math.*, 233(2):417–480, 2007.

[Man12a] Cristina Manolache. Virtual pull-backs. *J. Algebraic Geom.*, 21(2):201–245, 2012.

[Man12b] Cristina Manolache. Virtual push-forwards. *Geom. Topol.*, 16(4):2003–2036, 2012.

[Mor79] Shigefumi Mori. Projective manifolds with ample tangent bundles. *Ann. of Math. (2)*, 110(3):593–606, 1979.

[Par17] Keli Siqueiros Parker. Semistable modular compactifications of moduli spaces of genus one curves, 2017. PhD thesis, University of Colorado at Boulder.

[RSW17] D. Ranganathan, K. Santos-Parker, and J. Wise. Moduli of stable maps in genus one and logarithmic geometry I. *ArXiv e-prints*, August 2017.

[Sch91] David Schubert. A new compactification of the moduli space of curves. *Compositio Math.*, 78(3):297–313, 1991.

[Smy11a] David Ishii Smyth. Modular compactifications of the space of pointed elliptic curves I. *Compos. Math.*, 147(3):877–913, 2011.

[Smy11b] David Ishii Smyth. Modular compactifications of the space of pointed elliptic curves II. *Compos. Math.*, 147(6):1843–1884, 2011.

[Vak00] Ravi Vakil. The enumerative geometry of rational and elliptic curves in projective space. *J. Reine Angew. Math.*, 529:101–153, 2000.

[Vis12] Michael Viscardi. Alternate compactifications of the moduli space of genus one maps. *Manuscripta Math.*, 139(1-2):201–236, 2012.

[VZ07] Ravi Vakil and Aleksey Zinger. A natural smooth compactification of the space of elliptic curves in projective space. *Electron. Res. Announc. Amer. Math. Soc.*, 13:53–59, 2007.

[Zin09] Aleksey Zinger. The reduced genus one Gromov-Witten invariants of Calabi-Yau hypersurfaces. *J. Amer. Math. Soc*, (22):691–737, 2009.