Scientific Programming: Algorithms (part B)

Introduction

Luca Bianco - Academic Year 2019-20
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About me

Computer Science

Ph.D. at the University of Verona, Italy, with thesis on Simulation of Biological Systems

Research Fellow at Cranfield University - UK

Three years at Cranfield University working at proteomics projects (GAPP, MRMaid, X-Tracker...)

Module manager and lecturer in several courses of the MSc in Bioinformatics

Bioinformatician at IASMA – FEM

Currently bioinformatician in the Computational Biology Group at Istituto Agrario di San Michele all'Adige – Fondazione Edmund Mach, Trento, Italy

Collaborator uniTN - CiBio

I ran the Scienitific Programming Lab for QCB for the last couple of years

Organization

145540 Scientific Programming (12 ECTS, LM QCB)
145685 Scientific Programming (12 ECTS, LM Data Science)

Part A - Programming (23/9-31/10)

Introduction to the Python language and to a collection of programming libraries for data analysis.

- Mutuated as 145912 Scientific Programming (LM Math, 6 credits)

Part B - Algorithms (4/11-12/12)

Design and analysis of algorithmic solutions. Presentation of the most important classes of algorithms and evaluation of their performance.

Topics

- Introduction
  - Recursion
  - Algorithm analysis
  - Asymptotic notations
- Data structures
  - High level overview
  - Sequences, maps (ordered/unordered), sets
  - Data structure implementations in Python
- Trees
  - Data structure definition
  - Visits
- Graphs
  - Data structure definition
  - Visits
  - Algorithms on graphs
- Algorithmic techniques
  - Divide-et-impera
  - Dynamic programming
  - Greedy
  - Backtrack
  - NP class: brief overview

Learning outcomes

At the end of the module, students are expected to:

- evaluate algorithmic choices and select the ones that best suit their problems;
- analyze the complexity of existing algorithms and algorithms created on their own;
- design simple algorithmic solutions to solve basic problems.

Teaching team

- Instructor: Dr. Luca Bianco
  - Theory lectures, algorithmic exercises
  - luca.bianco [AT] fmach.it
- Teaching assistant: Dr. Massimiliano Luca
  - Lab sessions on algorithms (QCB)
  - massimiliano.luca [AT] unitn.it
- Teaching assistant: Dr. David Leoni
  - Lab sessions on algorithms (data science)
  - david.leoni [AT] unitn.it

Schedule

Week day	**Time**	**Room**	**Description**
Monday	14.30-16.30	A107	Lecture
Tuesday	15.30-17.30	A107	Lab. QCB
Tuesday	15.30-17.30	A103	Lab. Data Science
Wednesday	11.30-13.30	A107	Lecture
Thursday	15.30-17.30	A107	Lab. QCB
Thursday	15.30-17.30	A208	Lab. Data Science

Course material

Lectures:

Material and information: https://sciproalgo2019.readthedocs.io/en/latest/

Practicals:

QCB:

Data science: https://datasciprolab.readthedocs.io/en/latest/

[Thanks to Prof. Alberto Montresor for the material]

Where we stand...

So far...

we have learnt a bit of Python and we started doing some little examples of data analysis (saw some libraries, etc...)

From now on..

we will focus on:

- “Solving problems” providing solutions (**correctness**), possibly in an efficient way (**complexity**), organizing data in the most suitable ways (**data structures**)

Maximal sum problem

- **Input**: a list A containing n numbers
- **Output**: a slice (sublist) $A[i:j]$ of maximal sum, i.e. the slice whose element sum $\sum_{k=i}^{j-1} A[k]$ is larger or equal than the sum of any other slice

simpler problem

Find the maximal sum, rather than the interval that provides the maximal sum.

Is the problem clear?

Example:

1	3	4	-8	2	3	-1	3	4	-3	10	-3	2

Maximal sum problem

- **Input**: a list A containing n numbers
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Maximal sum problem

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$\Downarrow$ simpler problem

Find the maximal sum, rather than the interval that provides the maximal sum.

Is the problem clear?

Example:

1	3	4	-8	2	3	-1	3	4	-3	10	-3	2

Any ideas on how to solve this problem?

Solution 1 ~ N^3

- **Input**: a list A containing n numbers
- **Output**: a slice (sublist) $A[i:j]$ of maximal sum, i.e. the slice whose element sum $\sum_{k=i}^{j-1} A[k]$ is larger or equal than the sum of any other slice

Idea:

Given the list A with N elements

Consider all pairs (i,j) such that i ≤ j
Get the elements in A[i:j+1]
Compute the sum of all elements in A[i:j+1]
Update max_so_far if sum ≥ max_so_far

```
def max_sum_v1(A):
    max_so_far = 0
    N = len(A)
    for i in range(N):
        for j in range(i,N):
            tmp_sum = sum (A[i:j+1])
            max_so_far = max(tmp_sum, max_so_far)

    return max_so_far
```

```
A = [1,3,4,-8,2, 3,-1,3,4,-3,10,-3,2]
print(A)
print(max_sum_v1(A))
```

```
[1, 3, 4, -8, 2, 3, -1, 3, 4, -3, 10, -3, 2]
18
```

List comprehension… ?

- **Input**: a list A containing n numbers
- **Output**: a slice (sublist) $A[i : j]$ of maximal sum, i.e. the slice whose element sum $\sum_{k=i}^{j-1} A[k]$ is larger or equal than the sum of any other slice

```
def max_sum_v1_listc_1(A):
    N = len(A)
    sums = [sum(A[i:j+1]) for i in range(N) for j in range(i,N)]

    return max(sums)

A = [1,3,4,-8,2, 3,-1,3,4,-3,10,-3,2]
print(A)
print(max_sum_v1_listc_1(A))
```

```
[1, 3, 4, -8, 2, 3, -1, 3, 4, -3, 10, -3, 2]
18
```

List comprehension… ?

- **Input**: a list A containing n numbers
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    return max(sums)

A = [1,3,4,-8,2, 3,-1,3,4,-3,10,-3,2]
print(A)
print(max_sum_v1_listc_1(A))
```

```
[1, 3, 4, -8, 2, 3, -1, 3, 4, -3, 10, -3, 2]
18
```

How many elements?

List comprehension... ?

- **Input**: a list A containing n numbers
- **Output**: a slice (sublist) $A[i:j]$ of maximal sum, i.e. the slice whose element sum $\sum_{k=i}^{j-1} A[k]$ is larger or equal than the sum of any other slice

No thanks!

```
def max_sum_v1_listc_1(A):
    N = len(A)
    sums = [sum(A[i:j+1]) for i in range(N) for j in range(i,N)]

    return max(sums)

A = [1,3,4,-8,2, 3,-1,3,4,-3,10,-3,2]
print(A)
print(max_sum_v1_listc_1(A))
```

```
[1, 3, 4, -8, 2, 3, -1, 3, 4, -3, 10, -3, 2]
18
```

How many elements?

N*(N+1)/2 ~ N^2

[1, 4, 8, 0, 2, 5, 4, 7, 11, 8, 18, 15, 17, 3, 7, -1, 1, 4, 3, 6, 10, 7, 17, 14, 16, 4, -4, -2, 1, 0, 3, 7, 4, 14, 11, 13, -8, -6, -3, -4, -1, 3, 0, 10, 7, 9, 2, 5, 4, 7, 11, 8, 18, 15, 17, 3, 2, 5, 9, 6, 16, 13, 15, -1, 2, 6, 3, 13, 10, 12, 3, 7, 4, 14, 11, 13, 4, 1, 11, 8, 10, -3, 7, 4, 6, 10, 7, 9, -3, -1, 2]
→ 91 elements!

If A has 100,000 elements ➜ ~ 40 GB RAM!!!

List comprehension... ?

- **Input**: a list A containing n numbers
- **Output**: a slice (sublist) $A[i:j]$ of maximal sum, i.e. the slice whose element sum $\sum_{k=i}^{j-1} A[k]$ is larger or equal than the sum of any other slice

```
def max_sum_v1_listc(A):
    N = len(A)
    intervals = [A[i:j+1] for i in range(N) for j in range(i,N)]
    sums = [sum(vals) for vals in intervals]
    return max(sums)

A = [1,3,4,-8,2, 3,-1,3,4,-3,10,-3,2]
print(A)
print(max_sum_v1_listc(A))
```

```
[1, 3, 4, -8, 2, 3, -1, 3, 4, -3, 10, -3, 2]
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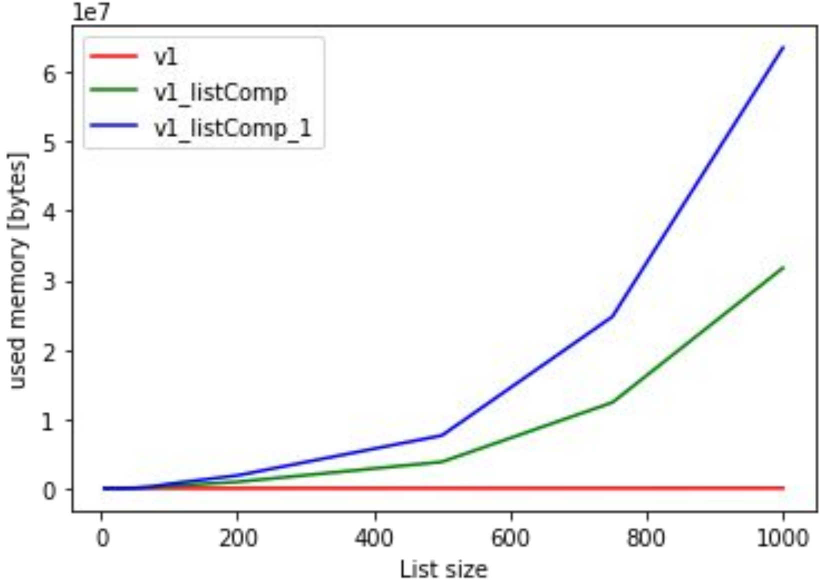
```

Stores intervals and sums!!!

If A has 100,000 elements → ~ 1.3 PB RAM!!!

List comprehension... ?

- **Input**: a list A containing n numbers
- **Output**: a slice (sublist) $A[i:j]$ of maximal sum, i.e. the slice whose element sum $\sum_{k=i}^{j-1} A[k]$ is larger or equal than the sum of any other slice



Important note:

Time and space (memory) are two important resources!

Solution 1 ~ N^3

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Idea:

Given the list A with N elements

Consider all pairs (i,j) such that i ≤ j
Get the elements in A[i:j+1]
Compute the sum of all elements in A[i:j+1]
Update max_so_far if sum ≥ max_so_far

```python
def max_sum_v1(A):
    max_so_far = 0
    N = len(A)
    for i in range(N):
        for j in range(i,N):
            tmp_sum = sum (A[i:j+1])
            max_so_far = max(tmp_sum, max_so_far)

    return max_so_far
```

Why N^3 ?

Intuitively,

We have N*(N+1)/2 pairs and the sum of N numbers takes N operations.

So: N*[(N+1)/2]*N ~ N^3

Solution 2 ~ N^2

Observation: There is no point in computing the same sums over and over again!

If S = sum(A[i:j]) ➔ sum(A[i:j+1]) = S + A[j+1]

```
def max_sum_v2(A):
    N = len(A)
    max_so_far = 0
    for i in range(N):
        tot = 0 #ACCUMULATOR!
        for j in range(i,N):
            tot = tot + A[j]
            max_so_far = max(max_so_far, tot)
    return max_so_far

A = [1,3,4,-8,2, 3,-1,3,4,-3,10,-3,2]
print(A)
print(max_sum_v2(A))
```

```
[1, 3, 4, -8, 2, 3, -1, 3, 4, -3, 10, -3, 2]
18
```

- **Input**: a list A containing n numbers
- **Output**: a slice (sublist) $A[i:j]$ of maximal sum, i.e. the slice whose element sum $\sum_{k=i}^{j-1} A[k]$ is larger or equal than the sum of any other slice

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A = [1,3,4,-8,2, 3,-1,3,4,-3,10,-3,2]
print(A)
print(max_sum_v2(A))
```

```
[1, 3, 4, -8, 2, 3, -1, 3, 4, -3, 10, -3, 2]
18
```

Tot
0, 1, 4, 8, 0, 2, 5, 4, 7, 11, 8, 18, 15, 17, ← (0, x)
0, 3, 7, -1, 1, 4, 3, 6, 10, 7, 17, 14, 16, ← (1, x)
0, 4, -4, -2, 1, 0, 3, 7, 4, 14, 11, 13, ← (2, x)
0, -8, -6, -3, -4, -1, 3, 0, 10, 7, 9,
0, 2, 5, 4, 7, 11, 8, 18, 15, 17,
0, 3, 2, 5, 9, 6, 16, 13, 15,
0, -1, 2, 6, 3, 13, 10, 12,
0, 3, 7, 4, 14, 11, 13,
0, 4, 1, 11, 8, 10,
0, -3, 7, 4, 6,
0, 10, 7, 9,
0, -3, -1,
0, 2 ← (N-1, x)

Maxes
[1, 4, 8, 8, 8, 8, 8, 8, 11, 11, 18, 18, 18, .., 18]

Solution 2 ~ N^2

Observation: There is no point in computing the same sums over and over again!

If S = sum(A[i:j]) → sum(A[i:j+1]) = S + A[j+1]

```
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A = [1,3,4,-8,2, 3,-1,3,4,-3,10,-3,2]
print(A)
print(max_sum_v2(A))
```

```
[1, 3, 4, -8, 2, 3, -1, 3, 4, -3, 10, -3, 2]
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```

- **Input**: a list A containing n numbers
- **Output**: a slice (sublist) $A[i:j]$ of maximal sum, i.e. the slice whose element sum $\sum_{k=i}^{j-1} A[k]$ is larger or equal than the sum of any other slice

Intuitively, we have to consider N*(N+1)/2 ~ N^2 intervals

The space required is just a couple of variables: **constant**!

Solution 2 ~ N^2

Tip: use itertools

Accumulate of itertools is done in C so it is faster

- **Input**: a list A containing n numbers
- **Output**: a slice (sublist) $A[i:j]$ of maximal sum, i.e. the slice whose element sum $\sum_{k=i}^{j-1} A[k]$ is larger or equal than the sum of any other slice

```
from itertools import accumulate

A = list(range(10))
print(A)
print(list(accumulate(A)))
```

```
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9]
[0, 1, 3, 6, 10, 15, 21, 28, 36, 45]
```

Solution 2 ~ N^2

- **Input**: a list A containing n numbers
- **Output**: a slice (sublist) $A[i:j]$ of maximal sum, i.e. the slice whose element sum $\sum_{k=i}^{j-1} A[k]$ is larger or equal than the sum of any other slice

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from itertools import accumulate

A = list(range(10))
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```

```
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9]
[0, 1, 3, 6, 10, 15, 21, 28, 36, 45]
```

```
from itertools import accumulate

def max_sum_v2_bis(A):
    N = len(A)
    max_so_far = 0
    for i in range(N):
        tot = max(accumulate(A[i:]))
        max_so_far = max(max_so_far,tot)
    return max_so_far


A = [1,3,4,-8,2, 3,-1,3,4,-3,10,-3,2]
print(A)
print(max_sum_v2_bis(A))
```

```
[1, 3, 4, -8, 2, 3, -1, 3, 4, -3, 10, -3, 2]
18
```

Similar as before but max computed on the accumulated sum

Important note: N intervals, sum of N elements each time: ~ N^2 operations

The improvement comes from implementation not algorithm! (code faster by a constant factor)

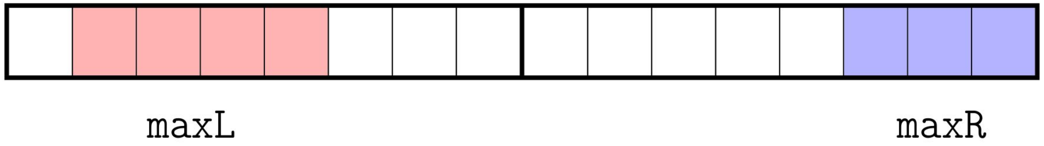
Solution 3 ~ N log(N)

- **Input**: a list A containing n numbers
- **Output**: a slice (sublist) $A[i:j]$ of maximal sum, i.e. the slice whose element sum $\sum_{k=i}^{j-1} A[k]$ is larger or equal than the sum of any other slice

Divide et impera (Divide and conquer)

Idea:

- Split it in two equally sized sublists
- Find maxL as the sum of the maximal sublist on the left part
- Find maxR as the sum of the maximal sublist on the right part



Is this correct? Do you see any problem with this?

Solution 3 ~ N log(N)

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Idea:

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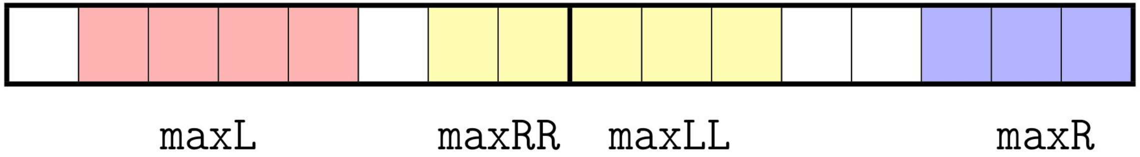
- maxLL+maxRR is the value of the maximal sublist accross the two parts



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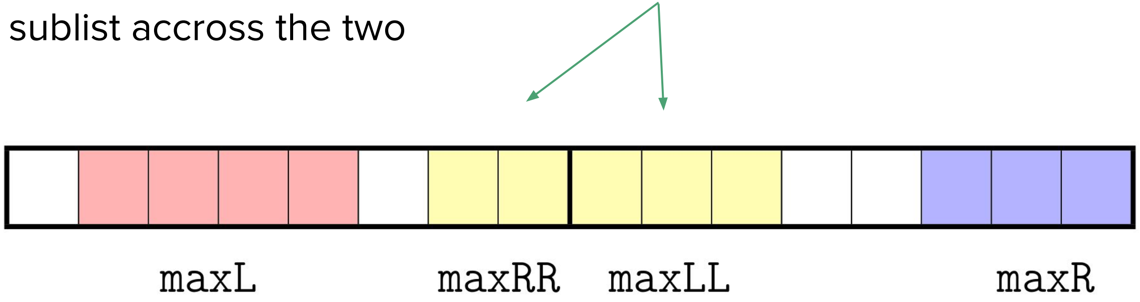
- Split it in two equally sized sublists
- Find maxL as the sum of the maximal sublist on the left part
- Find maxR as the sum of the maximal sublist on the right part
- maxLL+maxRR is the value of the maximal sublist accross the two parts

Get the point before the mid-point M and go to the left until the sum increases.
Repeat starting from M+1.
Result is: max(maxL, maxRR, maxLL, maxR)



Solution 3 ~ N log(N)

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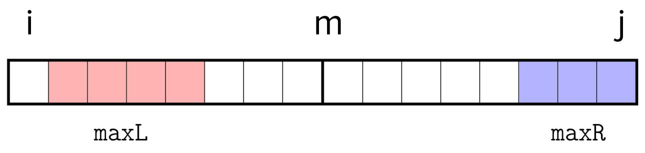
Divide et impera (Divide and conquer)

```python
def max_sum_v3_rec(A, i, j):
    if i == j:
        return max(0, A[i])
    m = (i+j)//2
    maxML = 0
    s = 0
    for k in range(m,i-1,-1):
        s = s + A[k]
        maxML = max(maxML, s)

    maxMR = 0
    s = 0
    for k in range(m+1, j+1):
        s = s + A[k]
        maxMR = max(maxMR, s)
    maxL = max_sum_v3_rec(A,i,m) #Left maximal subvector
    maxR = max_sum_v3_rec(A,m+1,j) #Right maximal subvector

    return max(maxL, maxR, maxML + maxMR)

def max_sum_v3(A):
    return max_sum_v3_rec(A,0,len(A) - 1)


A = [1,3,4,-8,2, 3,-1,3,4,-3,10,-3,2]
print(A)
print(max_sum_v3(A))
```

```
[1, 3, 4, -8, 2, 3, -1, 3, 4, -3, 10, -3, 2]
18
```

Recursive code: calls itself on a smaller sublist.

Runs in **N*log(N)** … more on this later



Solution 3 ~ N log(N)

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Divide et impera (Divide and conquer)

```
def max_sum_v3_rec_bis(A,i,j):
    if i == j:
        return max(0,A[i])
    m = (i+j)//2
    maxL = max_sum_v3_rec_bis(A,i,m)
    maxR = max_sum_v3_rec_bis(A, m+1, j)
    maxML = max(accumulate(A[m:-len(A) + i -1: -1]))
    maxMR = max(accumulate(A[m+1:j+1]))
    return max(maxL, maxR, maxML+ maxMR)

def max_sum_v3(A):
    return max_sum_v3_rec_bis(A,0,len(A) - 1)


A = [1,3,4,-8,2, 3,-1,3,4,-3,10,-3,2]
print(A)
print(max_sum_v3(A))
```

```
[1, 3, 4, -8, 2, 3, -1, 3, 4, -3, 10, -3, 2]
18
```

0	1	2	3	4	5	6	7	8	9	10	11	12	13
L	u	t	h	e	r		C	o	l	l	e	g	e
-14	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1

```
A = list(range(10))
print(A)

#interval 4-2 going to the left...
M = 4
print(-len(A) + 2 - 1)
A[M: -len(A) + 2 -1 : -1]
```

```
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9]
-9

[4, 3, 2]
```

Recursive code: can use itertools as before to accumulate the sum.

Runs in **N*log(N)** …just a little bit faster, more on this later

Solution 4 ~ N

- **Input**: a list A containing n numbers
- **Output**: a slice (sublist) $A[i:j]$ of maximal sum, i.e. the slice whose element sum $\sum_{k=i}^{j-1} A[k]$ is larger or equal than the sum of any other slice

Dynamic Programming

Let's define maxHere[i] as the maximum value of each sublist that ends in i.

The result is computed from the maximum slice that ends in any position.

$$maxHere[i] = \begin{cases} 0 & i < 0 \\ \max(maxHere[i-1] + A[i], 0) & i \geq 0 \end{cases}$$

Solution 4 ~ N

Dynamic Programming

Let's define maxHere[i] as the maximum value of each sublist that ends in i.

The result is computed from the maximum slice that ends in any position.

$$maxHere[i] = \begin{cases} 0 & i < 0 \\ \max(maxHere[i-1] + A[i], 0) & i \geq 0 \end{cases}$$

Goes through A once: runs in **N**

- **Input**: a list A containing n numbers
- **Output**: a slice (sublist) $A[i:j]$ of maximal sum, i.e. the slice whose element sum $\sum_{k=i}^{j-1} A[k]$ is larger or equal than the sum of any other slice

```
def max_sum_v4(A):
    max_so_far = 0 #Max found so far
    max_here = 0   #Max slice ending at cur pos
    for i in range(len(A)):
        max_here = max(A[i] + max_here, 0)
        max_so_far = max(max_so_far, max_here)
    return max_so_far

A = [1,3,4,-8,2, 3,-1,3,4,-3,10,-3,2]
print(A)
print(max_sum_v3(A))
```

```
[1, 3, 4, -8, 2, 3, -1, 3, 4, -3, 10, -3, 2]
18
```

Solution 4 ~ N

Dynamic Programming

- **Input**: a list A containing n numbers
- **Output**: a slice (sublist) $A[i:j]$ of maximal sum, i.e. the slice whose element sum $\sum_{k=i}^{j-1} A[k]$ is larger or equal than the sum of any other slice

```
def max_sum_v4(A):
    max_so_far = 0 #Max found so far
    max_here = 0   #Max slice ending at cur pos
    for i in range(len(A)):
        max_here = max(A[i] + max_here, 0)
        max_so_far = max(max_so_far, max_here)
    return max_so_far

A = [1,3,4,-8,2, 3,-1,3,4,-3,10,-3,2]
print(A)
print(max_sum_v3(A))
```

```
[1, 3, 4, -8, 2, 3, -1, 3, 4, -3, 10, -3, 2]
18
```

A: [1, 3, 4, -8, 2, 3, -1, 3, 4, -3, 10, -3, 2]
max_here: [0, 1, 4, 8, 0, 2, 5, 4, 7, 11, 8, 18, 15, 17]
max_so_far: [0, 1, 4, 8, 8, 8, 8, 8, 8, 11, 11, 18, 18, 18]

Solution 4 ~ N

Dynamic Programming

Stores also the indexes

```
def max_sum_v4_bis(A):
    max_so_far = 0 #Max found so far
    max_here = 0   #Max slice ending at cur pos
    start = 0 #start of cur maximal slice
    end = 0   #end of cur maximal slice
    last = 0  #beginning of max slice ending here
    for i in range(len(A)):
        max_here = A[i] + max_here
        if max_here <= 0:
            max_here = 0
            last = i + 1
        if max_here > max_so_far:
            max_so_far = max_here
            start = last
            end = i

    return (start,end,max_so_far)

A = [1,3,4,-8,2, 3,-1,3,4,-3,10,-3,2]
print("A: {}".format(A))
print(max_sum_v4_bis(A))
```

```
A: [1, 3, 4, -8, 2, 3, -1, 3, 4, -3, 10, -3, 2]
(4, 10, 18)
```

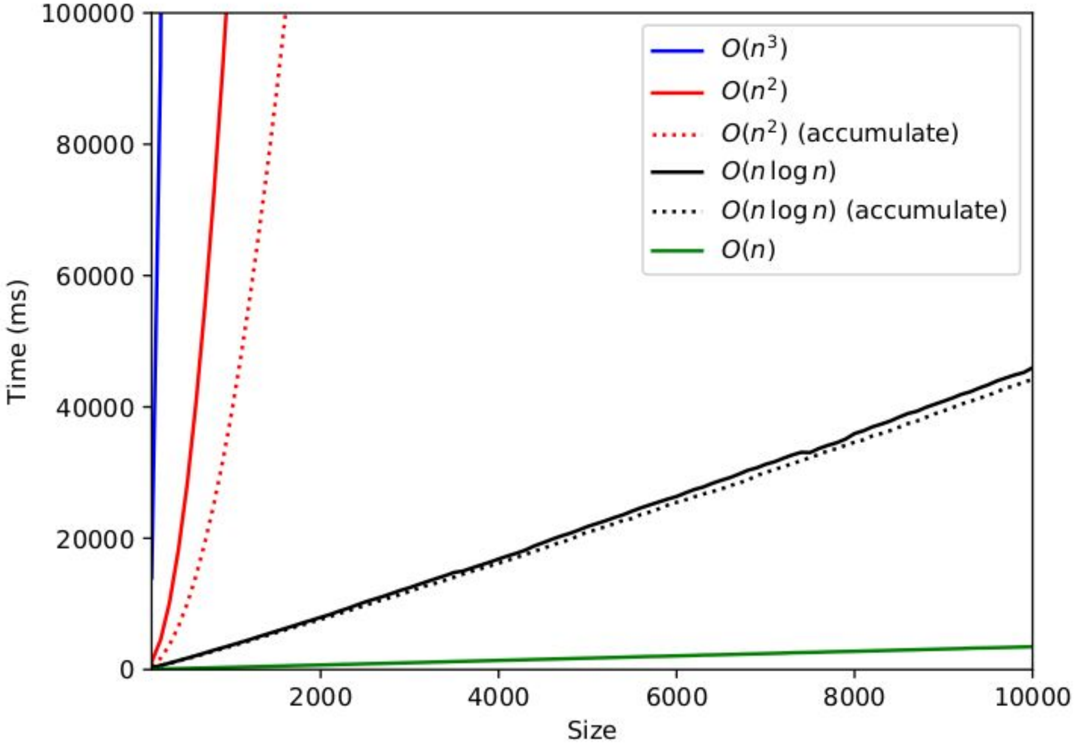
- **Input**: a list A containing n numbers
- **Output**: a slice (sublist) $A[i:j]$ of maximal sum, i.e. the slice whose element sum $\sum_{k=i}^{j-1} A[k]$ is larger or equal than the sum of any other slice

A: [1, 3, 4, -8, 2, 3, -1, 3, 4, -3, 10, -3, 2]

Max_so_far: [0, 1, 4, 8, 8, 8, 8, 8, 8, 11, 11, 18, 18, 18]
Max_here: [0, 1, 4, 8, 0, 2, 5, 4, 7, 11, 8, 18, 15, 17]
Last: [0, 0, 0, 0, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4]
Start: [0, 0, 0, 0, 0, 0, 0, 0, 0, 4, 4, 4, 4, 4]
End: [0, 0, 1, 2, 2, 2, 2, 2, 2, 8, 8, 10, 10, 10]

Running times...

- **Input**: a list A containing n numbers
- **Output**: a slice (sublist) $A[i:j]$ of maximal sum, i.e. the slice whose element sum $\sum_{k=i}^{j-1} A[k]$ is larger or equal than the sum of any other slice



Some definitions...

Computational problem

The formal relationship between the input and the desired output

Algorithm

- The description of the sequence of actions that an executor must execute to solve the problem
- Among their tasks, algorithms represent and organize the input, the output, and all the intermediate data required for the computation

Some history...

- Ahmes' Papyrus (1850 BC, peasant algorithm for multiplication)
- Numerical algorithms have been studied by Babylonians and Indian mathematicians
- Algorithms used even today have been studies by Greek mathematicians more than 2000 years ago
  - Euclid's Algorithm for the greatest common divisor
  - Geometrical algorithms (angle bisection and trisection, tangent drawing, etc)

Algorithms: the name...

Abu Abdullah Muhammad bin Musa al-Khwarizmi

- He was a Persian mathematician, astronomer, astrologer, geographer
- He introduced the indian numbers in the western world
- From his name: algorithm

Al-Kitab al-muhtasar fi hisab al-gabr wa-l-muqabala

- His most famous work (820 AC)
- Translated in Latin with the title: *Liber algebrae et almucabala*

Computational problems: examples

Minimum

The minimum of a set S is the element of S which is smaller or equal that any other element of S.

$$min(S) = a \Leftrightarrow \exists a \in S : \forall b \in S : a \leq b$$

Looukp

Let $S = s_0, s_1, \ldots, s_{n-1}$ be a sequence of distinct, sorted numbers, i.e. $s_0 < s_1 < \ldots < s_{n-1}$. To perform a lookup of the position of value v in S corresponds to returning the index i such that $0 \leq i < n$, if v is contained at position i, -1 otherwise.

$$lookup(S, v) = \begin{cases} i & \exists i \in \{0, \ldots, n-1\} : S_i = v \\ -1 & \text{otherwise} \end{cases}$$

Computational problems: examples

Minimum

The minimum of a set S is the element of S which is smaller or equal that any other element of S.

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Let $S = s_0, s_1, \ldots, s_{n-1}$ be a sequence of distinct, sorted numbers, i.e. $s_0 < s_1 < \ldots < s_{n-1}$. To perform a lookup of the position of value v in S corresponds to returning the index i such that $0 \leq i < n$, if v is contained at position i, -1 otherwise.

$$lookup(S, v) = \begin{cases} i & \exists i \in \{0, \ldots, n-1\} : S_i = v \\ -1 & \text{otherwise} \end{cases}$$

Note: we described a relationship between input and output. Nothing is said on how to compute the result (that's the difference of math to computer science :-))

Naive solutions

Minimum

To find the minimum of a set, compare each element with every other element; the element that is smaller than any other is the minimum.

Lookup

To find a value v in the sequence S, compare v with any other element of S, in order, and return the corresponding index if a correspondence is found; returns -1 if none of the elements is equal to v.

Computational Problem

↓

First, let's **translate** the computational problem into an algorithm to solve it.

Then, make it **more efficient** if possible!

Naive solutions: the code

```python
def my_min(S):
    for x in S:
        isMin = True
        for y in S:
            if x > y:
                isMin = False
        if isMin:
            return x

A = [7, -1, 9,121, -3, 4, 13]

print(A)
print("min: {}".format(my_min(A)))
```

```
[7, -1, 9, 121, -3, 4, 13]
min: -3
```

```python
def lookup(L, v):
    for i in range(len(L)):
        if L[i] == v:
            return i
    return -1

my_list = [1, 3, 5, 11, 17, 121, 443]
print(my_list)
print("{} in pos: {}".format(17,
                             lookup(my_list, 17)))
print("{} in pos: {}".format(4,
                             lookup(my_list, 4)))
```

```
[1, 3, 5, 11, 17, 121, 443]
17 in pos: 4
4 in pos: -1
```

This is a direct translation of the computational problem. Can we do better?

Algorithm evaluation

Does it solve the problem in a correct way?

- Mathematical proof vs informal description
- Some problems can only be solved in an approximate way
- Some problems cannot be solved at all

Does it solve the problem in an efficient way?

- How to measure efficiency
- Some solutions are optimal: you cannot find better solutions
- For some problems, there are no efficient solutions

Note on efficiency: algorithm efficiency has a bigger impact on performance than technical details (e.g. using Python vs. C, itertools vs sum etc...)

Efficiency: time and space

Algorithm complexity

Analysis of the resources employed by an algorithm to solve a problem, depending on the size and the type of input

Resources

- Time: time needed to execute the algorithm
  - Should we measure it with a cronometer?
  - Should I measure it by counting the number of elementary operations?
- Space: amount of used memory
- Bandwidth: amount of bit transmitted (distributed algorithms)

Normally, we focus on time because there is a relationship between TIME and SPACE. Intuitively, Using N^2 space will require at least N^2 time to read the input... Normally, TIME > SPACE...

Algorithm evaluation: minimum

How many comparisons do we perform?

```python
def my_min(S):
    for x in S:
        isMin = True
        for y in S:
            if x > y:
                isMin = False
        if isMin:
            return x

A = [7, -1, 9,121, -3, 4, 13]

print(A)
print("min: {}".format(my_min(A)))
```

```
[7, -1, 9, 121, -3, 4, 13]
min: -3
```

This is the most expensive operation (might work on ints, strings, files,...)

If len(S) = n:

for x in 1,...,n:

for y in 1,...,n:

x>y

...

➔ n*n comparisons

Naive algorithm has complexity: O(n^2)

Algorithm evaluation: minimum, a better solution

How many comparisons do we perform?

```
def my_faster_min(S):
    min_so_far = S[0] #first element
    i = 1
    while i < len(S):
        if S[i] < min_so_far:
            min_so_far = S[i]
        i = i +1
    return min_so_far

A = [7, -1, 9,121, -3, 4, 13]

print(A)
print("min: {}".format(my_min(A)))
```

```
[7, -1, 9, 121, -3, 4, 13]
min: -3
```

This is the most expensive operation (might work on ints, strings, files,...)

If len(S) = n:

i= 1,...,n-1

S[i] < min_so_far

➔ n-1 comparisons

Naive algorithm "has complexity": n^2

Better algorithm "has complexity": n-1

Algorithm evaluation: lookup

How many comparisons do we perform?

```
def lookup(L, v):
    for i in range(len(L)):
        if L[i] == v:
            return i
    return -1

my_list = [1, 3, 5, 11, 17, 121, 443]
print(my_list)
print("{} in pos: {}".format(17,
                             lookup(my_list, 17)))
print("{} in pos: {}".format(4,
                             lookup(my_list, 4)))
```

```
[1, 3, 5, 11, 17, 121, 443]
17 in pos: 4
4 in pos: -1
```

I compare v with first element, then to the second etc. when I find it I stop.

➜ n comparisons

Naive algorithm “has complexity”: n

Algorithm evaluation: lookup, better solution

How many comparisons do we perform?

```
def lookup(L, v):
    for i in range(len(L)):
        if L[i] == v:
            return i
        elif L[i] > v:
            return -1
    return -1

my_list = [1, 3, 5, 11, 17, 121, 443]
print(my_list)
print("{} in pos: {}".format(17,
                             lookup(my_list, 17)))
print("{} in pos: {}".format(4,
                             lookup(my_list, 4)))

print("{} in pos: {}".format(500,
                             lookup(my_list, 4)))
```

```
[1, 3, 5, 11, 17, 121, 443]
17 in pos: 4
4 in pos: -1
500 in pos: -1
```

I loop through the list, if I find value > v I can stop.

Generally faster, but worst case (es. 500 below)

➔ n comparisons

Naive algorithm “has complexity”: n
Better algorithm “has complexity”: n

Algorithm evaluation: best, worst and average case

What is the most important case?

Best: lookup(L,1) solved in 1 step.

Not interested. We are never lucky!!!

Worst: lookup(L,10) solved in 9 steps

Normally, the most informative case

Average: lookup(L,6) solved in 4 steps

Sometimes interesting

Lookup: more efficient algorithm

The list is sorted…

lookup(L,v)

ex. lookup(L,28)

1	7	12	15	21	27	29	41	57

Lookup: more efficient algorithm

The list is sorted…

lookup(L,v)

ex. lookup(L,28)

m

1	7	12	15	21	27	29	41	57

Let's start considering the **median value**, m.

If L[m] = v. Found it!

if L[m] > v. Search L[0:m]

if L[m] <v. Search L[m+1:]

Lookup: more efficient algorithm

The list is sorted…

lookup(L,v)

ex. lookup(L,28)

				m				
1	7	12	15	21	27	29	41	57

Let’s start considering the **median value**, m.

If L[m] = v. Found it!

if L[m] > v. Search L[0:m]

21 < **28** ➜ ignore L[0:m]

if L[m] <v. Search L[m+1:]

Lookup: more efficient algorithm

The list is sorted...

lookup(L,v)

ex. lookup(L,28)

						m		
1	7	12	15	21	27	29	41	57

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if L[m] <v. Search L[m+1:]

28 < 29 → ignore L[m+1:]

Lookup: more efficient algorithm

The list is sorted...

lookup(L,v)

ex. lookup(L,28)

						m		
1	7	12	15	21	27	29	41	57

Let's start considering the **median value**, m.

If L[m] = v. Found it!

if L[m] > v. Search L[0:m]

28 < 29 → ignore L[m+1:]

if L[m] <v. Search L[m+1:]

Lookup: more efficient algorithm

The list is sorted...

lookup(L,v)

ex. lookup(L,28)

					m			
1	7	12	15	21	27	29	41	57

Let’s start considering the **median value**, m.

If L[m] = v. Found it!

if L[m] > v. Search L[0:m]

if L[m] <v. Search L[m+1:]

27 != **28** → NOT FOUND

Lookup: the recursive code

```
def lookup_rec(L, v, start,end):
    if end < start:
        return -1
    else:
        m = (start + end)//2
        if L[m] == v: #found!
            return m
        elif v < L[m]: #look to the left
            return lookup_rec(L, v, start, m-1)
        else: #look to the right
            return lookup_rec(L, v, m+1, end)




my_list = [1, 3, 5, 11, 17, 121, 443]
print(my_list)
print("{} in pos: {}".format(17,
                             lookup_rec(my_list, 17, 0, len(my_list)-1)))
print("{} in pos: {}".format(4,
                             lookup_rec(my_list, 4, 0, len(my_list)-1)))

print("{} in pos: {}".format(443,
                             lookup_rec(my_list, 443, 0, len(my_list)-1)))
```

```
[1, 3, 5, 11, 17, 121, 443]
17 in pos: 4
4 in pos: -1
443 in pos: 6
```

can stop and check when end == start
but it is similar

Lookup: the recursive code

```python
def lookup_rec(L, v, start,end):
    if end < start:
        return -1
    else:
        m = (start + end)//2
        if L[m] == v: #found!
            return m
        elif v < L[m]: #look to the left
            return lookup_rec(L, v, start, m-1)
        else: #look to the right
            return lookup_rec(L, v, m+1, end)




my_list = [1, 3, 5, 11, 17, 121, 443]
print(my_list)
print("{} in pos: {}".format(17,
                             lookup_rec(my_list, 17, 0, len(my_list)-1)))
print("{} in pos: {}".format(4,
                             lookup_rec(my_list, 4, 0, len(my_list)-1)))

print("{} in pos: {}".format(443,
                             lookup_rec(my_list, 443, 0, len(my_list)-1)))
```

```
[1, 3, 5, 11, 17, 121, 443]
17 in pos: 4
4 in pos: -1
443 in pos: 6
```

How many comparisons?

2 comparisons (==, <) at each call

Anyone wants to try?

Lookup: the recursive code

```
def lookup_rec(L, v, start,end):
    if end < start:
        return -1
    else:
        m = (start + end)//2
        if L[m] == v: #found!
            return m
        elif v < L[m]: #look to the left
            return lookup_rec(L, v, start, m-1)
        else: #look to the right
            return lookup_rec(L, v, m+1, end)



my_list = [1, 3, 5, 11, 17, 121, 443]
print(my_list)
print("{} in pos: {}".format(17,
                             lookup_rec(my_list, 17, 0, len(my_list)-1)))
print("{} in pos: {}".format(4,
                             lookup_rec(my_list, 4, 0, len(my_list)-1)))

print("{} in pos: {}".format(443,
                             lookup_rec(my_list, 443, 0, len(my_list)-1)))
```

```
[1, 3, 5, 11, 17, 121, 443]
17 in pos: 4
4 in pos: -1
443 in pos: 6
```

How many comparisons?

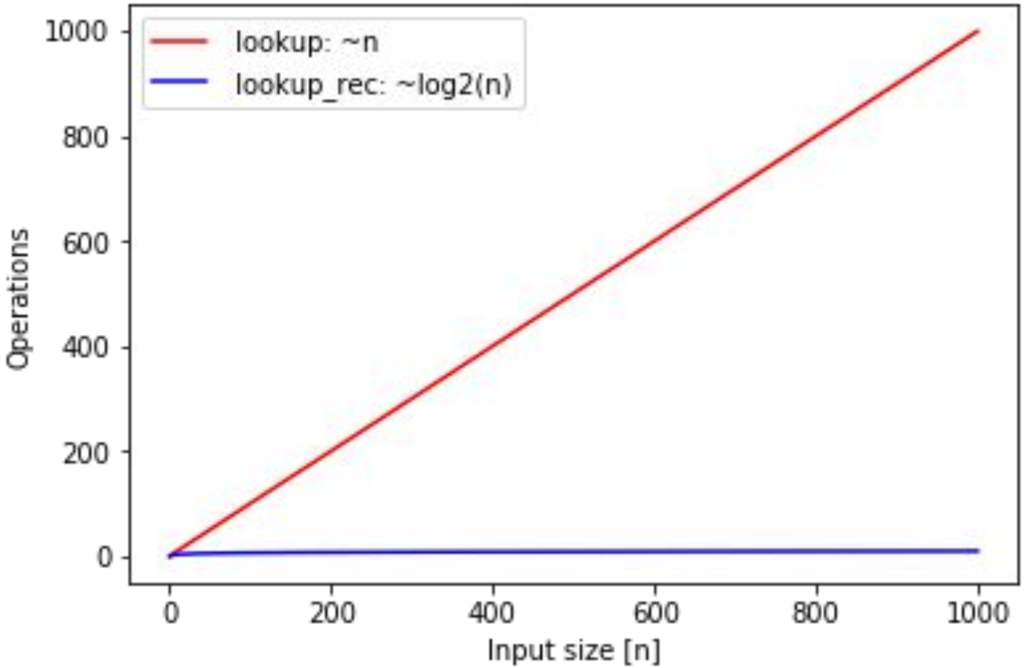
2 comparisons (==, <) at each call

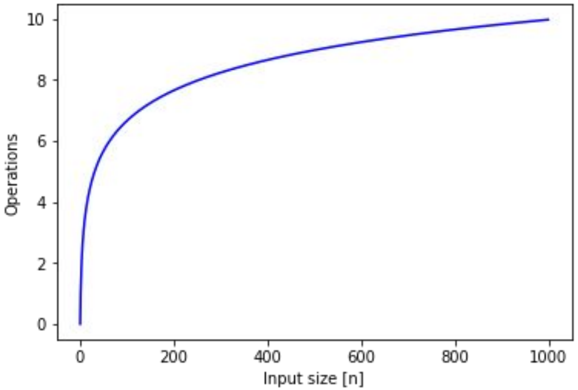
At beginning 1024 elements...
then 512...
then 256...
then 128...
then 64...
then 32...
then 16...
then 8...
then 4...
then 2...
then 1

→ log2(1024) +1 iterations

Complexity ~ log2 n

Lookup analysis





Correctness

Invariant

A condition that is always true in a specific point in an algorithm

Loop invariant

- A condition that is always true at the beginning of a loop iteration
- what is exactly the beginning of a loop iteration?

Class invariant

- A condition always true when the execution of a class method is completed

Correctness

The **loop invariant** helps us proving that the algorithm is **correct:**

By induction...

Initialization (base case):

The condition is true before the first iteration

Conservation (inductive step):

If the condition is true before the iteration of the loop, then it remains true at the end (before the next iteration)

Conclusion:

At the end **the invariant** must represent the "correctness" of the algorithm

Correctness of min

Invariant: At the beginning of each iteration of the while loop, min_so_far contains the partial minimum of the elements in S[0:i].

```
def my_faster_min(S):
    min_so_far = S[0] #first element
    i = 1
    while i < len(S):
        if S[i] < min_so_far:
            min_so_far = S[i]
        i = i +1
    return min_so_far

A = [7, -1, 9,121, -3, 4, 13]

print(A)
print("min: {}".format(my_min(A)))
```

```
[7, -1, 9, 121, -3, 4, 13]
min: -3
```

Base case:

min_so_far = S[0] **IS** the minimum of elements in S[0:1]

Induction step:

at each iteration i,
min_so_far is updated
IFF S[i] < min_so_far

min_so_far always contains min of elements S[0:i]

Correctness of lookup

Exercise: prove the correctness of lookup_rec

```python
def lookup_rec(L, v, start,end):
    if end < start:
        return -1
    else:
        m = (start + end)//2
        if L[m] == v: #found!
            return m
        elif v < L[m]: #look to the left
            return lookup_rec(L, v, start, m-1)
        else: #look to the right
            return lookup_rec(L, v, m+1, end)




my_list = [1, 3, 5, 11, 17, 121, 443]
print(my_list)
print("{} in pos: {}".format(17,
                             lookup_rec(my_list, 17, 0, len(my_list)-1)))
print("{} in pos: {}".format(4,
                             lookup_rec(my_list, 4, 0, len(my_list)-1)))

print("{} in pos: {}".format(443,
                             lookup_rec(my_list, 443, 0, len(my_list)-1)))
```

```
[1, 3, 5, 11, 17, 121, 443]
17 in pos: 4
4 in pos: -1
443 in pos: 6
```

What is the invariant?

Correctness of lookup

Exercise: prove the correctness of lookup_rec

```python
def lookup_rec(L, v, start,end):
    if end < start:
        return -1
    else:
        m = (start + end)//2
        if L[m] == v: #found!
            return m
        elif v < L[m]: #look to the left
            return lookup_rec(L, v, start, m-1)
        else: #look to the right
            return lookup_rec(L, v, m+1, end)




my_list = [1, 3, 5, 11, 17, 121, 443]
print(my_list)
print("{} in pos: {}".format(17,
                              lookup_rec(my_list, 17, 0, len(my_list)-1)))
print("{} in pos: {}".format(4,
                              lookup_rec(my_list, 4, 0, len(my_list)-1)))

print("{} in pos: {}".format(443,
                              lookup_rec(my_list, 443, 0, len(my_list)-1)))
```

```
[1, 3, 5, 11, 17, 121, 443]
17 in pos: 4
4 in pos: -1
443 in pos: 6
```

What is the invariant?

If v is in L, it is located in L[start:end+1]

Correctness of lookup

```
def lookup_rec(L, v, start,end):
    if end < start:
        return -1
    else:
        m = (start + end)//2
        if L[m] == v: #found!
            return m
        elif v < L[m]: #look to the left
            return lookup_rec(L, v, start, m-1)
        else: #look to the right
            return lookup_rec(L, v, m+1, end)
```

Exercise: prove the correctness of lookup_rec.
By induction on n = end-start

Base case (n = 0): if n == 0, this means that **end < start**.
The algorithm **returns −1**. Correct given that if n == 0, v is not present.

Inductive hypothesis: given a size n, let us assume that the algorithm is correct for all sizes n 0 < n

Inductive step: given a size n > 0, let m be the median element.

> If S[m]==v, then the algorithm returns m, because m is the actual position of v
>
> If v < S[m], then if v is present, since S is sorted, it must be located in S[start:m].
>
> By inductive hypothesis, lookup_rec(S, v,start, m-1) will return the correct position of v if present, or -1 if not present.
>
> if v > S[m] is symmetric.