

# Scientific Programming: Part B

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## Lecture 3

Luca Bianco - Academic Year 2019-20  
luca.bianco@fmach.it

# Problem vs. algorithm complexity

**Goal:** *reason about complexity of problems and algorithms*

- In some cases, it is possible to improve what is considered "normal"
- In other cases, it is impossible to improve the existing solutions
- What is the relation between a problem and the algorithms that solve it?

**Back to basics!**

- Sums
- Products

# Sum of binary (or any other base) numbers

## Basic sum algorithm – `sum()`

- each of the  $n$  bits have to be considered
- total cost is equal to  $cn$  ( $c \equiv$  the cost required to sum three bits and generate the carry-over)

$$\begin{array}{r} 1110111111111 \\ 101100110110111 \\ \hline 1101010100100010 \end{array} +$$

**Note:** in programming languages like python the sum is a basic operation, up to the biggest possible integer it is done by the CPU.

After that → arbitrary precision and therefore we would need an algorithm like this.

# Sum of binary (or any other base) numbers

## Question

Is there a better method?

$$\begin{array}{r} 111011111111 \\ 101100110110111 \\ 111101101101011 \\ \hline 1101010100100010 \end{array}$$



**Nope. There is no better way** (improvements like grouping bits just deliver better constants)

## Sketch of the proof. Reasoning by contradiction:

since to compute the result I have to consider all bits,  
if there existed a better method, it would skip some bits

→ hence the solution might not be correct if only those bits are changed!

# Lower bound to the complexity of a problem

## Notation $\Omega(f(n))$ – Lower bound

The computational complexity of a problem is equal to  $\Omega(f(n))$  if all possible algorithms solving the problem have a complexity which is equal to  $c \cdot f(n)$  or larger, where  $c$  is an appropriate constant.

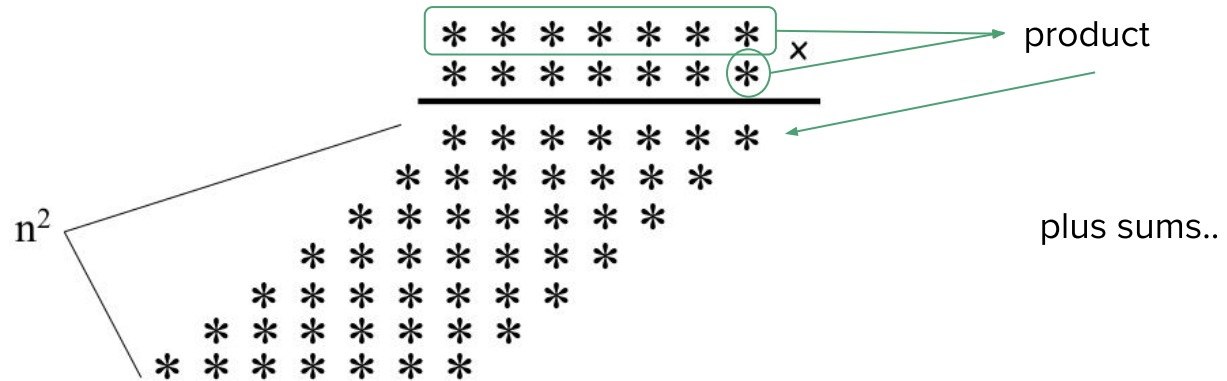
## Lower bound for the sum problem

The problem of summing two binary numbers with  $n$  bits is  $\Omega(n)$ .

# Product of binary (or any other base) numbers

## "Elementary school" algorithm – `prod()`

- bit-by-bit product
- total cost  $cn^2$



# Arithmetic algorithms

## Comparison of computational complexity

- Sum :  $T_{sum}(n) = O(n)$
- Products :  $T_{prod}(n) = O(n^2)$

We could conclude that:

- The product problem is inherently more costly than the sum problem
- This confirms our intuition



**Wrong.** We are comparing problems but we here in fact we have solutions. What I am saying is that **THIS solution** to compute the product is more costly than the sum!!!

# Arithmetic algorithms

## Comparing problems

To prove that the product problem is more costly than the sum problem, we must prove that **there is no solution** in linear time for the product.

We compared the algorithms, not the problems!

- We only know the "elementary school" algorithm for the sum is more efficient than the "elementary school" algorithm for the product.
- In 1960, during a conference, Kolmogorov claimed that the product has  $\Omega(n^2)$  lower bound
- A week later, he was proved wrong!



# Product of binary numbers - Divide-et-impera

## Binary number product

$$X = a \cdot 2^{n/2} + b$$

$$Y = c \cdot 2^{n/2} + d$$

$$XY = ac \cdot 2^n + (ad + bc) \cdot 2^{n/2} + bd$$

X	a	b
Y	c	d

Split the numbers in 2.  
(**n** is the number of digits).  
The most significant and least significant part.

## Decimal number product: $9977 \times 2348$

$$X = a \cdot 10^{n/2} + b$$

$$Y = c \cdot 10^{n/2} + d$$

$$XY = ac \cdot 10^n + (ad + bc) \cdot 10^{n/2} + bd$$

$$XY = 99 \times 23 \cdot 10^4 + (99 \times 48 + 77 \times 23) \cdot 10^2 + 77 \times 48$$

X	a=99	b=77
Y	c=23	d=48

Apparently we now have 4 multiplications (multiply by  $2^n$  or  $2^{n/2}$  is actually moving the digits: **shift**)

# Product of binary numbers - Divide-et-impera

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`pdi( $X$ ,  $Y$ , int  $n$ )`

---

**if**  $n == 1$  **then**

    | **return**  $X[0] \cdot Y[0]$

**else**

    | break  $X$  in two parts  $a; b$  and  $Y$  in two parts  $c; d$

    | **return**  $\text{pdi}(a, c, n/2) \cdot 2^n +$

$(\text{pdi}(a, d, n/2) + \text{pdi}(b, c, n/2)) \cdot 2^{n/2} +$

$\text{pdi}(b, d, n/2)$

---

Recursive code for  $n/2$  bits.

**Recombination of results:** sums and multiplications by  $2^{n/2} \rightarrow$  linear cost.

$$T(n) = \begin{cases} c_1 & n = 1 \\ 4T(n/2) + c_2 \cdot n & n > 1 \end{cases}$$

Note: multiplication by  $2^t$  is **linear (shift) of  $t$  positions**

# Product of binary numbers - Divide-et-impera

$$T(n) = \begin{cases} c_1 & n = 1 \\ 4T(n/2) + c_2 \cdot n & n > 1 \end{cases}$$

## Theorem

Let  $a$  and  $b$  two integer constants such that  $a \geq 1$  e  $b \geq 2$ , and let  $c, \beta$  be two real constants such that  $c > 0$  e  $\beta \geq 0$ . Let  $T(n)$  be defined by the following recurrence:

$$T(n) = \begin{cases} aT(n/b) + cn^\beta & n > 1 \\ \Theta(1) & n \leq 1 \end{cases}$$

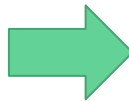
Given  $\alpha = \log a / \log b = \log_b a$ , then:

$$T(n) = \begin{cases} \Theta(n^\alpha) & \alpha > \beta \\ \Theta(n^\alpha \log n) & \alpha = \beta \\ \Theta(n^\beta) & \alpha < \beta \end{cases}$$

$$\alpha = \log_2 4 = 2$$

$$\beta = 1$$

$$T(n) = \Theta(n^2)$$



**all this mess pointless?!?**

# Product of binary numbers - Divide-et-impera

## Comparing the computational complexity

- Product :  $T_{prod}(n) = O(n^2)$
- Product :  $T_{pdi}(n) = O(n^2)$

All this, for nothing?

## Question: Is it possible to improve this idea?

Note that this recursive version calls itself 4 times



Can we call the function less than 4 times?

Sums and shifts cannot be faster than  $\Theta(n)$ .

[more on this later... let's step back a sec]

```
pdi(X, Y, int n)
if n == 1 then
    return X[0] · Y[0]
else
    break X in two parts a; b and Y in two parts c; d
    return pdi(a, c, n/2) · 2n +
           (pdi(a, d, n/2) + pdi(b, c, n/2)) · 2n/2 +
           pdi(b, d, n/2)
```

$$T(n) = \begin{cases} c_1 & n = 1 \\ 4T(n/2) + c_2 \cdot n & n > 1 \end{cases}$$

# Product of complex numbers (courtesy of Gauss)

## Multiplying complex numbers

- $(a + bi)(c + di) = [ac - bd] + [ad + bc]i$
- Input:  $a, b, c, d$
- Output:  $a \times c - b \times d, a \times d + b \times c$

$a$ : real part  
 $b * i$ : imaginary part

$$i * i = -1$$

## Questions

Let's consider a computing model where scalar multiplications cost 1, scalar sums/subtractions cost 0.01

- How much does it cost to multiply two complex numbers?
- Can you do better than this?

4 multiplications and 2 sums back then very expensive to multiply numbers (cost: 4.002)

# Product of complex numbers (courtesy of Gauss)

$$(a + bi)(c + di) = [ac - bd] + [ad + bc]i$$

Input:  $a, b, c, d$

Output:  $a \times c - b \times d, a \times d + b \times c$  \_

Multiplication: costs 1

Sum/subtraction: cost 0.01

- The naive algorithm associated to the definition cost 4.02
- Gauss solution (1805):

**Input:**  $a, b, c, d$ , **Output:**  $A1 = ac - bd, A2 = ad + bc$

$$m1 = a \times c$$

$$m2 = b \times d$$

$$A1 = m1 - m2 = ac - bd$$

$$m3 = (a + b) \times (c + d) = ac + ad + bc + bd$$

$$A2 = m3 - m1 - m2 = ad + bc$$



clever part.

In this case, the cost is 3.05.

This makes it 3  
multiplications and 5  
sums/subtractions  
(25% improvement).

# Karatsuba Algorithm (1962) (Inspired by Gauss)

$$(a + bi)(c + di) = [ac - bd] + [ad + bc]i$$

Input:  $a, b, c, d$

Output:  $a \times c - b \times d, a \times d + b \times c$

$$A_1 = a \times c$$

$$A_3 = b \times d$$

$$m = (a + b) \times (c + d) = ac + ad + bc + bd$$

$$A_2 = m - A_1 - A_3 = ad + bc$$

Three recursive calls that split the numbers in  $n/2$  digits.

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```
boolean [] KARATSUBA(boolean[] X, boolean[] Y, int n)
```

---

```
if n == 1 then
```

```
    return X[0] · Y[0]
```

```
else
```

```
    break X in a; b e Y in c; d
```

```
    A1 = KARATSUBA(a, c, n/2)
```

```
    A3 = KARATSUBA(b, d, n/2)
```

```
    m = KARATSUBA(a + b, c + d, n/2)
```

```
    A2 = m - A1 - A3
```

```
    return A1 · 2n + A2 · 2n/2 + A3
```

---

Recurrence:

$$T(n) = \begin{cases} c_1 & n = 1 \\ 3T(n/2) + c_2 \cdot n & n > 1 \end{cases}$$

# Karatsuba Algorithm (1962) (Inspired by Gauss)

Recurrence

$$T(n) = \begin{cases} c_1 & n = 1 \\ 3T(n/2) + c_2 \cdot n & n > 1 \end{cases}$$

Applying the master theorem

$$\alpha = \log_2 3 \approx 1.58$$

$$\beta = 1$$

$$T(n) = \Theta(n^{1.58})$$



# Take home message...

## Comparing the computational complexity

- Product :  $T_{prod}(n) = O(n^2)$       Es.  $T_{prod}(10^6) = 10^{12}$
- Product :  $T_{kara}(n) = O(n^{1.58...})$       Es.  $T_{kara}(10^6) = 3 \cdot 10^9$



1000x  
faster

## Conclusions

- The "naif" algorithm is not always the best...
- ... there is often space for improvement ...
- ... unless you can prove the the opposite!

# Extensions... Multiplication

- **Toom-Cook** (1963)
  - Also called Toom3, its complexity is  $O(n^{\log 5 / \log 3}) \approx O(n^{1.465})$
  - Karatsuba  $\equiv$  Toom2
  - "Elementary school" product  $\equiv$  Toom1
- **Schönhage-Strassen** (1971)
  - Complexity:  $O(n \cdot \log n \cdot \log \log n)$
  - Based on Fast Fourier Transforms
- **Martin Fürer** (2007)
  - Complexity:  $O(n \cdot \log n \cdot 2^{O(\log^* n)})$
- Lower bound:  $\Omega(n \log n)$  (conjecture)

## Iterated logarithm

$n$	$\log^* n$
$(-\infty, 1]$	0
$(1, 2]$	1
$(2, 4]$	2
$(4, 16]$	3
$(16, 2^{16}]$	4
$(2^{16}, 2^{65536}]$	5



still to be proven

# Logarithmic cost model

```
def fact(n):  
    res = 1  
    for i in range(1,n+1):  
        res = res * i  
    return res
```

```
print(fact(5))  
print(fact(10))  
print(fact(20))  
print(fact(30))  
print(fact(40))
```

```
120  
3628800  
2432902008176640000  
2652528598121910586363084800000000  
8159152832478977343456112695961158942720000000000
```

This algorithm performs  $n$  multiplications, so  $\theta(n)$

Is it correct?

Remember that a cost function goes from the **size of the input** to the **time**.

# Logarithmic cost model

```
def fact(n):  
    res = 1  
    for i in range(1,n+1):  
        res = res * i  
    return res
```

This algorithm performs **n** multiplications, so  **$\Theta(n)$**

What is the size of the input?

$$k = \lceil \log n \rceil$$

How many multiplications, in terms of  $k$ ?

$$n = 2^k$$

How many bits are necessary, to represent the output?

$$\begin{aligned} \lceil \log n! \rceil &= \Theta(n \log n) \\ &= 2^k \cdot k \end{aligned}$$

How much does it cost to multiply two numbers of  $2^k \cdot k$  bits?

$$\Theta(2^{2k} \cdot k^2)$$

What is the complexity of the multiplication?

$$\Theta(2^{3k} \cdot k^2) = \Theta(n^3 (\log n)^2)$$

Is it correct?

Remember that a cost function goes from the **size of the input** to the **time**.

# Sorting algorithms

**Goal:** *evaluate the algorithms based on the type of input*

- In some cases, algorithms behave differently depending on the characteristics of the input
- Knowing these characteristics in advance enable to choose the best algorithm for that particular scenario
- The **sorting problem** is a good school where to show such concepts



sorting algorithms are already implemented (in general, no need to reinvent the wheel) , but they are a great training ground.

# Sorting

## Sorting problem

- **Input:** A sequence  $A = a_1, a_2, \dots, a_n$  containing  $n$  values
- **Output:** A sequence  $B = b_1, b_2, \dots, b_n$  that is a permutation of  $A$  such that  $b_1 \leq b_2 \leq \dots \leq b_n$

Naive approach:

- Search for the minimum, put it in the correct position, reduce the problem to the  $n - 1$  elements that are left and continue until the sequence is finished
- This is called **selection sort**

# Selection Sort

Search for the minimum, put it in the correct position, reduce the problem to the  $n - 1$  elements that are left and continue until the sequence is finished

```
#returns the index of the minimum element in A[i:]
def argmin(A,i):
    min_pos = i
    for j in range(i+1, len(A)):
        if A[j] < A[min_pos]:
            min_pos = j
    return min_pos
```

```
def selection_sort(A):
    for i in range(len(A)-1):
        ind = argmin(A,i)
        tmp = A[i]
        A[i], A[ind] = A[ind], A[i] #swap
```

```
L = [7,4,2,1,8,3,5]
print("{}".format(L))
selection_sort(L)
print("{}".format(L))
```

```
[7, 4, 2, 1, 8, 3, 5]
[1, 2, 3, 4, 5, 7, 8]
```



This function repeatedly searches the **minimum in  $A[i:]$** , and **swaps** it with the element in  **$A[i]$**   
 $i: 0, \dots, n-1$  since the last value is already in the right position

# Selection Sort

Search for the minimum, put it in the correct position, reduce the problem to the  $n - 1$  elements that are left and continue until the sequence is finished

```
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```

	<i>j</i> = 0	<i>j</i> = 1	<i>j</i> = 2	<i>j</i> = 3	<i>j</i> = 4	<i>j</i> = 5	<i>j</i> = 6
<i>i</i> = 0	7	4	2	1	8	3	5
<i>i</i> = 1	1	4	2	7	8	3	5
<i>i</i> = 2	1	2	4	7	8	3	5
<i>i</i> = 3	1	2	3	7	8	4	5
<i>i</i> = 4	1	2	3	4	8	7	5
<i>i</i> = 5	1	2	3	4	5	7	8
<i>i</i> = 6	1	2	3	4	5	7	8

How much does this cost?



# Selection Sort

Search for the minimum, put it in the correct position, reduce the problem to the  $n - 1$  elements that are left and continue until the sequence is finished

```
#returns the index of the minimum element in A[i:]
def argmin(A,i):
    min_pos = i
    for j in range(i+1, len(A)):
        if A[j] < A[min_pos]:
            min_pos = j
    return min_pos
```

```
def selection_sort(A):
    for i in range(len(A)-1):
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```

```
[7, 4, 2, 1, 8, 3, 5]
[1, 2, 3, 4, 5, 7, 8]
```

How much does this cost?

How many comparisons in `argmin(A, i)`?

$$\text{len}(A) - 1 - i = n - 1 - i$$

How many comparisons in `selectionSort(A)`?

$$\begin{aligned} &= \sum_{i=0}^{n-2} (n - 1 - i) \\ &= (n - 1) + (n - 2) + \dots + 2 + 1 \\ &= \sum_{i=1}^{n-1} i = \frac{n(n-1)}{2} = n^2 - n/2 \end{aligned}$$

Complexity is  **$\theta(n^2)$**  in worst, average, best case (the algorithm works in the same way regardless of the input).

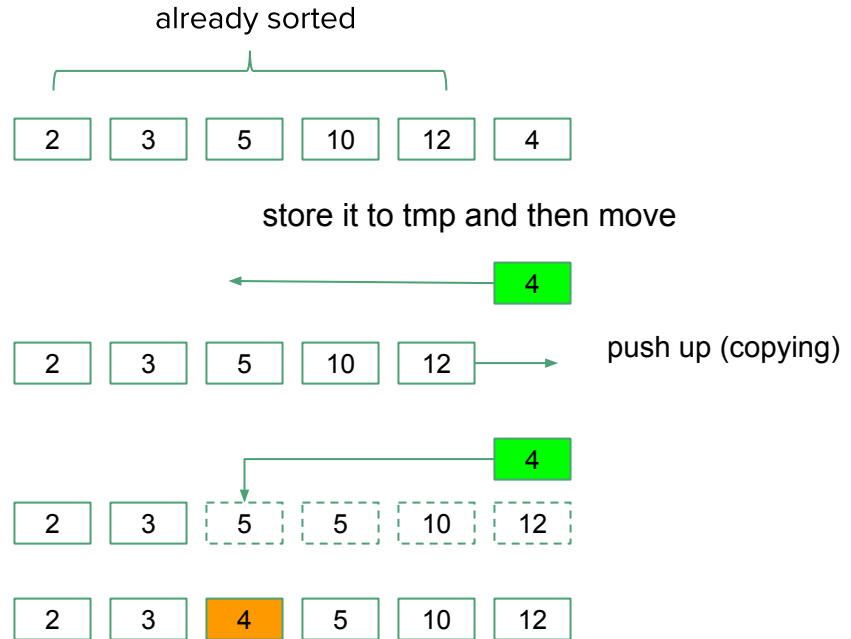
# Insertion Sort

The idea of **insertion sort** is to build a sorted list **step by step**. In each step, one element is placed in its correct position on the left-side part of the array.

At each iteration (i):

```
A[i] → temp
for j: i-1,...,0
    if A[j] < A[i]
        copy A[j] → A[j+1]
A[j] ← A[i].
```

Efficient algorithm to sort small sets of elements ~100s (small constants)  
It is “in-place” there is no need to copy the list (saves memory!)



# Insertion Sort

The idea of **insertion sort** is to build a sorted list **step by step**. In each step, one element is placed in its correct position on the left-side part of the array.

```
def insertion_sort(A):  
    for i in range(1, len(A)):  
        tmp = A[i]  
        j = i  
        while j > 0 and A[j-1] > tmp:  
            A[j] = A[j-1]  
            j = j - 1  
        A[j] = tmp
```

```
L = [7,4,2,1,8,3,5]  
print("{}".format(L))  
insertion_sort(L)  
print("{}".format(L))
```

```
[7, 4, 2, 1, 8, 3, 5]  
[1, 2, 3, 4, 5, 7, 8]
```

The first element is assumed to be a sorted list (with one element).

The first current element is the second in the list.

The current element is placed in a TMP variable, and I copy the values before in the list until they are lower than TMP.

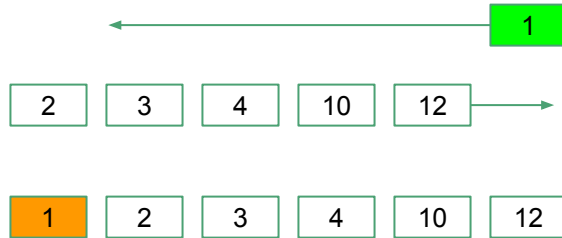
When I find a value that is lower than TMP, I place the value there

# Insertion Sort

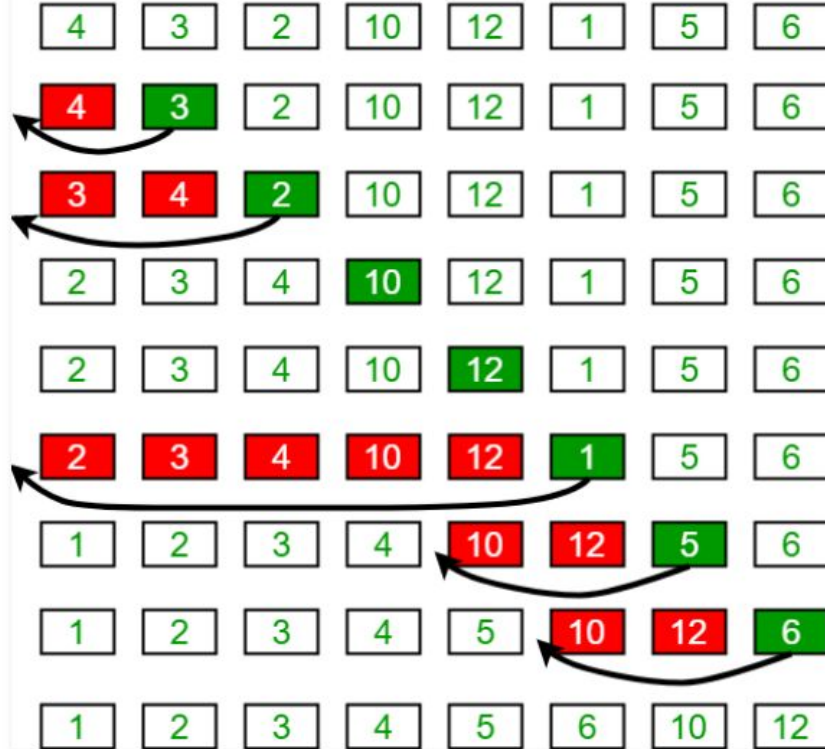
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[1, 2, 3, 4, 5, 7, 8]
```



## Insertion Sort Execution Example



# Insertion Sort: complexity

The cost does not depend only on the **size of the input** but also on how the values are sorted

What is the cost if the list is already sorted?

- the for is executed ( $n$  operations), never gets into the while:  $\theta(n)$

What is the cost if the list is sorted in reverse order?

- the for is executed ( $n$  operations), for each, all elements have to be pushed up ( $n$  operations):  $\theta(n^2)$

What is the cost on average? (informally, half list sorted)

- the for is executed ( $n$  operations), for each, half of the elements have to be pushed up ( $n$  operations):  $\theta(n^2)$

```
def insertion_sort(A):  
    for i in range(1, len(A)):  
        tmp = A[i]  
        j = i  
        while j > 0 and A[j-1] > tmp:  
            A[j] = A[j-1]  
            j = j - 1  
        A[j] = tmp
```

```
L = [7,4,2,1,8,3,5]  
print("{}".format(L))  
insertion_sort(L)  
print("{}".format(L))
```

```
[7, 4, 2, 1, 8, 3, 5]  
[1, 2, 3, 4, 5, 7, 8]
```

# Merge Sort

**IDEA:** Sorting two sublists already sorted is fast!

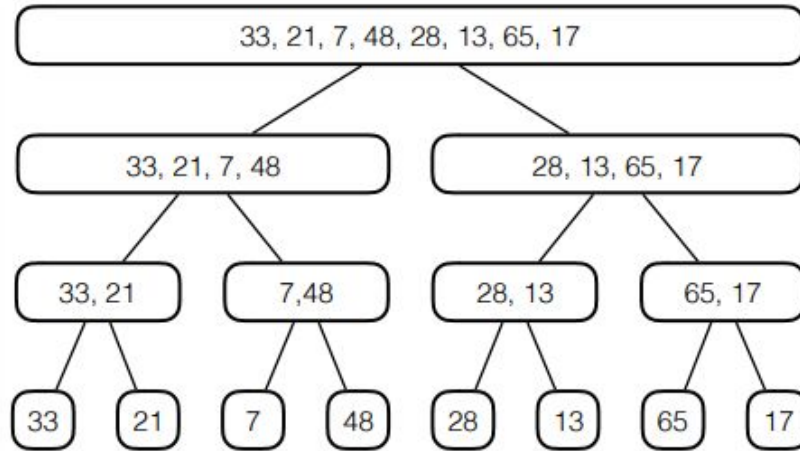
**MergeSort** is based on the divide-et-impera technique

**Divide:** Break (virtually) the sequence of  $n$  elements in two sub-sequences

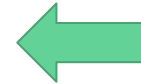
**Impera:** Call MergeSort recursively on both sub-sequences (note that sub-lists of one element are sorted lists!)

**Combine:** Join (merge) the two sorted sub-sequences

# Merge Sort

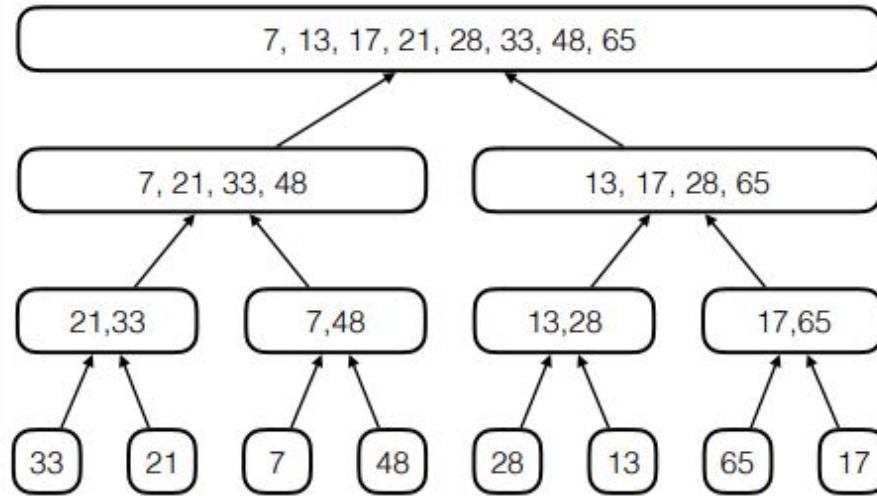


Keep dividing



1 sized lists are **ordered**!

# Merge Sort



1 List only: solution



merge sorted lists  
into bigger sorted  
lists



1 element only: sorted!



# Merge Sort

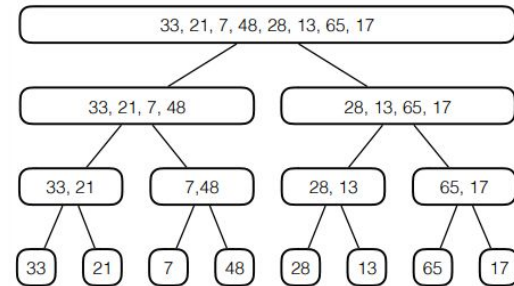
Merge sort requires three methods:

1. **merge**: gets **two sorted lists and produces a sorted list** with all the elements. Builds the return list by getting the minimum element of the two lists, “removing” it from the corresponding list and appending it to the list with the result. “removal” can be done by using two indexes pointing to the smallest elements of each of the two (sub)lists and incrementing the index of the minimum of the two (i.e. the element that is also copied to the result list);
2. **recursiveMergeSort**: gets **an unordered (sub)list**, the **index of the beginning** of the list and the index of the end of the list and recursively **splits it in two halves until it reaches lists with length 0 or 1**, at that point **it starts merging pairs of sorted lists to build the result (with merge)**;
3. **mergeSort** gets an **unordered list and applies the recursiveMergeSort** method to it starting **from position 0 to  $len-1$** .

# Merge sort: implementation

Merge sort requires three methods:

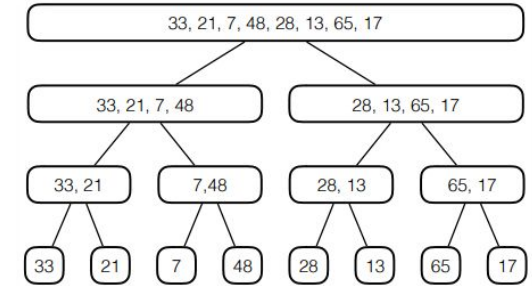
1. (**merge**): **gets two sorted lists and produces a sorted list** with all the elements. Builds the return list by getting the minimum element of the two lists, “removing” it from the corresponding list and appending it to the list with the result. “removal” can be done by using two indexes pointing to the smallest elements of each of the two (sub)lists and incrementing the index of the minimum of the two (i.e. the element that is also copied to the result list);
2. (**recursiveMergeSort**): gets **an unordered (sub)list**, the the **index of the beginning** of the list and the index of the end of the list and recursively **splits it in two halves until it reaches lists with length 0 or 1**, at that point it **starts merging pairs of sorted lists to build the result (with merge)**;
3. (**mergeSort**) gets an **unordered list and applies the recursiveMergeSort method** to it starting **from position 0 to len-1**..



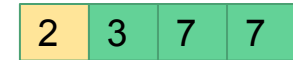
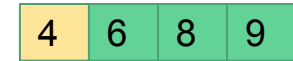
# Merge sort: implementation

The Merge method:

**gets two sorted lists and produces a sorted list** with all the elements. Builds the return list by getting the minimum element of the two lists, “removing” it from the corresponding list and appending it to the list with the result (using two indexes pointing to the minimum of each list).



i

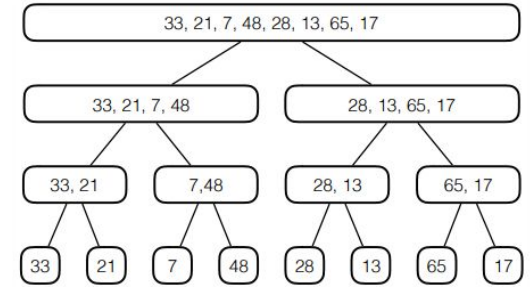


j

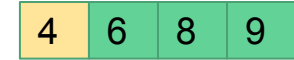
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i



2

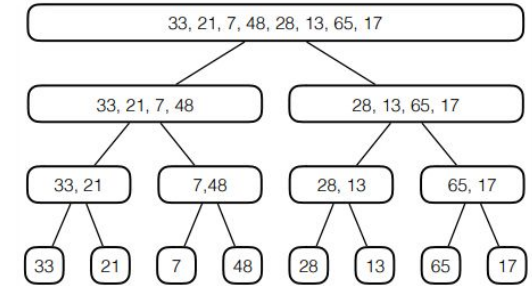


j

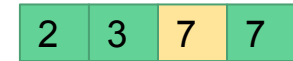
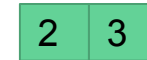
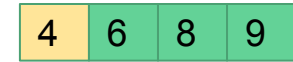
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The Merge method:

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i

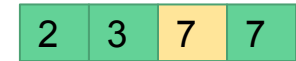
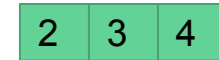
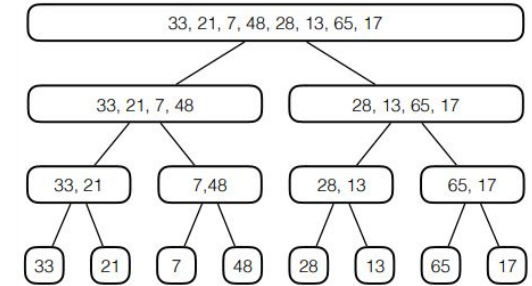


j

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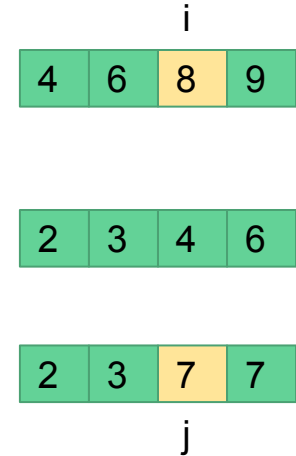
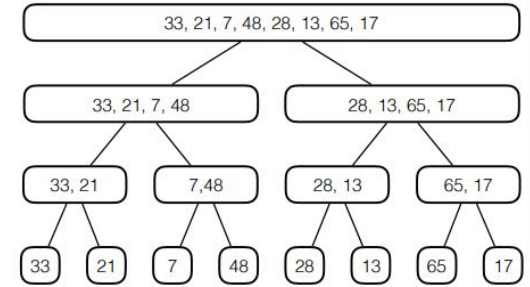


j

# Merge sort: implementation

The Merge method:

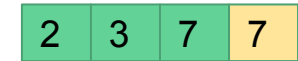
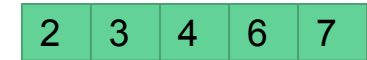
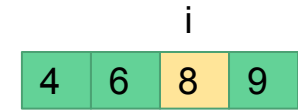
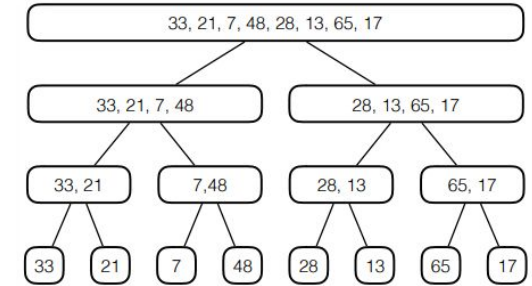
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# Merge sort: implementation

The Merge method:

**gets two sorted lists and produces a sorted list** with all the elements. Builds the return list by getting the minimum element of the two lists, “removing” it from the corresponding list and appending it to the list with the result (using two indexes pointing to the minimum of each list).



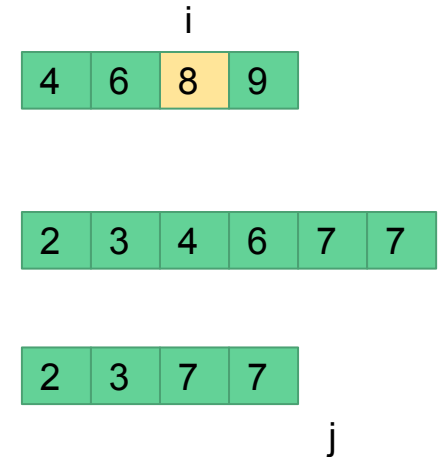
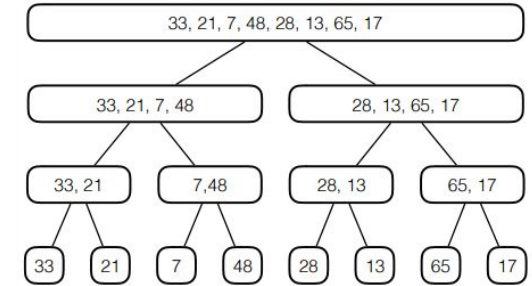
j



# Merge sort: implementation

The Merge method:

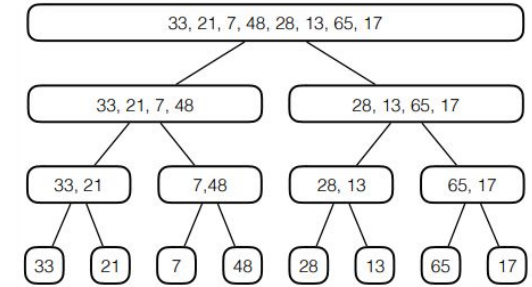
**gets two sorted lists and produces a sorted list** with all the elements. Builds the return list by getting the minimum element of the two lists, “removing” it from the corresponding list and appending it to the list with the result (using two indexes pointing to the minimum of each list).



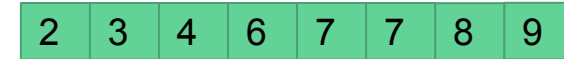
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i

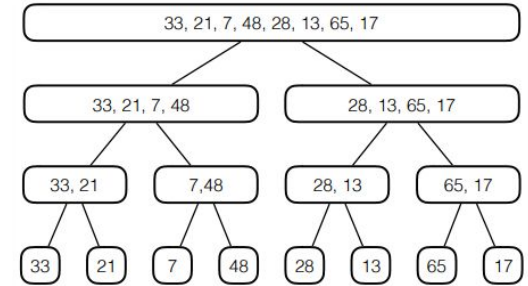


j

# Merge sort: merge

```
def merge(A, first, last, mid):  
    i = first  
    j = mid + 1  
    B = []  
  
    while i <= mid and j <= last:  
        if A[i] <= A[j]:  
            B.append(A[i])  
            i = i + 1  
        else:  
            B.append(A[j])  
            j = j + 1  
    while i <= mid:  
        B.append(A[i])  
        i = i + 1  
    for k in range(len(B)):  
        A[first + k] = B[k]  
  
A = [1, 7, 9, 11, 4, 5, 6, 13, 19]  
  
print(A)  
merge(A, 0, 8, 3)  
print(A)
```

```
[1, 7, 9, 11, 4, 5, 6, 13, 19]  
[1, 4, 5, 6, 7, 9, 11, 13, 19]
```



Variables  $i$  and  $j$  are used to scan the values of the two sublists ( $mid$  is the mid-point between the two sorted sublists).

The first while loop compares the elements when both  $A[i:mid+1]$  and  $A[j:last+1]$  are not empty, add the smaller to  $B$ , and increase either  $i$  or  $j$

The second while loop moves the remaining elements in  $A[i:mid+1]$  to  $B$

The elements in  $B$  are smaller than those in  $A[j:last+1]$   
They are moved back into  $A[first:first+len(B)]$

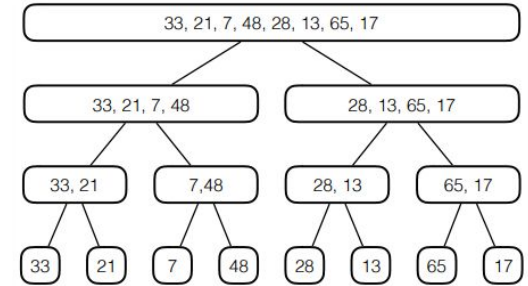
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            B.append(A[i])  
            i = i + 1  
        else:  
            B.append(A[j])  
            j = j + 1  
    while i <= mid:  
        B.append(A[i])  
        i = i + 1  
  
    for k in range(len(B)):  
        A[first + k] = B[k]
```

```
A = [1, 7, 9, 11, 4, 5, 6, 13, 19]
```

```
print(A)  
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Variables **i** and **j** are used to scan the values of the two sublists (mid is the mid-point between the two sorted sublists).

The first **while** loop compares the elements when both  $A[i:mid+1]$  and  $A[j:last+1]$  are not empty, add the smaller to **B**, and increase either **i** or **j**

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The elements in **B** are smaller than those in  $A[j:last+1]$   
They are moved back into  $A[first:first+len(B)]$

# Merge sort. Cost of merge

What is the computational cost of Merge ()?

Every time we place one element, n comparisons  $\Rightarrow$   
 **$O(n)$**

```
def merge(A, first, last, mid):
    i = first
    j = mid + 1
    B = []

    while i <= mid and j <= last:
        if A[i] <= A[j]:
            B.append(A[i])
            i = i + 1
        else:
            B.append(A[j])
            j = j + 1
    while i <= mid:
        B.append(A[i])
        i = i + 1

    for k in range(len(B)):
        A[first + k] = B[k]

A = [1, 7, 9, 11, 4, 5, 6, 13, 19]

print(A)
merge(A, 0, 8, 3)
print(A)
```

```
[1, 7, 9, 11, 4, 5, 6, 13, 19]
[1, 4, 5, 6, 7, 9, 11, 13, 19]
```

# Merge sort. Complete code

```
def merge(A, first, last, mid):
    i = first
    j = mid + 1
    B = []

    while i <= mid and j <= last:
        if A[i] <= A[j]:
            B.append(A[i])
            i = i + 1
        else:
            B.append(A[j])
            j = j + 1
    while i <= mid:
        B.append(A[i])
        i = i + 1
    while j <= last:
        B.append(A[j])
        j = j + 1

    for k in range(len(B)):
        A[first + k] = B[k]
```

```
A = [1,7, 9, 11, 4, 5, 6, 13,19]
print(A)
merge(A,0,8,3)
print(A)
```

```
[1, 7, 9, 11, 4, 5, 6, 13, 19]
[1, 4, 5, 6, 7, 9, 11, 13, 19]
```

```
def recursive_merge_sort(A, first, last):
    if last > first:
        m = (first + last) // 2
        recursive_merge_sort(A, first, m)
        recursive_merge_sort(A, m+1, last)
        merge(A, first, last, m)

def merge_sort(A):
    recursive_merge_sort(A, 0, len(A)-1)

L = [12, 71, 44, 33, 22, 9, 7, -1, 12,13]
print(L)
merge_sort(L)
print(L)
```

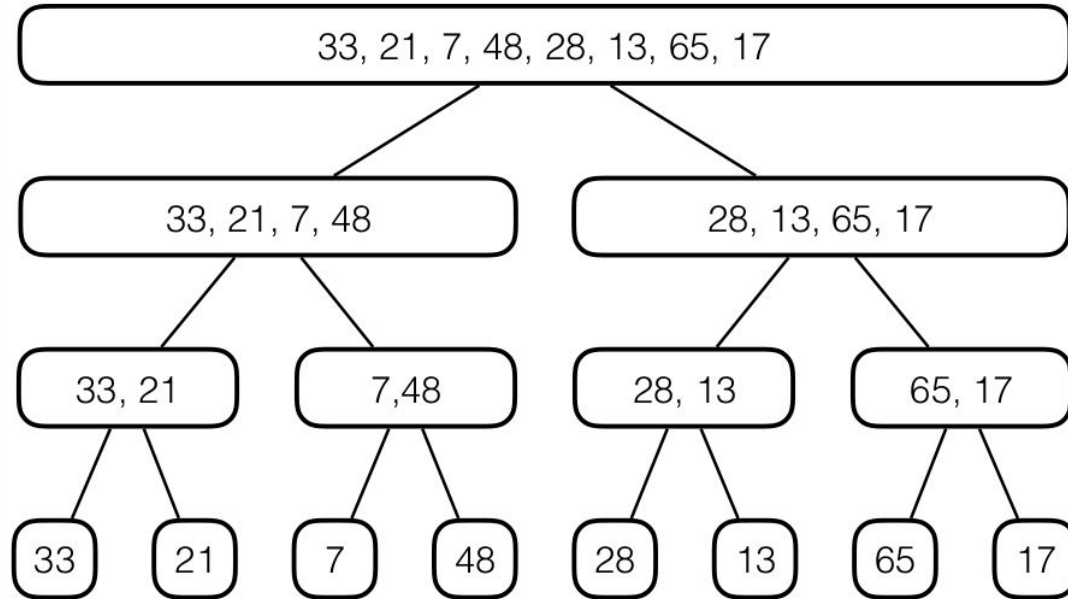
```
[12, 71, 44, 33, 22, 9, 7, -1, 12, 13]
[-1, 7, 9, 12, 12, 13, 22, 33, 44, 71]
```



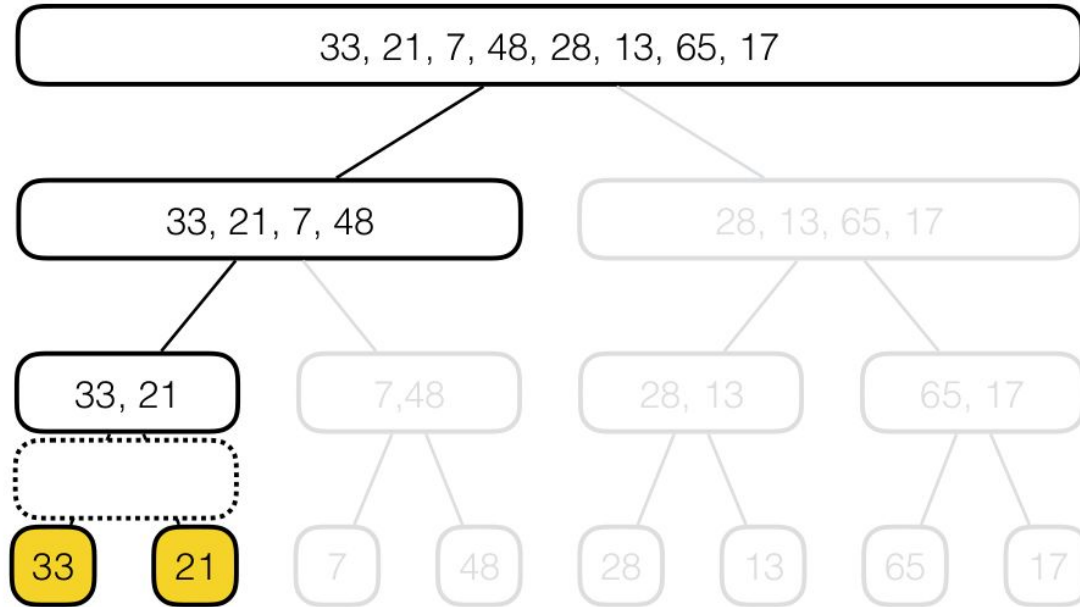
If we have some elements:

- sort from first to m
- sort from m+1 to last
- merge the two sublists

# Merge sort.

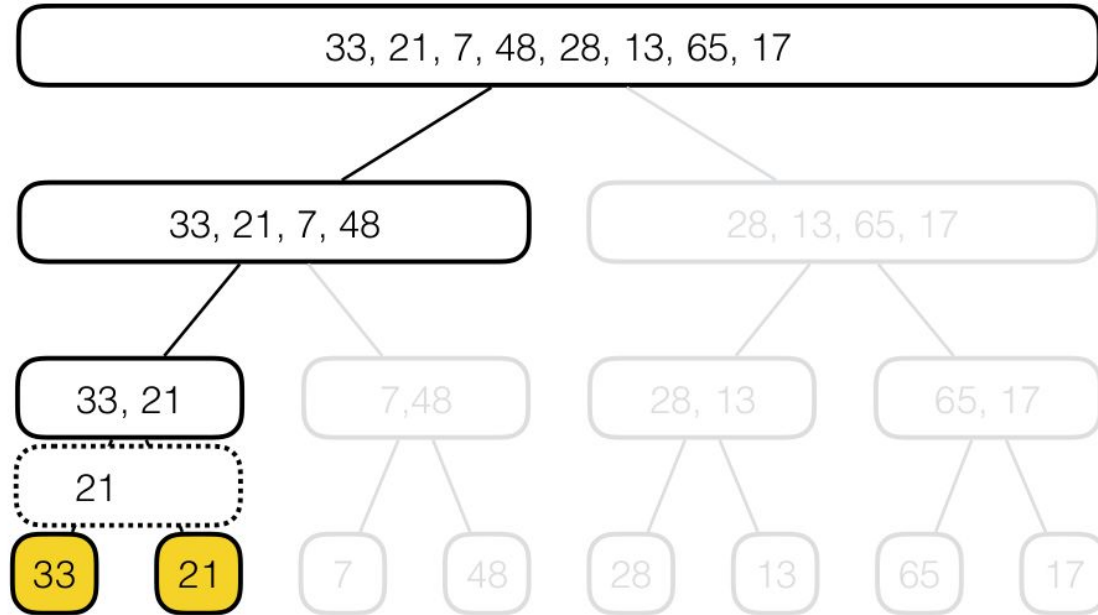


# Merge sort.

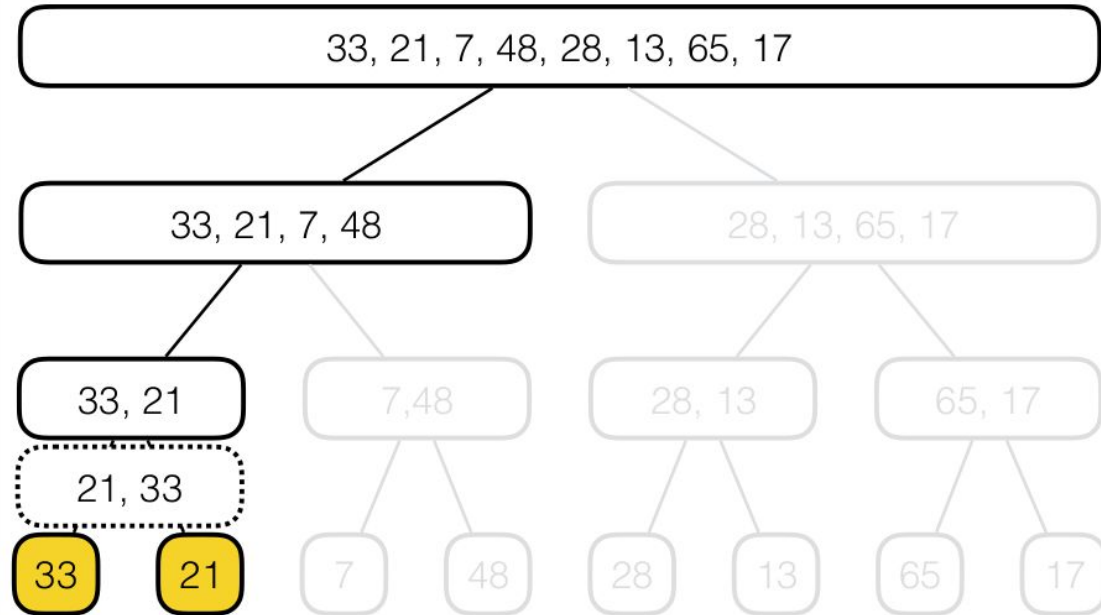




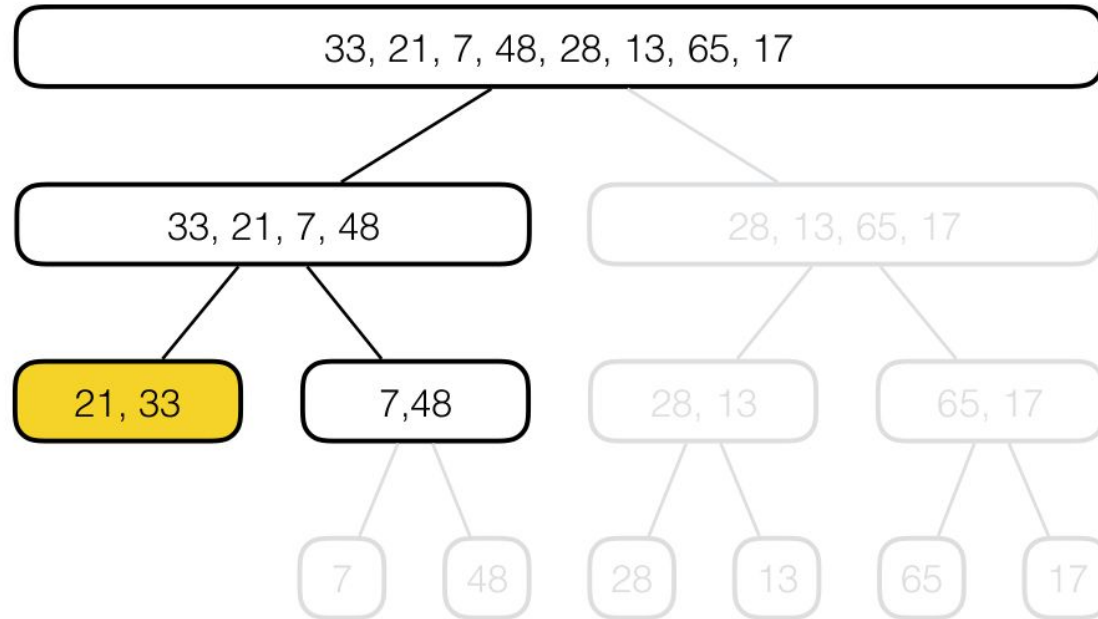
# Merge sort.



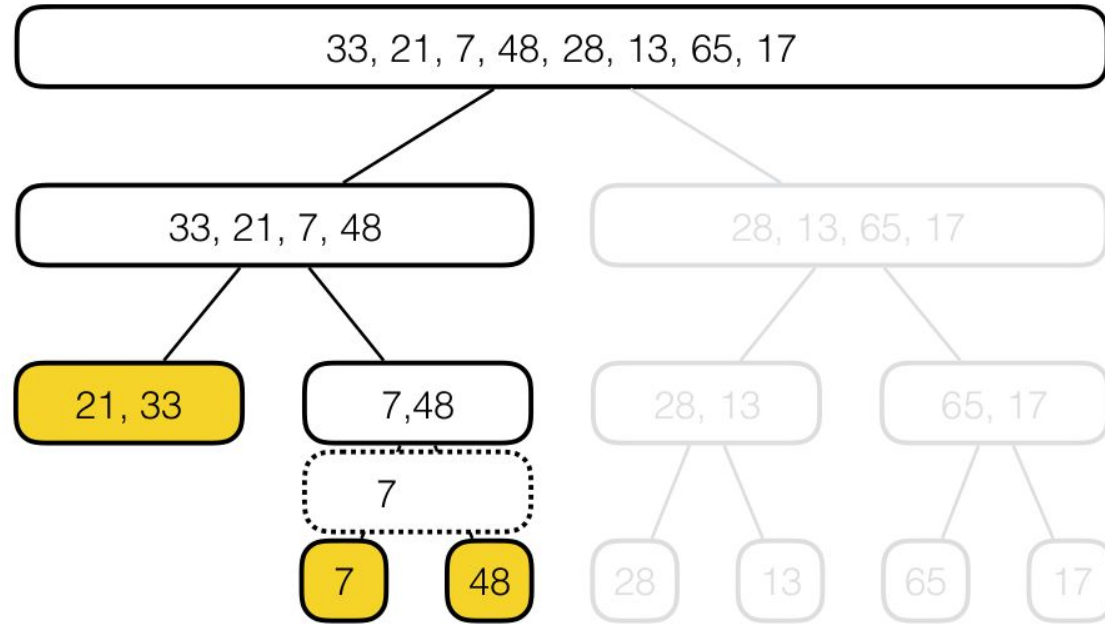
# Merge sort.



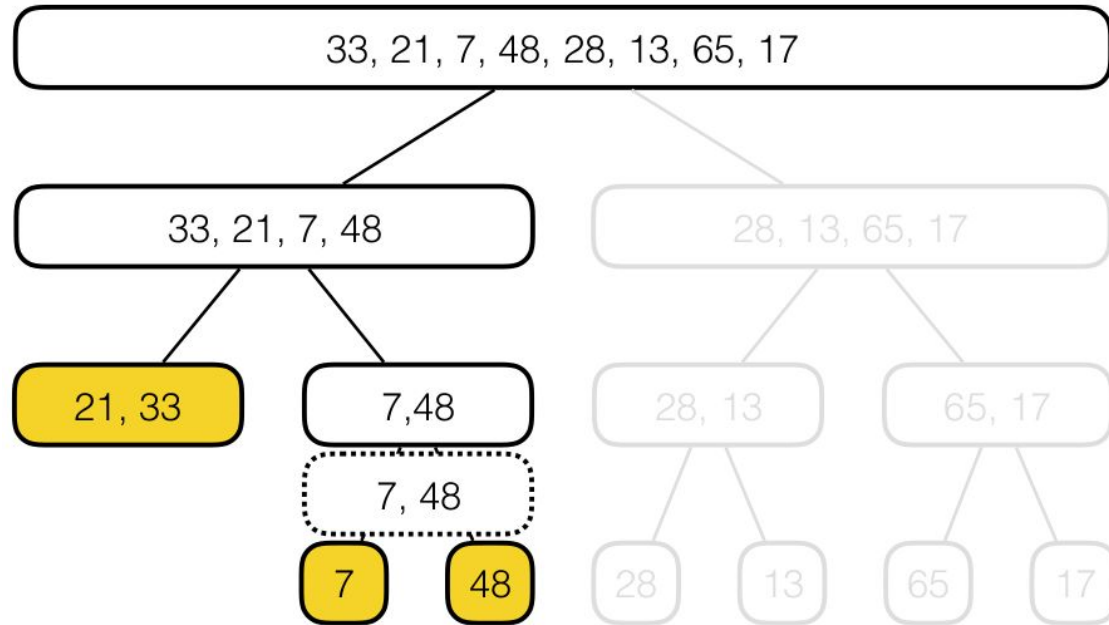
# Merge sort.



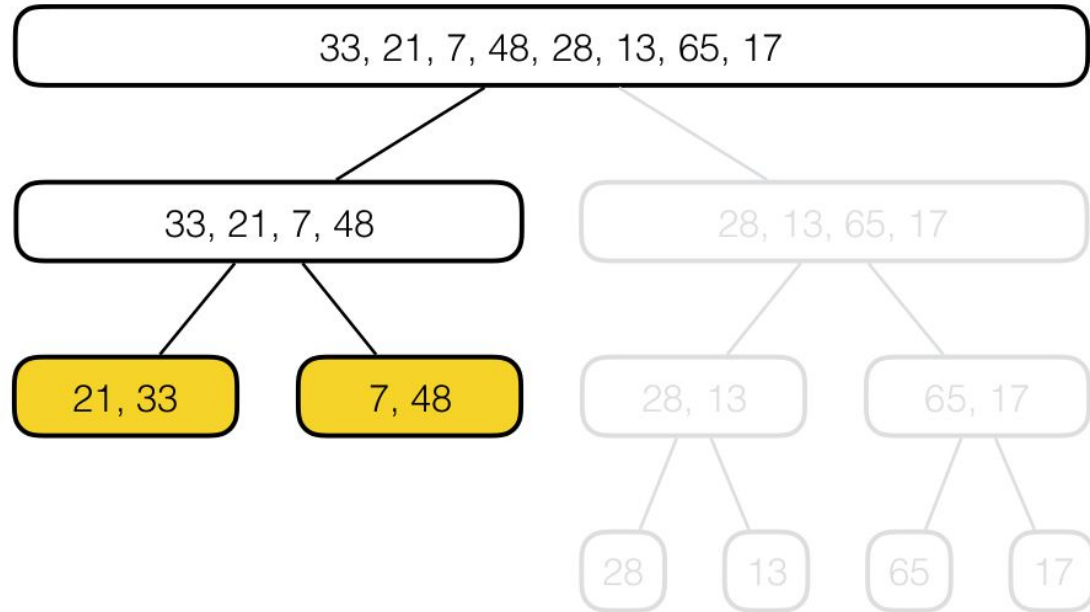
# Merge sort.



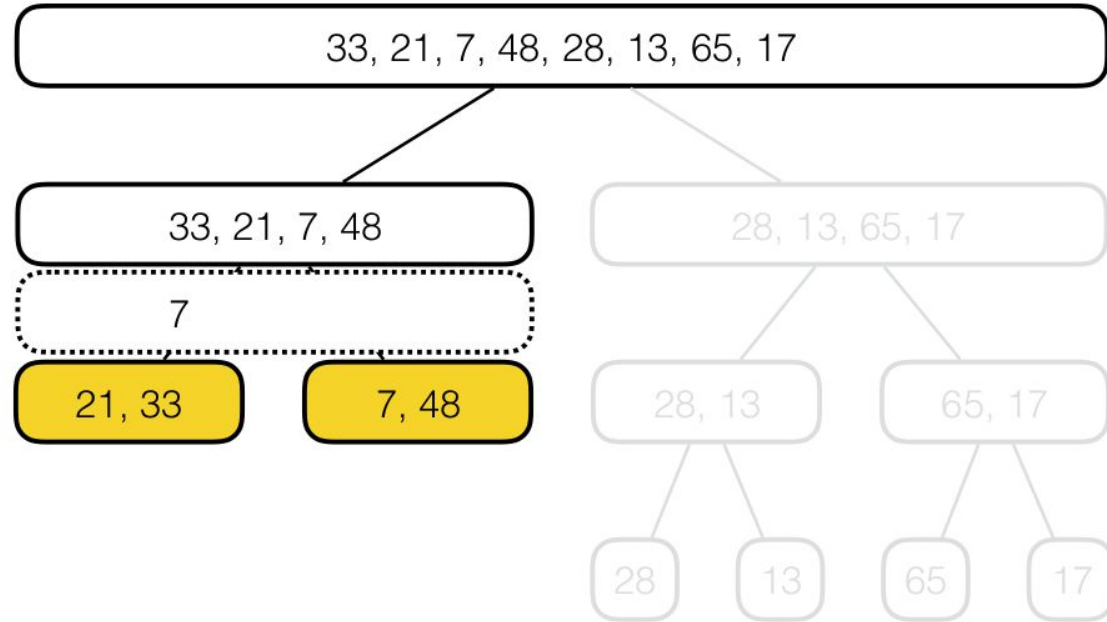
# Merge sort.



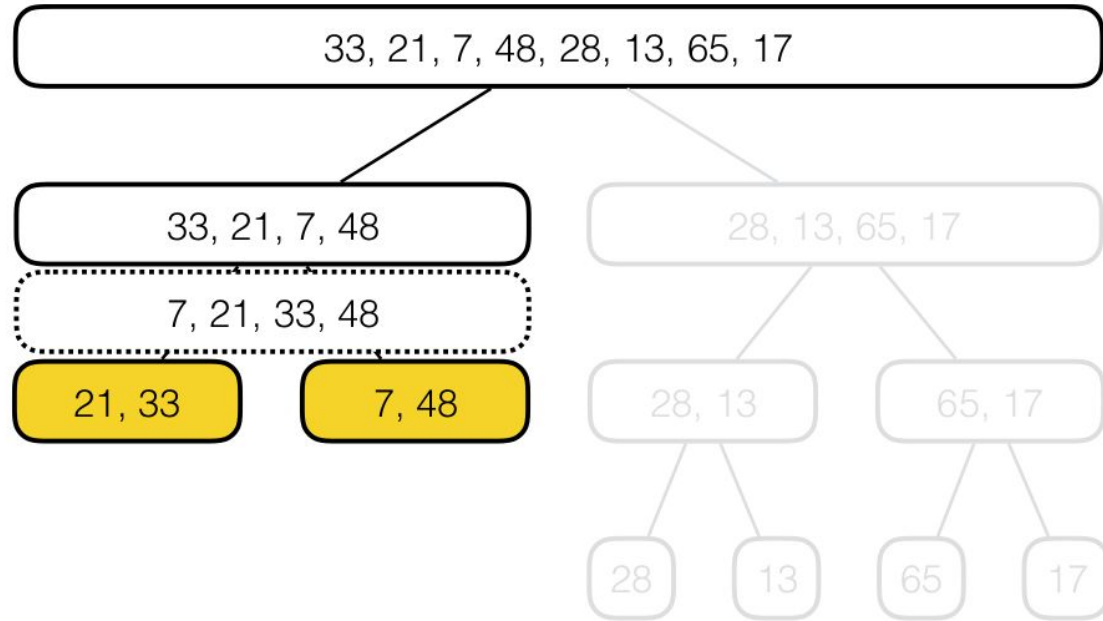
# Merge sort.



# Merge sort.

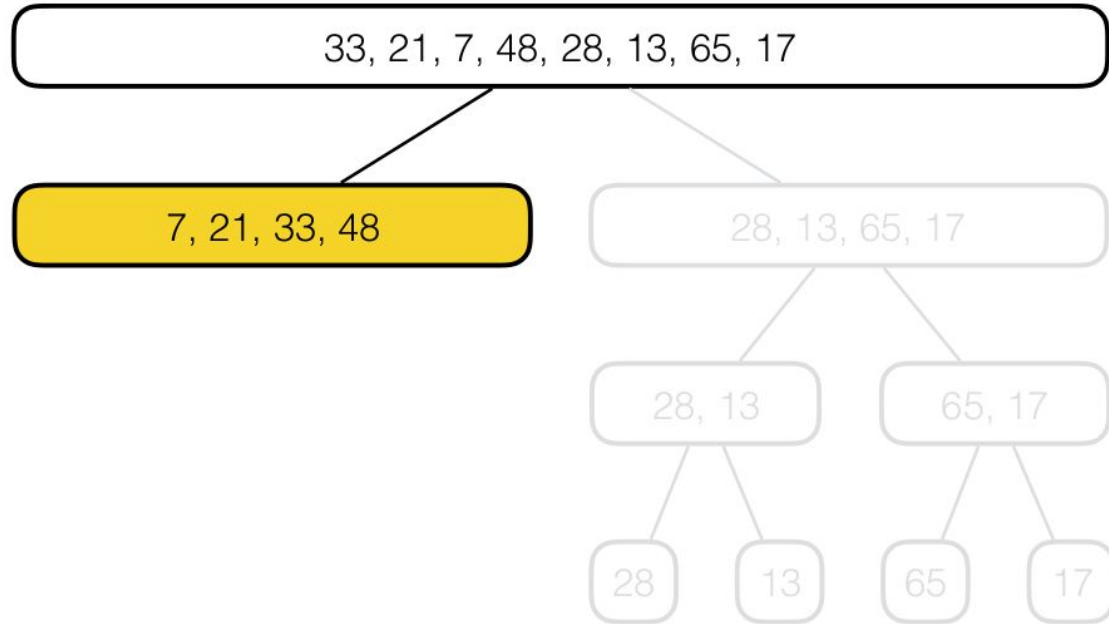


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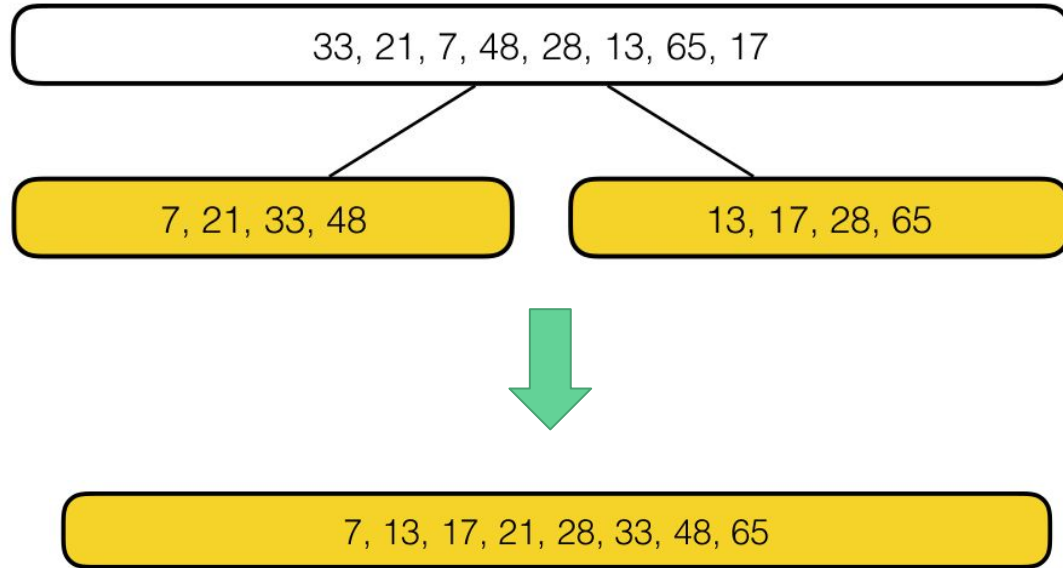




# Merge sort.



# Merge sort.



# Merge sort. Cost

Simplifying assumptions:

- $n = 2^k$ , i.e. the number of subdivisions is equal to  $k = \log n$ ;
- All the subsequences have size that are exact powers of 2

**Computational cost:**

$$T(n) = \begin{cases} c & n = 1 \\ 2T(n/2) + dn & n > 1 \end{cases}$$

# Merge sort. Cost

## Theorem

Let  $a$  and  $b$  two integer constants such that  $a \geq 1$  e  $b \geq 2$ , and let  $c, \beta$  be two real constants such that  $c > 0$  e  $\beta \geq 0$ . Let  $T(n)$  be defined by the following recurrence:

$$T(n) = \begin{cases} aT(n/b) + cn^\beta & n > 1 \\ \Theta(1) & n \leq 1 \end{cases}$$


Given  $\alpha = \log a / \log b = \log_b a$ , then:

$$T(n) = \begin{cases} \Theta(n^\alpha) & \alpha > \beta \\ \Theta(n^\alpha \log n) & \alpha = \beta \\ \Theta(n^\beta) & \alpha < \beta \end{cases}$$

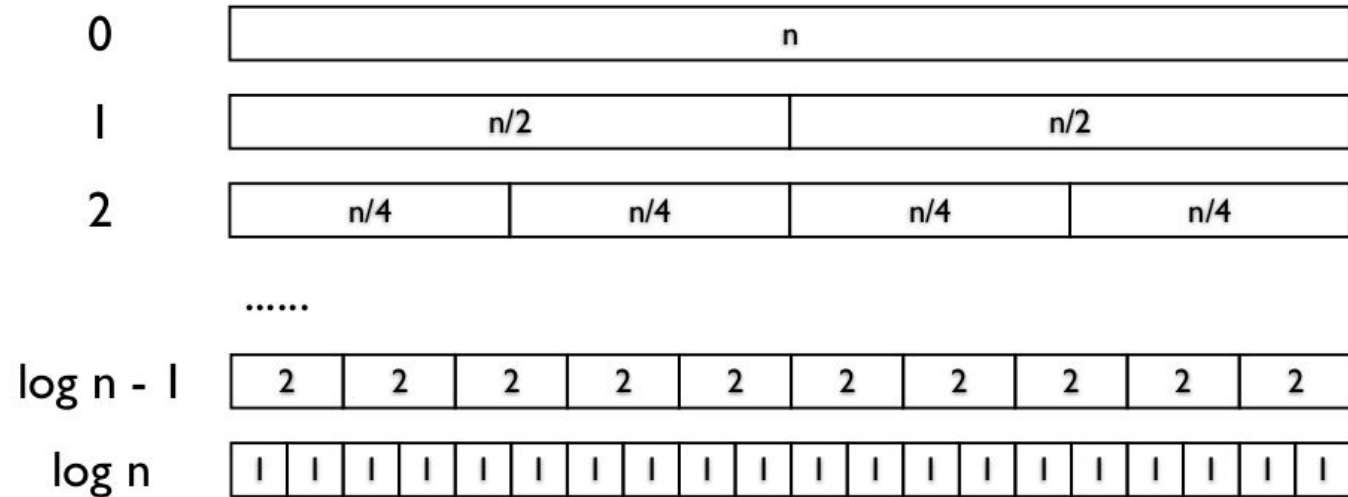
## Computational cost:

$$T(n) = \begin{cases} c & n = 1 \\ 2T(n/2) + dn & n > 1 \end{cases}$$

$$\alpha = \log_2 2 = 1 \\ \beta = 1$$

  $\Theta(n \log n)$

# Merge sort. Cost



# Quick sort

- This algorithm is based on a **divide-et-impera** strategy
- Average case:  $O(n \log n)$ , Worst case:  $O(n^2)$
- Average case vs worst case: the **multiplicative factor** of QuickSort is better than MergeSort and QuickSort is in-place (does not need a tmp list)
- It is possible to use “heuristic” techniques to avoid the worst case hence it is often preferred to other algorithms


[more info: R. Sedgewick, "Implementing Quicksort Programs". Communications of the ACM, 21(10):847-857, 1978. <http://portal.acm.org/citation.cfm?id=359631>]

# Quick sort

## Input

- Sequence  $A$  containing  $n$  values
- Indexes **first**, **last** such that  $0 \leq \text{first} \leq \text{last} < n$

## Divide


- Select a value  $p \in A[\text{first} : \text{last} + 1]$  called **pivot**
- Move all the elements in slice  $A[\text{first} : \text{last} + 1]$  in a way that
$$\forall i \in A[\text{first} : j] : A[i] \leq p$$
$$\forall i \in A[j + 1 : \text{last}] : A[i] \geq p$$
 at each iteration the pivot is put in its right place
- Index  $j$  is computed in a way that satisfies such condition
- The pivot is moved in position  $A[j]$

# Quick sort: divide step

## Input

- Sequence  $A$  containing  $n$  values
- Indexes **first**, **last** such that  $0 \leq \text{first} \leq \text{last} < n$

## Divide

- Select a value  $p \in A[\text{first} : \text{last} + 1]$  called **pivot**
- Move all the elements in slice  $A[\text{first} : \text{last} + 1]$  in a way that
$$\forall i \in A[\text{first} : j] : A[i] \leq p$$
$$\forall i \in A[j + 1 : \text{last}] : A[i] \geq p$$
 at each iteration the pivot is put in its right place
- Index  $j$  is computed in a way that satisfies such condition
- The pivot is moved in position  $A[j]$



# Quick sort: impera and combine steps

## Impera

Sort the slices `A[first:j]` and `A[j + 1:last+1]` by recursively calling Quicksort until only single elements are reached

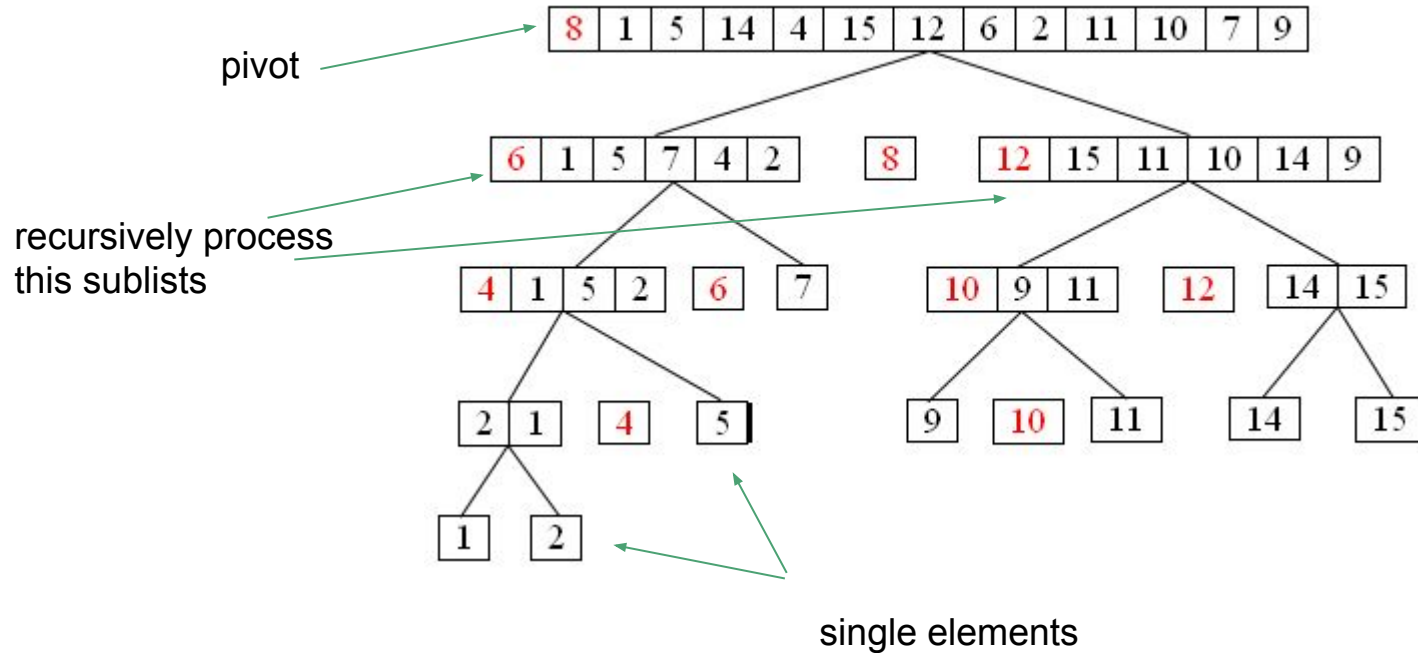
## Combine

Do nothing:

- the left subslice `A[first:j]`
- `A[j]`,
- the right subslice `A[j+1:last+1]`

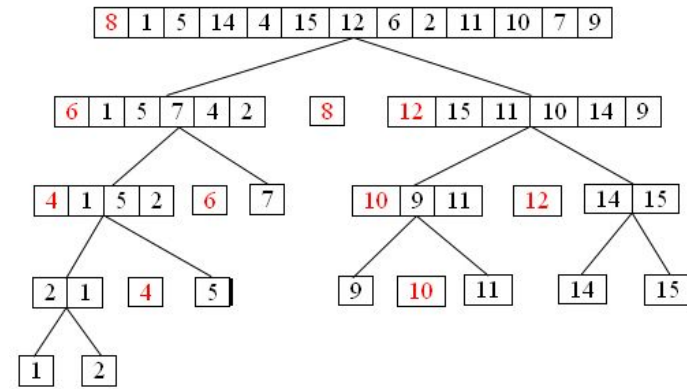
are already ordered

# Quick sort



**NOTE:** not great idea to pick the first element as pivot!!!

# Quick sort



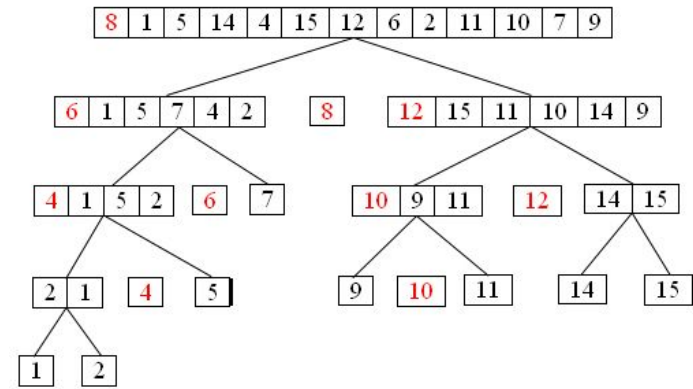
The algorithm makes use of the following methods:

1. **pivot** : gets the list, a **start** and **end** index, sets the **first element** as **pivot** and reorders all the elements in the list from **start** to **end** in such a way that **all the elements to the left of the pivot (i.e. having index lower) are smaller than the pivot and all the elements to the right (i.e. with index higher) are bigger than the pivot**. The function **returns the index of the pivot**;
2. **swap**: gets two indexes and swaps their values;
3. **recursiveQuickSort**: **gets an unordered (sub)list**, with **start** and **end** positions, **finds the pivot and recursively applies the same procedure to the sublists to the left and right of the pivot** (if sublist has size > 1);
4. **quickSort**: **gets an unordered list and applies the recursive quick sort procedure to it**.

# Pivot method

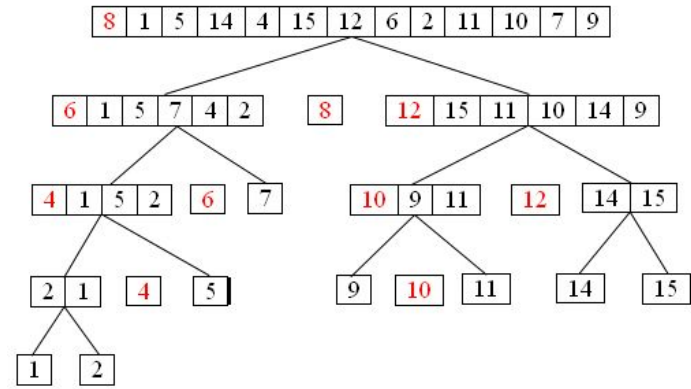
Pivot partitions the list in two: lower than 6 and higher than 6  
Pivot called on this list:

6	1	5	9	7	3	8	9
---	---	---	---	---	---	---	---



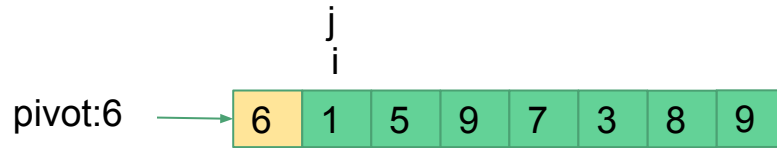
# Pivot method

Pivot called on this list:

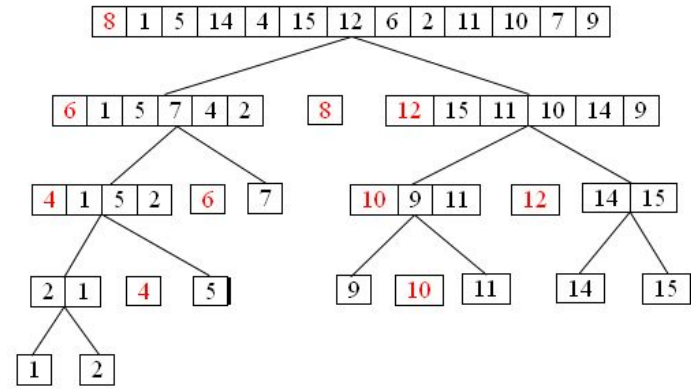


# Pivot method

Pivot called on this list:

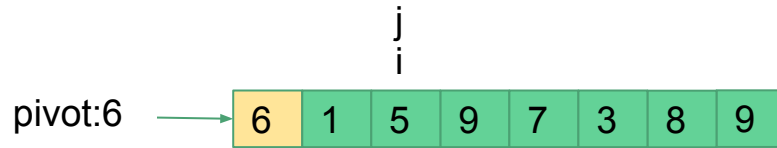


$6 > 1$   
increment j  
swap (i,j)

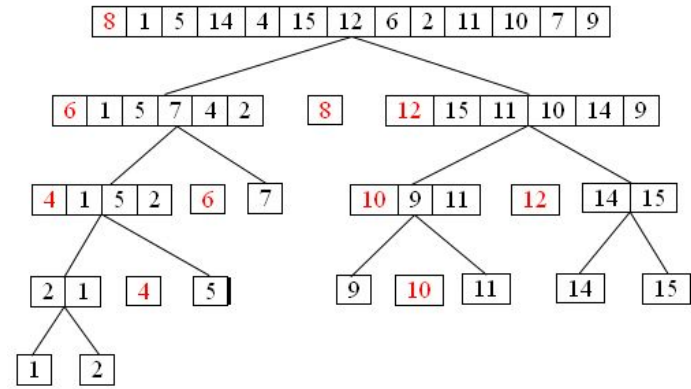


# Pivot method

Pivot called on this list:



6 > 5  
increment j  
swap (i,j)

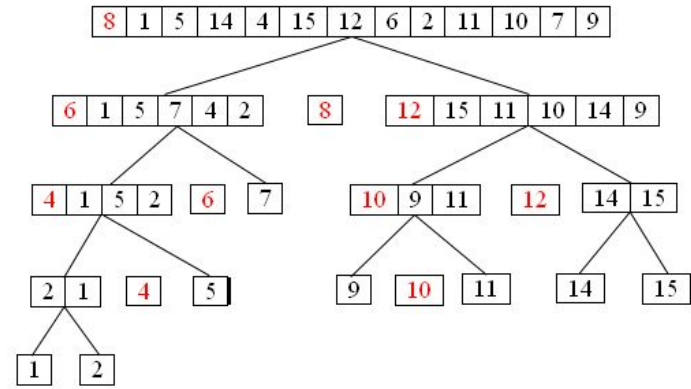


# Pivot method

Pivot called on this list:



$6 < 9$   
do nothing



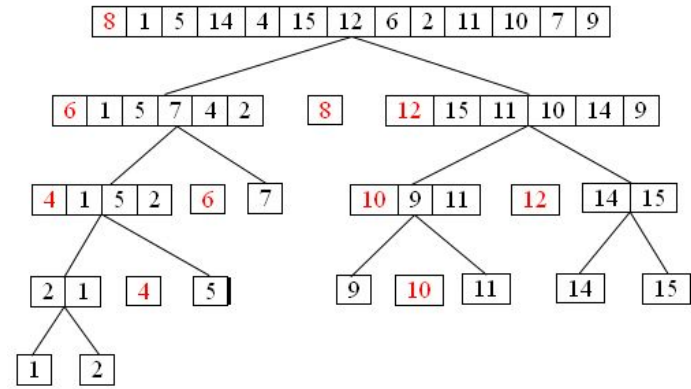


# Pivot method

Pivot called on this list:

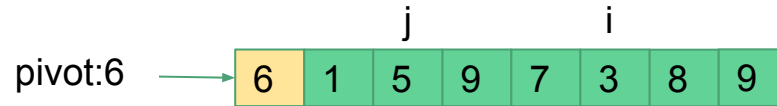


$6 < 7$   
do nothing



# Pivot method

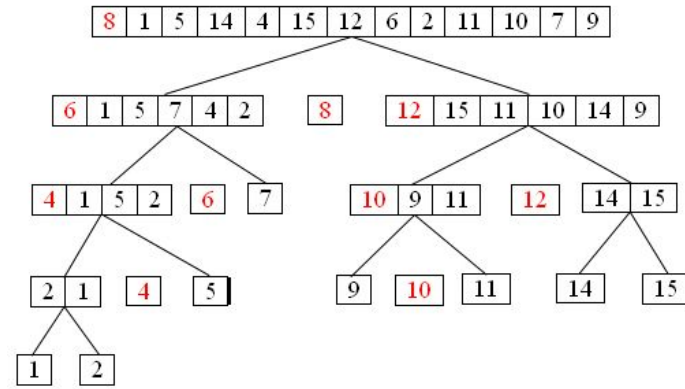
Pivot called on this list:



$6 > 3$   
increment j

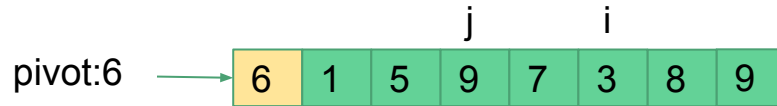


j always points to the last  
element smaller than pivot



# Pivot method

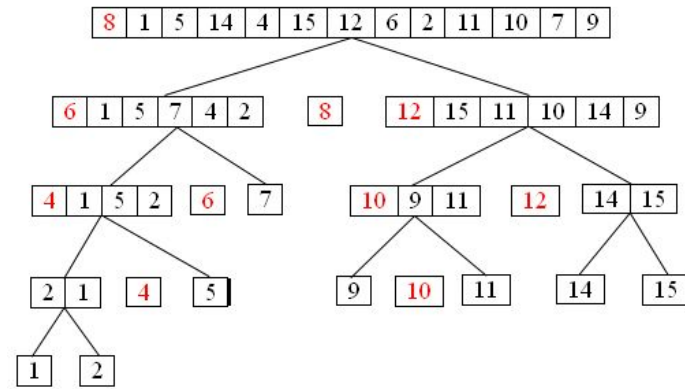
Pivot called on this list:



$6 > 3$   
increment j  
swap i,j

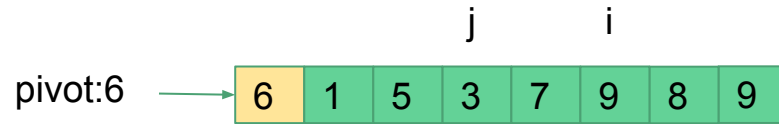


j always points to the last  
element smaller than pivot

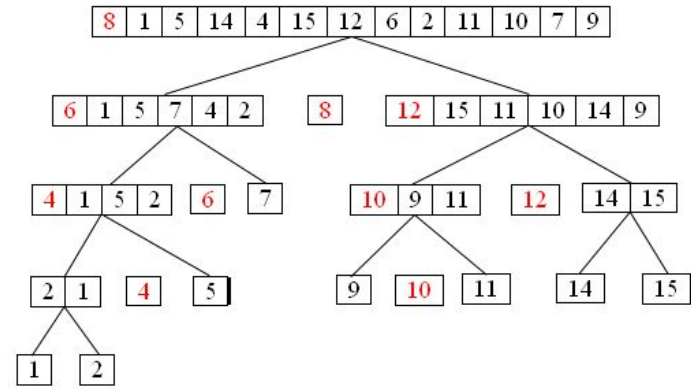


# Pivot method

Pivot called on this list:

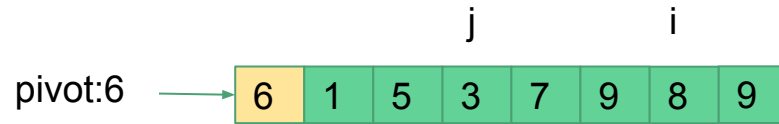


$6 < 9$   
do nothing

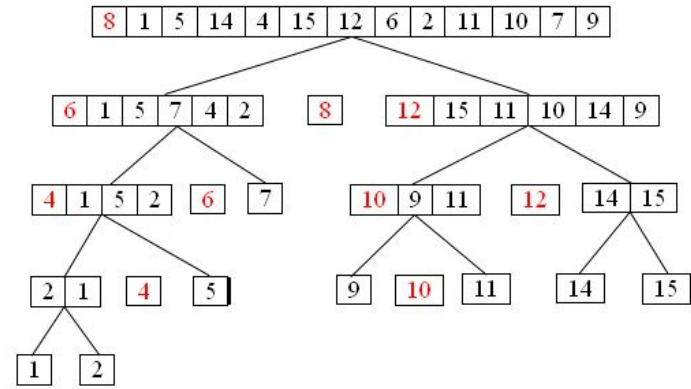


# Pivot method

Pivot called on this list:

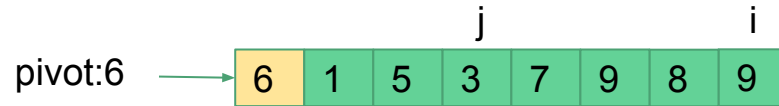


$6 < 8$   
do nothing

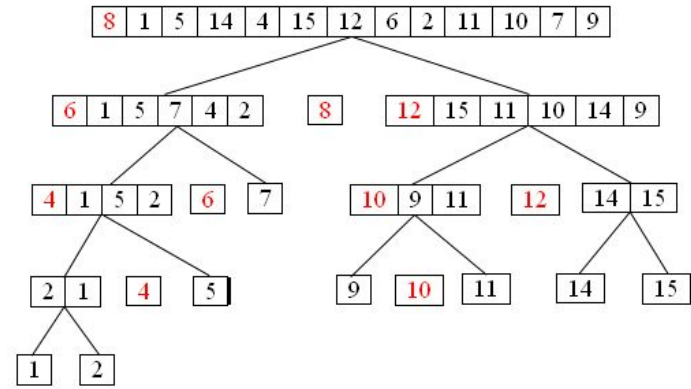


# Pivot method

Pivot called on this list:



$6 < 9$   
do nothing  
END!  
swap 0,j  
return j



# Quick sort: the code

```
def pivot(A, first, last):
    p = A[first]
    j = first
    for i in range(first+1, last+1):
        if A[i] < p: #A[i] should be to
                     #the right of p

            j = j + 1 #points to the last
                     #element smaller than p
            A[i],A[j] = A[j],A[i] #swap the values

    A[first] = A[j] #last smaller is placed at the beginning
    A[j] = p #p goes to position j
    return j #returns index

def quick_sort_rec(A,first,last):
    if first < last: #if we have more than 1 element
        p = pivot(A, first,last)
        quick_sort_rec(A,first,p-1) #pivot is in the
        quick_sort_rec(A,p+1,last) #right place

def quick_sort(A):
    quick_sort_rec(A,0,len(A)-1)

L = [12, 71, 44, 33, 22, 9, 7, -1, 12,13]
print(L)
quick_sort(L)
print(L)

[12, 71, 44, 33, 22, 9, 7, -1, 12, 13]
[-1, 7, 9, 12, 12, 13, 22, 33, 44, 71]
```

# Quick sort: complexity

Cost of `pivot()`

- $\Theta(n)$  [it is only one for loop]

Cost of Quicksort: depends on the partitioning

- **Worst partitioning**
  - Given a list of size  $n$ , the list is subdivided in two sublists of size 0 and  $n - 1$
  - $T(n) = T(n - 1) + T(0) + \Theta(n) = T(n - 1) + \Theta(n) = \Theta(n^2)$
  - Question: When do you get the worst case?
- **Best partitioning**
  - Given a list of size  $n$ , the list is subdivided in two sublists of size  $n/2$
  - $T(n) = 2T(n/2) + \Theta(n) = \Theta(n \log n)$



worst case, when list is already sorted  
(theorem not seen)

Pick pivot as random value to heuristically reduce the probability of worst case!



# Sorting methods: based on comparisons (?!?)

## Summary - Sorting algorithms

- SelectionSort -  $\Theta(n^2)$
- InsertionSort -  $\Omega(n)$ ,  $O(n^2)$
- ShellSort -  $\Omega(n)$ ,  $O(n^{3/2})$
- MergeSort -  $\Theta(n \log n)$
- HeapSort -  $\Theta(n \log n)$
- QuickSort -  $\Omega(n \log n)$ ,  $O(n^2)$

## Summary - Sorting algorithms

- All these algorithms are based on comparisons
  - Decisions about sorting are based on comparisons between two values ( $<$ ,  $=$ ,  $>$ )
- Best algorithms:  $O(n \log n)$  \_\_\_\_\_
  - InsertionSort and ShellSort are faster only in special cases

## Sorting problem – Lower bound

It is possible to show that any sorting algorithms based on comparisons is  $\Omega(n \log n)$ .

# Counting Sort: NOT based on comparisons

## Assumption

The numbers to be sorted are included in a range  $[0 \dots k - 1]$

## Idea

- Build a list B with k entries, where B[i] contains the number of times that i is contained in A (B is a list of counters).
- Put back together the elements in A, using the counters in B

## Possible improvements

The interval might not be limited to  $[0 \dots k - 1]$ ; any **known** interval  $[i, j]$  can work. In such case, you must subtract i from each number.

First, find the minimum value and subtract all the other values to it (remembering to add them back at the end)

Input list

4	6	4	1	2	1
---	---	---	---	---	---

Counter

0	1	2	3	4	5	6
0	2	1	0	2	0	1

Sorted list

1	1	2	4	4	6
---	---	---	---	---	---

# Counting Sort: code

```
#A is assumed to have all elements in [0,k-1]
def counting_sort(A, k):
    B = [0]*k
    for i in A:
        B[i] += 1

    j = 0
    for i in range(len(B)):
        while B[i] > 0:
            A[j] = i
            B[i] -= 1
            j += 1

L = [12, 71, 44, 33, 22, 9, 7, 0, 12, 13]
print(L)
counting_sort(L, max(L)+1)
print(L)
```

```
[12, 71, 44, 33, 22, 9, 7, 0, 12, 13]
[0, 7, 9, 12, 12, 13, 22, 33, 44, 71]
```

Input list

4	6	4	1	2	1
---	---	---	---	---	---

Counter

0	1	2	3	4	5	6
0	2	1	0	2	0	1

Sorted list

1	1	2	4	4	6
---	---	---	---	---	---

# Counting Sort: complexity

```
#A is assumed to have all elements in [0,k-1]
def counting_sort(A, k):
    B = [0]*k
    for i in A:
        B[i] += 1

    j = 0
    for i in range(len(B)):
        while B[i] > 0:
            A[j] = i
            B[i] -= 1
            j += 1

L = [12, 71, 44, 33, 22, 9, 7, 0, 12, 13]
print(L)
counting_sort(L, max(L)+1)
print(L)
```

```
[12, 71, 44, 33, 22, 9, 7, 0, 12, 13]
[0, 7, 9, 12, 12, 13, 22, 33, 44, 71]
```

Size of A:  $n$

adds  $k$  zeros to B :  $\theta(k)$

executed once for each  
element in A:  $\theta(n)$

Overall, complexity:  $\theta(n + k)$

# Counting Sort: complexity

## Complexity of CountingSort

- $\Theta(n + k)$
- If  $k$  is  $O(n)$ , then the complexity of Counting Sort is  $\Theta(n)$

## Counting Sort and lower bounds for sorting

- Counting Sort is not based on comparisons
- Se  $k \in \Omega(n^3)$ , this algorithm is worse than any other algorithm seen so far

```
#A is assumed to have all elements in [0,k-1]
def counting_sort(A, k):
    B = [0]*k
    for i in A:
        B[i] += 1

    j = 0
    for i in range(len(B)):
        while B[i] > 0:
            A[j] = i
            B[i] -= 1
            j += 1

L = [12, 71, 44, 33, 22, 9, 7, 0, 12, 13]
print(L)
counting_sort(L, max(L)+1)
print(L)
```

```
[12, 71, 44, 33, 22, 9, 7, 0, 12, 13]
[0, 7, 9, 12, 12, 13, 22, 33, 44, 71]
```

# Counting Sort

This can be used to sort every object that can be associated to a sortable key (can use lists of objects)

## Pigeonhole

- What happens when numbers are not integer, but tuples associated with a key to be sorted?
- We cannot use counters..
- But we can use lists instead!

