Introduction to DSGE Models

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Introduction to DSGE Models

Program

- DSGE Introductory course (6h)
  - Object: deriving DSGE models
- Computational Macroeconomics (10h) (Prof. L. Corrado)
  - Object: techniques to solve rational expectations linear models like DSGE (requires MATLAB)
- Topics:
  - DSGE History (Galì (2008) ch.1)
  - Real business cycle models (Galì (2008) ch.2)
  - New-Keynesian models (Galì (2008) ch.3)

Motivation

Why DSGE?

- Historical reason: Neo-Classical Synthesis
  - Real Business Cycle (RBC, “fresh water”) and New Keynesian (NK, “salt water”) literature (Blachard, 2000 and 2008)
- Theoretical reason: Robust to Lucas (1976), Lucas and Sargent (1978) Critique
  - Microfoundation of macroeconomic models
- Practical reason: CBs macroeconomic models
  - Bank of Canada (ToTEM), Bank of England (BEQM), European Central Bank (NAWM), US Federal Reserve (SIGMA), IMF (GEM), European Commission (QUEST III)

DSGE Model

What is a DSGE

- *Dynamic* means thare are intertemporal problems and agents rationally form expectations;
- *Stochastic* means exogenous stochastic process may shift aggregates
- *General Equilibrium* means that all markets are always in equilibrium
    - Exogenous/unpredictable shocks may temporally deviate the economy from the equilibrium

RBC Revolution

Main Points

- Seminal papers Kydland and Prescott (1982) and Prescott (1986)
- Efficiency of the business cycle (BC)
    - BC is the outcome of the real forces in an environment with perfect competition
- Technology is the main driver of the BC
    - Technology (Total factor productivity/Solow residual) is something exogenous
- No monetary policy references
    - Including money leads to "monetary neutrality". Money has no effects on real variables, thus CBs have no power

NK Features

Main Points

- Monopolistic Competition
  - Each firm have monopolistic power in the market she operates
- Nominal rigidities
  - Sticky price/wage
- Money is not neutral
  - Consequences of rigidities
  - However, money is neutral in the long-run

Neo-classical Synthesis

Main Points

- Use of the RBC way of modelling
  - Infinitely living agents maximize utility given by consumption and leisure
  - Firms have access to the same technology and are subjected to a random shift
- Implementation of NK Features
  - Stiky price/wage
  - Monopolistic Competition
  - Money is not neutral –> CBs have room for adjusting rigidities

RBC Model

Households

Assumptions:

- Perfect competition, homogeneous goods, zero profits
- Flexible price and wage
- No capital, no investments and no government
- Discrete time
- Rationally infinity-lived price taker agents
- Complete market and perfect information
- Money is unit of account (no medium of exchange or reserve of value)
- Regularity conditions on the utility function hold
- Additively separable consumption and leisure (CRRA functional form)
  - U differentiable and has continuous I, II derivatives
  - $\partial U/\partial C_t > 0$, $\partial U/\partial N_t < 0$, $\partial U/\partial C_t^2 < 0$ and $\partial U/\partial N_t^2 < 0$

RBC Model

Households

$$\max_{C_t, N_t} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\phi}}{1+\phi} \right) \tag{1}$$

s.t.

$$P_t C_t + Q_t B_t \leq B_{t-1} + W_t N_t + T_t \tag{2}$$

$$\lim_{T \to \infty} \mathbb{E}_t \left\{ B_T \right\} \geq 0, \quad \forall t \tag{3}$$

Variables: C_t: consumption; N_t: labor; B_t: bond; P_t: price; Q_t: bond price; W_t: wage; T_t: lump-sum transfer/tax.
Parameters: β: discount factor; σ: coef. of relative risk aversion/reciprocal of intertemporal elasticity of substitution; ϕ: inverse of the elasticity of work w.r.t. wage (inverse of Frish elasticity).

RBC Model

Households (cont'd)

F.O.C.

$$\frac{W_t}{P_t} = N_t^{\phi} C_t^{\sigma} \tag{4}$$

$$\mathbb{E}_t \left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{1}{\pi_{t+1}} \right] = Q_t \tag{5}$$

RBC Model

Firms

$$\max_{N_t} \quad P_t Y_t - W_t N_t \tag{6}$$

s.t.

$$Y_t = A_t N_t^{1-\alpha} \tag{7}$$

$$a_t = \rho_a a_{t-1} + \epsilon_{a,t}, \quad |\rho_a| < 1, \quad \epsilon_{a,t} \sim \mathcal{N}(0, \sigma_a) \tag{8}$$

Variables: Y_t: output; A_t: technology; N_t: labor; P_t: price; W_t: wage; $a_t \equiv log(A_t)$;
Parameters: α output elasticity w.r.t. labor (return to scale determinant).

RBC Model

Firms (cont'd)

F.O.C.

$$\frac{W_t}{P_t} = (1-\alpha)A_t N_t^{-\alpha} \tag{9}$$

RBC Model

Equilibrium

- Agents maximize utility subject to the budget constraint;
- Firms maximize profits subject to the production function;
- Goods and labor markets clear.

The last point in this setting without capital and government means

$$Y_t = C_t \tag{10}$$

RBC Model

Log-Linearization

Problem: systems of non-linear rational expectation difference equations are hard to solve.
A possible solution: take the log and linearize around the non-stochastic steady state using the F.O. Taylor expansion.

$$f(x) \approx f(x_{ss}) + \frac{\partial f(x)}{\partial x}\Big|_{x_{ss}}(x - x_{ss}) \tag{11}$$

RBC Model

Log-Linearization (cont'd)

An easy way to log-linearize (up to a constant) following Uhlig (1999):

- Set $X_t = Xe^{\hat{x}_t}$ (if $X_t^\alpha = X^\alpha e^{\alpha \hat{x}_t}$)
- Approximate $e^{\hat{x}_t} \approx (1 + \hat{x}_t)$ (if $e^{\alpha \hat{x}_t} \approx (1 + \alpha \hat{x}_t)$)
- $\hat{x}_t \hat{y}_t \approx 0$
- Use the Steady State relationships to remove the remaining constants

RBC Model

Non-Stochastic Steady State (NSSS)

$$Q = \beta \tag{12}$$

$$\frac{W}{P} = N^{\phi} C^{\sigma} \tag{13}$$

$$\frac{W}{P} = (1-\alpha) N^{-\alpha} \tag{14}$$

$$Y = N^{(1-\alpha)} \tag{15}$$

$$C = Y \tag{16}$$

RBC Model

Log-Linear Model

$$\hat{c}_t = \mathbb{E}_t \hat{c}_{t+1} - \sigma^{-1} \hat{r}_t \tag{17}$$

$$\hat{\omega} = \phi \hat{n}_t + \sigma \hat{c}_t \tag{18}$$

$$\hat{\omega} = -\alpha \hat{n}_t + a_t \tag{19}$$

$$\hat{y}_t = (1 - \alpha) \hat{n}_t + a_t \tag{20}$$

$$\hat{y}_t = \hat{c}_t \tag{21}$$

RBC Model

Log-Linear Model (cont'd)

$$\hat{r}_t = \hat{i}_t - \mathbb{E}_t \pi_{t+1} \tag{22}$$

$$\hat{\omega}_t = \hat{w}_t - \hat{p}_t \tag{23}$$

Results:

- Real variables are determined independently of monetary policy
- Not clear how conduct monetary policy (indeterminacy)
- Nominal variables may be pinned-down setting an interest rate rule

$$\hat{i}_t = \phi_\pi \pi_t \tag{24}$$

RBC Model

Linear Rational Expectation Model

$$A(\Theta)\mathbb{E}_t x_{t+1} = B(\Theta)x_t + C(\Theta)\epsilon_t \tag{25}$$

- The endogenous variables are $x_t \equiv \{\hat{c}_t, \hat{n}_t, \hat{w}_t, \hat{y}_t, \hat{r}_t, a_t\}$.
- The exogenous variable is $\epsilon_t \equiv \{\epsilon_{a,t}\}$.
- $A(\Theta)$, $B(\Theta)$ and $C(\Theta)$ are matrices containing time invariant structural parameters.
- The parameter space is $\Theta \equiv [\alpha, \beta, \phi, \sigma, \rho_a, \sigma_a]$

RBC Model

Linear Rational Expectation Model (cont'd)

There are many linear rational expectation solution methods:

- Balchard and Khan (1980)
- King and Watson (1998)
- Sims (2001)
- Uhlig (1999)

Returning (up to measurement errors)

$$x_{t+1} = D(\Theta)x_t + E(\Theta)\epsilon_t \tag{26}$$

Where $D(\Theta)$ and $E(\Theta)$ are matrices depending on paramenters Θ

RBC Model

Parameters

Two approaches to deal with the parameters $\Theta = [\alpha, \beta, \phi, \sigma, \rho_a, \sigma_a]$

- *Calibration*
  - Calibration IS NOT estimation!
  - Long-run relationship (Hours worked per Household)
  - Results obtained in microeconomic studies (risk aversion, discount factor)
- *Estimation*
  - Matching Moments (GMM, Simulated GMM, Indirect Inference)
  - Maximum Likelihood
  - Bayesian Estimation

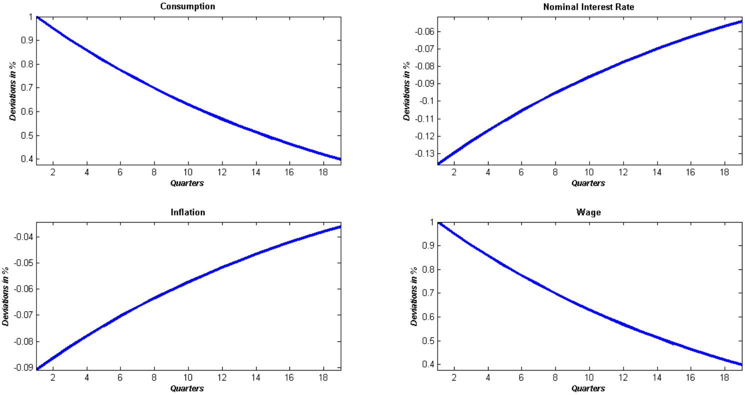
RBC Model

Standard Calibration

Parameter	Description	Value
σ	Intertemporal elasticity of substitution	1.0
β	Discount factor	0.99
ϕ	Frisch elasticity of labor supply	1.0
α	Labor elasticity in the production function	0.36
ϕ_π	Reaction coefficient on inflation	1.50
ρ_a	Persistence of TFP shock	0.95
σ_a	Volatility of TFP shock	0.0072

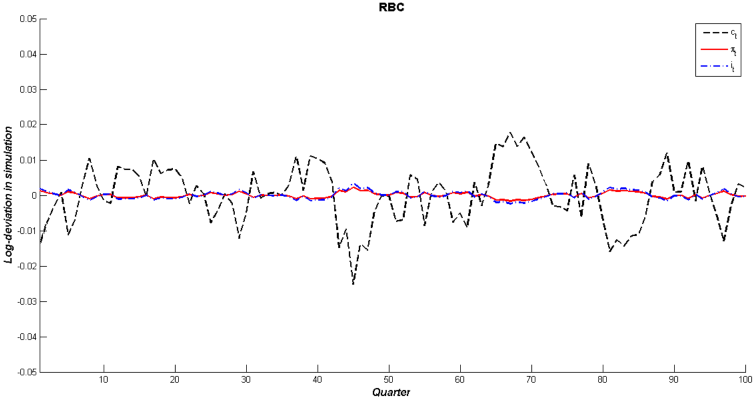
RBC Model

TFP shock Impulse Response Functions



RBC Model

Simulated data



New-Keynesian Model

Motivation

RBC model limitations

- Price adjust slowly (Christiano, Eichenbaum and Evans, 1999)
- Liquidity effect (negative comovements between money and interest rate, Galì, 2008 pag. 9)
- Monetary policy short-run effects

New-Keynesian Model

Setting

We use the RBC setting (and assumptions), introducing two frictions:

- Monopolistic competitive firms (no longer Perfect competition)
  - Implying no longer homogeneous good
- Price rigidity (no longer flexible)
  - Firms randomly adjust prices following Calvo (1983)

New-Keynesian Model

Households

$$\max_{C_t, N_t} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\phi}}{1+\phi} \right) \tag{27}$$

s.t.

$$\int_0^1 P_t(i) C_t(i) di + Q_t B_t \leq B_{t-1} + W_t N_t + T_t \tag{28}$$

$$\lim_{T \to \infty} \mathbb{E}_t \{B_T\} \geq 0, \quad \forall t \tag{29}$$

Variables: $C_t(i)$: consumption of good i; N_t: labor; B_t: bond; $P_t(i)$: price of good i; Q_t: bond price; W_t: wage; T_t: lump-sum transfer/tax.
Parameters: β: discount factor; σ: coef. of relative risk aversion/reciprocal of intertemporal elasticity of substitution; ϕ: inverse of the elasticity of work w.r.t. wage (inverse of Frish elasticity).

New-Keynesian Model

Households (cont'd)

Monopolistic competition assumption consequences:

- We have ruled-out the homogeneous goods assumption
- We have a continuum of goods $\in [0, 1]$
- Households deal with a dualistic problem
  - Minimize the cost of achieving a certain amount of composite good C_t
  - Given the above choice, maximizing utility

New-Keynesian Model

Households-Cost Minimization

$$\min_{C_t(i)} \int_0^1 P_t(i) C_t(i) di \tag{30}$$

s.t.

$$\left[\int_0^1 C_t(i)^{\frac{\epsilon-1}{\epsilon}} \right]^{\frac{\epsilon}{\epsilon-1}} \geq C_t \tag{31}$$

New-Keynesian Model

Households-Cost Minimization (cont'd)

$$C_t(i) = C_t \left(\frac{P_t(i)}{\psi_t} \right)^{-\epsilon} \tag{32}$$

Where ψ_t is the Lagrangian multiplier.
Plugging into the definition of composite good and solving for ψ_t get the aggregate price index

$$\psi_t = \left[\int_0^1 P_t(i)^{1-\epsilon} di \right]^{\frac{1}{1-\epsilon}} \equiv P_t \tag{33}$$

Than the demand for good i is

$$C_t(i) = C_t \left(\frac{P_t(i)}{P_t} \right)^{-\epsilon} \tag{34}$$

New-Keynesian Model

Households-Utility Maximization (cont'd)

$$\max_{C_t, N_t} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\phi}}{1+\phi} \right) \tag{35}$$

s.t.

$$P_t C_t + Q_t B_t \leq B_{t-1} + W_t N_t + T_t \tag{36}$$

New-Keynesian Model

Households-Utility Maximization (cont'd)

F.O.C.

$$\frac{W_t}{P_t} = N_t^{\phi} C_t^{\sigma} \tag{37}$$

$$\hat{\omega} = \phi \hat{n}_t + \sigma \hat{c}_t \tag{38}$$

$$\mathbb{E}_t \left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{1}{\pi_{t+1}} \right] = Q_t \tag{39}$$

$$\hat{c}_t = \mathbb{E}_t \hat{c}_{t+1} - \sigma^{-1} (\hat{i}_t - \mathbb{E}_t \pi_{t+1} - \rho) \tag{40}$$

New-Keynesian Model

Firms

- Operate in monopolistic competition
- Produce different goods with the same technology
- Solve a dualistic problem
  - Minimize their cost
  - Choose the optimal price following Calvo (1983)

New-Keynesian Model

Firms Price Dynamics

There are a continuum of firms $i \in [0,1]$ with identical technology

$$Y_t(i) = A_t N_t(i)^{1-\alpha} \tag{41}$$

$$a_t = \rho_a a_{t-1} + \epsilon_{a,t}, \quad |\rho_a| < 1, \quad \epsilon_{a,t} \sim \mathcal{N}(0, \sigma_a) \tag{42}$$

- According to Calvo (1983) $1-\theta$ firms reset the price and θ do not
- θ is the probability of resetting price (index of price stickiness)
- It also gives the frequency of adjustment

New-Keynesian Model

Firms Price Dynamics

$$P_t = \left(\int_{S(t)} P_{t-1}(i)^{1-\epsilon} di + (1-\theta) P_t^{*1-\epsilon} \right)^{\frac{1}{1-\epsilon}} \tag{43}$$

Plugging the aggregate price index and dividing by P_{t-1} get

$$\Pi_t^{1-\epsilon} = \theta + (1-\theta) \left(\frac{P_t^*}{P_{t-1}} \right)^{1-\epsilon} \tag{44}$$

$$\hat{\pi} = (1-\theta)(\hat{p}_t^* - \hat{p}_{t-1}) \tag{45}$$

Where $S(t)$ is the set of non resetting firms, P_t^* is the reset price and Π_t is inflation.

New-Keynesian Model

Optimal Price Setting

$$\max_{P_t^*} \quad \sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left\{ Q_{t,t+k} \left(P_t^* Y_{t+k|t} - \Psi_{t+k}\left(Y_{t+k|t}\right)\right)\right\} \tag{46}$$

$$s.t. \quad Y_{t+k|t} = \left(\frac{P_t^*}{P_{t+k}}\right)^{-\epsilon} Y_{t+k} \tag{47}$$

Where $Q_{t,t+k} = \beta^k \left(\frac{C_{t+1}}{C_t}\right)^{-\sigma} \frac{P_t}{P_{t+k}}$, $\Psi_{t+k}()$ is the cost function and $Y_{t+k|t}$ is output in period $t + k$ for firm reset price in t

New-Keynesian Model

Optimal Price Setting (cont'd)

F.O.C.

$$\sum_{t=1}^{\infty} \theta^k \mathbb{E}_t \left[Q_{t,t+k} Y_{t+k|t} \left(\frac{P_t^*}{P_{t-1}} - \mathcal{M} MC_{t+k} \frac{P_{t+k}}{P_{t-1}} \right) \right] = 0 \qquad (48)$$

$$\hat{p}_t^* - \hat{p}_{t-1} = (1 - \beta\theta) \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[\left(\hat{mc}_{t+k|t} + (\hat{p}_{t+k} - \hat{p}_{t-1}) \right) \right] \qquad (49)$$

Where $\mathcal{M} = \frac{\epsilon}{\epsilon - \mathbf{1}}$ is the mark-up, $MC_{t+k} = \frac{\Psi'_{t+k}}{P_{t+k}}$ is the real marginal cost and $\Pi_{t+k|t} \equiv \frac{P_{t+k}}{P_{t-\mathbf{1}}}$ is the inflation between t and $t+k$ obtained dividing by $P_{t-\mathbf{1}}$

New-Keynesian Model

Equilibrium

- Agents maximize utility subject to the budget constraint;
- Firms maximize profits subject to the production function;
- Goods and labor markets clear.

New-Keynesian Model

Market Clearing

Goods market clearing

$$Y_t = C_t \tag{50}$$

Labor market clearing

$$N_t = \int_0^1 N_t(i)di \tag{51}$$

Solving the production function for $N_t(i)$ and using aggregate good definition and good market clearing get

$$N_t = \left(\frac{Y_t}{A_t}\right)^{\frac{1}{1-\alpha}} \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{\frac{\epsilon}{1-\alpha}} di \tag{52}$$

New-Keynesian Model

Market Clearing

$$\hat{y}_t = \hat{c}_t \tag{53}$$

$$(1-\alpha)\hat{n}_t = y_t - a_t \tag{54}$$

New-Keynesian Model

Market Clearing

$$mc_t = w_t - p_t + mpn_t \tag{55}$$

[1]here $mpn_t = log(1-\alpha) + a_t - \alpha n_t$ which is the log of the derivative of Y_t w.r.t. N_t. Plugging mpn_t into Equation (55) we get

$$mc_t = w_t - p_t + log(1-\alpha) + a_t - \alpha n_t \tag{56}$$

[2]nd by the fact that $n_t = \frac{1}{1-\alpha}(a_t - \alpha y_t)$

$$mc_t = w_t - p_t - \frac{1}{1-\alpha}(a_t - \alpha y_t) - log(1-\alpha) \tag{57}$$

[1] w

[2] a