Advanced Macroeconomics
Notes on New Keynesian Model

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1 The Baseline New Keynesian Model

Derivation is mostly taken from Galì J. (2008). I tried to be consistent with standard notation mostly used in DSGE literature.

Household

There is a representative infinity-lived household maximising his expected life-utility at period $t = 0$. We assume that utility is function of consumption and leisure. Consumers has to minimise expenditure given the level of composite good C_t

$$\max_{C_t, N_t} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(C_t, N_t) \tag{1}$$

We assume that regularity conditions hold and $\partial U/\partial C_t > 0$, $\partial U/\partial N_t < 0$, $\partial U/\partial C_t^2 < 0$ and $\partial U/\partial N_t^2 < 0$. Moreover we assume a standard constant relative risk aversion (CRRA) functional form with separable consumption and leisure.

$$\max_{C,N} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\phi}}{1+\phi} \right) \tag{2}$$

We also assume there exist a continuous between $[0, 1]$ of different goods produced in monopolistic competition goods market.

$$C_t = \left(\int_0^1 C_t(i)^{1-\frac{1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}} \tag{3}$$

Utility is maximised subject to the budget constraint and the *No-Ponzi Game* condition. The last one is a solvency condition on government bonds.

$$\int_0^1 P_t(i)C_t(i)di + Q_tB_t \leq B_{t-1} + W_tN_t + T_t \tag{4}$$

$$\lim_{T\to\infty} \mathbb{E}_t\left\{B_T\right\} \geq 0, \quad \forall t \tag{5}$$

The representative consumer allocates wealth between consumption and saving: $P_t(i)$ are prices of different goods i, Q_t is the interest rate, W_t stands for wage and T_t is a lump-sum transfer which also captures the dividends coming from firms owned by households.

In order to derive the optimal allocation between goods, the representative agent maximises total consumption subject to any possible level of expenditure:

$$\min_{C_t(i)} \quad \int_0^1 P_t(i)C_t(i)di \tag{6}$$

s.t.

$$\left[\int_0^1 C_t(i)^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon-1}} \geq C_t \tag{7}$$

The Lagrangian takes the form

$$\min_{C_t(i)} \quad \int_0^1 P_t(i)C_t(i)di - \lambda_t\left(\left[\int_0^1 C_t(i)^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon-1}} - C_t\right) \tag{8}$$

From first order conditions we can recover the *demand schedule* and the *aggregate price*

$$\frac{\partial}{\partial C_t(i)} \equiv C_t(i) = C_t\left(\frac{P_t(i)}{\psi_t}\right)^{-\epsilon} \tag{9}$$

Where ψ_t is the Lagrangian multiplier.

Plugging into the definition of composite good and solving for ψ_t get the *aggregate price index*

$$\psi_t = \left[\int_0^1 P_t(i)^{1-\epsilon}di\right]^{\frac{1}{1-\epsilon}} \equiv P_t \tag{10}$$

Than the *demand for good* i is

$$C_t(i) = C_t\left(\frac{P_t(i)}{P_t}\right)^{-\epsilon} \tag{11}$$

Thus we get

$$\int_0^1 P_t(i)C_t(i)di = P_tC_t \tag{12}$$

which can be plugged into the original budget constraint yielding Equation (13)

$$P_tC_t + Q_tB_t \leq B_{t-1} + W_tN_t + T_t \tag{13}$$

Maximising the utility function w.r.t. (13) we can construct the Lagrangian:

$$\max_{C_t,N_t} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\phi}}{1+\phi} - \Lambda_t \left(P_tC_t + Q_tB_t - B_{t-1} - W_tN_t - T_t \right) \right) \tag{14}$$

The first order conditions are [1]

$$\frac{\partial}{\partial C_t} \equiv C_t^{-\sigma} = \nu_t P_t \tag{15}$$

$$\frac{\partial}{\partial N_t} \equiv N_t^{\phi} = \nu_t W_t \tag{16}$$

$$\frac{\partial}{\partial B_t} \equiv \beta \frac{\Lambda_{t+1}}{\Lambda_t} = Q_t \tag{17}$$

By solving the system we can recover the *Labour Supply* (41). Solving forward Equation (15) we get the *Euler Equation* (42).

$$\frac{W_t}{P_t} = N_t^{\phi} C_t^{\sigma} \tag{18}$$

$$\mathbb{E}_t \left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{1}{\pi_{t+1}} \right] = Q_t \tag{19}$$

The *Euler Equation* states how to allocate consumption between different periods by acquiring bonds at price Q_t.

[1] Remember that to show the budget constraint is binding, you need to show that the Lagrangian multiplier is positive $\Lambda_t > 0$. In this way the *complementary slackness condition* is satisfy $\Lambda_t(P_tC_t + Q_tB_t - B_{t-1} - W_tN_t - T_t) = 0$. From the derivative w.r.t. C_t is easy to show that $\Lambda_t = C_tP_t$ which is positive by assumption.

Firms

We assume firms operate in monopolistic competition and produce differentiated goods by using just labour and technology. Technology A_t is equal among firms. The production function is the following:

$$Y_t(i) = A_t N_t(i)^{1-\alpha} \tag{20}$$

Price levels adjust *à la Calvo* with fraction $1-\theta$ re-optimizing firms and θ non re-optimizing ($\theta \in [0,1]$). $S(t)$ is the set of non re-optimizing firms.

$$P_t = \left(\int_0^1 P_t(i)^{1-\epsilon} di \right)^{\frac{1}{1-\epsilon}} \tag{21}$$

$$P_t = \left(\theta \int_{S(t)} P_{t-1}(i)^{1-\epsilon} di + (1-\theta) P_t^{*1-\epsilon} \right)^{\frac{1}{1-\epsilon}} \tag{22}$$

$$P_t = \left(\theta P_{t-1}^{1-\epsilon} + (1-\theta) P_t^{*1-\epsilon} \right)^{\frac{1}{1-\epsilon}} \tag{23}$$

dividing both sides by P_{t-1} we can rewrite Equation (21) in terms of inflation

$$\Pi_t^{1-\epsilon} = \theta + (1-\theta) \left(\frac{P_t^*}{P_{t-1}} \right)^{1-\epsilon} \tag{24}$$

Re-optimizing firms solve the following profit maximisation subject to the *Demand Constraint*

$$\max_{P_t^*} \quad \sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left\{ Q_{t,t+k} \left(P_t^* Y_{t+k|t} - \Psi_{t+k} \left(Y_{t+k|t} \right) \right) \right\} \tag{25}$$

$$s.t. \quad Y_{t+k|t} = \left(\frac{P_t^*}{P_{t+k}} \right)^{-\epsilon} Y_{t+k} \tag{26}$$

directly plugging the *Demand Constraint* into the objective equation and maximising for P_t^* yields

$$\max_{P_t^*} \quad \sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left\{ Q_{t,t+k} \left(P_t^* \left(\frac{P_t^*}{P_{t+k}} \right)^{-\epsilon} Y_{t+k} - \Psi_{t+k} \left(\left(\frac{P_t^*}{P_{t+k}} \right)^{-\epsilon} Y_{t+k} \right) \right) \right\} \tag{27}$$

$$\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t Q_{t,t+k} \left((1-\epsilon) \left(\frac{P_t^*}{P_{t+k}} \right)^{-\epsilon} Y_{t+k} + \Psi'_{t+k|t} \frac{P_t^{*-\epsilon-1}}{P_{t+k}^{\epsilon}} Y_{t+k} \epsilon \right) = 0 \tag{28}$$

(29) $$\sum_{k=0}^{\infty}\theta^k \mathbb{E}_t\left[Q_{t,t+k}Y_{t+k|t}\left(P_t^* - \frac{\epsilon}{\epsilon-1}\Psi'_{t+k|t}\right)\right] = 0$$

dividing by P_{t-1} and plugging $Q_{t,t+k} = \beta^k\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\frac{P_t}{P_{t+k}}$ and $MC_{t+k} = \frac{\Psi'_{t+k}}{P_{t+k}}$ we get

(30) $$\sum_{k=0}^{\infty}(\theta\beta)^k \mathbb{E}_t\left[\left(\frac{C_{t+k}}{C_t}\right)^{-\sigma}\left(\frac{P_t}{P_{t+k}}\right)\left(\frac{P_t^*}{P_{t+k}}\right)^{-\epsilon} Y_{t+k}\left(\frac{P_t^*}{P_{t-1}} - \mathcal{M}MC_{t+k}\frac{P_{t+k}}{P_{t-1}}\right)\right] = 0$$

where $\mathcal{M} = \frac{\epsilon}{\epsilon-1}$. Notice that as $\theta = 0$ we are in the case of flexible price, thus the optimal price setting is given by $P_t^* = \mathcal{M}\Psi'_{t|t}$

Firms, as consumers, face a dualistic problem. They need to choose the optimal price in order to maximize profits and also has to choose the amount of labor to minimize cost.

(31) $$\min_{N_t(i)} \quad \frac{W_t}{P_t}N_t(i)$$

s.t.

(32) $$Y_t(i) = A_t N_t(i)_{1-\alpha}$$

Building-up the Lagrangian function we define the Lagrangian multiplier as the marginal cost of increasing the production.

(33) $$\min_{N_t(i)} \quad \frac{W_t}{P_t}N_t(i) - MC_t(Y_t(i) - A_t N_t(i)_{1-\alpha})$$

(34) $$\frac{\partial}{\partial N_t(i)}MC_t = \frac{W_t}{P_t}\frac{1}{(1-\alpha)A_t N_t(i)^{-\alpha}}$$

Market Clearing

Goods Market

The market clearing for the good market is

(35) $$Y_t(i) = C_t(i)$$

From which we get the *Aggregate Output Equation*

(36) $$Y_t = C_t$$

Labour Market

The *Aggregate Work* equation is

$$N_t = \int_0^1 N_t(i)di \tag{37}$$

Rewrite the *Production Function* (20) and solving for $N_t(i)$ we get

$$N_t(i) = \left(\frac{Y_t(i)}{A_t}\right)^{\frac{1}{1-\alpha}} \tag{38}$$

and by plugging into the *Aggregate Work* Equation (39) holds

$$N_t = \int_0^1 \left(\frac{Y_t(i)}{A_t}\right)^{\frac{1}{1-\alpha}} di \tag{39}$$

finally by using the definition of $Y_t(i)$, we arrive to Equation (48)

$$N_t = \left(\frac{Y_t}{A_t}\right)^{\frac{1}{1-\alpha}} \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\frac{\epsilon}{1-\alpha}} di \tag{40}$$

System of Equations

The non-linear system of equations is made-up by the following equations: labor supply, Euler equation, firms optimal price setting, firms cost minimization, price dynamics, inflation dynamics, goods market clearing, labor market clearing plus an exogenous law of motion for aggregate technology and a CBs rule to set the nominal rate of inflation.

$$\frac{W_t}{P_t} = N_t^{\phi} C_t^{\sigma} \tag{41}$$

$$\mathbb{E}_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\frac{1}{\pi_{t+1}}\right] = Q_t \tag{42}$$

$$\sum_{k=0}^{\infty}(\theta\beta)^k\mathbb{E}_t\left[\left(\frac{C_{t+k}}{C_t}\right)^{-\sigma}\left(\frac{P_t}{P_{t+k}}\right)\left(\frac{P_t^*}{P_{t+k}}\right)^{-\epsilon} Y_{t+k}\left(\frac{P_t^*}{P_{t-1}} - \mathcal{M}MC_{t+k}\frac{P_{t+k}}{P_{t-1}}\right)\right] = 0 \tag{43}$$

$$MC_t = \frac{W_t}{P_t}\frac{1}{(1-\alpha)A_tN_t(i)^{-\alpha}} \tag{44}$$

$$P_t = \left(\theta P_{t-1}^{1-\epsilon} + (1-\theta)P_t^{*1-\epsilon}\right)^{\frac{1}{1-\epsilon}} \tag{45}$$

(46) $$\Pi_t^{1-\epsilon} = \theta + (1-\theta)\left(\frac{P_t^*}{P_{t-1}}\right)^{1-\epsilon}$$

(47) $$Y_t = C_t$$

(48) $$N_t = \left(\frac{Y_t}{A_t}\right)^{\frac{1}{1-\alpha}} \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\frac{\epsilon}{1-\alpha}} di$$

Flexible Price Equilibrium

From the firms price optimization we get the flexible price mark-up $P_t^* = \mathcal{M}\Psi'_{t|t}$. Given that under flexible prices $P_t^* = P_t$ we have $MC_t = \frac{1}{\mathcal{M}}$. Now we would like to find an expression for outpu under flexible price, in order to build-up the Dynamic IS equation and the NKPC. Plugging the flexible price mark-up into the labor supply $MC_t = \frac{W_t}{P_t}\frac{1}{A_t(1-\alpha)N_t^{-\alpha}}$ and using the goods market clearing condition $C_t = Y_t$, the labor demand $\frac{W_t}{P_t} = C_t^\sigma N_t^\phi$ and the labor market clearing condition $N_t = \left(\frac{Y_t}{A_t}\right)^{\frac{1}{1-\alpha}}$ get

(49) $$Y_t^F = \left(\frac{1}{\mathcal{M}}(1-\alpha)A_t^{\frac{1+\phi}{1-\alpha}}\right)^{\frac{1-\alpha}{\phi+\sigma(1-\alpha)+\alpha}}$$

2 Steady State Relationship

In steady state we obtain the following relationships by dropping time indicator and assuming steady state inflation equal to one $\Pi = 1$.

Steady State Labor Supply

(50) $$\frac{W}{P} = N^\phi C^\sigma$$

Steady State Euler Equation

(51) $$Q = \beta$$

By assuming in steady state $P^* = P_t$ we get the *Steady State Price Setting*

(52) $$MC = \frac{1}{\mathcal{M}}$$

Steady State Goods Market Clearing

$$C = Y \tag{53}$$

Steady State Labor Market Clearing

$$N = \left(\frac{Y}{A}\right)^{\frac{1}{1-\alpha}} \int_0^1 \left(\frac{P(i)}{P}\right)^{-\frac{\epsilon}{1-\alpha}} di \tag{54}$$

where the RHS integral is equal to one.

Steady State Firms Cost Minimization

$$MC = \frac{W}{P} \frac{1}{(1-\alpha)N^{-\alpha}} \tag{55}$$

3 Log-Linearization

In this section I will follow mainly two different approaches to log-linearise the equations around the non-stochastic steady state of the model. The first method uses a first order Taylor expansion, thus it reduces to compute a bunch of first order derivatives, while the second uses some results from Uhlig (1995) and can be applied to the equations without directly compute any derivatives. In my opinion, the first method is more general and it is really useful when one would like to compute higher order approximations (second, third, etc.). Instead, the second method is faster for first order approximation. Following this reasoning I will just apply the first method to some equations, but I will carry on the analysis of the model using the second one.

3.1 Taylor expansion

Let's start remarking that the reason why we use the logarithm is to interpret parameters as elasticity (percentage change). However, using the log and the Taylor expansion leads to two different approximations; the first coming from the fact that log-differences are approximations of the percentage change (negligible when the percentage change is small), while the second coming from the fact that we are using a linear function to approximate a non-linear one (at least using a first order Taylor expansion). This is negligible only in a small interval around the steady-state.

The equation below represents a first order Taylor expansion with n variables. Notice that the function is approximated in a particular point x_0 which is a vector in $\mathbb{R}^n$. Thus the function will be approximated to its value computed at that particular point plus the slope in the different dimensions represented by the first order partial derivatives computed at that particular point. In a two dimesion simple case this means that I am approximating a non-linear function with a line, while in a higher dimensional space I am approximating it with some planes ($\mathbb{R}^3$) or hyperplanes ($\mathbb{R}^{n>3}$)

$$f(x_1, x_2, \ldots, x_n) \approx f(x_{1,0}, x_{2,0}, \ldots, x_{n,0}) + \frac{\partial f(x_1, x_2, \ldots, x_n)}{\partial x_1}\Big|_{x_1=x_{1,0}} (x - x_{1,0})$$
$$+ \frac{\partial f(x_1, x_2, \ldots, x_n)}{\partial x_2}\Big|_{x_2=x_{2,0}} (x - x_{2,0}) + \ldots + \frac{\partial f(x_1, x_2, \ldots, x_n)}{\partial x_n}\Big|_{x_n=x_{n,0}} (x - x_{n,0})$$

Now, I would like to express my function as a log-deviation from the steady state. Recalling the properties of the log, for our proof is useful to highlight that the log-deviation is an approximation of the percentage change.

$$\hat{x} = log(x) - log(x_0) \approx \frac{x - x_0}{x_0} \tag{56}$$

The basic idea behind the prof is to rewrite the Taylor expansion in percentage change and use the log-deviation to approximate it. In order to comply with this idea it is useful to work with a two variable case.

$$f(x_1, x_2) - f(x_{1,0}, x_{2,0}) \approx \frac{\partial f(x_1, x_2)}{\partial x_1}\Big|_{x_1=x_{1,0}} (x - x_{1,0}) + \frac{\partial f(x_1, x_2)}{\partial x_2}\Big|_{x_2=x_{2,0}} (x - x_{2,0}) \tag{57}$$

We multiply and divide the three different pieces by $f(x_{1,0}, x_{2,0})$, $x_{1,0}$ and $x_{2,0}$ holding:

$$f(x_{1,0}, x_{2,0})\frac{f(x_1, x_2) - f(x_{1,0}, x_{2,0})}{f(x_{1,0}, x_{2,0})} \approx x_{1,0}\frac{\partial f(x_1, x_2)}{\partial x_1}\Big|_{x_1=x_{1,0}} \frac{x - x_{1,0}}{x_{1,0}} + x_{2,0}\frac{\partial f(x_1, x_2)}{\partial x_2}\Big|_{x_2=x_{2,0}} \frac{x - x_{2,0}}{x_{2,0}}$$

Finally by using Equation 56 we can rewrite the general formula to linearise any function as a log-deviation from the steady state.

$$f(x_{1,0}, x_{2,0})\hat{f}(x_1, x_2) \approx x_{1,0}\frac{\partial f(x_1, x_2)}{\partial x_1}\Big|_{x_1=x_{1,0}} \hat{x}_1 + x_{2,0}\frac{\partial f(x_1, x_2)}{\partial x_2}\Big|_{x_2=x_{2,0}} \hat{x}_2 \tag{58}$$

As an illustrative example consider the labor supply (Equation 41) repeated here for convenience:

$$\frac{W_t}{P_t} = N_t^{\phi} C_t^{\sigma}$$

Using Equation 58 we can rewrite it in the following way

$$W_{ss}\hat{w}_t = P_{ss}\frac{\partial W_t}{\partial P_t}\Big|_{P_t=P_{ss}} \hat{p}_t + N_{ss}\frac{\partial W_t}{\partial N_t}\Big|_{N_t=N_{ss}} \hat{n}_t + C_{ss}\frac{\partial W_t}{\partial C_t}\Big|_{C_t=C_{ss}} \hat{c}_t \tag{59}$$

by computing the derivatives of W_t w.r.t. price, hours worked and consumption an plugging into the above equation and using the exponential properties we get

(60) $$W\hat{w}_t = PN^{\phi}C^{\sigma}\hat{p}_t + \phi PN^{\phi}C^{\sigma}\hat{n}_t + \sigma PN^{\phi}C^{\sigma}\hat{c}_t$$

where in order to simplify the notation I have written the steady state value without the subscript ss. Finally solwing for $\hat{w}_t$ and using the steady state relation $\frac{PN^{\phi}C^{\sigma}}{W} = 1$ we get the log-linearised labor supply.

(61) $$\hat{w}_t = \hat{p}_t + \phi\hat{n}_t + \sigma\hat{c}_t$$

all the other equation in the system can be easily derived following the same reasoning.

3.2 Uhlig 1995 method

The secon method of log-linearisation mainly follows the three Uhlig's building blocks[2].

(62) $$e^{x_t + ay_t} \approx 1 + x_t + ay_t$$

(63) $$x_t y_t \approx 0$$

(64) $$\mathbb{E}_t\left[ae^{x_{t+1}}\right] \approx \mathbb{E}_t\left[ax_{t+1}\right]$$

Where x_t and y_t are real variables close to zero ($x_t = log(X_t) - log(\bar{X})$, in our notation this will be $\hat{x}_t$), $\bar{X}$ is the steady state value of the variable X_t (in our notation this will be just X) and a is a constant (the second and third building blocks are up to a constant). As suggested by Uhlig we replace each variables by $\bar{X}e^{x_t}$, than applying the three building blocks. After some manipulations, all the constants drop out to each equations.

Labour Supply

(65) $$\hat{w}_t - \hat{p}_t = \phi\hat{n}_t + \sigma\hat{c}_t$$

Euler Equation

(66) $$\hat{c}_t = \mathbb{E}_t\hat{c}_{t+1} - \frac{1}{\sigma}\left(\hat{i}_t - \mathbb{E}_t\pi_{t+1}\right)$$

[2] Reported here as in the original notation of Uhlig (1995)

Inflation Dynamics

$$\pi_t = (1-\theta)(\hat{p}_t^* - \hat{p}_{t-1}) \tag{67}$$

Price Setting

$$\sum_{k=0}^{\infty} \theta^k \beta^k \left(\hat{p}_t^* - \hat{p}_{t-1}\right) = \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[\left(\hat{mc}_{t+k|t} + (\hat{p}_{t+k} - \hat{p}_{t-1})\right)\right] \tag{68}$$

Goods Market Clearing

$$\hat{c}_t = \hat{y}_t \tag{69}$$

Labor Market Clearing[3]

$$\hat{n}_t = \frac{1}{(1-\alpha)}(\hat{y}_t - \hat{a}_t) \tag{72}$$

Firms Cost Minimization

$$\hat{mc}_t = \hat{w}_t - \hat{p}_t - \hat{a}_t + \alpha \hat{n}_t \tag{73}$$

Price Dynamics

$$\hat{p}_t = (1-\theta)\hat{p}_t^* + \theta \hat{p}_{t-1} \tag{74}$$

3.3 Minimum set of Equationa

By using the fact that $\frac{1}{\sum_{k=0}^{\infty} \theta^k \beta^k} = (1+\beta\theta)$, from Equation (68) follows

$$\hat{p}_t^* - \hat{p}_{t-1} = (1-\beta\theta) \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[\left(\hat{mc}_{t+k|t} + (\hat{p}_{t+k} - \hat{p}_{t-1})\right)\right] \tag{75}$$

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[3] Taking the log-deviation of the *Market Clearing Condition* we get

$$N\left(1+\hat{n}\right) = \left(\frac{Y}{A}\right)^{\frac{1}{1-\alpha}} \left(1 + \left(\frac{1}{1-\alpha}\right)(\hat{y}_t - \hat{a}_t)\right) + RES \tag{70}$$

where RES is something very small quantity in a neighborhood of the zero inflation steady state and can be not considered in a first order Taylor expansion. See Galì (2008) chapter 3, Appendix 3.3.

$$\hat{y}_t = (1-\alpha)\,\hat{n}_t + \hat{a}_t \tag{71}$$

When $\alpha \neq 0$ we rule out the constant return to scale hypothesis, meaning $\hat{mc}_{t+k|t} \neq \hat{mc}_{t+k}$. We than need to find an equation for $\hat{mc}_{t+k|t}$. Starting from the marginal cost equation and plugging $\hat{n}_t = \frac{1}{1-\alpha}(\hat{a}_t - \alpha\hat{y}_t)$ we get

$$\hat{mc}_{t+k} = \hat{w}_{t+k} - \hat{p}_{t+k} - \frac{1}{1-\alpha}(\hat{a}_{t+k} - \alpha\hat{y}_{t+k}) \tag{76}$$

and

$$\hat{mc}_{t+k|t} = \hat{w}_{t+k} - \hat{p}_{t+k} - \frac{1}{1-\alpha}(\hat{a}_{t+k} - \alpha\hat{y}_{t+k|t}) \tag{77}$$

Thus

$$\hat{mc}_{t+k|t} - \hat{mc}_{t+k} = \frac{\alpha}{1-\alpha}(\hat{y}_{t+k|t} + \hat{y}_{t+k}) \tag{78}$$

And by plugging the goods market clearing condition combined with the demand schedule we get [4]

$$\hat{mc}_{t+k|t} = \hat{mc}_{t+k} + \frac{\alpha\epsilon}{1-\alpha}\left(\hat{p}_t^* - \hat{p}_{t+k}\right) \tag{79}$$

By plugging into (75) we get

$$\hat{p}_t^* - \hat{p}_{t-1} = (1-\beta\theta)\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left(mc_{t+k} - \frac{\alpha\epsilon}{1-\alpha}\left(\hat{p}_t^* - \hat{p}_{t+k}\right) + \left(\hat{p}_{t+k} - \hat{p}_{t-1}\right)\right) \tag{80}$$

and after some algebraic manipulations we have

$$\hat{p}_t^* - \hat{p}_{t-1} = (1-\beta\theta)\,\Theta\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t mc_{t+k} + \sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left(\hat{p}_{t+k} - \hat{p}_{t-1}\right) \tag{81}$$

Where $\Theta = \frac{1-\alpha}{1-\alpha-\alpha\epsilon}$. By rewriting (82) as different equation, and using $\pi = (1-\theta)(\hat{p}_t^* - \hat{p}_{t-1})$ we get the *New Keynesian Philips Curve*

$$\hat{p}_t^* - \hat{p}_{t-1} = (1-\beta\theta)\,\Theta\frac{1}{(1-\theta\beta F)}mc_t + \frac{1}{(1-\theta\beta F)}\left(\hat{p}_t - \hat{p}_{t-1}\right) \tag{82}$$

Where F is the forward operator

[4] By taking log of the *Demand Constraint* $Y_{t+k|t} = \left(\frac{P_t^*}{P_{t+k}}\right)^{-\epsilon}(Y_{t+k})$ we have $\hat{y}_{t+k|t} = \hat{y}_{t+k} + \epsilon\left(\hat{p}_t^* - \hat{p}_{t+k}\right)$. Then $\hat{mc}_{t+k} = \hat{mc}_{t+k|t} + \frac{\alpha\epsilon}{1-\alpha}\left(\hat{p}_t^* - \hat{p}_{t+k}\right)$.

$$\hat{p}_t^* - \hat{p}_{t-1} = \beta\theta \mathbb{E}_t(\hat{p}_{t+1}^* - \hat{p}_t) + (1-\beta\theta)\,\Theta \hat{mc}_t + \pi_t \tag{83}$$

$$\hat{\pi}_t = \beta \mathbb{E}_t \hat{\pi}_{t+1} + \lambda \hat{mc}_t \tag{84}$$

where $\lambda = \frac{(1-\theta)(1-\beta\theta)\Theta}{\theta}$.

Finally the marginal cost equation is given by plugging the *Labour Supply* $\hat{w}_t - \hat{p}_t = \sigma \hat{y}_t + \phi \hat{n}_t$ and log of market clearing conditions $\hat{n}_t = \frac{1}{1-\alpha}(\hat{a}_t - \alpha \hat{y}_t)$ after some manipulations we get Equation (85)

$$\hat{mc}_t = \left(\frac{\sigma(1-\alpha)+\phi+\alpha}{1-\alpha}\right)\hat{y}_t - \left(\frac{\phi+1}{1-\alpha}\right)\hat{a}_t \tag{85}$$

Arranging in a convenient way

$$\hat{mc}_t = \left(\frac{\sigma(1-\alpha)+\phi+\alpha}{1-\alpha}\right)\left(\hat{y}_t - \left(\frac{1-\alpha}{\sigma(1-\alpha)+\phi+\alpha}\right)\left(\frac{\phi+1}{1-\alpha}\right)\hat{a}_t\right) \tag{86}$$

And by plugging the log-linear equation of flexible equilibrium output we get [5]

$$\hat{mc}_t = \left(\frac{\sigma(1-\alpha)+\phi+\alpha}{1-\alpha}\right)\left(\hat{y}_t - \hat{y}_t^F\right) \tag{87}$$

By plugging into the NKPC and defining $\tilde{y}_t = \hat{y}_t - \hat{y}_t^F$we get

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \phi_{y\pi} \tilde{y}_t \tag{88}$$

where $\phi_{y\pi} = \lambda \frac{\sigma(1-\alpha)+\phi+\alpha}{1-\alpha}$

In order to find a functional form for the output-gap we need to exploit the *Euler Equation.* Recalling the log form of the *Euler Equation* we have $\hat{c}_t = \mathbb{E}_t \hat{c}_{t+1} - \frac{1}{\sigma}(\hat{i}_t - \mathbb{E}_t \pi_{t+1})$. Plugging the market-clearing condition for the goods market we have the following relationships:

$$\hat{y}_t = \mathbb{E}_t \hat{y}_{t+1} - \frac{1}{\sigma}(\hat{i}_t - \mathbb{E}_t \pi_{t+1}) \tag{89}$$

In order to write it as function of the output-gap, just sum and subtract the flexible price output y_t^F

[5] $y_t^F = \left(\frac{1-\alpha}{\sigma(1-\alpha)+\phi+\alpha}\right)\left(\frac{\phi+1}{1-\alpha}\right)\hat{a}_t$

$$\tilde{y}_t = \mathbb{E}_t\tilde{y}_{t+1} - \frac{1}{\sigma}(\hat{i}_t - \mathbb{E}_t\pi_{t+1}) + \epsilon_{y,t} \tag{90}$$

Where $\epsilon_{y,t} = \hat{y}^F_{t+1} - \hat{y}^F_t$

Finally in order to close the model we assume that central bank responds to change in inflation, output gap and interest rate following a feedback rule

$$\hat{i}_t = \phi_\pi\pi_t + \phi_y\tilde{y}_t + \theta_{i,t} \tag{91}$$

Where $\theta_{i,t} = \rho_\theta\theta_{i,t-1} + \eta_{\theta,t}$ is an exogenous shock on interest rate which follows an AR(1) process.

4 The log-linearized model

The log-linearized version of the model is reported here for convenience.

$$\tilde{y}_t = \mathbb{E}_t\tilde{y}_{t+1} - \frac{1}{\sigma}(\hat{i}_t - \mathbb{E}_t\pi_{t+1}) + \epsilon_{y,t} \tag{92}$$

$$\pi_t = \beta\mathbb{E}_t\pi_{t+1} + \phi_{y\pi}\tilde{y}_t \tag{93}$$

$$\hat{i}_t = \phi_\pi\pi_t + \phi_y\tilde{y}_t + \theta_{i,t} \tag{94}$$

In matrix notation

$$A(\Omega)\mathbb{E}_tX_{t+1} = B(\Omega)X_t + C(\Omega)Z_t \tag{95}$$

Where $A(\Omega)$, $B(\Omega)$ and $C(\Omega)$ are matrices depending on the time-invariant structural parameters $\Omega \equiv [\sigma, \phi_\pi, \phi_y, \beta, \alpha, \phi, \theta, \epsilon]$.

$$\begin{bmatrix} 1 & -\frac{1}{\sigma(1-\phi_\pi)} \\ 0 & -\beta \end{bmatrix} \begin{bmatrix} \mathbb{E}_t y_{t+1} \\ \mathbb{E}_t\pi_{t+1} \end{bmatrix} = \begin{bmatrix} -1 & -\frac{\phi\pi}{\sigma(1-\phi_\pi)} \\ \phi_{y\pi} & -1 \end{bmatrix} \begin{bmatrix} y_t \\ \pi_t \end{bmatrix} + \begin{bmatrix} 1 & -\frac{1}{\sigma(1-\phi_\pi)} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \epsilon_{y,t} \\ \theta_{i,t} \end{bmatrix} \tag{96}$$

References

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