A Crash Course on DSGE Models

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1 The baseline New-Keynesian model

The model derivation follows Brugnolini and Corrado (2018) and Gali (2008). I tried to be consistent with standard notation mostly used in DSGE literature. The model I use for the analysis is a New-Keynesian model embedding infinitely life-time utility maximiser agents and monopolistically competitive firms producing differentiated goods using only labor and technology. There is no capital, and no investment. Each period, households choose between consumption and saving, and the only asset in the model is a risk-free bond. Finally, a Central Bank is in charge of maintaining price and output stability.

Household

The household sector is made by a representative household maximising his expected lifetime utility $U(C_t, N_t)$ at period $t = 0$. I assume a utility function depending only on consumption C_t and normalised leisure $1 - N_t$. Consumers minimise expenditure given the consumption level of composite good C_t. I assume that regularity conditions on the utility function hold and that $\partial U/\partial C_t > 0$, $\partial U/\partial N_t < 0$, $\partial U/\partial C_t^2 < 0$ and $\partial U/\partial N_t^2 < 0$. Moreover, I assume a standard constant relative risk aversion (CRRA) functional form of additively separable consumption and labor.

$$\max_{C_t,N_t,B_t} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\phi}}{1+\phi} \right) \tag{1}$$

Where β is the intertemporal discount factor, σ is the coefficient of the relative risk aversion, and ϕ is the inverse of the Frisch elasticity. $\mathbb{E}_0$ is the expectation operator conditional to the information set at time zero $\mathbb{E}\left[\cdot|\mathbb{I}_0\right]$ I also assume that there is a continuum (in the $[0,1]$ interval) of different goods produced with constant elasticity of substitution (CES) technology.

$$C_t = \left(\int_0^1 C_t(i)^{1-\frac{1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}} \tag{2}$$

Where ϵ is the parameter controlling the degree of substitutability among goods. The utility is maximised subject to the household's budget constraint and a *no-Ponzi game* condition in the government bonds market.

$$\int_0^1 P_t(i)C_t(i)di + B_t \leq (1+R_t)B_{t-1} + W_tN_t - T_t + \Pi_t^F \tag{3}$$

$$\lim_{T\to\infty} \mathbb{E}_t \{B_T\} \geq 0, \quad \forall t \tag{4}$$

The representative consumer allocates wealth between consumption and saving. In the equations, $P_t(i)$ denotes the prices of different goods i, $(1 + R_t)$ is the gross interest rate on risk-free bonds B_t purchased in the previous period. T_t is a lump-sum tax/transfer. W_t stands for labor price (wage). To derive the optimal allocation between goods, the representative agent maximises total consumption subject to any possible level of expenditure. The Lagrangian of the described maximisation problem is displayed in equation (5), and the associate multiplier is denoted by ψ_t.

$$\min_{C_t(i)} \quad \mathcal{L}_1 \equiv \int_0^1 P_t(i)C_t(i)di - \psi_t \left(\left[\int_0^1 C_t(i)^{\frac{\epsilon-1}{\epsilon}} \right]^{\frac{\epsilon}{\epsilon-1}} - C_t \right) \tag{5}$$

From the first order conditions (FOC) I can recover the *demand schedule* and the *aggregate price index* described by equations (6) and (7).

$$P_t \equiv \psi_t = \left[\int_0^1 P_t(i)^{1-\epsilon} di \right]^{\frac{1}{1-\epsilon}} \tag{6}$$

$$C_t(i) = C_t \left(\frac{P_t(i)}{P_t} \right)^{-\epsilon} \tag{7}$$

After some algebraic manipulations and by plugging equation (7) into the consumer's budget constraint, I can write the second-step of the household mazimisation problem as a current value Lagrangian.

$$\max_{C_t, N_t, B_t, \Lambda_t} \mathcal{L}_2 \equiv \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\phi}}{1+\phi} - \Lambda_t \left(P_t C_t + B_t - (1 + R_t) B_{t-1} - W_t N_t + T_t \right) \right)$$

Where Λ_t is the Lagrangian multiplier. By solving the system of first-order conditions I can recover the *labour supply* (8) and the *Euler equation* (9).

$$\frac{W_t}{P_t} = N_t^{\phi} C_t^{\sigma} \tag{8}$$

$$\mathbb{E}_t \left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{1}{\Pi_{t+1}} \right] = \frac{1}{(1 + R_t)} \tag{9}$$

Where $\Pi_{t+1} \equiv P_{t+1}/P_t$ is the gross inflation rate.

Firms

I assume that firms operate under monopolistic competition and produce differentiated goods by using labour N_t as their only source of input. Technology A_t is equal among firms, and the production function takes the following form:

$$Y_t(i) = A_t N_t(i)^{1-\alpha} \tag{10}$$

Where $Y_t(i)$ stands for the production of output i and α is the output elasticity concerning labor input. In the paper, I assume that price levels adjust *à la Calvo* (Calvo, 1983) with a fraction $1-\theta$ of re-optimizing firms and a fraction θ of non-re-optimizing firms with $\theta \in [0, 1]$. This assumption is extremely useful because it allows to compute a proxy for the potential output of the economy by setting to zero the fraction of non-re-optimizing firms. Imposing $\theta = 0$ allows removing the only source of inefficiency in the production sector by releasing the price rigidity assumption. Equation (11) displays the aggregate price index under the Calvo price assumption.

$$P_t = \left(\theta P_{t-1}^{1-\epsilon} + (1-\theta) P_t^{*1-\epsilon}\right)^{\frac{1}{1-\epsilon}} \tag{11}$$

Where P_t^* is the optimal price chosen by the optimizing firms. As $\theta \to 0$, $P_t = P_t^*$ implying that all the firms can reset their prices as in a flexible price economy. By dividing both sides by P_{t-1}, equation (11) can also be rewritten in terms of gross inflation.

$$\Pi_t^{1-\epsilon} = \theta + (1-\theta)\left(\frac{P_t^*}{P_{t-1}}\right)^{1-\epsilon} \tag{12}$$

Re-optimizing firms solve a profit maximisation problem subject to a downward sloping *demand constraint*. The Lagrangian for this problem can be written as in equation (13).

$$\max_{P_t^*} \quad \mathcal{L}_3 \equiv \sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left(Q_{t,t+k} \left(P_t^* \left(\frac{P_t^*}{P_{t+k}}\right)^{-\epsilon} Y_{t+k} - \Psi_{t+k}\left(\left(\frac{P_t^*}{P_{t+k}}\right)^{-\epsilon} Y_{t+k} \right) \right) \right) \tag{13}$$

Where $Q_{t,t+k} = \beta^k \left(\frac{C_{t+k}}{C_t}\right)^{-\sigma} \frac{P_t}{P_{t+k}}$ is the stochastic discount factor. As I assume that households own firms, coerently I assume that the two agents have the same discount factor. Ψ_{t+k} is a cost function depending on the production level. I assume that the regularity conditions on the cost function hold. Maximising for P_t^*, I retrieve equation (14).

$$P_t^* = \mathcal{M} \frac{\sum_{k=0}^{\infty} (\theta\beta)^k \mathbb{E}_t \left[C_{t+k}^{-\sigma} Y_{t+k} P_{t+k} MC_{t+k} \right]}{\sum_{k=0}^{\infty} (\theta\beta)^k \mathbb{E}_t \left[C_{t+k}^{-\sigma} Y_{t+k} \right]} \tag{14}$$

where $\mathcal{M} = \frac{\epsilon}{\epsilon - 1}$ is the firms' mark-up. Notice that when $\theta = 0$ the optimal price setting is given by $P_t^* = \mathcal{M}\Psi'_{t|t}$. As a second-step, firms choose the optimal amount of labor to minimise their total costs subject to the resource constraint. Equations (15) and (16) display the Lagrangian function for the firms' minimisation problem and the associated first-order condition for N_t.

$$\min_{N_t(i)} \quad \mathcal{L}_4 \equiv \frac{W_t}{P_t} N_t(i) - MC_t \left(Y_t(i) - A_t N_t(i)^{1-\alpha}\right) \tag{15}$$

$$\frac{\partial \mathcal{L}_4}{\partial N_t(i)} \equiv MC_t = \frac{W_t}{P_t} \frac{1}{(1-\alpha) A_t N_t(i)^{-\alpha}} \tag{16}$$

Where MC_t is the Lagrangian multiplier, which also represents the marginal cost of increasing the production by one unit (*shadow price*). A_t is a forcing variable representing a structural shock hitting the available technology. I assume that $ln(A_t)$ is distributed as an autoregressive process of order one with normal Gaussian innovation, as showed in equation (17).

$$ln(A_t) = \rho_a ln(A_{t-1}) + \epsilon_{a,t}, \quad \epsilon_{a,t} \sim \mathbb{N}\left(0, \sigma_a^2\right) \tag{17}$$

Where ρ_a is the autoregressive parameter with $|\rho_a| < 1$, and $\sigma_a^2 \in \mathbb{R}^+$ is the variance of the Gaussian innovation.

Central Bank

I assume that a Central Bank is in charge of maintaining price and output stability by responding to output and inflation deviations from their respective targets. The feedback rule employed by the Central Bank is described in equation (18).

$$\frac{R_t}{R} = \left(\frac{\Pi_t}{\Pi^*}\right)^{\phi_\pi} \left(\frac{Y_t}{Y^T}\right)^{\phi_y} \Theta_t \tag{18}$$

Where Π^* and Y^T are the Central Bank targets in term of inflation and output, R is the steady-state level of the net interest rate, ϕ_π and ϕ_y are the reaction coefficients of inflation and output deviations from the targets. Finally, Θ_t is the monetary policy shock. I assume that $ln(\Theta_t)$ is distributed as an autoregressive process of order one with normal Gaussian innovation, as showed in equation (19).

$$ln(\Theta_t) = \rho_a ln(\Theta_{t-1}) + \epsilon_{\theta,t}, \quad \epsilon_{\theta,t} \sim \mathbb{N}\left(0, \sigma_\theta^2\right) \tag{19}$$

Where ρ_θ is the autoregressive parameter with $|\rho_\theta| < 1$, and $\sigma_\theta^2 \in \mathbb{R}^+$ is the variance of the Gaussian innovation.

1.1 Equilibrium

In the model defined by equations (8) to (14) and (16) to (19), a competitive equilibrium is defined as a vector of prices $\{P_t, R_t, W_t\}$, allocations $\{C_t, Y_t, N_t, B_t, T_t, MC_t, \Lambda_t\}$ and a process for the exogenous state variables $\{A_t, \Theta_t\}$ such that:

- Households maximise the utility function subject to the budget constraint.
- Firms maximise the profit function subject to the resource constraint.
- The Central Bank abide by the feedback rule.
- The labor and good markets clear.

The last point implies that all the output produced is also consumed (equation 20), and that labor supply is equal to labor demand (equation 21).

$$Y_t = C_t \tag{20}$$

$$N_t = \left(\frac{Y_t}{A_t}\right)^{\frac{1}{1-\alpha}} \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\frac{\epsilon}{1-\alpha}} di \tag{21}$$

2 System of non-linear equations

The non-linear system of equations is made-up by equations (22) to (31). These are the labor supply, Euler equation, household's budget constraint, firms optimal price setting, labor demand, price dynamics, inflation dynamics, goods market clearing, labor market clearing, Central Bank feedback rule. I also model the exogenous variables as autoregressive processes of order one. The endogenous variables are $W_t, P_t, N_t, C_t, R_t, \Pi_t, P_t^*, Y_t, MC_t,$

B_t, T_t while A_t, Θ_t are exogenous forcing variables.

$$\frac{W_t}{P_t} = N_t^{\phi} C_t^{\sigma} \tag{22}$$

$$\mathbb{E}_t \left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{1}{\Pi_{t+1}} \right] = \frac{1}{(1+R_t)} \tag{23}$$

$$P_t C_t + B_t = (1+R_t) B_{t-1} + W_t N_t - T_t \tag{24}$$

$$P_t^* = \mathcal{M} \frac{\sum_{k=0}^{\infty} (\theta\beta)^k \mathbb{E}_t \left[C_{t+k}^{-\sigma} Y_{t+k} P_{t+k} MC_{t+k} \right]}{\sum_{k=0}^{\infty} (\theta\beta)^k \mathbb{E}_t \left[C_{t+k}^{-\sigma} Y_{t+k} \right]} \tag{25}$$

$$MC_t = \frac{W_t}{P_t} \frac{1}{(1-\alpha) A_t N_t(i)^{-\alpha}} \tag{26}$$

$$P_t = \left(\theta P_{t-1}^{1-\epsilon} + (1-\theta) P_t^{*1-\epsilon} \right)^{\frac{1}{1-\epsilon}} \tag{27}$$

$$\Pi_t^{1-\epsilon} = \theta + (1-\theta) \left(\frac{P_t^*}{P_{t-1}} \right)^{1-\epsilon} \tag{28}$$

$$Y_t = C_t \tag{29}$$

$$N_t = \left(\frac{Y_t}{A_t} \right)^{\frac{1}{1-\alpha}} \int_0^1 \left(\frac{P_t(i)}{P_t} \right)^{-\frac{\epsilon}{1-\alpha}} di \tag{30}$$

$$\frac{R_t}{R} = \left(\frac{\Pi_t}{\Pi^*} \right)^{\phi_\pi} \left(\frac{Y_t}{Y^T} \right)^{\phi_y} \Theta_t \tag{31}$$

3 Steady-state relationships

In steady-state I obtain the relationships described by equations (32) to (41) by dropping the time subscript. The detailed steady-state relationships of the model are reported in the appendix. In what follows, I assume that variables without the time t index denote the steady-state counterpart of the dynamic variables. Thus $P_t = P$, $R_t = R$, $W_t = W$, $C_t = C$, $Y_t = Y$, $N_t = N$, $B_t = B$, $T_t = T$, $MC_t = MC$, $\Lambda_t = \Lambda$, $A_t = A$, and $\Theta_t = \Theta$.

$$\frac{W}{P} = N^{\phi} C^{\sigma} \tag{32}$$

$$R = \frac{1-\beta}{\beta} \tag{33}$$

$$PC = RB + WN - T + \Pi^F \tag{34}$$

$$MC = \frac{1}{\mathcal{M}} \tag{35}$$

$$MC = \frac{W}{P} \frac{1}{(1-\alpha)N^{-\alpha}} \tag{36}$$

$$P^* = P \tag{37}$$

$$\Pi = 1 \tag{38}$$

$$Y = C \tag{39}$$

$$N = \left(\frac{Y}{A}\right)^{\frac{1}{1-\alpha}} \tag{40}$$

$$\Theta_t = 1 \tag{41}$$

3.1 Flexible price equilibrium

A useful feature of the NK model is the possibility to derive a final sistem which resemble a dynamic form of well-known economic relationship as the IS curve and the New-Keynesian Phillips curve. To derive these relationship we need to o find an equation for the potential output Y_t^F, as these relationships are written in terms of output-gap. Thus, I need to retrive a functional form for this variable. Starting from the firms' price maximisation, I get the flexible price mark-up $P_t^* = \mathcal{M}\Psi'_{t|t}$. Given that under flexible prices $P_t^* = P_t$ I have $MC_t = \frac{1}{\mathcal{M}}$. By plugging the flexible price mark-up into the labor demand equation $MC_t = \frac{W_t}{P_t}\frac{1}{A_t(1-\alpha)N_t^{-\alpha}}$, and using the market clearing condition $C_t = Y_t$, the labor supply $\frac{W_t}{P_t} = C_t^{\sigma} N_t^{\phi}$, and the labor market clearing condition $N_t = \left(\frac{Y_t}{A_t}\right)^{\frac{1}{1-\alpha}}$ I derive equation (42).

$$Y_t^F = \left(\frac{1}{\mathcal{M}}(1-\alpha)A_t^{\frac{1+\phi}{1-\alpha}}\right)^{\frac{1-\alpha}{\phi+\sigma(1-\alpha)+\alpha}} \tag{42}$$

Where variables (Y_t^F) with the superscript F denotes variables in flexible price equilibrium.

4 Log-Linearization

In this section I will follow mainly two different approaches to log-linearise the equations around the non-stochastic steady state of the model. The first method uses a first order Taylor expansion, thus it reduces to compute a bunch of first order derivatives, while the second uses some results from Uhlig (1995) and can be applied to the equations without directly compute any derivatives. In my opinion, the first method is more general and it is really useful when one would like to compute higher order approximations (second, third, etc.). Instead, the second method is faster for first order approximation. Following this reasoning I will just apply the first method to some equations, but I will carry on the analysis of the model using the second one.

4.1 Taylor expansion

Let's start remarking that the reason why we use the logarithm is to interpret parameters as elasticity (percentage change). However, using the log and the Taylor expansion leads to two different approximations; the first coming from the fact that log-differences are approximations of the percentage change (negligible when the percentage change is small), while the second coming from the fact that we are using a linear function to approximate a non-linear one (at least using a first order Taylor expansion). This is negligible only in a small interval around the steady-state.

Equation (43) shows a first order Taylor expansion with n variables. Notice that the function is approximated in a particular point x_0 which is a vector in $\mathbb{R}^n$. Thus the function will be approximated to its value computed at that particular point plus the slope in the different dimensions represented by the first order partial derivatives computed at that particular point. In a two dimesion simple case this means that I am approximating a non-linear function with a line, while in a higher dimensional space I am approximating it with some planes ($\mathbb{R}^3$) or hyperplanes ($\mathbb{R}^{n>3}$)

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) \approx f(x_{1,0}, x_{2,0}, \ldots, x_{n,0}) &+ \frac{\partial f(x_1, x_2, \ldots, x_n)}{\partial x_1}\Big|_{x_1 = x_{1,0}} (x - x_{1,0}) \\ &+ \frac{\partial f(x_1, x_2, \ldots, x_n)}{\partial x_2}\Big|_{x_2 = x_{2,0}} (x - x_{2,0}) + \ldots + \frac{\partial f(x_1, x_2, \ldots, x_n)}{\partial x_n}\Big|_{x_n = x_{n,0}} (x - x_{n,0}) \end{aligned} \tag{43}$$

Now, I would like to express my function as a log-deviation from the steady state. Recalling the properties of the log, for our proof is useful to highlight that the log-deviation is an approximation of the percentage change.

$$\hat{x} = log(x) - log(x_0) \approx \frac{x - x_0}{x_0} \tag{44}$$

The basic idea behind the prof is to rewrite the Taylor expansion in percentage change and use the log-deviation to approximate it. In order to comply with this idea it is useful to work with a two variable case as in equation (45).

$$f(x_1, x_2) - f(x_{1,0}, x_{2,0}) \approx \frac{\partial f(x_1, x_2)}{\partial x_1}\Big|_{x_1 = x_{1,0}} (x - x_{1,0}) + \frac{\partial f(x_1, x_2)}{\partial x_2}\Big|_{x_2 = x_{2,0}} (x - x_{2,0}) \tag{45}$$

We multiply and divide the three different pieces by $f(x_{1,0}, x_{2,0})$, $x_{1,0}$ and $x_{2,0}$ holding:

$$f(x_{1,0}, x_{2,0}) \frac{f(x_1, x_2) - f(x_{1,0}, x_{2,0})}{f(x_{1,0}, x_{2,0})} \approx x_{1,0} \frac{\partial f(x_1, x_2)}{\partial x_1}\Big|_{x_1 = x_{1,0}} \frac{x - x_{1,0}}{x_{1,0}} + x_{2,0} \frac{\partial f(x_1, x_2)}{\partial x_2}\Big|_{x_2 = x_{2,0}} \frac{x - x_{2,0}}{x_{2,0}}$$

Finally by using Equation (44) we can rewrite the general formula to linearise any function as a log-deviation from the steady state.

$$f(x_{1,0}, x_{2,0}) \hat{f}(x_1, x_2) \approx x_{1,0} \frac{\partial f(x_1, x_2)}{\partial x_1}\Big|_{x_1 = x_{1,0}} \hat{x}_1 + x_{2,0} \frac{\partial f(x_1, x_2)}{\partial x_2}\Big|_{x_2 = x_{2,0}} \hat{x}_2 \tag{46}$$

As an illustrative example consider the labor supply, as in equation (8), repeated here for convenience:

$$\frac{W_t}{P_t} = N_t^{\phi} C_t^{\sigma}$$

Using Equation (46) we can rewrite it in the following way

$$W_{ss}\hat{w}_t = P_{ss}\frac{\partial W_t}{\partial P_t}\Big|_{P_t=P_{ss}}\hat{p}_t + N_{ss}\frac{\partial W_t}{\partial N_t}\Big|_{N_t=N_{ss}}\hat{n}_t + C_{ss}\frac{\partial W_t}{\partial C_t}\Big|_{C_t=C_{ss}}\hat{c}_t \tag{47}$$

Where subscritp ss is added to highlight steady-state varaibles, but in what follow will be dropped for notation simplicity. By computing the derivatives of W_t w.r.t. price, hours worked and consumption an plugging into the above equation and using the properties of the exponential functions we get

$$W\hat{w}_t = PN^{\phi}C^{\sigma}\hat{p}_t + \phi PN^{\phi}C^{\sigma}\hat{n}_t + \sigma PN^{\phi}C^{\sigma}\hat{c}_t \tag{48}$$

where in order to simplify the notation I have written the steady state value without the subscript ss. Finally solwing for $\hat{w}_t$ and using the steady state relation $\frac{PN^{\phi}C^{\sigma}}{W} = 1$ we get the log-linearised labor supply.

$$\hat{w}_t = \hat{p}_t + \phi\hat{n}_t + \sigma\hat{c}_t \tag{49}$$

all the other equation in the system can be easily derived following the same reasoning.

4.2 Uhlig 1995 method

The secon method of log-linearisation mainly follows the three Uhlig's building blocks[1].

$$e^{x_t + ay_t} \approx 1 + x_t + ay_t \tag{50}$$

$$x_t y_t \approx 0 \tag{51}$$

$$\mathbb{E}_t\left[ae^{x_{t+1}}\right] \approx \mathbb{E}_t\left[ax_{t+1}\right] \tag{52}$$

Where x_t and y_t are real variables close to zero ($x_t = log(X_t) - log(\bar{X})$, in our notation this will be $\hat{x}_t$), $\bar{X}$ is the steady state value of the variable X_t (in our notation this will be just X) and a is a constant (the second and third building blocks are up to a constant). As suggested by Uhlig we replace each variables by $\bar{X}e^{x_t}$, than applying the three building blocks. After some manipulations, all the constants drop out to each equations.

[1]Reported here as in the original notation of Uhlig (1995).

5 The log-linearized model

The final model described by equations (8) to (14) and (16) to (20) is fully non-linear and cannot be solved analytically. The most straightforward route to solve the model is by fully log-linearising it around the non-stochastic steady-state, and taking a first-order perturbation approach. The fully-log-linearised model is reported in equations (53) to (63).

$$\hat{w}_t - \hat{p}_t = \phi \hat{n}_t + \sigma \hat{c}_t \tag{53}$$

$$\hat{c}_t = \mathbb{E}_t \hat{c}_{t+1} - \frac{1}{\sigma}\left(\hat{i}_t - \mathbb{E}_t \hat{\pi}_{t+1}\right) \tag{54}$$

$$\hat{\pi}_t = (1-\theta)(\hat{p}_t^* - \hat{p}_{t-1}) \tag{55}$$

$$\sum_{k=0}^{\infty} \theta^k \beta^k \left(\hat{p}_t^* - \hat{p}_{t-1}\right) = \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[\left(\hat{mc}_{t+k|t} + (\hat{p}_{t+k} - \hat{p}_{t-1})\right)\right] \tag{56}$$

$$\hat{c}_t = \hat{y}_t \tag{57}$$

$$\hat{n}_t = \frac{1}{(1-\alpha)}(\hat{y}_t - \hat{a}_t)^{2} \tag{60}$$

$$\hat{mc}_t = \hat{w}_t - \hat{p}_t - \hat{a}_t + \alpha \hat{n}_t \tag{61}$$

$$\hat{p}_t = (1-\theta)\hat{p}_t^* + \theta \hat{p}_{t-1} \tag{62}$$

$$\hat{i}_t = \phi_\pi \hat{\pi}_t + \phi_y \tilde{y}_t + \theta_t \tag{63}$$

Where $\kappa = \lambda \frac{\sigma(1-\alpha)+\phi+\alpha}{1-\alpha}$, $\lambda = \Theta^\lambda \frac{(1-\theta)(1-\beta\theta)}{\theta}$, and $\Theta^\lambda = \frac{1-\alpha}{1-\alpha-\alpha\epsilon}$.

[2] Taking the log-deviation of the market clearing condition I get

$$N\left(1+\hat{n}\right) = \left(\frac{Y}{A}\right)^{\frac{1}{1-\alpha}} \left(1 + \left(\frac{1}{1-\alpha}\right)\left(\hat{y}_t - \hat{a}_t\right)\right) + RES \tag{59}$$

where RES is a very small quantity in a neighborhood of the zero inflation steady-state and can be neglected in a first order Taylor expansion. See Gali (2008) chapter 3, Appendix 3.3.

$$\hat{y}_t = (1-\alpha)\,\hat{n}_t + \hat{a}_t \tag{60}$$

6 System reduction – IS and Phillips curve derivation

To simplify the model to a minimum set of equations, I mainly follow Gali (2008). Recalling the convergence result for geometric series $\frac{1}{\sum_{k=0}^{\infty}\theta^k\beta^k} = (1+\beta\theta)$, equation (56) can be rewritten as (64).

$$\hat{p}_t^* - \hat{p}_{t-1} = (1-\beta\theta)\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left[\left(\hat{mc}_{t+k|t} + (\hat{p}_{t+k} - \hat{p}_{t-1})\right)\right] \tag{64}$$

As $\alpha \neq 0$, I rule out the constant return to scale hypothesis, meaning $\hat{mc}_{t+k|t} \neq \hat{mc}_{t+k}$. I than need to find an equation for $\hat{mc}_{t+k|t}$. Starting from the marginal cost equation and plugging $\hat{n}_t = \frac{1}{1-\alpha}(\hat{a}_t - \alpha\hat{y}_t)$ I derive an equation for $\hat{mc}_{t+k}$, and accordingly for $\hat{mc}_{t+k|t}$ – equations (65) and (66).

$$\hat{mc}_{t+k} = \hat{w}_{t+k} - \hat{p}_{t+k} - \frac{1}{1-\alpha}(\hat{a}_{t+k} - \alpha\hat{y}_{t+k}) \tag{65}$$

$$\hat{mc}_{t+k|t} = \hat{w}_{t+k} - \hat{p}_{t+k} - \frac{1}{1-\alpha}(\hat{a}_{t+k} - \alpha\hat{y}_{t+k|t}) \tag{66}$$

Thus, equating the two expressions, and by plugging the demand schedule, I derive an expression for $\hat{mc}_{t+k|t}$[3].

$$\hat{mc}_{t+k|t} - \hat{mc}_{t+k} = \frac{\alpha}{1-\alpha}(\hat{y}_{t+k|t} + \hat{y}_{t+k}) \tag{67}$$

$$\hat{mc}_{t+k|t} = \hat{mc}_{t+k} + \frac{\alpha\epsilon}{1-\alpha}\left(\hat{p}_t^* - \hat{p}_{t+k}\right) \tag{68}$$

Finally, by plugging (68) into (64), and after some algebraic manipulations, using $\hat{\pi}_t = (1-\theta)(\hat{p}_t^* - \hat{p}_{t-1})$, I can retrieve an equation for the New-Keynesian Phillips curve.

$$\hat{p}_t^* - \hat{p}_{t-1} = (1-\beta\theta)\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left(mc_{t+k} - \frac{\alpha\epsilon}{1-\alpha}\left(\hat{p}_t^* - \hat{p}_{t+k}\right) + (\hat{p}_{t+k} - \hat{p}_{t-1})\right) \tag{69}$$

$$\hat{p}_t^* - \hat{p}_{t-1} = (1-\beta\theta)\,\Theta\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t mc_{t+k} + \sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left(\hat{p}_{t+k} - \hat{p}_{t-1}\right) \tag{70}$$

$$\hat{p}_t^* - \hat{p}_{t-1} = (1-\beta\theta)\,\Theta\frac{1}{(1-\theta\beta F)}mc_t + \frac{1}{(1-\theta\beta F)}\left(\hat{p}_t - \hat{p}_{t-1}\right) \tag{71}$$

$$\hat{p}_t^* - \hat{p}_{t-1} = \beta\theta\mathbb{E}_t(\hat{p}_{t+1}^* - \hat{p}_t) + (1-\beta\theta)\,\Theta\hat{mc}_t + \hat{\pi}_t \tag{72}$$

$$\hat{\pi}_t = \beta\mathbb{E}_t\hat{\pi}_{t+1} + \lambda\hat{mc}_t \tag{73}$$

Where $\Theta = \frac{1-\alpha}{1-\alpha-\alpha\epsilon}$, $\lambda = \frac{(1-\theta)(1-\beta\theta)\Theta}{\theta}$. and F is the forward operator. Finally, after some manipulations, the marginal cost equation is given by plugging into (62) the labour supply $\hat{w}_t - \hat{p}_t = \sigma\hat{y}_t + \phi\hat{n}_t$ and the log of the

[3] By taking log of the *demand constraint* $Y_{t+k|t} = \left(\frac{P_t^*}{P_{t+k}}\right)^{-\epsilon}(Y_{t+k})$ I have $\hat{y}_{t+k|t} = \hat{y}_{t+k} + \epsilon\left(\hat{p}_t^* - \hat{p}_{t+k}\right)$. Then $\hat{mc}_{t+k} = \hat{mc}_{t+k|t} + \frac{\alpha\epsilon}{1-\alpha}\left(\hat{p}_t^* - \hat{p}_{t+k}\right)$.

market clearing conditions $\hat{n}_t = \frac{1}{1-\alpha}\left(\hat{y}_t - \hat{a}_t\right)$.

$$\hat{mc}_t = \left(\frac{\sigma(1-\alpha)+\phi+\alpha}{1-\alpha}\right)\hat{y}_t - \left(\frac{\phi+1}{1-\alpha}\right)\hat{a}_t \tag{74}$$

Re-arranging in a convenient way, and by plugging the log-linear equation of flexible equilibrium output I derive equation (77)[4].

$$\hat{mc}_t = \left(\frac{\sigma(1-\alpha)+\phi+\alpha}{1-\alpha}\right)\left(\hat{y}_t - \left(\frac{1-\alpha}{\sigma(1-\alpha)+\phi+\alpha}\right)\left(\frac{\phi+1}{1-\alpha}\right)\hat{a}_t\right) \tag{75}$$

$$\hat{mc}_t = \left(\frac{\sigma Y(1-\alpha)+C(\phi+\alpha)}{C(1-\alpha)}\right)\left(\hat{y}_t - \hat{y}_t^F\right) \tag{76}$$

$$\hat{\pi}_t = \beta \mathbb{E}_t \hat{\pi}_{t+1} + \kappa \tilde{y}_t \tag{77}$$

where $\kappa = \lambda \frac{\sigma(1-\alpha)+\phi+\alpha}{1-\alpha}$.

Secondly, to find a functional form for the output-gap on the form of the New-Keynesian IS curve, I exploit the Euler equation. Recalling the log form of the Euler equation, I have $\hat{c}_t = \mathbb{E}_t\hat{c}_{t+1} - \frac{1}{\sigma}(\hat{i}_t - \mathbb{E}_t\hat{\pi}_{t+1})$. Plugging the market-clearing condition for the goods market I have the following relationships:

$$\hat{y}_t = \mathbb{E}_t\hat{y}_{t+1} - \frac{1}{\sigma}(\hat{i}_t - \mathbb{E}_t\hat{\pi}_{t+1}) \tag{78}$$

To write it as function of the output-gap, I sum and subtract the flexible price output y_t^F.

$$\tilde{y}_t = \mathbb{E}_t\tilde{y}_{t+1} - \frac{1}{\sigma}(\hat{i}_t - \mathbb{E}_t\hat{\pi}_{t+1}) + \epsilon_{y,t} \tag{79}$$

Where $\epsilon_{y,t} = \hat{y}_{t+1}^F - \hat{y}_t^F$.

7 Minimum system of equations

The log-linearized version of the model is reported here for convenience.

$$\tilde{y}_t = \mathbb{E}_t\tilde{y}_{t+1} - \frac{1}{\sigma}(\hat{i}_t - \mathbb{E}_t\hat{\pi}_{t+1}) + \epsilon_{y,t} \tag{80}$$

$$\hat{\pi}_t = \beta \mathbb{E}_t \hat{\pi}_{t+1} + \kappa \tilde{y}_t \tag{81}$$

$$\hat{i}_t = \phi_\pi \hat{\pi}_t + \phi_y \tilde{y}_t + \theta_{i,t} \tag{82}$$

[4] $y_t^F = \left(\frac{(1-\alpha)}{\sigma(1-\alpha)+\phi+\alpha}\right)\left(\frac{\phi+\alpha}{1-\alpha}\right)\hat{a}_t$

This system can be rewritten in matrix notation (*companion form*) as (83).

$$\mathbf{A}\left(\mathbf{\Theta}\right)\mathbb{E}_t\mathbf{Y}_{t+1} = \mathbf{B}\left(\mathbf{\Theta}\right)\mathbf{Y}_t + \mathbf{C}\left(\mathbf{\Theta}\right)\mathbf{X}_t \tag{83}$$

Where $\mathbf{A}\left(\mathbf{\Theta}\right)$ is the matrix of coefficients related to the forward-looking endogenous variables, $\mathbf{B}\left(\mathbf{\Theta}\right)$ is the matrix of coefficients related to the predetermined variables and backward-looking variables; finally, $\mathbf{C}\left(\mathbf{\Theta}\right)$ is the matrix of coefficients of the exogenous variables. These matrices depend on structural parameters, steady-state values and reduced form parameters collected in the vector $\mathbf{\Theta} \equiv [\sigma, \phi_\pi, \phi_y, \beta, \alpha, \phi, \theta, \epsilon]$. Also, $\mathbf{Y}_{t+1}$ is a vector of forward-looking endogenous variables, while $\mathbf{Y}_t$ is a vector of backward-looking and predetermined variables. $\mathbf{X}_t$ is a vector of exogenous variables distributed as an AR(1) and dependent on a set of i.i.d. exogenous shock. The system described by equations (80) to (82) can be accomodated as in (83) as follows:

$$\begin{bmatrix} -1 & -1 & 0 \\ 0 & -\beta & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbb{E}_t\tilde{y}_{t+1} \\ \mathbb{E}_t\hat{\pi}_{t+1} \\ \mathbb{E}_t\hat{i}_{t+1} \end{bmatrix} = \begin{bmatrix} -1 & 0 & -\frac{1}{\sigma} \\ \kappa & -1 & 0 \\ \phi_y & \phi_\pi & -1 \end{bmatrix} \begin{bmatrix} \tilde{y}_t \\ \hat{\pi}_t \\ \hat{i}_t \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \epsilon_{y,t} \\ \theta_{i,t} \end{bmatrix} \tag{84}$$

Notice that the matrix $\mathbf{A}\left(\mathbf{\Theta}\right)$ in this system is singulas, as it displays a row of zeros. Therefore the Blanchard and Kahn (1980)'s conditions are not satisfied. An easy way to circunvent this issue is by noticing that the equation corresponding to the row of zeros – i.e. the feedback rule – is a linear combination of other contemporaneous variables ($\tilde{y}_t$,$\hat{\pi}_t$), embedding no dynamics (dependence from past/future variables). Thus, we can directly plug it into equation (80), and rewrite the minimum system of equation only for $\tilde{y}_t$ and $\hat{\pi}_t$. This procedure, called *system reduction*, is the departure idea of King and Watson (1998, 2002). Equation (85) shows the system of equation after we solve for $\hat{i}_t$ and substitute into the other equations.

$$\begin{bmatrix} 1 & -\frac{1}{\sigma(1-\phi_\pi)} \\ 0 & -\beta \end{bmatrix} \begin{bmatrix} \mathbb{E}_t\tilde{y}_{t+1} \\ \mathbb{E}_t\hat{\pi}_{t+1} \end{bmatrix} = \begin{bmatrix} -1 & -\frac{\phi\pi}{\sigma(1-\phi_\pi)} \\ \kappa & -1 \end{bmatrix} \begin{bmatrix} \tilde{y}_t \\ \hat{\pi}_t \end{bmatrix} + \begin{bmatrix} 1 & -\frac{1}{\sigma(1-\phi_\pi)} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \epsilon_{y,t} \\ \theta_{i,t} \end{bmatrix} \tag{85}$$

Where the matrix $\mathbf{A}\left(\mathbf{\Theta}\right)$ is now full-rank as $|\mathbf{A}\left(\mathbf{\Theta}\right)| = -\beta$. After checking that $\mathbf{A}\left(\mathbf{\Theta}\right)$ is invertible, the model can be solved, provided that some other conditions are satisfied.

8 Solution method

The log-linearised model can be solved via different methodologies, as it results in a system of linear stochastic difference equations under rational expectations (Blanchard and Kahn, 1980; King and Watson, 1998, 2002; Sims, 2002). Although these methodologies return approximately the same solution, they are tailored for different issues that can arise in the model specification. With some minor differences, all the methodologies start rewriting the system of difference equations in companion form, as shown in equation (86).

$$\mathbf{A}\left(\mathbf{\Theta}\right)\mathbb{E}_t\mathbf{Y}_{t+1} = \mathbf{B}\left(\mathbf{\Theta}\right)\mathbf{Y}_t + \mathbf{C}\left(\mathbf{\Theta}\right)\mathbf{X}_t \tag{86}$$

Where $\mathbf{A}\left(\mathbf{\Theta}\right)$ is the matrix of coefficients related to the forward-looking endogenous variables, $\mathbf{B}\left(\mathbf{\Theta}\right)$ is the matrix of coefficients related to the predetermined variables and backward-looking variables; finally, $\mathbf{C}\left(\mathbf{\Theta}\right)$ is the matrix of coefficients of the exogenous variables. These matrices depend on structural parameters, steady-state values and reduced form parameters collected in the vector $\mathbf{\Theta}$. Also, $\mathbf{Y}_{t+1}$ is a vector of forward-looking endogenous variables, while $\mathbf{Y}_t$ is a vector of backward-looking and predetermined variables. $\mathbf{X}_t$ is a vector of exogenous variables distributed as an AR(1) and dependent on a set of i.i.d. exogenous shock. Blanchard and Kahn (1980)'s conditions require the invertibility of the leading matrix $\mathbf{A}(\mathbf{\Theta})$ to solve the companion form for $\mathbb{E}_t\mathbf{Y}_{t+1}$, as showed in equation (87).

$$\mathbb{E}_t\mathbf{Y}_{t+1} = \mathbf{A}(\mathbf{\Theta})^{-1}\mathbf{B}(\mathbf{\Theta})\mathbf{Y}_t + \mathbf{A}(\mathbf{\Theta})^{-1}\mathbf{C}(\mathbf{\Theta})\mathbf{X}_t \tag{87}$$

This condition implies that $|\mathbf{A}(\mathbf{\Theta})| \neq 0$, therefore the matrix $\mathbf{A}(\mathbf{\Theta})$ must be non-singular[5]. When this condition is not fullfiled, more general methods based on weaker requirements have to be used. One example is (King and Watson, 1998, 2002), which requires $|\mathbf{A}(\mathbf{\Theta})z - \mathbf{B}(\mathbf{\Theta})| \neq 0$[6].The model has three endogenous variables, no predetermined variables and two shocks. These variables are then collected in the following vectors:

- $\mathbb{E}_tY_{t+1} \equiv \mathbb{E}_t\left[\hat{y}_{t+1}, \hat{\pi}_{t+1}, \hat{i}_{t+1}\right]$
- $Y_t \equiv \left[\hat{y}_t, \hat{\pi}_t, \hat{i}_t\right]$
- $X_t \equiv [a_t, \theta_t]$

Following Blanchard and Kahn (1980), a solution exists if the number of backward-looking variables is equal to the number of stable roots, while the number of forward-looking variables must be equal to the number of unstable roots. On the contrary, when the number of stable roots is greater than the number of predetermined variables there are multiple solutions. While, when the number of unstable roots is greater than the number of forward-looking variables, I have no solutions. If this condition is met, the variables will return to their long-run equilibrium path after the model has been shocked. Since the stability of the solution crucially depends on the calibrated parameters, which are collected in the vector $\mathbf{\Theta}$, in the next section, I briefly describe our calibration procedure.

9 Calibration

Calibration is the procedure through which

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[5]This condition is usually called *rank condition*, due to the fact that any matrix to be non-singular has to be full-rank.

[6]As the system in (86) is transformed by the forward operator F in $(\mathbf{A}(\mathbf{\Theta})F - \mathbf{B}(\mathbf{\Theta}))\mathbf{Y}_t = \mathbf{C}(\mathbf{\Theta})\mathbf{X}_t$.

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