Technical Appendix

Fiscal Compact and Debt Consolidation Dynamics

1 The Baseline Model

Household

There is a representative infinity-lived household maximising his expected life-utility at period $t = 0$. We assume a utility function depending on consumption and leisure. Consumers have to minimise expenditure given the level of composite good C_t.

$$\max_{C_t, N_t} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(C_t, N_t) \tag{1}$$

We assume that regularity conditions hold and that $\partial U/\partial C_t > 0$, $\partial U/\partial N_t < 0$, $\partial U/\partial C_t^2 < 0$ and $\partial U/\partial N_t^2 < 0$. Moreover, we assume a standard constant relative risk aversion (CRRA) functional form with separable consumption and leisure.

$$\max_{C,N} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\phi}}{1+\phi} \right) \tag{2}$$

We also assume that there is a continuum (in the $[0, 1]$ interval) of different goods produced with monopolistic competition in the goods market.

$$C_t = \left(\int_0^1 C_t(i)^{1-\frac{1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}} \tag{3}$$

Utility is maximised subject to the household's budget constraint.

$$\int_0^1 P_t(i) C_t(i) di + B_t \leq (1 + R_t) B_{t-1} + W_t N_t - T_t \tag{4}$$

And to a *No-Ponzi Game* condition in the government bonds market.

$$\lim_{T \to \infty} \mathbb{E}_t \left\{ B_T \right\} \geq 0, \quad \forall t \tag{5}$$

The representative consumer allocates wealth between consumption and saving: $P_t(i)$ denote the prices of different goods i, Q_t is the interest rate, W_t stands for wage and T_t is a lump-sum transfer which also captures the dividends coming from firms owned by households.

In order to derive the optimal allocation between goods, the representative agent maximises total consumption subject to any possible level of expenditure:

$$\min_{C_t(i)} \int_0^1 P_t(i)C_t(i)di \tag{6}$$

s.t.

$$\left[\int_0^1 C_t(i)^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon-1}} \geq C_t \tag{7}$$

The Lagrangian takes the form

$$\min_{C_t(i)} \int_0^1 P_t(i)C_t(i)di - \psi_t \left(\left[\int_0^1 C_t(i)^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon-1}} - C_t\right) \tag{8}$$

From the first order conditions we can recover the *demand schedule* and the *aggregate price*

$$\frac{\partial}{\partial C_t(i)} \equiv C_t(i) = C_t \left(\frac{P_t(i)}{\psi_t}\right)^{-\epsilon} \tag{9}$$

Where ψ_t is the Lagrangian multiplier. From F.O.C it is possible tho show that $\psi_t > 0$ implying a binding constraint.

Plugging into the definition of a composite good and solving for ψ_t we obtain the *aggregate price index*

$$\psi_t = \left[\int_0^1 P_t(i)^{1-\epsilon}di\right]^{\frac{1}{1-\epsilon}} \equiv P_t \tag{10}$$

Than the *demand for good i* is

$$C_t(i) = C_t \left(\frac{P_t(i)}{P_t}\right)^{-\epsilon} \tag{11}$$

Thus we get

$$\int_0^1 P_t(i)C_t(i)di = P_tC_t \tag{12}$$

which can be plugged into the original budget constraint yielding Equation (13)

$$P_tC_t + B_t \leq (1+R_t)B_{t-1} + W_tN_t - T_t \tag{13}$$

Maximising the utility function w.r.t. (13) we can construct the Lagrangian:

$$\max_{C_t,N_t} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\phi}}{1+\phi} - \Lambda_t \left(P_tC_t + B_t - (1+R_t)B_{t-1} - W_tN_t + T_t \right) \right) \tag{14}$$

The first order conditions are [1]

$$\frac{\partial}{\partial C_t} \equiv C_t^{-\sigma} = \Lambda_t P_t \tag{15}$$

$$\frac{\partial}{\partial N_t} \equiv N_t^{\phi} = \Lambda_t W_t \tag{16}$$

$$\frac{\partial}{\partial B_t} \equiv \beta \frac{\Lambda_{t+1}}{\Lambda_t} = \frac{1}{(1+R_t)} \tag{17}$$

$$\frac{\partial}{\partial \Lambda_t} \equiv P_tC_t + B_t - (1+R_t)B_{t-1} - W_tN_t + T_t = 0 \tag{18}$$

By solving the system we can recover the *Labour Supply* (19). Solving forward Equation (15) we get the *Euler Equation* (20).

$$\frac{W_t}{P_t} = N_t^{\phi} C_t^{\sigma} \tag{19}$$

$$\mathbb{E}_t \left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{1}{\Pi_{t+1}} \right] = \frac{1}{(1+R_t)} \tag{20}$$

The *Euler Equation* states how to allocate consumption between different periods by acquiring bonds at price Q_t.

[1] The Lagrangian multiplier is positive $\Lambda_t > 0$. In this way the *complementary slackness condition* is satisfy and $\Lambda_t(P_tC_t + Q_tB_t - B_{t-1} - W_tN_tT_t) = 0$. From the derivative w.r.t. C_t is easy to show that $\Lambda_t = C_tP_t$ is positive by assumption.

Firms

We assume firms operate with monopolistic competition and produce differentiated goods by using just labour and technology. Technology A_t is equal among firms. The production function has the following form:

$$Y_t(i) = A_t N_t(i)^{1-\alpha} \tag{21}$$

Price levels adjust *à la Calvo* with a fraction $1-\theta$ of re-optimizing firms and a fraction θ of non re-optimizing firms with $\theta \in [0,1]$. $S(t)$ is the set of non re-optimizing firms.

$$P_t = \left(\int_0^1 P_t(i)^{1-\epsilon} di \right)^{\frac{1}{1-\epsilon}} \tag{22}$$

$$P_t = \left(\int_{S(t)} \theta P_{t-1}(i)^{1-\epsilon} di + (1-\theta) P_t^{*1-\epsilon} \right)^{\frac{1}{1-\epsilon}} \tag{23}$$

$$P_t = \left(\theta P_{t-1}^{1-\epsilon} + (1-\theta) P_t^{*1-\epsilon} \right)^{\frac{1}{1-\epsilon}} \tag{24}$$

dividing both sides by P_{t-1} we can rewrite Equation (22) in terms of inflation

$$\Pi_t^{1-\epsilon} = \theta + (1-\theta) \left(\frac{P_t^*}{P_{t-1}} \right)^{1-\epsilon} \tag{25}$$

Re-optimizing firms solve the following profit maximisation subject to the *Demand Constraint*

$$\max_{P_t^*} \quad \sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left\{ Q_{t,t+k} \left(P_t^* Y_{t+k|t} - \Psi_{t+k} \left(Y_{t+k|t} \right) \right) \right\} \tag{26}$$

where $Q_{t,t+k}$ is the discount factor. It is assumed to be equal to the consumers' discount factor. Ψ_{t+k} is a cost function depending on the production level. We assume regularity condition on the cost function.

$$s.t. \quad Y_{t+k|t} = \left(\frac{P_t^*}{P_{t+k}} \right)^{-\epsilon} Y_{t+k} \tag{27}$$

directly plugging the *Demand Constraint* into the objective equation and maximising for P_t^* yields

$$\max_{P_t^*} \quad \sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left\{ Q_{t,t+k} \left(P_t^* \left(\frac{P_t^*}{P_{t+k}} \right)^{-\epsilon} Y_{t+k} - \Psi_{t+k} \left(\left(\frac{P_t^*}{P_{t+k}} \right)^{-\epsilon} Y_{t+k} \right) \right) \right\} \tag{28}$$

$$\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t Q_{t,t+k} \left((1-\epsilon) \left(\frac{P_t^*}{P_{t+k}} \right)^{-\epsilon} Y_{t+k} + \epsilon \Psi'_{t+k|t} \frac{P_t^{*-\epsilon-1}}{P_{t+k}^{\epsilon}} Y_{t+k} \right) = 0 \tag{29}$$

$$\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left[Q_{t,t+k} Y_{t+k} \left(P_t^* - \frac{\epsilon}{\epsilon - 1} \Psi'_{t+k|t} \right) \right] = 0 \tag{30}$$

dividing by P_{t-1} and plugging $Q_{t,t+k} = \beta^k \left(\frac{C_{t+k}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+k}}$ and $MC_{t+k} = \frac{\Psi'_{t+k}}{P_{t+k}}$ we get

$$\sum_{k=0}^{\infty} (\theta\beta)^k \mathbb{E}_t \left[\left(\frac{C_{t+k}}{C_t} \right)^{-\sigma} \left(\frac{P_t}{P_{t+k}} \right) \left(\frac{P_t^*}{P_{t+k}} \right)^{-\epsilon} Y_{t+k} \left(\frac{P_t^*}{P_{t-1}} - \mathcal{M} MC_{t+k} \frac{P_{t+k}}{P_{t-1}} \right) \right] = 0 \tag{31}$$

and after some manipulations

$$P_t^* = \mathcal{M} \frac{\sum_{k=0}^{\infty} (\theta\beta)^k \mathbb{E}_t \left[C_{t+k}^{-\sigma} Y_{t+k} P_{t+k} MC_{t+k} \right]}{\sum_{k=0}^{\infty} (\theta\beta)^k \mathbb{E}_t \left[C_{t+k}^{-\sigma} Y_{t+k} \right]} \tag{32}$$

where $\mathcal{M} = \frac{\epsilon}{\epsilon - 1}$. Notice that as $\theta = 0$ we are in the case of flexible price, thus the optimal price setting is given by $P_t^* = \mathcal{M} \Psi'_{t|t}$.

Firms, as consumers, face a dualistic problem. They need to choose the optimal price in order to maximize profits and also have to choose the amount of labor to minimize the costs.

$$\min_{N_t(i)} \quad \frac{W_t}{P_t} N_t(i) \tag{33}$$

s.t.

$$Y_t(i) = A_t N_t(i)^{1-\alpha} \tag{34}$$

Building-up the Lagrangian function we define the Lagrangian multiplier as the marginal cost of increasing the production by one unit (*shadow price*).

$$\min_{N_t(i)} \quad \frac{W_t}{P_t} N_t(i) - MC_t (Y_t(i) - A_t N_t(i)^{1-\alpha}) \tag{35}$$

$$\frac{\partial}{\partial N_t(i)} \equiv MC_t = \frac{W_t}{P_t} \frac{1}{(1-\alpha) A_t N_t(i)^{-\alpha}} \tag{36}$$

Government

The government purchases a continuum of different public goods in the $[0,1]$ range produced by firms with monopolistic competition. The objective function is the following [2]:

$$\max_{G_t(i)} \quad G_t = \left(\int_0^1 G_t(i)^{1-\frac{1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}} \tag{37}$$

maximising the consumption of public goods for any level of expenditure.

$$s.t. \quad \int_0^1 P_t(i) G_t(i) di \tag{38}$$

The Lagrangian takes the form

$$\max_{G_t(i)} \quad \left(\int_0^1 G_t(i)^{1-\frac{1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}} - \nu_t \left(\int_0^1 P_t(i) G_t(i) di \right) \tag{39}$$

computing the F.O.C. with respect to $G_t(i)$ yields Equation (40)

$$\left(\int_0^1 G_t(i)^{1-\frac{1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}} G_t(i)^{-\frac{1}{\epsilon}} - \nu_t P_t(i) = 0 \tag{40}$$

from which we can recover the *Demand Schedule for Public Goods*

$$G_t(i) = G_t \left(\frac{P_t(i)}{P_t} \right)^{-\epsilon} \tag{41}$$

The Government is financed by *Lump-Sum Taxes* and abide by the *Government Budget Constraint* (GBC).[3]

$$B_t = (1 + R_t) B_{t-1} + G_t - T_t \tag{42}$$

Finally, in our model the public spending is not set as a purely exogenous process but is made by two parts; one part is deterministic and abide by the *Fiscal Compact Rule* (FCR). The other is an exogenous process and can be considered as an unexpected government spending shock Ω.[4]

$$G_t = T_t + \alpha_g \left(\frac{B^*}{Y^*} - \frac{B_t}{Y_t^F} \right) + \Omega_t \tag{43}$$

[2] We assume private and public goods having the same elasticities.

[3] We set T_t positive because we put it positive into the agents budget interpreting it as transfer as in Galì (2008). Having it negative would make no difference.

[4] The most straight example could be a war or an earthquake.

Where $\frac{B^*}{Y^*}$ is the government debt/GDP target exogenously assigned by the fiscal compact rule. We assume that the target level is equal to the steady state level B/Y. This assumption will be useful in the log-linearized version of the model, where variables are described as deviation from the steady state. Assuming the target level equal to the steady state level implies that the log-deviation from the steady state is also the deviation from the target. Making this assumption makes the debt return rules be equivalent to imposing a constraint on the timing of system shock absorption. In fact, changing the size of α_g changes the velocity with which the shock hitting debt is absorbed by the system. As α_g increases the shock is absorbed faster, while in the opposite case it is slower.

Market Clearing

Goods Market

The market clearing condition in the good market is

$$Y_t(i) = C_t(i) + G_t(i) \tag{44}$$

From which we get the *Aggregate Output Equation*

$$Y_t = C_t + G_t \tag{45}$$

Labour Market

The *Aggregate labour* equation is

$$N_t = \int_0^1 N_t(i)di \tag{46}$$

Rewriting the *Production Function* (21) and solving for $N_t(i)$ we get

$$N_t(i) = \left(\frac{Y_t(i)}{A_t}\right)^{\frac{1}{1-\alpha}} \tag{47}$$

and by plugging into the *Aggregate Labour*

$$N_t = \int_0^1 \left(\frac{Y_t(i)}{A_t}\right)^{\frac{1}{1-\alpha}} di \tag{48}$$

Finally by using the definition of $Y_t(i)$, we get Equation (49)

$$N_t = \left(\frac{Y_t}{A_t}\right)^{\frac{1}{1-\alpha}} \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\frac{\epsilon}{1-\alpha}} di \tag{49}$$

The System of Equations

The non-linear system of equations is made-up by the following equations: labor supply, Euler equation, firms optimal price setting, labor demand, price dynamics, inflation dynamics, goods market clearing, labor market clearing and fiscal compact rule. We also model the exogenous variables as autoregressive processes and we consider a standard *Feedback Rule* for the policy interest rate. We specify explicitly these equations in the next sections, where the log-linearised model is derived. The endogenous variables are $W_t, P_t, N_t, C_t, R_t, \Pi_t, Y_t, MC_t, B_t, G_t$ while A_t and Ω_t are exogenous shocks.

$$\frac{W_t}{P_t} = N_t^{\phi} C_t^{\sigma} \tag{50}$$

$$\mathbb{E}_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\frac{1}{\Pi_{t+1}}\right] = \frac{1}{(1+R_t)} \tag{51}$$

$$P_tC_t + B_t = (1+R_t)B_{t-1} + W_tN_t - T_t \tag{52}$$

$$\sum_{k=0}^{\infty}(\theta\beta)^k\mathbb{E}_t\left[\left(\frac{C_{t+k}}{C_t}\right)^{-\sigma}\left(\frac{P_t}{P_{t+k}}\right)\left(\frac{P_t^*}{P_{t+k}}\right)^{-\epsilon}Y_{t+k}\left(\frac{P_t^*}{P_{t-1}} - \mathcal{M}MC_{t+k}\frac{P_{t+k}}{P_{t-1}}\right)\right] = 0 \tag{53}$$

$$MC_t = \frac{W_t}{P_t}\frac{1}{(1-\alpha)A_tN_t(i)^{-\alpha}} \tag{54}$$

$$P_t = \left(\theta P_{t-1}^{1-\epsilon} + (1-\theta)P_t^{*1-\epsilon}\right)^{\frac{1}{1-\epsilon}} \tag{55}$$

$$\Pi_t^{1-\epsilon} = \theta + (1-\theta)\left(\frac{P_t^*}{P_{t-1}}\right)^{1-\epsilon} \tag{56}$$

$$Y_t = C_t + G_t \tag{57}$$

$$N_t = \left(\frac{Y_t}{A_t}\right)^{\frac{1}{1-\alpha}}\int_0^1\left(\frac{P_t(i)}{P_t}\right)^{-\frac{\epsilon}{1-\alpha}}di \tag{58}$$

$$B_t = (1+R_t)B_{t-1} + G_t - T_t \tag{59}$$

$$G_t = T_t + \alpha_g\left(\frac{B^*}{Y^*} - \frac{B_t}{Y_t^F}\right) + \Omega_t \tag{60}$$

Flexible Price Equilibrium

From the firm price optimization setting we get the flexible price mark-up $P_t^* = \mathcal{M}\Psi'_{t|t}$. Given that under flexible prices $P_t^* = P_t$ we have $MC_t = \frac{1}{\mathcal{M}}$. Now we would like to find an expression for output under flexible price, in order to build the Dynamic IS equation and the NKPC. Plugging the flexible price mark-up into labor demand $MC_t = \frac{W_t}{P_t}\frac{1}{A_t(1-\alpha)N_t^{-\alpha}}$ and using the goods market clearing condition $C_t = Y_t + G_t$, the labor supply $\frac{W_t}{P_t} = C_t^{\sigma}N_t^{\phi}$ and the labor market clearing condition $N_t = \left(\frac{Y_t}{A_t}\right)^{\frac{1}{1-\alpha}}$ we get:

$$(Y_t - G_t)^{\sigma} Y_t^{\frac{\phi+\alpha}{1-\alpha}} = \mathcal{M}(1-\alpha)A_t^{\frac{1+\phi}{1-\alpha}} \tag{61}$$

In order to disentangle Y_t we log-linearise the equation around the steady state.

$$\hat{y}_t^F = \left(\frac{(1-\alpha)C}{(1-\alpha)\sigma Y + C(\phi+\alpha)}\right)\left(\left(\frac{1+\phi}{1-\alpha}\right)\hat{a}_t - \sigma\frac{G}{C}\hat{g}_t\right) \tag{62}$$

2 Steady State Relationship

In steady state we obtain the following relationships by dropping time and assuming steady state inflation equal to one $\Pi = 1$.

Steady State Labor Supply

$$\frac{W}{P} = N^{\phi}C^{\sigma} \tag{63}$$

Steady State Euler Equation

$$\frac{1}{1+R} = \beta \tag{64}$$

Steady State Consumers Budget Constraint

$$PC = RB + WN - T \tag{65}$$

By assuming in steady state $P^* = P_t$ we get the *Steady State Price Setting*

$$MC = \frac{1}{\mathcal{M}} \tag{66}$$

Steady State Goods Market Clearing

$$Y = C + G \tag{67}$$

Steady State Labor Market Clearing

$$N = \left(\frac{Y}{A}\right)^{\frac{1}{1-\alpha}} \int_0^1 \left(\frac{P(i)}{P}\right)^{-\frac{\epsilon}{1-\alpha}} di \tag{68}$$

where the integral on the RHS is equal to one.

Steady State Firms Cost Minimization

$$MC = \frac{W}{P}\frac{1}{(1-\alpha)N^{-\alpha}} \tag{69}$$

Steady State Government Budget Constraint

$$B = \frac{1}{R}(G - T) \tag{70}$$

Steady State FCR

$$G = T + \Omega \tag{71}$$

3 Log-Linearization

Model log-linearisation mainly follows the three Uhlig's building blocks[5].

$$e^{x_t + a y_t} \approx 1 + x_t + a y_t \tag{72}$$

$$x_t y_t \approx 0 \tag{73}$$

$$\mathbb{E}_t \left[a e^{x_{t+1}}\right] \approx \mathbb{E}_t \left[a x_{t+1}\right] \tag{74}$$

Where x_t and y_t are real variables close to zero ($x_t = log(X_t) - log(\bar{X})$, in our notation this will be $\hat{x}_t$), $\bar{X}$ is the steady state value of the variable X_t (in our notation this will be just X) and a is a constant (the second and third building blocks are up to a constant). As suggested by Uhlig we replace each variables by $\bar{X}e^{x_t}$, than applying the three building blocks. After some manipulations, all the constants drop out to each equations.

[5]Reported here as in the original notation of Uhlig (1995)

Labour Supply

$$\hat{w}_t - \hat{p}_t = \phi \hat{n}_t + \sigma \hat{c}_t \tag{75}$$

Euler Equation

$$\hat{c}_t = \mathbb{E}_t \hat{c}_{t+1} - \frac{1}{\sigma}\left(\hat{i}_t - \mathbb{E}_t \pi_{t+1}\right) \tag{76}$$

Inflation Dynamics

$$\pi_t = (1-\theta)(\hat{p}_t^* - \hat{p}_{t-1}) \tag{77}$$

Price Setting

$$\sum_{k=0}^{\infty} \theta^k \beta^k \left(\hat{p}_t^* - \hat{p}_{t-1}\right) = \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left[\left(\hat{mc}_{t+k|t} + (\hat{p}_{t+k} - \hat{p}_{t-1})\right)\right] \tag{78}$$

Goods Market Clearing

$$\hat{c}_t = \frac{Y}{C}\hat{y}_t - \frac{G}{C}\hat{g}_t \tag{79}$$

Labor Market Clearing[6]

$$\hat{n}_t = \frac{1}{(1-\alpha)}(\hat{y}_t - \hat{a}_t) \tag{82}$$

Firms Cost Minimization

$$\hat{mc}_t = \hat{w}_t - \hat{p}_t - \hat{a}_t + \alpha \hat{n}_t \tag{83}$$

Price Dynamics

[6]Taking the log-deviation of the *Market Clearing Condition* we get

$$N\left(1+\hat{n}\right) = \left(\frac{Y}{A}\right)^{\frac{1}{1-\alpha}} \left(1 + \left(\frac{1}{1-\alpha}\right)(\hat{y}_t - \hat{a}_t)\right) + RES \tag{80}$$

where RES is something very small quantity in a neighborhood of the zero inflation steady state and can be not considered in a first order Taylor expansion. See Galì (2008) chapter 3, Appendix 3.3.

$$\hat{y}_t = (1-\alpha)\,\hat{n}_t + \hat{a}_t \tag{81}$$

$$\hat{p}_t = (1-\theta)\hat{p}_t^* + \theta \hat{p}_{t-1} \tag{84}$$

Government budget constraint

$$\hat{b}_t = (1+R)\hat{b}_{t-1} + R\hat{i}_t + \frac{G}{B}\hat{g}_t - \frac{T}{B}\hat{\tau}_t \tag{85}$$

FCR

$$\hat{g}_t = \hat{\tau}_t - \frac{\alpha_g B}{GY}(\hat{b}_t - \hat{y}_t^F) + \frac{1}{G}\omega_t \tag{86}$$

3.1 Minimum set of Equations

By using the fact that $\frac{1}{\sum_{k=0}^{\infty}\theta^k\beta^k} = (1+\beta\theta)$, from Equation (78) follows

$$\hat{p}_t^* - \hat{p}_{t-1} = (1-\beta\theta)\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left[\left(\hat{mc}_{t+k|t} + (\hat{p}_{t+k} - \hat{p}_{t-1})\right)\right] \tag{87}$$

When $\alpha \neq 0$ we rule out the constant return to scale hypothesis, meaning $\hat{mc}_{t+k|t} \neq \hat{mc}_{t+k}$. We than need to find an equation for $\hat{mc}_{t+k|t}$. Starting from the marginal cost equation and plugging $\hat{n}_t = \frac{1}{1-\alpha}(\hat{a}_t - \alpha\hat{y}_t)$ we get

$$\hat{mc}_{t+k} = \hat{w}_{t+k} - \hat{p}_{t+k} - \frac{1}{1-\alpha}(\hat{a}_{t+k} - \alpha\hat{y}_{t+k}) \tag{88}$$

and

$$\hat{mc}_{t+k|t} = \hat{w}_{t+k} - \hat{p}_{t+k} - \frac{1}{1-\alpha}(\hat{a}_{t+k} - \alpha\hat{y}_{t+k|t}) \tag{89}$$

Thus

$$\hat{mc}_{t+k|t} - \hat{mc}_{t+k} = \frac{\alpha}{1-\alpha}(\hat{y}_{t+k|t} + \hat{y}_{t+k}) \tag{90}$$

And by plugging the demand schedule we get [7]

$$\hat{mc}_{t+k|t} = \hat{mc}_{t+k} + \frac{\alpha\epsilon}{1-\alpha}\left(\hat{p}_t^* - \hat{p}_{t+k}\right) \tag{91}$$

[7] By taking log of the *Demand Constraint* $Y_{t+k|t} = \left(\frac{P_t^*}{P_{t+k}}\right)^{-\epsilon}(Y_{t+k})$ we have $\hat{y}_{t+k|t} = \hat{y}_{t+k} + \epsilon\left(\hat{p}_t^* - \hat{p}_{t+k}\right)$. Then $\hat{mc}_{t+k} = \hat{mc}_{t+k|t} + \frac{\alpha\epsilon}{1-\alpha}\left(\hat{p}_t^* - \hat{p}_{t+k}\right)$.

By plugging into (87) we get

$$\hat{p}_t^* - \hat{p}_{t-1} = (1-\beta\theta)\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left(mc_{t+k} - \frac{\alpha\epsilon}{1-\alpha}\left(\hat{p}_t^* - \hat{p}_{t+k}\right) + \left(\hat{p}_{t+k} - \hat{p}_{t-1}\right)\right) \tag{92}$$

and after some algebraic manipulations we have

$$\hat{p}_t^* - \hat{p}_{t-1} = (1-\beta\theta)\,\Theta\sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t mc_{t+k} + \sum_{k=0}^{\infty}\theta^k\beta^k\mathbb{E}_t\left(\hat{p}_{t+k} - \hat{p}_{t-1}\right) \tag{93}$$

Where $\Theta = \frac{1-\alpha}{1-\alpha-\alpha\epsilon}$. By rewriting (93) as different equation, and using $\pi = (1-\theta)(\hat{p}_t^* - \hat{p}_{t-1})$ we get the *New Keynesian Philips Curve*

$$\hat{p}_t^* - \hat{p}_{t-1} = (1-\beta\theta)\,\Theta\frac{1}{(1-\theta\beta F)}mc_t + \frac{1}{(1-\theta\beta F)}\left(\hat{p}_t - \hat{p}_{t-1}\right) \tag{94}$$

Where F is the forward operator

$$\hat{p}_t^* - \hat{p}_{t-1} = \beta\theta\mathbb{E}_t(\hat{p}_{t+1}^* - \hat{p}_t) + (1-\beta\theta)\,\Theta\hat{mc}_t + \pi_t \tag{95}$$

$$\hat{\pi}_t = \beta\mathbb{E}_t\hat{\pi}_{t+1} + \lambda\hat{mc}_t \tag{96}$$

where $\lambda = \frac{(1-\theta)(1-\beta\theta)\Theta}{\theta}$.

Finally the marginal cost equation is given by plugging into (84) the *Labour Supply* $\hat{w}_t - \hat{p}_t = \frac{\sigma Y}{C}\hat{y}_t - \frac{\sigma G}{C} + \phi\hat{n}_t$ and log of market clearing conditions $\hat{n}_t = \frac{1}{1-\alpha}\left(\hat{y}_t - \hat{a}_t\right)$ after some manipulations we get Equation (97)

$$\hat{mc}_t = \left(\frac{\sigma Y(1-\alpha) + C(\phi+\alpha)}{C(1-\alpha)}\right)\hat{y}_t - \left(\frac{\phi+1}{1-\alpha}\right)\hat{a}_t - \frac{\sigma G}{C}\hat{g}_t \tag{97}$$

Arranging in a convenient way

$$\hat{mc}_t = \left(\frac{\sigma Y(1-\alpha) + C(\phi+\alpha)}{C(1-\alpha)}\right)\left(\hat{y}_t - \left(\frac{C(1-\alpha)}{\sigma Y(1-\alpha) + C(\phi+\alpha)}\right)\left(\left(\frac{\phi+1}{1-\alpha}\right)\hat{a}_t - \frac{\sigma G}{C}\hat{g}_t\right)\right) \tag{98}$$

And by plugging the log-linear equation of flexible equilibrium output we get [8]

$$\hat{mc}_t = \left(\frac{\sigma Y(1-\alpha) + C(\phi+\alpha)}{C(1-\alpha)}\right)\left(\hat{y}_t - \hat{y}_t^F\right) \tag{99}$$

[8] $y_t^F = \left(\frac{C(1-\alpha)}{\sigma Y(1-\alpha)+C(\phi+\alpha)}\right)\left(\left(\frac{\phi+1}{1-\alpha}\right)\hat{a}_t + \frac{\sigma G}{C}\hat{g}_t\right)$

By plugging into the NKPC and defining $\tilde{y}_t = \hat{y}_t - \hat{y}_t^F$ we get

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa \tilde{y}_t + \zeta_{\pi,t} \tag{100}$$

where $\kappa = \lambda \frac{\sigma Y(1-\alpha)+C(\phi+\alpha)}{C(1-\alpha)}$, while $\zeta_{\pi,t} = \rho_\zeta \zeta_{\pi,t-1} + \eta_{\zeta,t}$ is a price mark-up shock distributed as an AR(1).

In order to find a functional form for the output-gap we need to exploit the *Euler Equation*. Recalling the log form of the *Euler Equation* we have $\hat{c}_t = \mathbb{E}_t \hat{c}_{t+1} - \frac{1}{\sigma}(\hat{i}_t - \mathbb{E}_t \pi_{t+1})$. Plugging the market-clearing condition for the goods market we have the following relationships:

$$\hat{y}_t = \mathbb{E}_t \hat{y}_{t+1} - \frac{C}{Y\sigma}(\hat{i}_t - \mathbb{E}_t \pi_{t+1}) - \frac{G}{Y}(\mathbb{E}_t \hat{g}_{t+1} - \hat{g}_t) \tag{101}$$

In order to write it as function of the output-gap, just sum and subtract the flexible price output y_t^F.

$$\tilde{y}_t = \mathbb{E}_t \tilde{y}_{t+1} - \frac{C}{Y\sigma}(\hat{i}_t - \mathbb{E}_t \pi_{t+1}) - \frac{G}{Y}(\mathbb{E}_t \hat{g}_{t+1} - \hat{g}_t) + \epsilon_{y,t} \tag{102}$$

Where $\epsilon_{y,t} = \hat{y}_{t+1}^F - \hat{y}_t^F$ and y_t^F can be rewritten as function of shocks and past variables by making some assumptions and plugging the equation for g_t after some manipulations. Fist plug the government budget constraint and re-arrange the terms.

$$\hat{g}_t = \hat{\tau}_t - \frac{\alpha_g BR}{G(Y+\alpha_g)}\hat{i}_t - \frac{\alpha_g B(1+R)}{G(Y+\alpha_g)}\hat{b}_{t-1} + \frac{\alpha_g B}{G(Y+\alpha_g)}\hat{y}_t^F + \frac{1}{G}\omega_t \tag{103}$$

Now given that in flexible price equilibrium $y_t = y^F \to \tilde{y} = 0$ and inflation is equal to inflation set by CB $i_t = \theta_{i,t}$. Finally plugging into y_t^F and rearranging terms we have

$$\hat{y}_t^F = \Xi\left(\left(\frac{\phi+1}{1-\alpha}\right)\hat{a}_t + \frac{\sigma G}{C}\hat{\tau}_t + \frac{\sigma\alpha_g BR}{C(Y+\alpha_g)}\theta_{i,t} - \frac{\sigma\alpha_g B(1+R)}{C(Y+\alpha_g)}\hat{b}_{t-1} + \frac{\sigma}{C}\omega_t\right) \tag{104}$$

where $\Xi = \left(\frac{(Y+\alpha_g)C(1-\alpha)}{\sigma Y(1-\alpha)+C(\phi+\alpha)(Y+\alpha_g)-(1-\alpha)\sigma\alpha_g B}\right)$.

Finally in order to close the model we assume that central bank responds to change in inflation, output gap and interest rate following a feedback rule

$$\hat{i}_t = \phi_\pi \pi_t + \phi_y \tilde{y}_t + \theta_{i,t} \tag{105}$$

Where $\theta_{i,t} = \rho_\theta \theta_{i,t-1} + \eta_{\theta,t}$ is an exogenous shock on interest rate which follows an AR(1) process.

4 The log-linearized model

The log-linearized version of the model is reported here for convenience.

$$\tilde{y}_t = \mathbb{E}_t \tilde{y}_{t+1} - \alpha_y(\hat{i}_t - \mathbb{E}_t \pi_{t+1}) - b_y(\mathbb{E}_t \hat{g}_{t+1} - \hat{g}_t) + \epsilon_{y,t} \tag{106}$$

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa \tilde{y}_t + \zeta_{\pi,t} \tag{107}$$

$$\hat{i}_t = \phi_\pi \pi_t + \phi_y \tilde{y}_t + \theta_{i,t} \tag{108}$$

$$\hat{b}_t = (1 + a_{b1})\hat{b}_{t-1} + a_{b1}\hat{i}_t + a_{b2}(\hat{g}_t - \hat{\tau}_t) \tag{109}$$

$$\hat{g}_t = \hat{\tau}_t - a_{d1}(\hat{b}_t - \hat{y}_t) + a_{d2}\omega_t \tag{110}$$

$$\hat{y}_t^F = \Xi \left(a_{e1}\hat{a}_t + a_{e2}\hat{\tau}_t + a_{e3}\theta_{i,t} - a_{e4}\hat{b}_{t-1} + a_{e5}\omega_t \right) \tag{111}$$

Where $a_y = C/Y\sigma$; $b_y = G/Y$; $\kappa = \lambda \frac{\sigma Y(1-\alpha)+C(\phi+\alpha)}{C(1-\alpha)}$; $\lambda = \frac{(1-\theta)(1-\beta\theta)\Theta}{\theta}$; $\Theta = \frac{1-\alpha}{1-\alpha-\alpha\epsilon}$; $a_{b1} = R$; $a_{b2} = G/B$; $a_{d1} = \alpha_g B/GY$; $a_{d2} = 1/G$; $\Xi = \left(\frac{(Y+\alpha_g)C(1-\alpha)}{\sigma Y(1-\alpha)+C(\phi+\alpha)(Y+\alpha_g)-(1-\alpha)\sigma\alpha_g B} \right)$; $a_{e1} = \frac{\phi+1}{1-\alpha}$; $a_{e2} = \frac{\sigma G}{C}$; $a_{e3} = \frac{\sigma\alpha_g BR}{C(Y+\alpha_g)}$; $a_{e4} = \frac{\sigma\alpha_g B(1+R)}{C(Y+\alpha_g)}$; $a_{e5} = \frac{\sigma}{C}$.

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