# Operations Research

## Sensitivity and Reoptimization

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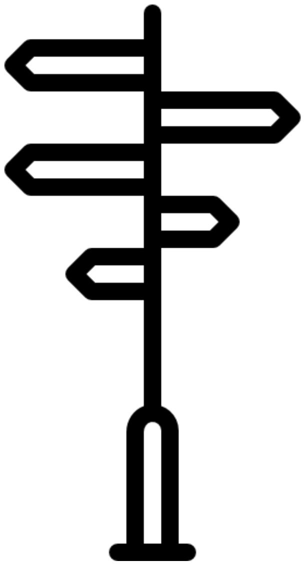
# Course outline



- **Linear Optimization**
  - Modeling with linear programming and graphical solution
  - Solving linear programs, convexity
  - Simplex algorithm, 2-phase Simplex
  - Simplex algorithm in matrix notation, game theory
  - **Sensitivity analysis, reoptimization**
  - Duality theory, zero-sum games
- **Integer Linear Optimization**
  - Modeling integer programming problems (IPs)
  - Solving IPs: branch & bound, cutting planes
  - Advanced solution methods: column generation, approximation algorithms
  - Fast solvable integer problems: total unimodularity, matroids
  - Network flow problems
  - Traveling salesperson problem
- **Nonlinear Optimization**
  - Optimization of multidimensional functions
  - Convex optimization

# Today you will learn…

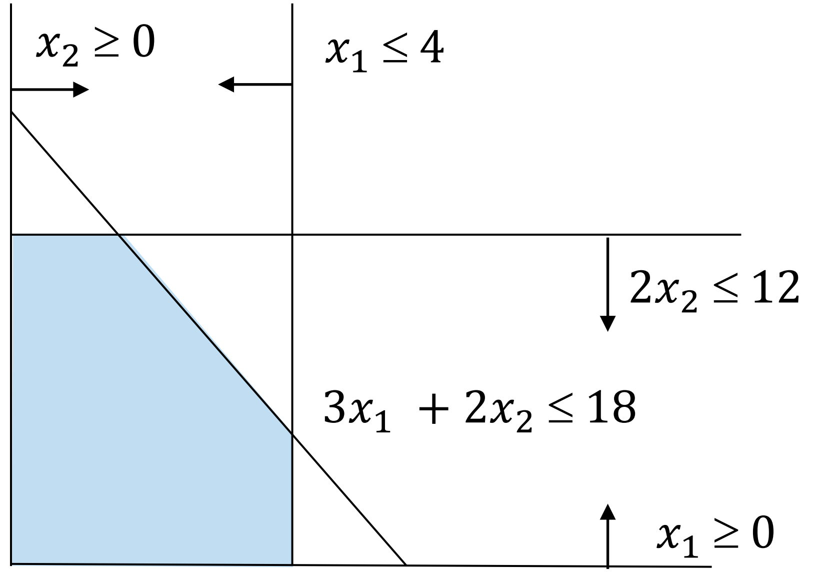
- how to analyze the **effect of minor changes in the parameters of an LP**.
- how to determine the **value of your resources**.
- how to **reoptimize** LPs after adding new variables.
- how to perform **sensitivity analysis with Gurobi**.

# Example - production

| | Product | | |
|---|---|---|---|
| | **1 (Profit: 5)** | **2 (Profit: 3)** | |
| **Machine** | **Production hours per truckload** | | **Available hours/week** |
| 1 | 1 | 0 | 4 |
| 2 | 0 | 2 | 12 |
| 3 | 3 | 2 | 18 |

# Linear program

$$
\begin{array}{llllll}
\text{max} & 5x_1 & + & 3x_2 & & \\
\text{s.t.} & x_1 & & & \leq & 4 \\
 & & & 2x_2 & \leq & 12 \\
 & 3x_1 & + & 2x_2 & \leq & 18 \\
 & & & x_1, x_2 & \geq & 0
\end{array}
$$

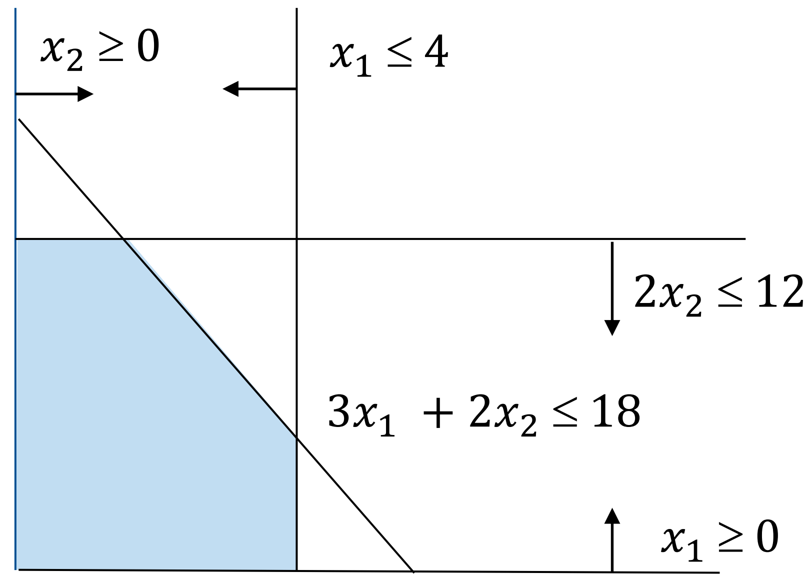


# Linear program in standard form

$$
\begin{array}{llllllll}
\max & 5x_1 + & 3x_2 & & & & & \\
\text{s.t.} & x_1 & & + \; x_3 & & & = & 4 \\
 & & 2x_2 & & + \; x_4 & & = & 12 \\
 & 3x_1 + & 2x_2 & & & + \; x_5 & = & 18 \\
 & & & & x_1, x_2, x_3, x_4, x_5 & & \geq & 0
\end{array}
$$

b

B := Basis




The LP solution is $\mathbf{(4, 3, 0, 6, 0)}$ with basic variables $\{\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_4}\}$ and objective value $\mathbf{29}$.

# Solution via (revised) Simplex

$$\begin{array}{llllllll}
\max & 5x_1 & + & 3x_2 & = z & & & \\
\text{s.t.} & x_1 & & & + x_3 & & = & 4 \\
 & & & 2x_2 & & + x_4 & = & 12 \\
 & 3x_1 & + & 2x_2 & & + x_5 & = & 18 \\
 & & & & x_1, x_2, x_3, x_4, x_5 & & \geq & 0
\end{array}$$

| Basic variables | Coefficients | | | | | | Solution |
|---|---|---|---|---|---|---|---|
| | $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | |
| $z$ | 1 | | | $1/2$ | | $3/2$ | 29 |
| $x_1$ | | 1 | | 1 | | 0 | 4 |
| $x_2$ | | | 1 | $-3/2$ | | $1/2$ | 3 |
| $x_4$ | | | | 3 | 1 | -1 | 6 |

in of

$\left\{ \begin{array}{l} c_B = \text{coefficients for basis} \\ c_N = \text{coefficients for non-basis} \end{array} \right.$

# Formulas from the previous class

$$
\begin{array}{llcl}
\max & 5x_1 + 3x_2 & & \\
\text{s.t.} & x_1 + x_3 & = & 4 \\
& 2x_2 + x_4 & = & 12 \\
& 3x_1 + 2x_2 + x_5 & = & 18 \\
& x_1, x_2, x_3, x_4, x_5 & \geq & 0
\end{array}
$$

Basic variables $\{x_1, x_2, x_4\}$

$$
B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ 3 & 1 & -1 \end{pmatrix}
$$

| Basic variable | Coefficients | | | | | | | Solution |
|---|---|---|---|---|---|---|---|---|
| | $Z$ | $x_{B_1}$ | … | $x_{B_m}$ | $x_{N_1}$ | … | $x_{N_{n-m}}$ | |
| $Z$ | 1 | 0 | | 0 | $\boldsymbol{c_N'^T} = -(\boldsymbol{c_N} - \boldsymbol{N'^T c_B})^T$ | | | $\boldsymbol{c_B^T B^{-1} b}$ |
| $x_{B_1}$ | | 1 | | | $\boldsymbol{N'} = \boldsymbol{B^{-1} N}$ | | | $\boldsymbol{b'} = \boldsymbol{B^{-1} b}$ |
| … | | | … | | | | | |
| $x_{B_m}$ | | | | 1 | | | | |

# Sensitivity analysis

How do changes in the data affect the feasibility and optimality of the basis?

- What if machine 1 can run longer?

**Change in the vector b**

- What happens if the profit of the first product is not 5 € but only 3 €?

**Change in the vector c**

It holds:

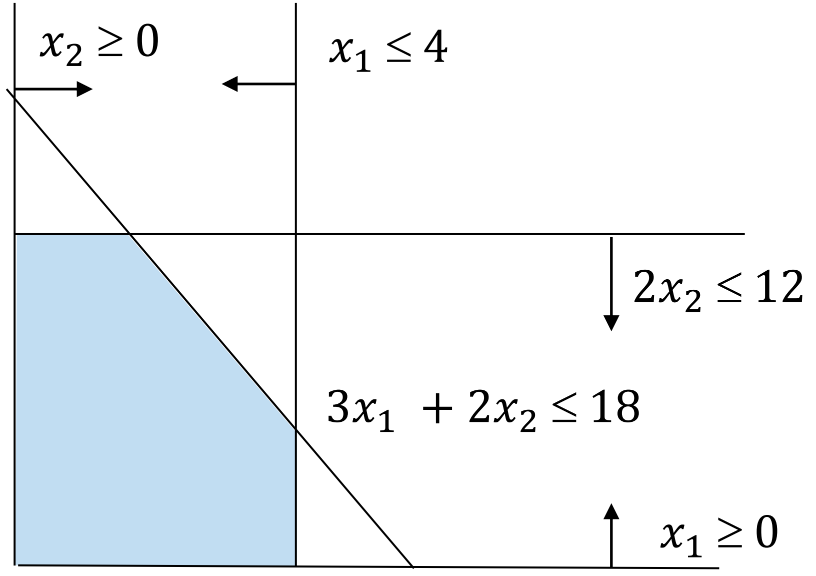
The basis is feasible if $\boldsymbol{B}^{-1}\boldsymbol{b} \geq \mathbf{0}$

The basis is optimal if $-\left(\boldsymbol{c}_{\boldsymbol{N}} - \boldsymbol{N}'^{\boldsymbol{T}}\boldsymbol{c}_{\boldsymbol{B}}\right) \geq \mathbf{0}$

# Change on the right hand side

Example: If machine 1 can run for more hours during the week?

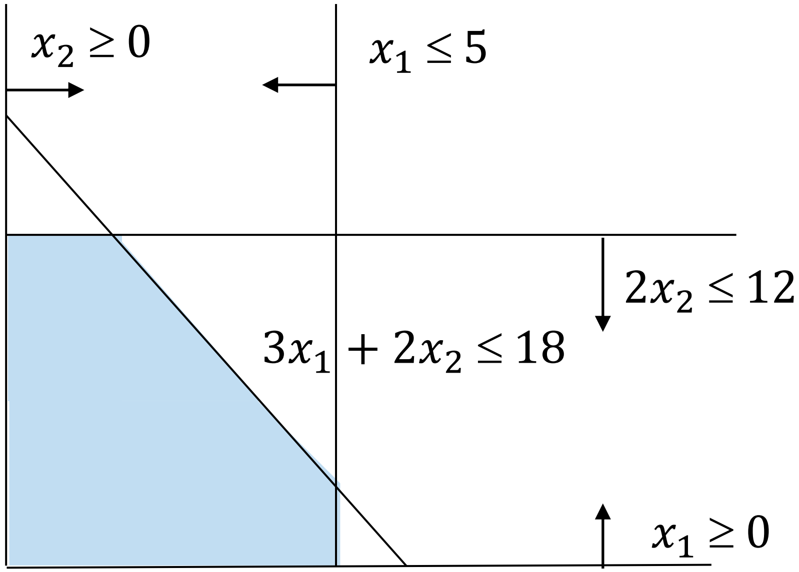
| | | | | | | |
|---|---|---|---|---|---|---|
| max | $5x_1$ | $+$ | $3x_2$ | | | |
| s.t. | $x_1$ | | | $\leq$ | $4$ |
| | | | $2x_2$ | $\leq$ | $12$ |
| | $3x_1$ | $+$ | $2x_2$ | $\leq$ | $18$ |
| | | | $x_1, x_2$ | $\geq$ | $0$ |



# Change on the right hand side

Example: If machine 1 can run for more hours during the week?

| | | | | | | |
|---|---|---|---|---|---|---|
| max | $5x_1$ | $+$ | $3x_2$ | | | |
| s.t. | $x_1$ | | | $\leq$ | $\mathbf{5}$ |
| | | | $2x_2$ | $\leq$ | $12$ |
| | $3x_1$ | $+$ | $2x_2$ | $\leq$ | $18$ |
| | | | $x_1, x_2$ | $\geq$ | $0$ |



# What is going to change in the result?

$$
\begin{array}{llllllll}
\text{max} & 5x_1 + 3x_2 & & & & & & \\
\text{s.t.} & x_1 & & + x_3 & & & = & \mathbf{5} \\
 & & 2x_2 & & + x_4 & & = & \mathbf{12} \\
 & 3x_1 + & 2x_2 & & & + x_5 & = & \mathbf{18} \\
 & & & x_1, x_2, x_3, x_4, x_5 & & & \geq & 0
\end{array}
$$

| Basic variable | Coefficients | | | | | | | Solution |
|---|---|---|---|---|---|---|---|---|
| | $Z$ | $x_{B_1}$ | … | $x_{B_m}$ | $x_{N_1}$ | … | $x_{N_{n-m}}$ | |
| $Z$ | 1 | 0 | | 0 | $c_N'^T = -(c_N - N'^T c_B)^T$ | | | $c_B^T B^{-1} \boldsymbol{b}$ |
| $x_{B_1}$ | | 1 | | | $N' = B^{-1} N$ | | | $b' = B^{-1} \boldsymbol{b}$ |
| … | | | … | | | | | |
| $x_{B_m}$ | | | | 1 | | | | |

# Solution via (revised) Simplex

$$\begin{array}{llcccccl} \text{max} & 5x_1 & + & 3x_2 & & & & \\ \text{s.t.} & x_1 & & & + x_3 & & & = \mathbf{5} \\ & & & 2x_2 & & + x_4 & & = \mathbf{12} \\ & 3x_1 & + & 2x_2 & & & + x_5 & = \mathbf{18} \\ & & & & & & x_1, x_2, x_3, x_4, x_5 & \geq 0 \end{array}$$

Basic variables $\{\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_4}\}$

$$\boldsymbol{c_B^T B^{-1}} = (5\ 3\ 0) * \begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ 3 & 1 & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{3}{2} \end{pmatrix}$$

$$\boldsymbol{c_B^T B^{-1} b} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{3}{2} \end{pmatrix} * \begin{pmatrix} 5 \\ 12 \\ 18 \end{pmatrix} = 29.5$$

| Basic variable | Coefficient | | | | | | Solution |
|---|---|---|---|---|---|---|---|
| | $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | |
| $z$ | 1 | | | $^1/_2$ | | $^3/_2$ | $\mathbf{29.5}$ |
| $x_1$ | | 1 | | 1 | | 0 | $\mathbf{5}$ |
| $x_2$ | | | 1 | $-^3/_2$ | | $^1/_2$ | $\mathbf{1.5}$ |
| $x_4$ | | | | 3 | 1 | -1 | $\mathbf{9}$ |

# Shadow prices

The entries under the slack variables contained in the result row of each simplex tableau are called opportunity costs (shadow prices).

| Basic variable | Coefficient | | | | | | Solution |
|---|---|---|---|---|---|---|---|
| | $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | |
| $z$ | 1 | | | $^1/_2$ | | $^3/_2$ | 29.5 |
| $x_1$ | | 1 | | 1 | | 0 | 5 |
| $x_2$ | | | 1 | $-{}^3/_2$ | | $^1/_2$ | 1.5 |
| $x_4$ | | | | 3 | 1 | -1 | 9 |

# Shadow prices

$$
\begin{array}{lllllllllll}
\text{max} & 5x_1 & + & 3x_2 & & & & & & & \\
\text{s.t.} & x_1 & & & + & \boldsymbol{x_3} & & & & = & 4 \\
 & & & 2x_2 & & & + & x_4 & & = & 12 \\
 & 3x_1 & + & 2x_2 & & & & & + \; x_5 & = & 18 \\
 & & & & & & & x_1, x_2, x_3, x_4, x_5 & & \geq & 0
\end{array}
$$

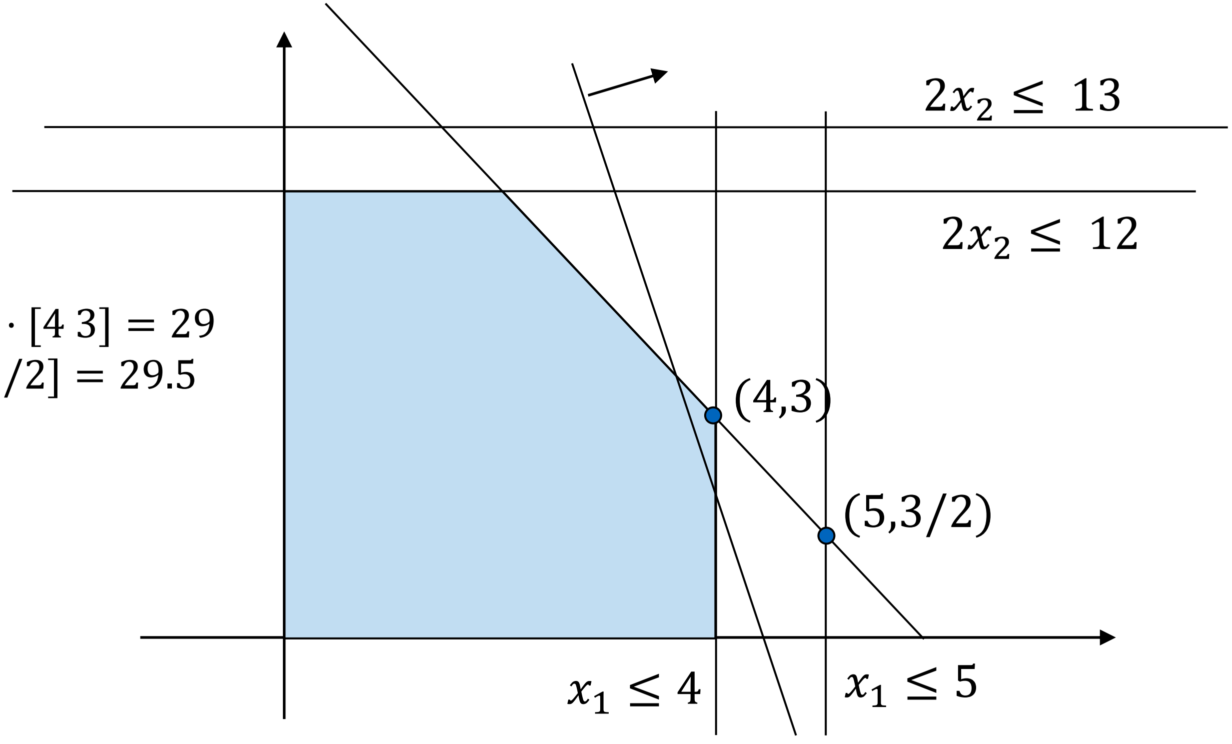
- $x_3$ is the slack variable in the capacity constraint for machine $1$.
- If the capacity is increased from 4 to 5, we obtain the point $(5, 3/2)$ as the optimal solution with an objective function value of $29.5$.
- The profit gain for an additional unit of time on $M1$ corresponds to the value ½ that can be read in the simplex tableau.

# Shadow prices

Resource constraints with shadow prices are called binding constraints.

$Z = c^T x = [5\ 3] \cdot [4\ 3] = 29$

$Z' = [5\ 3] \cdot [5\ 3/2] = 29.5$



# Shadow prices

Many LPs are used to allocate scarce resources to activities.

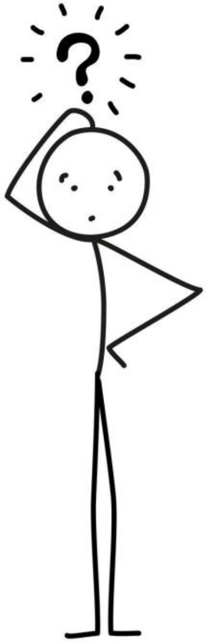
The shadow price of a resource is the marginal value of an additional unit of this resource to the firm.

The shadow price of a resource with non-binding outcomes is 0.

Binding constraints correspond to scarce resources and therefore have positive shadow prices/opportunity costs.

This information is important in deciding which resources are best to invest in.

You are doing shift scheduling at an airport (as in one of the earlier modeling examples) and have a constraint per hour that specifies how many employees must be present over multiple shifts. How should the shadow price of this constraint be interpreted?



# Roster

| Time period | Time frame | | | | | Minimum number of required workers |
|---|---|---|---|---|---|---|
| | Shift | | | | | |
| | 1 | 2 | 3 | 4 | 5 | |
| 6 am to 8 am | x | | | | | 48 |
| 8 am to 10 am | x | x | | | | 79 |
| 10 am to 12 am | x | x | | | | 65 |
| 12 am to 2 pm | x | x | x | | | 87 |
| 2 pm to 4 pm | | x | x | | | 64 |
| 4 pm to 6 pm | | | x | x | | 73 |
| 6 pm to 8 pm | | | x | x | | 82 |
| 8 pm to 10 pm | | | | x | | 43 |
| 10 pm to 12 pm | | | | x | x | 52 |
| 12 pm to 6 am | | | | | x | 15 |

# Check feasibility for changes in $b$

For which values on the right hand side does the current basis remain feasible?
It must hold that $\boldsymbol{B^{-1}b} \geq \mathbf{0}$!

$$B^{-1}b \geq 0 \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ 3 & 1 & -1 \end{pmatrix} \begin{pmatrix} 4+\delta \\ 12 \\ 18 \end{pmatrix} \geq 0$$

$$\begin{pmatrix} 4+\delta \\ 3-3\delta/2 \\ 6+3\delta \end{pmatrix} \geq 0 \Rightarrow -2 \leq \delta \leq 2$$

The optimal basis is feasible as long as the number of hours on machine 1 is between 2 and 6.

→ i.e. the shadow price of 0.5 €/hour corresponding to machine 1 is valid as long as the number of hours is between **2** and **6**.

# Shadow prices for the given $b$

$$B^{-1}b \geq 0 \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ 3 & 1 & -1 \end{pmatrix} \begin{pmatrix} 4+\delta \\ 12 \\ 18 \end{pmatrix} \geq 0 \qquad \begin{pmatrix} 4+\delta \\ 3-3\delta/2 \\ 6+3\delta \end{pmatrix} \geq 0 \Rightarrow -2 \leq \delta \leq 2$$

$$N' = B^{-1}N = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ 3 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{3}{2} & \frac{1}{2} \\ 3 & -1 \end{pmatrix}$$

$$c_N'^T = -(c_N - N'^T c_B)^T = -\left( (0\ 0) - \begin{pmatrix} 1 & -\frac{3}{2} & 3 \\ 0 & \frac{1}{2} & -1 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} \frac{1}{2} & \frac{3}{2} \end{pmatrix} = (x_3\ x_5)$$

# Shadow price of machine 1 for < 2 hours

$$
\begin{array}{llllllll}
\max & 5x_1 & + & 3x_2 & & & & \\
\text{s.t.} & x_1 & & & + x_3 & & = & \mathbf{1} \\
& & & 2x_2 & & + x_4 & = & 12 \\
& 3x_1 & + & 2x_2 & & + x_5 & = & 18 \\
& & & & & x_1, x_2, x_3, x_4, x_5 & \geq & 0
\end{array}
$$

Old basis variables $\{x_1, x_2, x_4\}$

$$B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ 3 & 1 & -1 \end{pmatrix}$$

Now another basis is optimal. The shadow prices for the basis $x_1, x_2, x_5$ are

$$B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & {}^{1}\!/_{2} & 0 \\ -3 & -1 & 1 \end{pmatrix}, N' = B^{-1} N = \begin{pmatrix} 1 & 0 \\ 0 & {}^{1}\!/_{2} \\ -3 & -1 \end{pmatrix}$$

$$c_N'^T = -(c_N - N'^T c_B)^T = \left(5, {}^{3}\!/_{2}\right) \Rightarrow \{x_3, x_4\}$$

The shadow price for machine 1 is now 5 €, while the shadow price for machine 2 is 3/2.

# Shadow prices in Gurobi

Just like variables, attributes of constraints can be queried after optimization:

```
model.addConstr(… , name=‘c1‘)
c1 = model.getConstrByName(‘c1‘)
```

Query shadow prices:

```
c1.Pi
```

Querying the bounds of the shadow prices:

```
c1.SARHSLow
c1.SARHSUp
```

# Gurobi Example

```python
model = gp.Model('ProductionPlan')

# Create variables
x1 = model.addVar(lb=0, ub=GRB.INFINITY, vtype=GRB.CONTINUOUS, name='Product 1')
x2 = model.addVar(lb=0, ub=GRB.INFINITY, vtype=GRB.CONTINUOUS, name='Product 2')

# Add constraint: x_1 <= 4
model.addConstr(x1 <= 4)

# Add constraint: 2x_2 <= 12
model.addConstr(2 * x2 <= 12)

# Add constraint: 3x_1 + 2x_2 <= 18
model.addConstr(3 * x1 + 2 * x2 <= 18)

# Set objective
model.setObjective(5 * x1 + 3 * x2, GRB.MAXIMIZE)

# Optimize model
model.optimize()
```

# Gurobi Example

```python
print(f'Objective = {model.getObjective().getValue()}')
print(f'{x1.VarName} = {x1.X}')
print(f'{x2.VarName} = {x2.X}')

# Get shadow prices
m1 = model.getConstrByName('Machine 1')
m2 = model.getConstrByName('Machine 2')
m3 = model.getConstrByName('Machine 3')

print(f'Shadow price for constraint 1: {m1.Pi} between {m1.SARHSLow} and {m1.SARHSUp}')
print(f'Shadow price for constraint 2: {m2.Pi} between {m2.SARHSLow} and {m2.SARHSUp}')
print(f'Shadow price for constraint 3: {m3.Pi} between {m3.SARHSLow} and {m3.SARHSUp}')
```

```
Objective = 29.0
Product 1 = 4.0
Product 2 = 3.0
Shadow price for constraint 1: 0.5 between 2.0 and 6.0
Shadow price for constraint 2: 0.0 between 6.0 and inf
Shadow price for constraint 3: 1.5 between 12.0 and 24.0
```

# Parametric programming in Gurobi

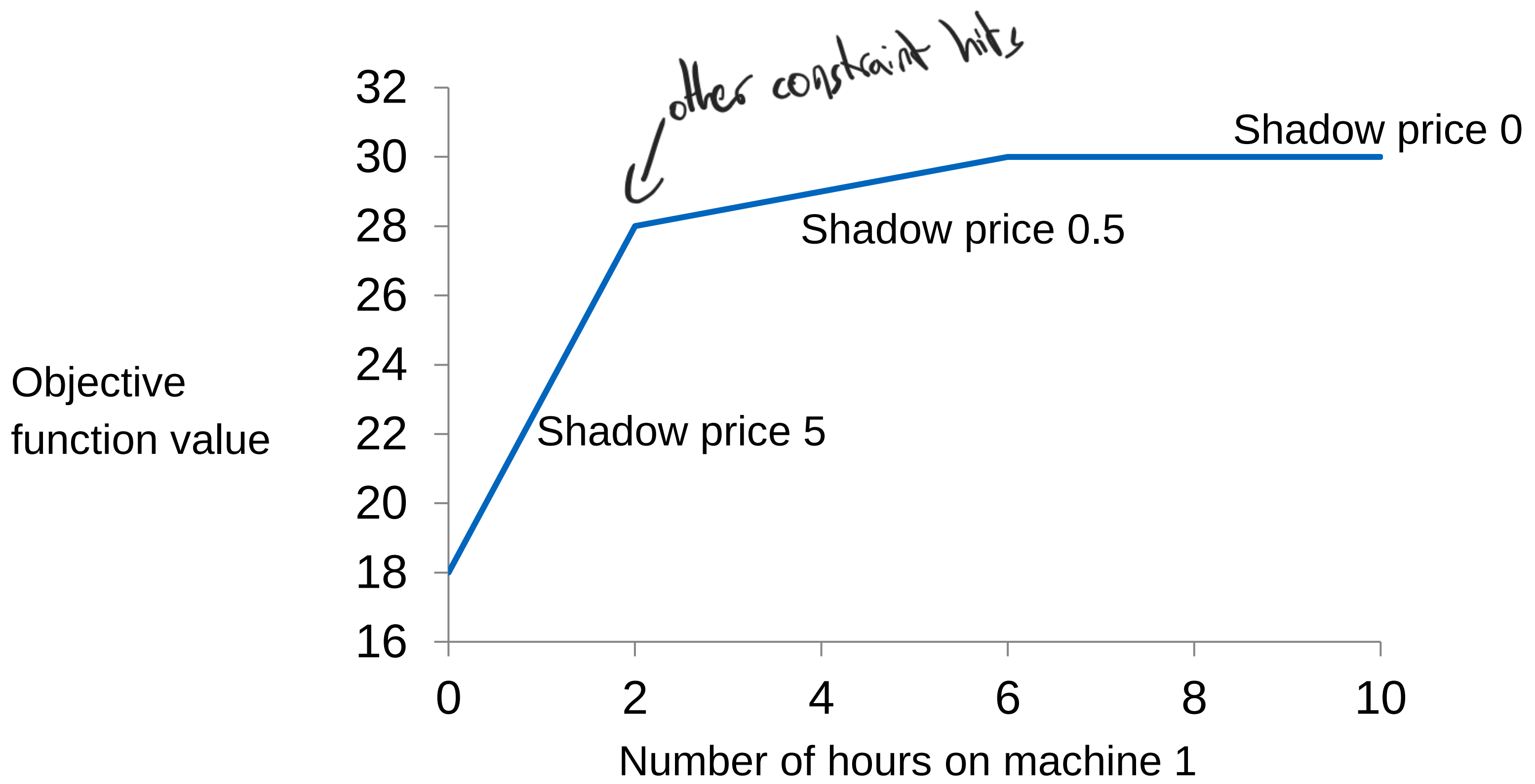
```python
rhs_val = 0
m1.rhs = rhs_val
model.optimize()
shadow = m1.Pi

while shadow > 0:
    print(f'Shadow price for constraint 1: {m1.Pi} between {m1.SARHSLow} and {m1.SARHSUp}')
    rhs_val = m1.SARHSUp
    m1.rhs = rhs_val + 0.01
    model.update()
    model.optimize()
    shadow = m1.Pi

print(f'Shadow price for constraint 1: {m1.Pi} between {m1.SARHSLow} and {m1.SARHSUp}')
```

- Shadow price of constraint 1: 5.0 for RHS between 0.0 and 2.0
- Shadow price of constraint 1: 0.5 for RHS between 2.0 and 6.0
- Shadow price of constraint 1: 0.0 for RHS between 6.0 and 1.0E100

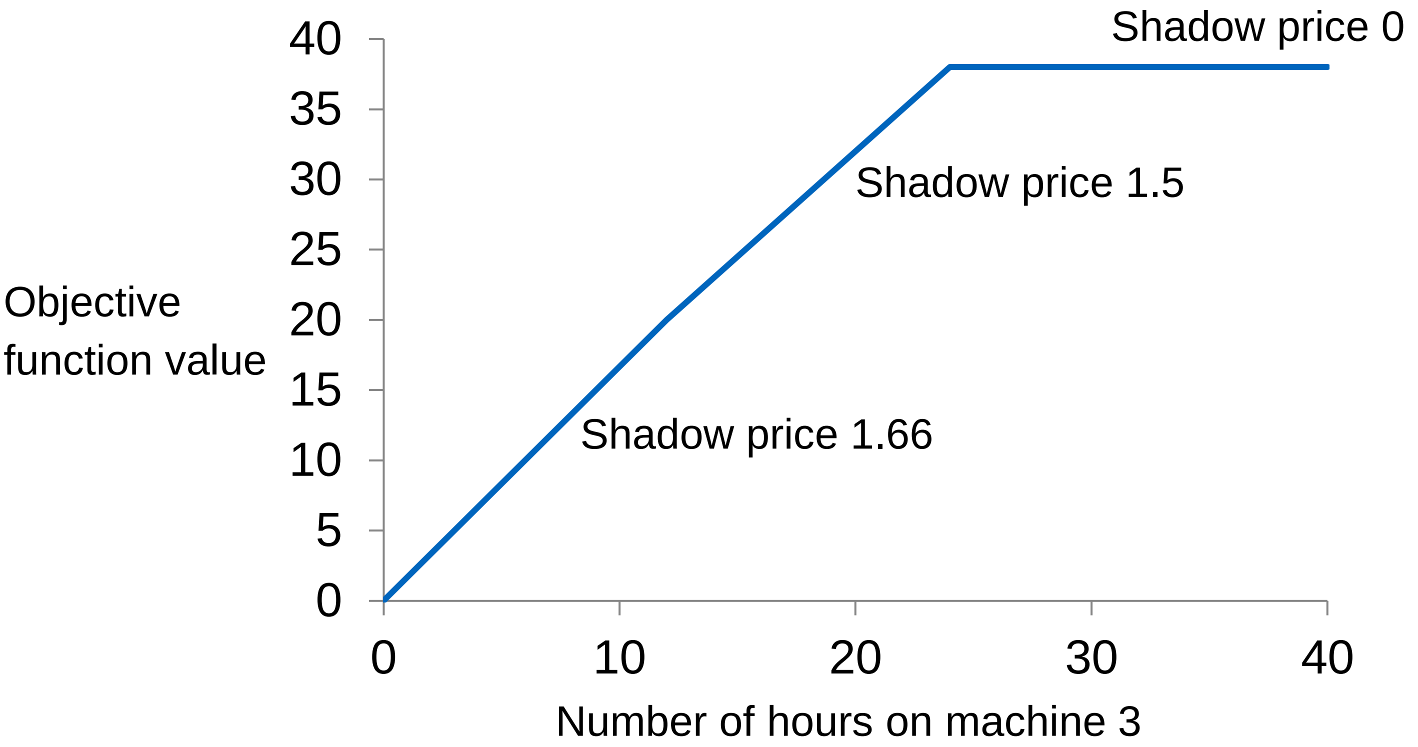
# Parametric programming

$x_1 \leq 2$:

- What is the objective function value depending on the available hours on machine 1?
- This can be computed by iterative analysis analogously to the previous slides.



# Machine 3


Shadow price 0
Shadow price 1.5
Shadow price 1.66
Objective function value
40
35
30
25
20
15
10
5
0
0
10
20
30
40
Number of hours on machine 3

# Change in two constraints

Suppose you can reallocate hours between machines 1 and 3.
What would be the best distribution of hours?

$$B^{-1}b \geq 0 \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ -{}^{3}/_{2} & 0 & {}^{1}/_{2} \\ 3 & 1 & -1 \end{pmatrix} \begin{pmatrix} 4+\delta \\ 12 \\ 18-\delta \end{pmatrix} \geq 0$$

$$\begin{pmatrix} 4+\delta \\ 3-2\delta \\ 6+4\delta \end{pmatrix} \geq 0 \Rightarrow -{}^{3}/_{2} \leq \delta \leq {}^{3}/_{2}$$

Change in the objective function value in this range: $-{}^{3}/_{2}$ is optimal

$$c_B^T B^{-1} b = (B^{-1}b)^T\, c_B = \begin{pmatrix} 4+\delta \\ 3-2\delta \\ 6+4\delta \end{pmatrix}^T \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix} = 29-\delta$$

# Change in two constraints

Old basis variables $\{x_1, x_2, x_4\}$

$$\begin{array}{c} \boldsymbol{x_1} \\ \boldsymbol{x_2} \\ \boldsymbol{x_4} \end{array} \begin{pmatrix} 4+\delta \\ 3-2\delta \\ 6+4\delta \end{pmatrix} \geq 0 \Rightarrow -{}^{3}/_{2} \leq \delta \leq {}^{3}/_{2}$$

$$B^{-1}b$$

$$N' = \begin{array}{cc} \boldsymbol{x_3} & \boldsymbol{x_5} \\ \end{array}$$

$$N' = \begin{pmatrix} 1 & 0 \\ -{}^{3}/_{2} & {}^{1}/_{2} \\ 3 & -1 \end{pmatrix} \begin{array}{c} \boldsymbol{x_1} \\ \boldsymbol{x_2} \\ \boldsymbol{x_4} \end{array}$$

$$N' = B^{-1}N$$

For $\delta < -{}^{3}/_{2}$, $x_4$ is negative: basis change of $x_4$ and $x_5$

$$B^{-1}b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & {}^{1}/_{2} & 0 \\ -3 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 4+\delta \\ 12 \\ 18-\delta \end{pmatrix} \geq 0 \quad \begin{pmatrix} 4+\delta \\ 6 \\ -6-4\delta \end{pmatrix} \geq 0 \Rightarrow -4 \leq \delta \leq -{}^{3}/_{2}$$

Change in the objective value in this range:

$$c_B^T B^{-1} b = (B^{-1}b)^T \, c_B = \begin{pmatrix} 4+\delta \\ 6 \\ -6-4\delta \end{pmatrix}^T \cdot \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix} = 38 + 5\delta$$

Since only 4 hours are available on machine 1, the problem becomes infeasible for $\delta < -4$.

# Change in two constraints

Old basis variables $\{x_1, x_2, x_4\}$

$$\begin{array}{c} x_1 \\ x_2 \\ x_4 \end{array} \begin{pmatrix} 4+\delta \\ 3-2\delta \\ 6+4\delta \end{pmatrix} \geq 0 \Rightarrow -{}^3/_2 \leq \delta \leq {}^3/_2$$

$$B^{-1}b$$

$$N' = \begin{array}{c} \begin{array}{cc} x_3 & x_5 \end{array} \\ \begin{pmatrix} 1 & 0 \\ -{}^3/_2 & {}^1/_2 \\ 3 & -1 \end{pmatrix} \end{array} \begin{array}{c} x_1 \\ x_2 \\ x_4 \end{array}$$

$$N' = B^{-1}N$$

For $\delta > {}^3/_2$, $x_2$ is negative: basis change of $x_2$ and $x_3$

$$\begin{pmatrix} 0 & 0 & {}^1/_3 \\ 1 & 0 & -{}^1/_3 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 4+\delta \\ 12 \\ 18-\delta \end{pmatrix} \geq 0$$

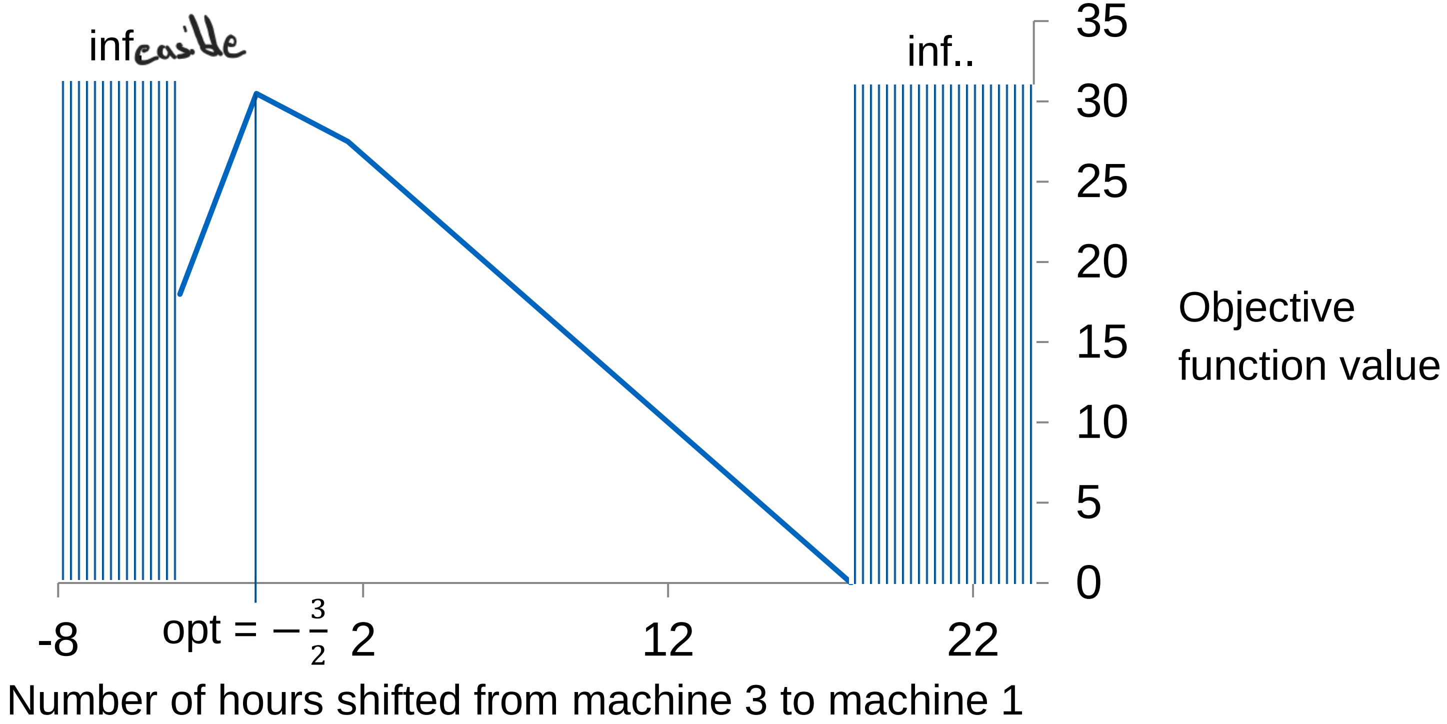
$$\begin{pmatrix} 6 - {}^1/_3\,\delta \\ -2 + {}^4/_3\,\delta \\ 12 \end{pmatrix} \geq 0 \Rightarrow \frac{3}{2} \leq \delta \leq 18$$

Change in the objective value in this range:

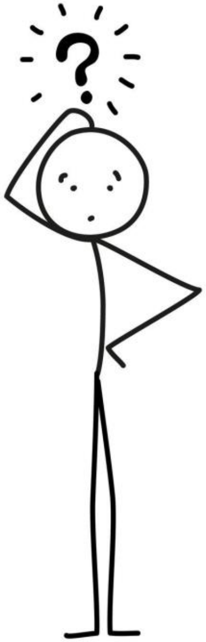
$$c_B^T B^{-1} b = (B^{-1}b)^T\, c_B \;=\; \begin{pmatrix} 6 - {}^1/_3\,\delta \\ -2 + {}^4/_3\,\delta \\ 12 \end{pmatrix}^T \cdot \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} = 30 - {}^5/_3\,\delta$$

Since only 18 hours are available on machine 3, the problem becomes infeasible for $\delta > 18$.

# Parametric programming



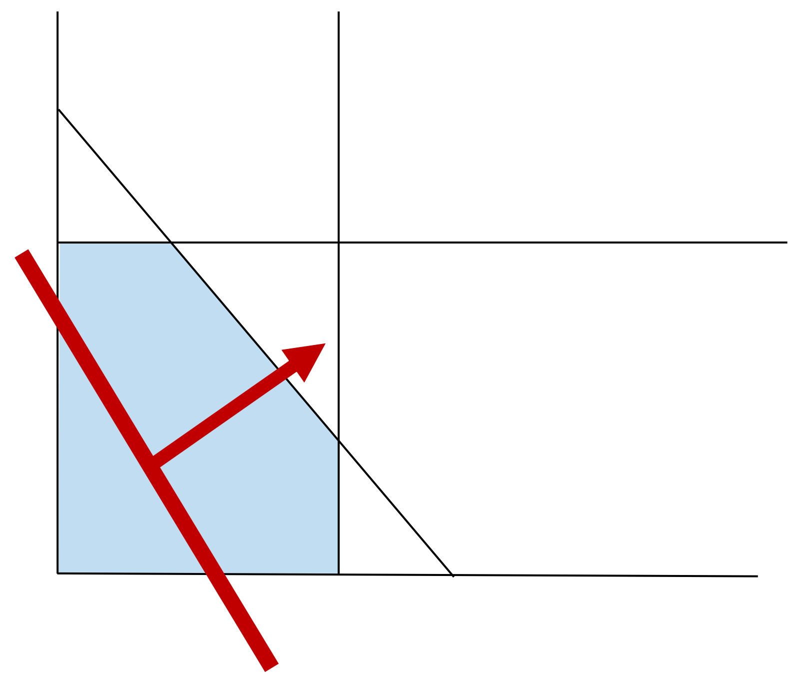
What are shadow prices and when are they valid?



# Change in the objective function

Example: The profit of the first product drops from 5 € to 3 €.

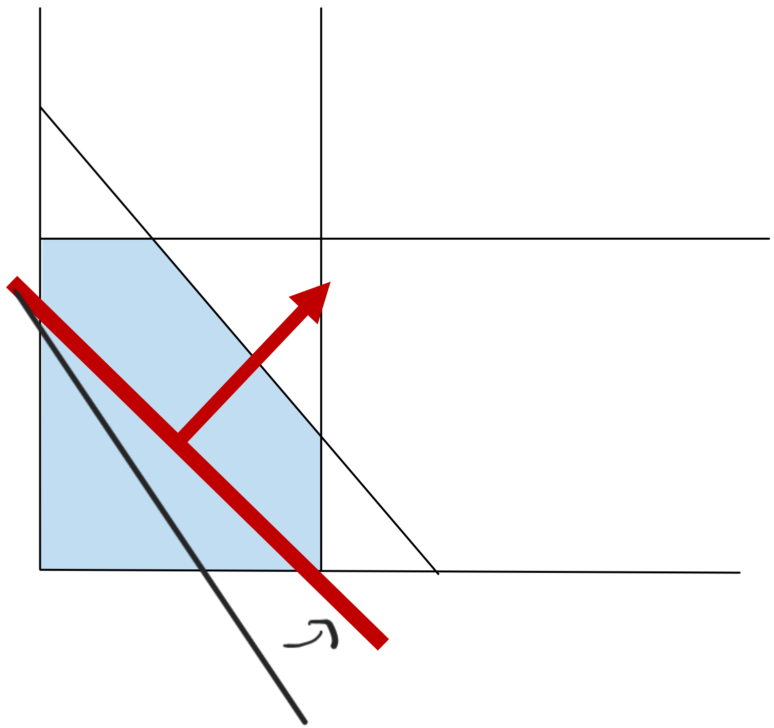
$$
\begin{array}{lcccccc}
\text{max} & 5x_1 & + & 3x_2 & & \\
\text{s.t.} & x_1 & & & \leq & 4 \\
 & & & 2x_2 & \leq & 12 \\
 & 3x_1 & + & 2x_2 & \leq & 18 \\
 & & & x_1, x_2 & \geq & 
\end{array}
$$



# Change in the objective function

Example: The profit of the first product drops from 5 € to 3 €.

$$
\begin{array}{llcll}
\text{max} & \mathbf{3}x_1 & + & 3x_2 & & \\
\text{s.t.} & x_1 & & & \leq & 4 \\
 & & & 2x_2 & \leq & 12 \\
 & 3x_1 & + & 2x_2 & \leq & 18 \\
 & & & x_1, x_2 & \geq & 
\end{array}
$$



# What changes in the result?

| Basic variables | Coefficients: | | | | | | | Solution |
|---|---|---|---|---|---|---|---|---|
| | $Z$ | $x_{B_1}$ | … | $x_{B_m}$ | $x_{N_1}$ | … | $x_{N_{n-m}}$ | |
| $Z$ | 1 | 0 | | 0 | $\mathbf{c}_{\mathbf{N}}^{\prime\mathbf{T}} = -(c_N - N'^T c_B)^T$ | | | $\mathbf{c}_{\mathbf{B}}^{\mathbf{T}} B^{-1} b$ |
| $x_{B_1}$ | | 1 | | | $N' = B^{-1}N$ | | | $b' = B^{-1}b$ |
| … | | | … | | | | | |
| $x_{B_m}$ | | | | 1 | | | | |

# Change in the objective function

$x_B = (x_1 \; x_2 \; x_4)^T$, $\boldsymbol{c_B = (3 \; 3 \; 0)}$, $x_N = (x_3 \; x_5)^T$, $c_N = (0 \; 0)$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 3 & 2 & 0 \end{pmatrix}, N = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Is the current basis $\{x_1, x_2, x_4\}$ still optimal?

Compute $-(c_N - N'^T c_B)$

# Example

Compute $c'_N = -(c_N - N'^T c_B)$

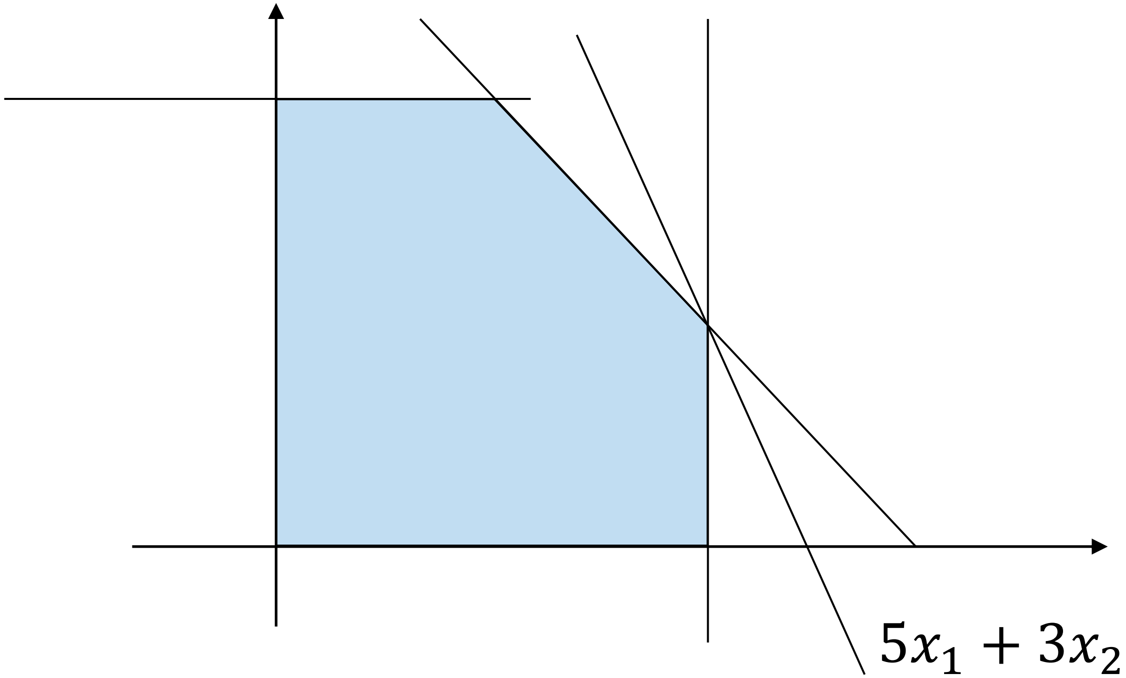
$$N' = \begin{pmatrix} 1 & 0 \\ -{}^{3}\!/_{2} & {}^{1}\!/_{2} \\ 3 & -1 \end{pmatrix}$$

$$c'_N = -\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & -{}^{3}\!/_{2} & 3 \\ 0 & {}^{1}\!/_{2} & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} -{}^{3}\!/_{2} \\ {}^{3}\!/_{2} \end{pmatrix} \begin{matrix} \boldsymbol{x_3} \\ \boldsymbol{x_5} \end{matrix}$$

→ The solution is no longer optimal.

# Example

What values can the profit of the first product assume such that the current production level remains optimal?



# Sensitivity analysis for the solution optimality

Suppose we change the cost of the first product from $5$ € to $5$ € $+\ \delta$ €, i.e. $c$ changes.

What conditions must $\delta$ satisfy?

$-\left(c_N - N'^T c_B\right) \geq 0$ ($B^{-1}b$ is unchanged)

$$c'_N = -\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & -{}^3/_2 & 3 \\ 0 & {}^1/_2 & -1 \end{pmatrix} \begin{pmatrix} 5+\delta \\ 3 \\ 0 \end{pmatrix} \geq 0$$

$$\Leftrightarrow 5 + \delta - 4.5 \geq 0$$

$$\Leftrightarrow \delta \geq -0.5$$

# Reduced costs

Positive coefficients corresponding to the structural variables are also called reduced costs.

- Amount by which the coefficient of the nonbasic variable in the objective function must be reduced (minimization) / increased (maximization) so that the structural variable (e.g., product) is included in the optimal solution.

| Basic variables | Coefficients | | | | | | | Solution |
|---|---|---|---|---|---|---|---|---|
| | $z$ | $x_1$ | $x_2$ | $x_3$ | $s_4$ | $s_5$ | $s_6$ | |
| $z$ | 1 | | 5 | | | 10 | 10 | 280 |
| $s_4$ | | | -2 | | 1 | 2 | -8 | 24 |
| $x_3$ | | | -2 | 1 | | 2 | -4 | 8 |
| $x_1$ | | 1 | 1 | | | -1 | 2 | 2 |

# Reduced costs in Gurobi

Query the reduced costs:

x1.get(**GRB.DoubleAttr.*RC*)**

Query the smallest objective function value for which the basis remains optimal:

x1.get(**GRB.DoubleAttr.*SAObjLow*)**

# Example for reoptimization

$$
\begin{array}{llllllll}
\text{max} & 5x_1 & + & 3x_2 & & & & \\
\text{s.t.} & x_1 & & & + & x_3 & & = 4 \\
 & & & 2x_2 & & & + \; x_4 & = 12 \\
 & 3x_1 & + & 2x_2 & & & + \; x_5 & = 18 \\
 & & & & & & x_1, x_2, x_3, x_4, x_5 & \geq 0
\end{array}
$$

The solution to this LP is $(4, 3, 0, 6, 0)$ with basic variables $\{x_1, x_2, x_4\}$ (i.e. products 1 and 2 are produced) and objective function value of $29$.

The company is now considering producing a third product:

A window with

- a price of $4$ €
- 4 hours working time on machine 2
- 3 hours working time on machine 3

# Reoptimization

By introducing a new variable $x_6$ we can formulate a new LP with three products:

$$
\begin{array}{llllllllllll}
\max & 5x_1 & + & 3x_2 & & & & & & + & \mathbf{4x_6} & & \\
\text{s.t.} & x_1 & & & + & x_3 & & & & & & = & 4 \\
 & & & 2x_2 & & & + & x_4 & & + & \mathbf{4x_6} & = & 12 \\
 & 3x_1 & + & 2x_2 & & & & & + \; x_5 & + & \mathbf{3x_6} & = & 18 \\
 & & & & & & & & & & x_1, x_2, x_3, x_4, x_5, \mathbf{x_6} & \geq & 0
\end{array}
$$

Do we need to solve the LP from scratch?

Possibility: Use the current optimal basis for the new problem.

# Reusing the old tableau

We know the optimal tableau for the production problem with two products:

| Basic variables | Coefficients | | | | | | | Solution |
|---|---|---|---|---|---|---|---|---|
| | $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | |
| $z$ | 1 | | | 0.5 | | 1.5 | ? | 29 |
| $x_1$ | | 1 | | 1 | | 0 | | 4 |
| $x_4$ | | | | 3 | 1 | -1 | | 6 |
| $x_2$ | | | 1 | -1.5 | | 0.5 | | 3 |

What are $x_B, B, B^{-1}$ and $N$?

# Computing the reduced costs

We should just check if the variable $x_6$ is an entering basic variable:

- We compute $c'_N = -(c_N - N'^T c_B)$, where $B$ includes the current basic variables (the current optimal solution) $x_1, x_2, x_4$ and $N' = B^{-1}N$.

$$x_B = (x_1\ x_2\ x_4)^T$$
$$c_B = (5\ \ 3\ \ 0)^T$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 3 & 2 & 0 \end{pmatrix}$$

$$x_N = (x_3\ x_5\ x_6)^T$$
$$c_N = (0\ 0\ 4)$$

$$N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 1 & 3 \end{pmatrix}$$

# Computing the reduced costs

$c_B^T B^{-1} N - c_N^T =$

$$\begin{pmatrix} 5 & 3 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 3 & 2 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 1 & 3 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 3 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ 3 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 1 & 3 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 4 \end{pmatrix} =$$

$$\begin{pmatrix} \frac{1}{2} & 0 & \frac{3}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 1 & 3 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

# Solution

The tableau leads to:

$c_N'^T = (0.5\ 1.5\ \ 0.5)$

$z\ +\ 0.5x_3 + 1.5x_5 + 0.5x_6 = 29$

$(\Rightarrow\ c_B^T B^{-1} b\ =\ [0.5\ 0\ 1.5] \cdot [4\ 12\ 18]^T = 29)$

**$x_3, x_5$ and $x_6$ remain non-basic variables.**

The current basis is optimal, even after introducing the variable for the new product.

→ The new product is not profitable enough to remove resources for product 1 and 2 and allocate them to the new product.

# Reoptimization

In general, the introduction of a new variable does not require a completely new optimization.

One can take the optimal basis of the first solution as the initial basis for the new formulation and run the Simplex algorithm.

The introduction of new constraints is more difficult, since they can cause infeasibilities. However, using the dual (see next class) allows you to treat the new constraint as a new column in the dual program.

# Computing profitable products

We can immediately determine the reduced cost of the product at the current basis based on the characteristics of the product (contribution margin + machine hours requirement).

For new product $i$: $c_N - (B^{-1}N)^T c_B$

$$c_i - \left( \begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ 3 & 1 & -1 \end{pmatrix} \begin{pmatrix} a_{1i} \\ a_{2i} \\ a_{3i} \end{pmatrix} \right)^T \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix}$$

Only products, for which this value is positive, increase the profit and are included in the production plan.

Column Generation

# Summary

(Revised) simplex can help us answer many questions:

- What is the impact of changes on the right side?
- How much money should we invest in expanding our resources?
- What is the best use of our resources?
- How sensitive is our solution to changes in the objective function?
- Is it worthwhile to produce a new product?