Dualities for abelian ℓ-groups and vector lattices beyond archimedeanity

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Abelian ℓ-groups and vector lattices

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Abelian ℓ-groups and vector lattices form varieties.

ℓ-ideals

Congruences in abelian ℓ-groups and vector lattices correspond to ℓ-ideals.

Definition

- An ℓ-ideal in an abelian ℓ-group is a subgroup I that is convex, i.e. $|a| \leq |b|$ and $b \in I$ imply $a \in I$.
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Definition

- A proper ℓ-ideal is called maximal if it is maximal wrt inclusion.
- A nontrivial abelian ℓ-group/vector lattice A is simple if $\{0\}$ and A are the only ℓ-ideals of A.

Archimedeanity

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An abelian ℓ-group/vector lattice is semisimple if the intersection of all its maximal ℓ-ideals is $\{0\}$.

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- A/I is simple iff I is maximal.
- A/I is semisimple iff I is intersection of maximal ℓ-ideals.

Baker-Beynon duality

Piecewise linear functions

Definition

A continuous function $f : \mathbb{R}^\kappa \to \mathbb{R}$ is piecewise linear if there exist $g_1, \ldots, g_n$ linear homogeneous polynomials in the variables $(x_\alpha)_{\alpha<\kappa}$ such that for each $x \in \mathbb{R}^\kappa$ we have $f(x) = g_i(x)$ for some $i = 1, \ldots, n$.

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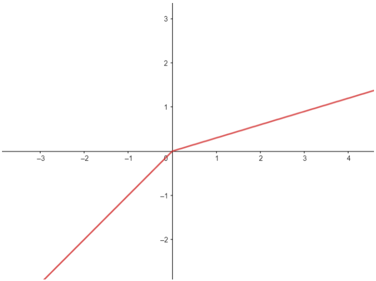

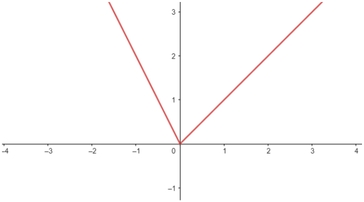

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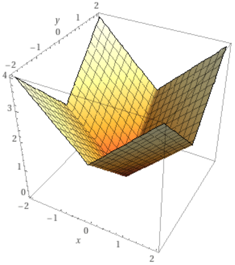

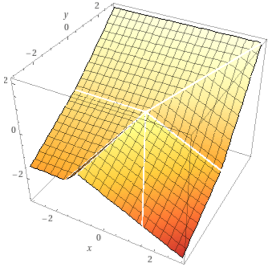

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Theorem

- $\mathrm{PWL}_{\mathbb{R}}(\mathbb{R}^\kappa)$ *is iso to the free vector lattice on* κ *generators.*
- $\mathrm{PWL}_{\mathbb{Z}}(\mathbb{R}^\kappa)$ *is iso to the free abelian* ℓ*-group on* κ *generators.*

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Theorem (Baker 1968)

- *Every κ-generated semisimple vector lattice is isomorphic to* $\mathrm{PWL}_{\mathbb{R}}(C)$ *where C is a cone that is closed in $\mathbb{R}^\kappa$.*
- *Every κ-generated semisimple abelian ℓ-group is isomorphic to* $\mathrm{PWL}_{\mathbb{Z}}(C)$ *where C is a cone that is closed in $\mathbb{R}^\kappa$.*

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Theorem (Beynon 1974)

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Theorem (Beynon 1974)

- *The category of finitely generated archimedean vector lattices is dually equivalent to the category of closed cones in $\mathbb{R}^n$ for $n \in \mathbb{N}$ and piecewise linear maps with real coefficients.*
- *The category of finitely generated archimedean abelian ℓ-groups is dually equivalent to the category of closed cones in $\mathbb{R}^n$ for $n \in \mathbb{N}$ and piecewise linear maps with integer coefficients.*

General affine duality approach

Basic Galois connection

Let V be the variety of abelian ℓ-groups or the variety of vector lattices. Let $A \in V$, κ a cardinal, and $\mathscr{F}_\kappa$ be the free algebra in V over κ generators.

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For any $T \subseteq \mathscr{F}_\kappa$ and $S \subseteq A^\kappa$, we define the following operators.

$$\begin{aligned} \mathbb{V}_A(T) =& \{x \in A^\kappa \mid t(x) = 0 \text{ for all } t \in T\} \\ \mathbb{I}_A(S) =& \{t \in \mathscr{F}_\kappa \mid t(x) = 0 \text{ for all } x \in S\}. \end{aligned}$$

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$$T \subseteq \mathbb{I}_A(S) \quad \text{iff} \quad S \subseteq \mathbb{V}_A(T)\ .$$

Fixpoints of the basic Galois connection

Algebraic Nullstellensatz (Caramello, Marra, and Spada 2021)

- Let I be an ℓ-ideal of $\mathscr{F}_\kappa$. We have $I = \mathbb{I}_A(x)$ for some $x \in A^\kappa$ iff $\mathscr{F}_\kappa/I$ embeds into A.

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The fixpoints of the Galois connection are:

- the intersections of ideals I of $\mathscr{F}_\kappa$ such that $\mathscr{F}_\kappa / I$ embeds into A,
- the Zariski closed subsets of A^κ.

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$$\mathscr{F}_\kappa / I \longrightarrow \mathbb{V}_A(I)$$

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Applying the general affine duality approach with $A = \mathbb{R}$

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Thus, this approach yields Baker-Beynon duality.

Beyond Baker-Beynon duality

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Theorem

Let γ be a cardinal. There exists an ultrapower $\mathcal{U}$ of $\mathbb{R}$ such that every κ-generated linearly ordered abelian ℓ-group/vector lattice with $\kappa \leq \gamma$ embeds into $\mathcal{U}$.

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Let γ be a cardinal. There exists an ultrapower $\mathcal{U}$ of $\mathbb{R}$ such that:

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The Zariski topology on $\mathcal{U}^\kappa$ depends on whether we work with abelian ℓ-groups or vector lattices.

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We define:

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If $X \subseteq \mathcal{U}^\kappa$, we can consider ${}^*\mathrm{PWL}_{\mathbb{R}}(X)$ and ${}^*\mathrm{PWL}_{\mathbb{Z}}(X)$.

Proposition

Let C be a Zariski closed subset of $\mathcal{U}^\kappa$.

- $\mathscr{F}_\kappa / \mathbb{I}_{\mathcal{U}}(C) \cong {}^*\mathrm{PWL}_{\mathbb{R}}(C)$ (vector lattices).
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The Zariski topology on $\mathcal{U}^n$

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The irreducible Zariski-closed subsets of $\mathbb{R}^n$ are the semilines starting from the origin ($\mathbb{V}_{\mathbb{R}}(I)$ with I maximal) and the origin ($\mathbb{V}_{\mathbb{R}}(I)$ with $I = \mathscr{F}_n$).

Orthogonal decomposition theorem (Goze 1995)

If $x \in \mathcal{U}^n$, then x can be written in a unique way as $\alpha_1 v_1 + \cdots + \alpha_k v_k$ with $v_1, \ldots, v_k$ orthonormal vectors of $\mathbb{R}^n$ and $0 < \alpha_1, \ldots, \alpha_n \in \mathcal{U}$ such that α_{i+1}/α_i is infinitesimal.

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Theorem (C., Lapenta, Spada)

In the Zariski topology of $\mathcal{U}^n$ relative to vector lattices each irreducible closed of $\mathcal{U}^n$ is $\mathrm{Cone}(\mathbf{v})$ for some index $\mathbf{v}$.

Indices and cones

Every subset $X \subseteq \mathbb{R}^n$ can be associated with a subset ${}^*X$ of $\mathcal{U}^n$ called the enlargement of X. Every predicate $P \subseteq \mathbb{R}^n$ and function $f : \mathbb{R}^n \to \mathbb{R}$ can be enlarged to ${}^*P \subseteq \mathcal{U}^n$ and ${}^*f : \mathcal{U}^n \to \mathcal{U}$.

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Let φ be a first order formula and ${}^*\varphi$ the formula obtained by replacing every predicate symbol P and every function symbol f with ${}^*P$ and ${}^*f$. Then φ is true in $\mathbb{R}$ iff ${}^*\varphi$ is true in $\mathcal{U}$.

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If $\mathbf{v}$ is an index, we say that a closed cone of $\mathbb{R}^n$ is a **v**-cone if there exist real numbers $r_2, \ldots, r_k > 0$ such that the cone is generated by $\{v_1, v_1 + r_2 v_2, \ldots, v_1 + r_2 v_2 + \cdots + r_k v_k\}$.

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Proposition

Cone($\mathbf{v}$) is the intersection of the enlargements of all the $\mathbf{v}$-cones.

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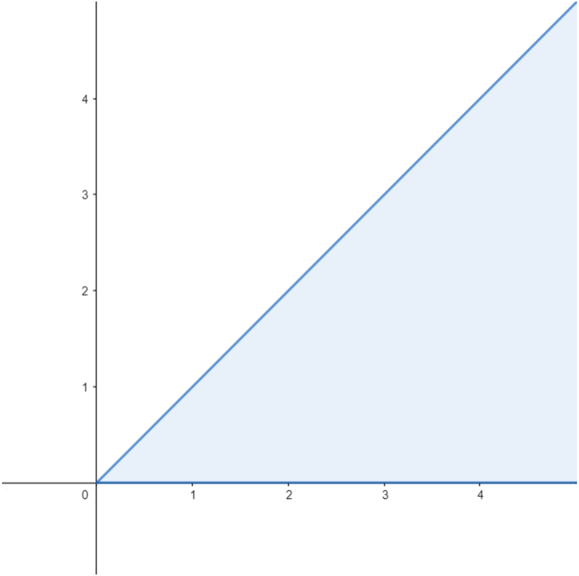

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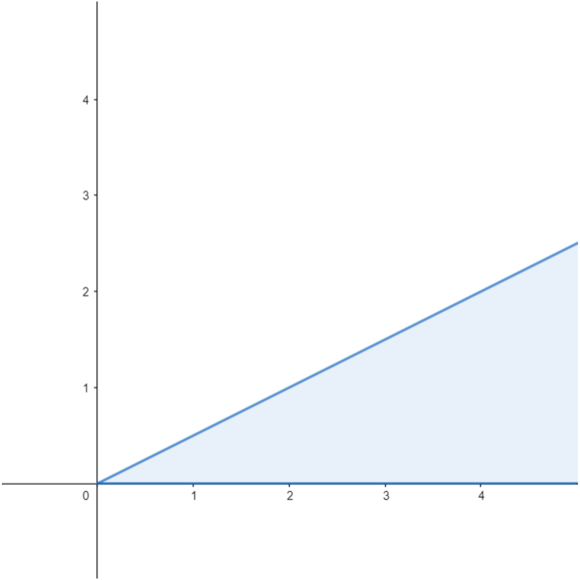

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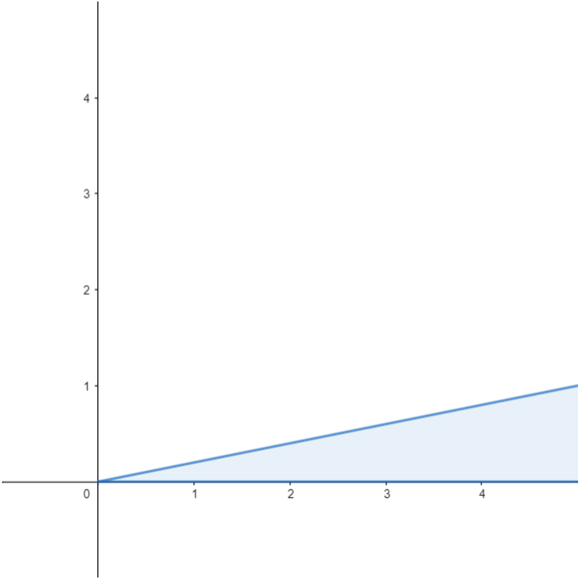

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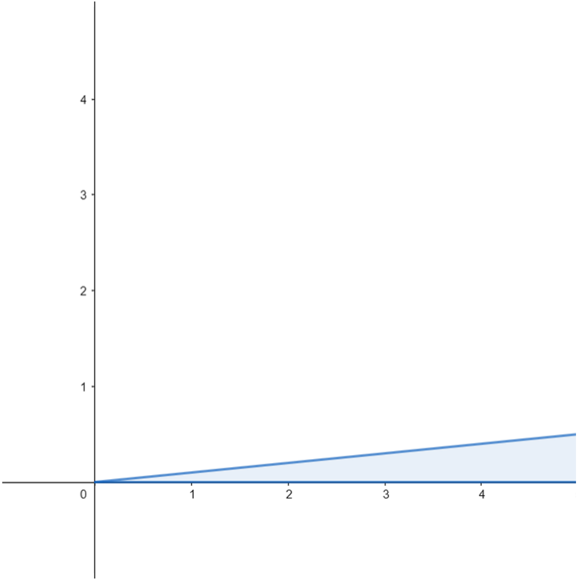

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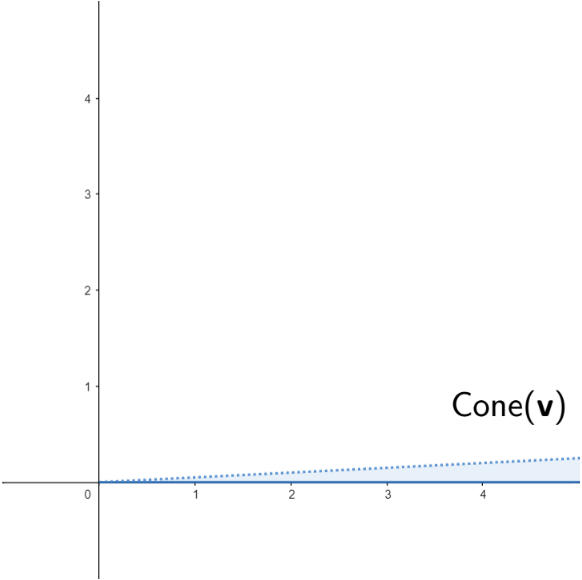

Primes and indices

Theorem (C., Lapenta, and Spada)

If $f \in \mathrm{PWL}_{\mathbb{R}}(\mathbb{R}^n)$*, then* ${}^*f$ *vanishes on* $\mathrm{Cone}(\mathbf{v})$ *iff* f *vanishes on some* $\mathbf{v}$*-cone.*

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Theorem (Panti 1999)

Each prime ℓ-ideal of the vector lattice $\mathscr{F}_n$ is of the form $\{f \in \mathrm{PWL}_{\mathbb{R}}(\mathbb{R}^n) \mid f \text{ vanishes on a } \mathbf{v}\text{-cone}\}$ for some index $\mathbf{v}$.

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Fix a positive infinitesimal $\varepsilon \in \mathcal{U}$. If I is the prime ℓ-ideal of the vector lattice $\mathscr{F}_n$ associated with the index $\mathbf{v} = (v_1, \ldots, v_k)$, then $\mathbb{V}_{\mathcal{U}}(I) = \mathrm{cl}\{v_1 + \varepsilon v_2 + \cdots + \varepsilon^{k-1} v_k\}$.

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This allows to embed the spectrum of a finitely generated vector lattice V into its dual cone so that $V \cong {}^*\mathrm{PWL}_{\mathbb{R}}(\mathtt{Spec}(V))$.

Abelian ℓ-groups and $\mathbb{Z}$-reduced indices

Definition

If $w \in \mathbb{R}^n$, let $\langle w \rangle$ be the smallest subspace containing w that admits a basis in $\mathbb{Z}^n$.

An index $\mathbf{v} = (v_1, \ldots, v_k)$ is $\mathbb{Z}$-reduced if $\langle v_i \rangle$ and $\langle v_j \rangle$ are orthogonal for each $i \neq j$.

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Theorem (C., Lapenta, and Spada)

In the Zariski topology of $\mathcal{U}^n$ relative to abelian ℓ-groups each irreducible closed of $\mathcal{U}^n$ is of the form

$$\bigcup \{\mathrm{Cone}(\mathbf{w}) \mid \mathrm{red}(\mathbf{w}) = \mathbf{v}\}.$$

for some $\mathbb{Z}$-reduced index $\mathbf{v}$.

Theorem (C., Lapenta, and Spada)

Let γ be a cardinal. There exists an ultrapower $\mathcal{U}$ of $[0,1]$ such that:

- *The category of κ-generated MV-algebras for some $\kappa \leq \gamma$ is dually equivalent to the category of Zariski closed subsets of $\mathcal{U}^\kappa$ for some $\kappa \leq \gamma$.*
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This is an affine version of the dualities for abelian ℓ-groups and vector lattices.

THANK YOU!