A temporal interpretation of intuitionistic quantifiers

Guram Bezhanishvili and Luca Carai

New Mexico State University

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Thank you: S.Ghilardi, V.Goranko, I.Shapirovsky, and V.Shehtman for fruitful discussions.

Predicate Gödel translation

IQC = intuitionistic predicate calculus
QS4 = predicate S4

$$(-)^t : \mathrm{IQC} \longrightarrow \mathrm{QS4}$$

$$\begin{array}{rcl}
\bot^t & = & \bot \\
P(x_1, \ldots, x_n)^t & = & \Box P(x_1, \ldots, x_n) \\
(A \wedge B)^t & = & A^t \wedge B^t \\
(A \vee B)^t & = & A^t \vee B^t \\
(A \to B)^t & = & \Box(A^t \to B^t) \\
(\forall x A)^t & = & \Box \forall x A^t \\
(\exists x A)^t & = & \exists x A^t
\end{array}$$

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For any intuitionistic formula A, we have

$$\mathsf{IQC} \vdash A \quad \textit{iff} \quad \mathsf{QS4} \vdash A^t$$

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A standard way to prove this result is to use syntax to show faithfulness an semantics to show fullness.

Kripke semantics for IQC

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- R is a partial order on W.

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An IQC-*frame* is a triple $\mathfrak{F} = (W, R, D)$ where

- W is a nonempty set whose elements are called the *worlds* of $\mathfrak{F}$.
- R is a partial order on W.
- D is a function that associates to each $w \in W$ a nonempty set D_w such that wRv implies $D_w \subseteq D_v$ for each $w, v \in W$. The set D_w is called the *domain* of w.

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- A model $\mathfrak{M}$ is given by a frame together with an interpretation of each predicate symbol.
- An n-ary predicate symbol is interpreted in each $w \in W$ as an n-ary relation on D_w such that if wRv, then the relation on D_v extends the relation on D_w.

Kripke semantics for IQC

Truth in a model is defined in the usual inductive way. When a formula A is true in a world w of a model $\mathfrak{M}$, we write $\mathfrak{M} \vDash_w A$. We recall the truth conditions for the quantifiers:

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$\mathfrak{M} \vDash_w \forall x A$	iff	*A is true for every object of the domain of every world accessible from w.*
$\mathfrak{M} \vDash_w \exists x A$	iff	*A is true for some object in the domain of w.*

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The definition of validity in a frame is the usual one. When a formula A is valid in $\mathfrak{F}$, we write $\mathfrak{F} \vDash A$.

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Theorem (Kripke 1965)

The intuitionistic predicate logic IQC *is sound and complete with respect to Kripke semantics; that is, for each formula A,*

$$\mathsf{IQC} \vdash A \quad \textit{iff} \quad \mathfrak{F} \vDash A \textit{ for each } \mathsf{IQC}\textit{-frame } \mathfrak{F}.$$

Removing asymmetry via the temporal interpretation

$\mathfrak{M} \vDash_w \forall x A$ iff A is true for every object of the domain of every world accessible from w.

$\mathfrak{M} \vDash_w \exists x A$ iff A is true for some object of the domain of some world from which w is accessible.

Removing asymmetry via the temporal interpretation

$\mathfrak{M} \vDash_w \forall x A$ iff A is true for every object of the domain of every world accessible from w.

A is true for every object in the future.

$\mathfrak{M} \vDash_w \exists x A$ iff A is true for some object of the domain of some world from which w is accessible.

A is true for some object in the past.

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Consequently

- $\Diamond_F := \neg\Box_F\neg$ is interpreted as "sometime in the future", and
- $\Diamond_P := \neg\Box_P\neg$ is interpreted as "sometime in the past".

Modified Gödel translation

We then modify the Gödel translation accordingly.

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\begin{array}{rcl}
\bot^t & = & \bot \\
P(x_1, \ldots, x_n)^t & = & \Box_F P(x_1, \ldots, x_n) \\
(A \wedge B)^t & = & A^t \wedge B^t \\
(A \vee B)^t & = & A^t \vee B^t \\
(A \to B)^t & = & \Box_F (A^t \to B^t) \\
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We need to find the tense predicate logic that is the right target of this modified translation.

S4.t

The standard tense extension of S4 is S4.t.

Definition

The logic S4.t is the least set of propositional temporal formulas containing all substitution instances of S4-axioms for both $\Box_F$ and $\Box_P$, the axiom schemes

1. $A \to \Box_P \Diamond_F A$
2. $A \to \Box_F \Diamond_P A$

and closed under the inference rules

$$\frac{A \quad A \to B}{B} \qquad \text{Modus Ponens (MP)}$$

$$\frac{A}{\Box_F A} \quad \Box_F\text{-Necessitation } (\mathrm{N_F}) \qquad \frac{A}{\Box_P A} \quad \Box_P\text{-Necessitation } (\mathrm{N_P})$$

Kripke semantics for S4.t

Kripke frames and models for S4.t coincide with the ones for S4. The truth conditions for the classical connectives are standard and for the temporal modalities we have the following conditions:

Definition

$$
\begin{array}{lcl}
\mathfrak{M} \vDash_w \Box_F A & \text{iff} & (\forall v \in W)(wRv \Rightarrow \mathfrak{M} \vDash_v A) \\
\mathfrak{M} \vDash_w \Box_P A & \text{iff} & (\forall v \in W)(vRw \Rightarrow \mathfrak{M} \vDash_v A)
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QS4.t

By adding standard classical predicate axioms we obtain the predicate extension QS4.t of S4.t. This is a natural candidate to be the target of the modified translation.

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By adding standard classical predicate axioms we obtain the predicate extension QS4.t of S4.t. This is a natural candidate to be the target of the modified translation.
Among these axioms there is the *universal instantiation axiom* which will be fundamental in our considerations.

$$\forall x A \to A(y/x) \qquad\qquad \text{Universal instantiation (UI)}$$

Barcan and converse Barcan formulas for tense logics

Two formula schemes play an important role in predicate modal logic. They are called *converse Barcan formula* and *Barcan formula*.

$\Box\forall x A \to \forall x \Box A$	converse Barcan formula	(CBF)
$\forall x \Box A \to \Box\forall x A$	Barcan formula	(BF)

They are valid in frames with increasing and decreasing domains, respectively.

Barcan and converse Barcan formulas for tense logics

We can consider the analogous formula schemes in the temporal language.

$\Box_F \forall x A \to \forall x \Box_F A$	converse Barcan formula for $\Box_F$	$(\mathrm{CBF_F})$
$\forall x \Box_F A \to \Box_F \forall x A$	Barcan formula for $\Box_F$	$(\mathrm{BF_F})$
$\Box_P \forall x A \to \forall x \Box_P A$	converse Barcan formula for $\Box_P$	$(\mathrm{CBF_P})$
$\forall x \Box_P A \to \Box_P \forall x A$	Barcan formula for $\Box_P$	$(\mathrm{BF_P})$

$\mathrm{CBF_F}$ and $\mathrm{BF_P}$ are valid in frames with increasing domains while $\mathrm{CBF_P}$ and $\mathrm{BF_F}$ are valid in frames with decreasing domains.

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Proposition

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Indeed, QS4.t is complete with respect to the class of Kripke frames with constant domains.

QS4.t cannot be the target

$$\forall x(A \vee B) \to (A \vee \forall x B) \quad \text{with } x \text{ not free in } A \tag{CD}$$

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- IQC $\nvdash$ CD

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- Let $\mathfrak{F}$ be an IQC-frame. $\mathfrak{F} \models \mathrm{CD}$ iff $\mathfrak{F}$ has constant domains.
- $\mathrm{IQC} \nvdash \mathrm{CD}$
- $\mathrm{QS4.t} \vdash (\mathrm{CD})^t$ because $\mathrm{QS4.t} \vdash \mathrm{BF_F}, \mathrm{CBF_P}$

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Therefore, QS4.t is **not** the right candidate to be the target of our temporal translation.

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The formulas $\mathrm{CBF_F}, \mathrm{CBF_P}, \mathrm{BF_F}, \mathrm{BF_P}$ all need the universal instantiation axiom (UI) to be proved.

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The formulas $\mathrm{CBF_F}, \mathrm{CBF_P}, \mathrm{BF_F}, \mathrm{BF_P}$ all need the universal instantiation axiom (UI) to be proved.

Thus, we consider logics where UI is replaced by its weaker version

$$\forall y(\forall x A \to A(y/x))$$

History

- Kripke (1963) was the first to consider the weak universal instantiation axiom. His goal was to have a predicate modal logic whose semantics did not require increasing domains. He gave a semantics for this logic but he did not prove completeness.

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- Hughes and Cresswell (1996) introduced a similar predicate modal logic and proved its completeness with respect to a generalized Kripke semantics.
- Fitting and Mendelsohn (1998) gave an alternate axiomatization of this logic.
- Corsi (2002) defined the system $Q^{\circ}.K$ and proved its completeness with respect to a generalized Kripke semantics.

Generalized Kripke semantics

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Definition

$\mathfrak{M} \vDash_w \forall x A$ iff A is true for every object of the inner domain of w.
$\mathfrak{M} \vDash_w \exists x A$ iff A is true for some object of the inner domain of w.

Corsi's completeness results

Variables are interpreted in the outer domains and quantifiers in the inner domains. Thus, the universal instantiation axiom $\forall x A \to A(y/x)$ is not valid in these frames. On the other hand, its weaker version $\forall y(\forall x A \to A(y/x))$ is.

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Theorem (Corsi 2002)

- $\mathrm{Q}^\circ.\mathrm{K}$ *is sound and complete with respect to the class of all generalized Kripke frames.*
- $\mathrm{Q}^\circ.\mathrm{K} + \mathrm{CBF}$ *is sound and complete with respect to the class of generalized Kripke frames with increasing inner domains.*
- $\mathrm{Q}^\circ.\mathrm{K} + \mathrm{CBF} + \mathrm{BF}$ *is sound and complete with respect to the class of generalized Kripke frames with constant inner domains.*

As far as we know, it is still an open problem whether $\mathrm{Q}^\circ.\mathrm{K} + \mathrm{BF}$ is complete with respect to the class of generalized Kripke frames with decreasing inner domains.

Generalized Kripke semantics for $\mathsf{Q}^\circ\mathsf{S4.t}$

We want to define a predicate tense logic that we call $\mathsf{Q}^\circ\mathsf{S4.t}$ whose intended semantics is given by the following generalized Kripke frames.

Generalized Kripke semantics for Q°S4.t

We want to define a predicate tense logic that we call Q°S4.t whose intended semantics is given by the following generalized Kripke frames.

Definition

A Q°S4.t-*frame* is a quadruple $\mathfrak{F} = (W, R, D, U)$ where

- W is a nonempty set of *worlds*.
- R is a quasi-order on W.
- D is a function that associates to each $w \in W$ a nonempty set D_w such that wRv implies $D_w \subseteq D_v$ for each $w, v \in W$. The set D_w is called the *inner domain* of w.
- U is a set containing D_w for all $w \in W$. It is called the *outer domain* of $\mathfrak{F}$.

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- U is a set containing D_w for all $w \in W$. It is called the *outer domain* of $\mathfrak{F}$.

We want to interpret predicate symbols and variables in the outer domain U while the scopes of the quantifiers are the inner domains.

$Q^{\circ}S4.t$

We define the tense predicate logic $Q^{\circ}S4.t$ by combining S4.t and $Q^{\circ}.K$.

Definition

The logic $Q^{\circ}S4.t$ is the least set of temporal formulas containing all the substitution instances of the S4.t-axioms, the axiom schemes

1. $\forall y(\forall x A \to A(y/x))$ (UI°)
2. $\forall x(A \to B) \to (\forall x A \to \forall x B)$
3. $\forall x \forall y A \leftrightarrow \forall y \forall x A$
4. $A \to \forall x A$ with x not free in A
5. $\forall x A \to A$ with x not free in A (NID)
6. $\Box_F \forall x A \to \forall x \Box_F A$ $(\mathrm{CBF_F})$

and closed under (MP), (Gen), $(\mathrm{N_F})$, and $(\mathrm{N_P})$.

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and closed under (MP), (Gen), ($\mathrm{N_F}$), and ($\mathrm{N_P}$).

We add the axioms NID (nonempty inner domains) and $\mathrm{CBF_F}$ (converse Barcan for $\Box_F$) because we want nonemtpy increasing inner domains.

Generalized Kripke semantics for Q°S4.t

Theorem

Q°S4.t *is sound with respect to the class of* Q°S4.t*-frames; that is, for each formula* A

$$\text{if } \mathrm{Q}^\circ\mathrm{S4.t} \vdash A \text{ then } \mathfrak{F} \vDash A \text{ for each } \mathrm{Q}^\circ\mathrm{S4.t}\text{-frame } \mathfrak{F}.$$

Generalized Kripke semantics for $\mathrm{Q}^{\circ}\mathrm{S4.t}$

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$$\textit{if } \ \mathrm{Q}^{\circ}\mathrm{S4.t} \vdash A \ \ \textit{then } \ \mathfrak{F} \models A \textit{ for each } \mathrm{Q}^{\circ}\mathrm{S4.t}\textit{-frame } \mathfrak{F}.$$

Completeness is still an open problem. It seems to be related to the open problem of the completeness of $\mathrm{Q}^{\circ}.\mathrm{K} + \mathrm{BF}$.

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- *For any intuitionistic formula A, we have*

$$\mathsf{IQC} \vdash A \quad \textit{iff} \quad \mathsf{Q}^\circ\mathsf{S4.t} \vdash \forall x_1 \cdots \forall x_n A^t$$

where $x_1, \ldots, x_n$ are the free variables in A.

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where $x_1, \ldots, x_n$ are the free variables in A.

If A contains constants, they first need to be replaced with fresh variables.

Problem with faithfulness

It is not true in general that

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for example when A is the universal instantiation axiom. Thus, the translation is not faithful in the standard sense.

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Lemma

$$\mathrm{IQC} \vdash A \quad \Rightarrow \quad \mathrm{Q}^{\circ}\mathrm{S4.t} \vdash \forall x_1 \cdots \forall x_n A^t$$

The lemma is proved syntactically by induction on the length of the IQC-proof of A.

Fullness

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The lemma is proved by transforming each IQC-model into a Q°S4.t-model.

Definition

- For an IQC-model $\mathfrak{M}$ based on the frame (W, R, D), let $\overline{\mathfrak{M}}$ be the Q°S4.t-model based on the generalized Kripke frame (W, R, D, U) where $U = \bigcup\{D_w \mid w \in W\}$.

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Theorem

If $\mathfrak{M} \nvDash A$ *then* $\overline{\mathfrak{M}} \nvDash \forall x_1 \cdots \forall x_n A^t$.

Open problems and future directions

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4. Study of logics with weak universal instantiation axiom.

Thanks for your attention!