Subordination algebras and closed relations between compact Hausdorff spaces

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De Vries duality is a generalization of Stone duality to the category KHaus of compact Hausdorff spaces and continuous functions.

Definition

An open subset U is called regular open if $\mathrm{int}(\mathrm{cl}(U)) = U$. The set $\mathcal{RO}(X)$ of regular open subsets a topological space X forms a complete boolean algebra.

- $X \in \mathsf{KHaus}$ is sent to $(\mathcal{RO}(X), \prec)$, where $U \prec V$ iff $\mathrm{cl}(U) \subseteq V$.
- A continuous function $f\colon X \to Y$ is sent to $\mathcal{RO}(f)\colon \mathcal{RO}(Y) \to \mathcal{RO}(X)$ given by $\mathcal{RO}(f)(V) = \mathrm{int}(\mathrm{cl}(f^{-1}(V)))$.

Theorem (de Vries 1962)

KHaus *is dually equivalent to the category* DeV *of de Vries algebras and de Vries morphisms.*

De Vries algebras are complete boolean algebras equipped with binary relations satisfying some properties.

Our goal is to extend de Vries duality to closed relations between compact Hausdorff spaces.

Definition

A binary relation $R\colon X \to Y$ is closed if $R \subseteq X \times Y$ is a closed subset.

Proposition

Let $R\colon X \to Y$ be a relation between compact Hausdorff spaces. R is closed iff $R[C]$ and $R^{-1}[D]$ are closed for any $C \subseteq X$ and $D \subseteq Y$ closed subsets.

Closed relations are natural generalizations of continuous functions and continuous relations.

The composition of two closed relations is closed.

If $R_1\colon X \to Y$ and $R_2\colon Y \to Z$, then $x\,(R_2 \circ R_1)\,z$ iff there exists $y \in Y$ such that $x\,R_1\,y$ and $y\,R_2\,z$.

Let's first generalize Stone duality to closed relations between Stone spaces.

Definition

Let Stone$^{\mathsf{R}}$ be the category of Stone spaces and closed relations.

Celani in 2018 showed that Stone$^{\mathsf{R}}$ is dually equivalent to the category of boolean algebras and quasi-semi-homomorphisms, where $\Delta \colon A \to B$ is a quasi-semi-homomorphism if it is a meet-semilattice homomorphisms from A to $\mathrm{Id}(B)$.

A duality for Stone$^{\mathsf{R}}$ can also be derived from a duality due to Jung, Kurz, and Moshier (2019) that uses order enriched categories.

Since our goal is to generalize de Vries duality, we prefer to work with subordination relations between boolean algebras.

Definition

A binary relation $S\colon A \to B$ between boolean algebras is a subordination if:

- $0 \, S \, 0$ and $1 \, S \, 1$;
- $a \, S \, c$ and $b \, S \, c$ imply $(a \vee b) \, S \, c$;
- $a \, S \, c$ and $a \, S \, d$ imply $a \, S \, (c \wedge d)$;
- $a \leq b \, S \, c \leq d$ implies $a \, S \, d$.

Let BA^{S} be the category of boolean algebras and subordination relations, where the identity on B is $\leq$, and composition is relation composition.

Theorem

$\mathsf{Stone}^{\mathsf{R}}$ *is equivalent to* BA^{S}*.*

The functors behave as the ones of Stone duality on objects and as follows on morphisms:

- A closed relation $R\colon X \to Y$ is sent to $S_R\colon \mathrm{Clop}(X) \to \mathrm{Clop}(Y)$ defined by $U\, S_R\, V$ iff $R[U] \subseteq V$.
- A subordination relation $S\colon A \to B$ is sent to $R_S\colon \mathrm{Uf}(A) \to \mathrm{Uf}(B)$ defined by $x\, R_S\, y$ iff $S[x] \subseteq y$.

The hom-set $\mathsf{Stone}^{\mathsf{R}}(X, Y) = \{R \colon X \to Y \text{ closed}\}$ ordered by inclusion is a complete lattice (actually a coframe).

We can also order $\mathsf{BA}^{\mathsf{S}}(A, B) = \{S \colon A \to B \text{ subordination}\}$ by inclusion.

Proposition

- *$R_1 \subseteq R_2$ iff $S_{R_2} \subseteq S_{R_1}$.*
- *$S_1 \subseteq S_2$ iff $R_{S_2} \subseteq R_{S_1}$.*

Corollary

$\mathsf{BA}^{\mathsf{S}}(A, B)$ is a complete lattice (actually a frame).

It follows that $\mathsf{Stone}^{\mathsf{R}}$ and BA^{S} are locally ordered 2-categories, a particular kind of 2-categories. The functors establishing the equivalence are covariant on 1-cells and contravariant on 2-cells.

There is additional structure on $\mathsf{Stone}^{\mathsf{R}}$ and BA^{S}.

- If $R\colon X \to Y$ is a closed relation, then there is a closed relation $R^{\smile}\colon Y \to X$ given by $y \, R^{\smile} \, x$ iff $x \, R \, y$.
- If $S\colon A \to B$ is a subordination relation, then there is a subordination $\widehat{S}\colon B \to A$ given by $b \, \widehat{S} \, a$ iff $\neg a \, S \, \neg b$.

Proposition

- $\mathsf{Stone}^{\mathsf{R}}$ *and* BA^{S} *are dagger categories. In particular, they are isomorphic to their opposite category.*
- $S_{R^{\smile}} = \widehat{S_R}$ *and* $R_{\widehat{S}} = (R_S)^{\smile}$.

Thus, we could also have obtained a dual equivalence between $\mathsf{Stone}^{\mathsf{R}}$ and BA^{S}.

Theorem

$\mathsf{Stone}^{\mathsf{R}}$ *and* BA^{S} *are allegories and the functors between them are morphisms of allegories.*

If you like category theory:

- By Stone duality Stone is equivalent to $\mathsf{BA}^{\mathsf{op}}$.
- Stone (and hence $\mathsf{BA}^{\mathsf{op}}$) are regular categories. So their categories of internal relations (subobjects of binary products) $\mathsf{Rel}(\mathsf{Stone})$ and $\mathsf{Rel}(\mathsf{BA}^{\mathsf{op}})$ are allegories.
- Stone duality lifts to an equivalence of allegories between $\mathsf{Rel}(\mathsf{Stone})$ and $\mathsf{Rel}(\mathsf{BA}^{\mathsf{op}})$.
- $\mathsf{Stone}^{\mathsf{R}} \cong \mathsf{Rel}(\mathsf{Stone})$ because internal relations in Stone correspond to Stone subspaces of $X \times Y$, which are closed relations.
- $\mathsf{BA}^{\mathsf{S}} \cong \mathsf{Rel}(\mathsf{BA}^{\mathsf{op}})$ because internal relations in $\mathsf{BA}^{\mathsf{op}}$ correspond to filters of $A \oplus B$, which correspond to subordinations.
- The compositions are exactly our functors.

$$\mathsf{Stone}^{\mathsf{R}} \cong \mathsf{Rel}(\mathbf{Stone}) \simeq \mathsf{Rel}(\mathbf{BA}^{\mathbf{op}}) \cong \mathsf{BA}^{\mathsf{S}}$$

Let's move to compact Hausdorff spaces.

Definition

Let $\mathsf{KHaus}^{\mathsf{R}}$ be the category of compact Hausdorff spaces and closed relations.

The idea is to treat compact Hausdorff spaces as quotients of Stone spaces.

Proposition

Compact Hausdorff spaces are, up to homeomorphism, the quotients of Stone spaces over closed equivalence relations.

Definition

Let $\mathsf{StoneE^R}$ be the category defined as follows:

- objects of $\mathsf{StoneE^R}$ are pairs (X, E), where X is a Stone space and E is a closed equivalence relation on X;
- a morphism $R\colon (X_1, E_1) \to (X_2, E_2)$ is a closed relation $R\colon X_1 \to X_2$ that is compatible, i.e. $E_2 \circ R = R = R \circ E_1$.

Theorem

$\mathsf{KHaus^R}$ *and* $\mathsf{StoneE^R}$ *are equivalent (as allegories).*

The equivalence is given by the functor $\mathcal{Q}\colon \mathsf{StoneE^R} \to \mathsf{KHaus^R}$ that maps:

- a pair (X, E) to the quotient X/E;
- a compatible closed relation $R\colon (X_1, E_1) \to (X_2, E_2)$ to the induced relation $\mathcal{Q}(R)\colon X_1/E_1 \to X_2/E_2$.

Equivalence relations in $\mathsf{Stone}^{\mathsf{R}}$ can be characterized in the language of allegories:

- $\mathrm{id}_X \subseteq R$ (reflexivity)
- $R = R^{\smile}$ (symmetry)
- $R \circ R \subseteq R$ (transitivity)

Definition

A subordination $S\colon B \to B$ is an S5-subordination if for all $a, b \in B$:

- $a\ S\ b$ implies $a \leq b$; $(S \subseteq \leq)$
- $a\ S\ b$ implies $\neg b\ S\ \neg a$; $(S = \widehat{S})$
- $a\ S\ b$ implies there is $c \in B$ such that $a\ S\ c$ and $c\ S\ b$. $(S \subseteq S \circ S)$

Let $\mathsf{SubS5}^{\mathsf{S}}$ the category defined as follows:

- objects of $\mathsf{SubS5}^{\mathsf{S}}$ are pairs (B, S), where B is a boolean algebra and S is an S5-subordination (called S5-subordination algebras);
- a morphism $T\colon (B_1, S_1) \to (B_2, S_2)$ is a subordination $T\colon B_1 \to B_2$ that is compatible, i.e. $T \circ S_1 = T = S_2 \circ T$.

The equivalence between $\mathsf{Stone}^{\mathsf{R}}$ and BA^{S} can be lifted.

Theorem

$\mathsf{StoneE}^{\mathsf{R}}$ *and* $\mathsf{SubS5}^{\mathsf{S}}$ *are equivalent (as allegories).*

This can also be seen in the language of allegories: $\mathsf{StoneE}^{\mathsf{R}}$ and $\mathsf{SubS5}^{\mathsf{S}}$ are obtained by splitting the equivalences of $\mathsf{Stone}^{\mathsf{R}}$ and BA^{S}.

Corollary

$\mathsf{KHaus}^{\mathsf{R}}$ *is equivalent to* $\mathsf{SubS5}^{\mathsf{S}}$ *(as allegories).*

Let's look at de Vries algebras more in detail.

Definition

A de Vries algebra is a pair $(B, \prec)$, where B is a complete boolean algebra and $\prec$ a binary relation on B such that

- $0 \prec 0$ and $1 \prec 1$;
- $a \prec c$ and $b \prec c$ imply $(a \vee b) \prec c$;
- $a \prec c$ and $a \prec d$ imply $a \prec (c \wedge d)$;
- $a \leq b \prec c \leq d$ implies $a \prec d$.
- $a \prec b$ implies $a \leq b$;
- $a \prec b$ implies $\neg b \prec \neg a$;
- $a \prec b$ implies there is $c \in B$ such that $a \prec c$ and $c \prec b$;
- $a \neq 0$ implies there is $b \neq 0$ such that $b \prec a$.

De Vries algebras are S5-subordination algebras!

There are two additional conditions: they are complete boolean algebras and satisfy an additional axiom.

Since S5-subordination algebras can be thought of as a generalization of de Vries algebras, the equivalence between $\mathsf{KHaus}^{\mathsf{R}}$ and $\mathsf{SubS5}^{\mathsf{S}}$ can be thought of as a generalization of de Vries duality.

We can consider the full subcategory $\mathsf{DeV}^{\mathsf{S}}$ of $\mathsf{SubS5}^{\mathsf{S}}$ consisting of de Vries algebras. What is the subcategory of $\mathsf{StoneE}^{\mathsf{R}}$ corresponding to $\mathsf{DeV}^{\mathsf{S}}$?

Definition

- A Gleason space is an object (X, E) of $\mathsf{StoneE}^{\mathsf{R}}$ such that:
  - X is extremally disconnected (closure of each open set is open);
  - E is irreducible (if a closed subset $C \subseteq X$ is proper, then so is $E[C]$).
- We let $\mathsf{Gle}^{\mathsf{R}}$ denote the full subcategory of $\mathsf{StoneE}^{\mathsf{R}}$ whose objects are Gleason spaces.

Theorem

The equivalence between $\mathsf{StoneE}^{\mathsf{R}}$ *and* $\mathsf{SubS5}^{\mathsf{S}}$ *restricts to an equivalence (of allegories) between* $\mathsf{Gle}^{\mathsf{R}}$ *and* $\mathsf{DeV}^{\mathsf{S}}$.

$$
\begin{tikzcd}
\mathsf{KHaus}^{\mathsf{R}} \arrow[r, leftrightarrow] & \mathsf{StoneE}^{\mathsf{R}} \arrow[r, leftrightarrow] & \mathsf{SubS5}^{\mathsf{S}} \\
& \mathsf{Gle}^{\mathsf{R}} \arrow[u, hook] \arrow[r, leftrightarrow] & \mathsf{DeV}^{\mathsf{S}} \arrow[u, hook]
\end{tikzcd}
$$

Theorem

The inclusion $\mathsf{Gle}^{\mathsf{R}} \hookrightarrow \mathsf{StoneE}^{\mathsf{R}}$ *is an equivalence (of allegories).*

Corollary

All the maps in the diagram above are equivalences (of allegories).

Theorem

$\mathsf{KHaus}^{\mathsf{R}}$ *and* $\mathsf{DeV}^{\mathsf{S}}$ *are equivalent (as allegories).*

The functors establishing this equivalence behave as the ones of de Vries duality on objects.

We describe the functor $\mathcal{RO}\colon \mathsf{KHaus}^{\mathsf{R}} \to \mathsf{DeV}^{\mathsf{S}}$:

- To a compact Hausdorff space X it is associated the de Vries algebra $(\mathcal{RO}(X), \prec)$ of regular opens of X equipped with the relation $\prec$ given by $U \prec V$ iff $\mathrm{cl}(U) \subseteq V$.
- If $R\colon X \to Y$ is a closed relation, we associate the subordination $\mathcal{RO}(R)$ given by $U \mathrel{\mathcal{RO}(R)} V$ iff $R[U] \subseteq V$.

How does the equivalence between $\mathsf{KHaus}^{\mathsf{R}}$ and $\mathsf{DeV}^{\mathsf{S}}$ restricts to KHaus?

Proposition

Let $R\colon (X_1, E_1) \to (X_2, E_2)$ be a compatible closed relation between Gleason spaces.
The closed relation $\mathcal{Q}(R)\colon X_1/E_1 \to X_2/E_2$ is a continuous function iff $E_1 \subseteq R^{\smile} \circ R$ and $R \circ R^{\smile} \subseteq E_2$.

If $E_1 \subseteq R^{\smile} \circ R$ and $R \circ R^{\smile} \subseteq E_2$, then we say that R is functional.

Definition

Let $\mathsf{Gle}^{\mathsf{F}}$ be the category of Gleason spaces and functional compatible closed relations.

Theorem

$\mathcal{Q}\colon \mathsf{Gle}^{\mathsf{F}} \to \mathsf{KHaus}$ *is an equivalence.*

Definition

We say that a compatible subordination $T\colon (B_1, \prec_1) \to (B_2, \prec_2)$ is functional if $\widehat{T} \circ T \subseteq \prec_1$ and $\prec_2 \subseteq T \circ \widehat{T}$.

Let $\mathsf{DeV^F}$ be the category of de Vries algebras and functional subordinations.

Theorem

- $\mathsf{Gle^F}$ *is equivalent to* $\mathsf{DeV^F}$.
- KHaus *is equivalent to* $\mathsf{DeV^F}$.
- DeV *is dually isomorphic to* $\mathsf{DeV^F}$.

$$
\begin{tikzcd}
\mathsf{KHaus^R} \arrow[r, leftrightarrow] & \mathsf{Gle^R} \arrow[r, leftrightarrow] & \mathsf{DeV^S} & \\
\mathsf{KHaus} \arrow[r, leftrightarrow] \arrow[u, hook] & \mathsf{Gle^F} \arrow[r, leftrightarrow] \arrow[u, hook] & \mathsf{DeV^F} \arrow[r, leftrightarrow, "d"] \arrow[u, hook] & \mathsf{DeV}
\end{tikzcd}
$$

Isbell duality establishes a dual equivalence between KHaus and the category of compact regular frames and frame homomorphisms.

Isbell duality has been generalized to closed relations:

Theorem (Townsend 1996, Jung, Kegelmann, Moshier 2001)

$\mathsf{KHaus}^{\mathsf{R}}$ *is dually equivalent to the category* $\mathsf{KRFrm}^{\mathsf{P}}$ *of compact regular frames and preframe homomorphisms.*

- To each compact Hausdorff space X it is associated the frame $\mathcal{O}(X)$ of open subsets of X.
- To each closed relation $R\colon X \to Y$ it is associated the map $\mathcal{O}(R)\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ given by $\mathcal{O}(R)(V) = X \setminus R^{-1}[Y \setminus V]$.

$$
\begin{tikzcd}
 & \mathsf{KRFrm}^{\mathsf{P}} & \\
 & \mathsf{KHaus}^{\mathsf{R}} \arrow[u, "\mathcal{O}"] \arrow[dl, "\mathcal{RO}"] \arrow[dr, "\mathcal{RO}"'] & \\
\mathsf{DeV}^{\mathsf{S}} \arrow[rr, hook] & & \mathsf{SubS5}^{\mathsf{S}}
\end{tikzcd}
$$

Definition

A subordination ideal of an S5-subordination algebra (B, S) is an ideal I of B such that $S^{-1}[I] = I$. The subordination ideals of (B, S) form a compact regular frame, denoted $\mathcal{SI}(B, S)$.

$\mathcal{SI}$ becomes a contravariant functor by mapping a compatible subordination T to the map given by $I \mapsto T^{-1}[I]$.

Theorem

$\mathcal{SI} \colon \mathsf{SubS5^S} \to \mathsf{KRFrm^P}$ *is a dual equivalence.*

$\mathcal{SI}$ is a generalization of the ideal completion. Indeed, if $S = \leq$ on B, then $\mathcal{SI}(B, \leq)$ is exactly the ideal completion of B.

$$
\begin{tikzcd}
 & \mathsf{KRFrm}^{\mathsf{P}} & \\
 & \mathsf{KHaus}^{\mathsf{R}} \arrow[u, "\mathcal{O}"] \arrow[dl, "\mathcal{RO}"'] \arrow[dr, "\mathcal{RO}"] & \\
\mathsf{DeV}^{\mathsf{S}} \arrow[rr, hook] & & \mathsf{SubS5}^{\mathsf{S}} \arrow[uul, "\mathcal{SI}"']
\end{tikzcd}
$$

Definition

If L is a compact regular frame, then the set $\mathfrak{B}(L) = \{a \in L \mid \neg\neg a = a\}$ form a boolean algebra, called the booleanization of L. Moreover, $(\mathfrak{B}(L), \prec)$ is a de Vries algebra with $a \prec b$ iff $\neg a \vee b = 1$.

$\mathfrak{B}$ becomes a contravariant functor by mapping a preframe homomorphism $f\colon L_1 \to L_2$ to the compatible subordination relation $\mathfrak{B}(f)\colon (\mathfrak{B}(L_2), \prec_1) \to (\mathfrak{B}(L_1), \prec_2)$ given by $a\, \mathfrak{B}(f)\, b$ iff $b \prec f(a)$.

Theorem

$\mathfrak{B}\colon \mathrm{KRFrm}^{\mathrm{P}} \to \mathrm{DeV}^{\mathrm{S}}$ *is a dual equivalence.*

$$
\begin{tikzcd}
 & & \mathsf{KRFrm}^{\mathsf{P}} \arrow[ddll, "\mathfrak{B}"'] & & \\
 & & \mathsf{KHaus}^{\mathsf{R}} \arrow[u, "\mathcal{O}"] \arrow[dll, "\mathcal{RO}"] \arrow[drr, "\mathcal{RO}"'] & & \\
\mathsf{DeV}^{\mathsf{S}} \arrow[rrrr, hook] & & & & \mathsf{SubS5}^{\mathsf{S}} \arrow[uull, "\mathcal{SI}"']
\end{tikzcd}
$$

We denote by $\mathcal{NI}$ the composition $\mathfrak{B} \circ \mathcal{SI}$.

Corollary

$\mathcal{NI}\colon \mathsf{SubS5^S} \to \mathsf{DeV^S}$ *is an equivalence.*

If (B, S) is an S5-subordination algebra, then $\mathcal{NI}(B, S)$ is a de Vries algebra whose objects are the regular elements of the frame of subordination ideals of (B, S) that we call normal subordination ideals.

Proposition

A subordination ideal I of (B, S) is normal iff $I = S^{-1}[L(S[U(I)])]$, where $L(X)$ and $U(X)$ denote the sets of lower and upper bounds of a subset $X \subseteq B$.

$\mathcal{NI}$ is a generalization of the MacNeille completion. Indeed, if $S = \leq$, then $\mathcal{NI}(B, \leq)$ is the MacNeille completion of B.

$$
\begin{tikzcd}
 & \mathsf{KRFrm}^{\mathsf{P}} \arrow[ddl, "\mathfrak{B}"'] & \\
 & \mathsf{KHaus}^{\mathsf{R}} \arrow[u, "\mathcal{O}"] \arrow[dl, "\mathcal{RO}"] \arrow[dr, "\mathcal{RO}"'] & \\
\mathsf{DeV}^{\mathsf{S}} \arrow[rr, hook] & & \mathsf{SubS5}^{\mathsf{S}} \arrow[uul, "\mathcal{SI}"'] \arrow[ll, bend left, "\mathcal{NI}"]
\end{tikzcd}
$$

THANK YOU!