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Kinetic Theory and the Vlasov Equation
Methods and Models for Statistical Mechanics - 097660

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Contents

1	Introduction	4
2	The Vlasov Equation	5
2.1	Properties of the Vlasov Equation	12
3	The Effect of Binary Collisions	17
3.1	Relaxation Models	17
3.2	Fokker-Planck Model	20
4	Fluid Models	23
A	Appendix: Hamiltonian Mechanics	28

Abstract

This project is based on the paper [1], with the aim to explain in detail the theory involved and to write down clearly all the methods and calculations used by the author.

This work is structured in the following way: other than this abstract, the structure of [1] in terms of sections and content has been preserved. In order to do that, in each section the content of [1] has been copied as it is in the original text and it is characterized by the colour **violet**. Equations are always in black to improve readability. The text in black is instead completely original and includes computations, proofs, comments, insights about the mathematical reasoning or the physical meaning and everything else I considered relevant in order to provide a complete work.

I made extensive use of other references, which were either quoted by [1] or that I consulted on my own initiative. However, I included direct citation in **violet** only of the main paper [1]. All the other references are listed in the bibliography, they have been quoted when necessary and their content has always been elaborated.

With the aim to provide a self-sufficient work, I decided to include an appendix (A) about Hamiltonian mechanics. It covers the definition of the Hamilton's equations for a mechanical system starting from the Lagrangian formulation and concludes with the Liouville's theorem. The content of this appendix has been genuinely written by me, based mainly on the reference book [2].

The physical explanations about plasma physics concepts have been written using as references the books [3], [4] and [5], as well as my lecture notes from the course 097609-*Plasma Physics*.

In conclusion, obviously, I made an extensive use of my lecture notes of this course 097660-*Methods and Models for Statistical Mechanics*, together with the reference book [6].

1 Introduction

Plasma is considered the fourth state of matter after solid, liquid and gas. The name *plasma* was introduced in 1923 by Langmuir. At nonzero temperature any gas has a certain number of ionized atoms. As the concentration of ionized particles in the gas increases, their motion becomes more and more constrained. At sufficiently high concentrations the interaction of charged particles results in persistent macroscopic neutrality, any disturbances induce strong electric fields in order to restore the neutrality ([4]). Roughly speaking, a plasma is a gas which is so ionized that electromagnetic interactions dominate its behavior.

The kinetic theory is the most fundamental aspect of the mathematical description of plasmas. The aims of the theory are to start from an assembly of N charged particles moving under the influence of their own charges and whatever external fields might be present and to derive the dynamical properties of plasmas. There is a vast body of research on the foundations of the kinetic theory of plasmas and an even vaster body of work which rests on these results. The interest of [1] is to focus on the physical arguments; I tried to improve the mathematical detail by stressing the derivations. Moreover, I introduced and explained some concepts about plasma physics that the author does not stress much as they are probably assumed to be known by the reader.

The kinetic theory of plasmas is a truly N -body problem since due to the long range nature of the Coulomb interaction between the charged particles making up the plasma every particle interacts with every other particle. Thus, the main task for a *satisfactory* kinetic theory of plasma is to find a convincing way of dealing with this situation. The *chapter* will be divided into three parts.

In the first part the derivation of the Vlasov equation will be outlined and a brief description of the physical content and properties of the equation will be given. This is the largest section of the work and also the one in which my contribution was wider. Among other computations, I stressed the derivation of the BBGKY hierarchy and the H-theorem proof for a system of charged particles, which were omitted by the author. Also, I provided some definitions and explanations about basic plasma physics concepts such as the Debye shielding and the plasma parameter.

In the second part, the way in which binary collision processes are incorporated into the kinetic theory will be discussed. In this section the BGK and the Fokker-Planck models for the collision operator have been discussed. I tried to support the physical reasonings of the authors by extending all the computations.

In the last part we shall show how the fluid treatment of plasmas can be derived from the kinetic description. In this section the author completely omits the averaging procedure to obtain a macroscopic fluid description. For this reason I added a detailed derivation/definition for the macroscopic quantities and fluid equations used by [1].

2 The Vlasov Equation

The Vlasov equation is at the base of the kinetic theory of plasmas. It is the Boltzmann equation for a kinetic, non-collisional plasma. The Vlasov equation is indeed derived after the BBGKY hierarchy of equations, by neglecting the terms related to interactions among particles. The name BBGKY comes from the names Bogoliubov, Born, Green, Kirkwood and Yvone. This theory yields a model for many-particles problems, so it matches perfectly with the aim of providing a kinetic description of plasmas. Indeed, quoting directly [7], in a plasma *many particles will interact simultaneously with each other. It is a many-body problem par excellance.*

We shall confine the present discussion to a one-component plasma, in which the ions can be thought of as immobile serving only to neutralize the electron charge in equilibrium.

This can be done since the mass of electrons is way lower than proton (the latter is about 1836 times the first one). As a consequence, under the same electric field, the velocity of the ions is significantly lower than the one of the electrons, so that in first approximation it is allowed to consider ions at rest with respect to the electrons.

We shall also ignore magnetic interactions and consider only the Coulomb interaction.

This assumption simplifies a lot the procedure. Indeed, the action of a magnetic field on an electron is described by the Lorentz force. In particular, the Lorentz force acting on the i^{th} electron can be expressed as:

$$\mathbf{F}_{L_i} = -\frac{e}{m} \mathbf{p}_i \times \mathbf{B}_i$$

where e and m are, respectively, the unit charge and mass of the electron and \mathbf{B}_i is the magnetic field acting on the particle. It is remarkable how \mathbf{F}_{L_i} directly depends on the electron velocity \mathbf{p}_i/m . Since we are operating in the phase space, momenta of the particles are independent variables.

We begin by writing down the Hamiltonian for the N-electron system where the i^{th} electron has six phase-space coordinates $\mathbf{x} = (\mathbf{q}, \mathbf{p})$, where \mathbf{q} and \mathbf{p} are canonical Hamiltonian variables.

This was the first time I had to deal with an Hamiltonian description for a mechanical system. For this reason, I considered pertinent to include in this report a short appendix (A) on Hamiltonian mechanics.

Here, the starting point is the Hamiltonian for an N-electron system:

$$H_N = \sum_{i=1}^N \frac{p_i^2}{2m} + \sum_{i=1}^N \sum_{j>i} \frac{e^2}{4\pi\epsilon_0 |\mathbf{q}_i - \mathbf{q}_j|}. \quad (1)$$

In fact, the whole energy of the system is given by the total kinetic energy of the system and the total electrostatic potential energy between particles. The former is clearly represented in (1) by the first term. The latter is, on the other hand, represented by the second term: the electrostatic potential $\phi_{i,j}$ between positions \mathbf{q}_i and \mathbf{q}_j is indeed defined as¹:

$$\phi_{i,j} = \frac{e^2}{4\pi\epsilon_0 |\mathbf{q}_i - \mathbf{q}_j|}. \quad (2)$$

The sum over the indexes " $i = 1, \dots, N, j > i$ " ensures that (1) accounts for the electrostatic potential between every couple of electrons in the system. Since we are not interested in the detailed motion of all N-particles we need some averaging procedure to enable us to reduce the complexity of the system. We now introduce the N-electron probability distribution $D_N(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N, t)$ normalized to unity which describes the state of the N-electron system at any instant of time. The evolution of the system in time and the evolution of D_N is given by the Liouville's theorem.

$$\frac{\partial D_N}{\partial t} + \{D_N, H_N\} = 0. \quad (3)$$

¹Here, [1] adopts the *Gauss* system of units, i.e. ϕ is defined without the coefficient $(4\pi\epsilon_0)^{-1}$. I decided to be coherent with the Hamiltonian definition (1) and to adopt the *International System of Units*. I do not understand why [1] changes system of units, but it is just a matter of multiplicative coefficients so it does not change much.

I remark how the Liouville's theorem requires the force to be independent on momenta. Hence, this treatment is meaningful only because the effect of the Lorentz force (2) is neglected, otherwise (3) should have included an extra term that involves the velocity gradient of the force.

The ultimate aim of the appendix was indeed to explain in detail the theory behind equation (3), which coincides with (152). Starting from this equation, there is a systematic technique for reducing equation (3) for the N-electron probability to a one- or two-electron distribution function. This procedure leads to the BBGKY hierarchy of equations.

Paper [1] does not provide the derivation of the BBGKY, however a similar procedure is better shown in [7]. Nevertheless, [7] includes in the Hamiltonian also potentials due to boundary effects, having as a result different computations.

By writing out the Poisson brackets in (3) I obtain:

$$\frac{\partial D_N}{\partial t} + \sum_{i=1}^N \frac{\partial D_N}{\partial \mathbf{p}_i} \cdot \frac{\partial H_N}{\partial \mathbf{q}_i} - \sum_{i=1}^N \frac{\partial D_N}{\partial \mathbf{q}_i} \cdot \frac{\partial H_N}{\partial \mathbf{p}_i} = 0. \quad (4)$$

$$\frac{\partial D_N}{\partial t} + \sum_{i=1}^N \frac{\partial D_N}{\partial \mathbf{q}_i} \cdot \frac{\mathbf{p}_i}{m} - \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N \frac{\partial D_N}{\partial \mathbf{p}_i} \cdot \frac{1}{4\pi\epsilon_0} \frac{\partial}{\partial \mathbf{q}_i} \left(\frac{e^2}{|\mathbf{q}_i - \mathbf{q}_j|} \right) = 0. \quad (5)$$

To give a better expression, it is useful to introduce the electric field through the following relation

$$e\mathbf{E}_i = \sum_{\substack{j=1 \\ j \neq i}}^N \frac{\partial \phi_{i,j}}{\partial \mathbf{q}_i} = \sum_{\substack{j=1 \\ j \neq i}}^N \frac{1}{4\pi\epsilon_0} \frac{\partial}{\partial \mathbf{q}_i} \left(\frac{e^2}{|\mathbf{q}_i - \mathbf{q}_j|} \right). \quad (6)$$

In particular, \mathbf{E}_i is the electric field acting on the i^{th} electron.

As a result, (5) can be equivalently written down in compact form:

$$\frac{\partial D_N}{\partial t} + \sum_{i=1}^N \frac{\partial D_N}{\partial \mathbf{q}_i} \cdot \frac{\mathbf{p}_i}{m} - e \sum_{i=1}^N \mathbf{E}_i \cdot \frac{\partial D_N}{\partial \mathbf{p}_i} = 0. \quad (7)$$

The first step toward semplification is the definition of a *reduced probability distribution* f_s/V^s , where V is the configuration space volume of the system and where the funcion f_s is now a distribution involving only s-particles.

$$\frac{f_s}{V^s} := \int D_N d\mathbf{x}_{s+1} \dots d\mathbf{x}_N. \quad (8)$$

By integrating D_N over every variable $\mathbf{x}_i = (\mathbf{p}_i, \mathbf{q}_i) \forall i > s$, the s-particles distribution function is obtained. f_s distributions are the unknowns of the BBGKY hierarchy. It follows that, to derive the BBGKY, I integrate (5) over the subspace " $\mathbf{x}_{s+1}, \mathbf{x}_{s+2}, \dots, \mathbf{x}_N$ ".

$$\frac{\partial}{\partial t} \int D_N d\mathbf{x}_{s+1} \dots d\mathbf{x}_N + \sum_{i=1}^N \int \frac{\partial D_N}{\partial \mathbf{q}_i} \cdot \frac{\mathbf{p}_i}{m} d\mathbf{x}_{s+1} \dots d\mathbf{x}_N - e \sum_{i=1}^N \int \mathbf{E}_i \cdot \frac{\partial D_N}{\partial \mathbf{p}_i} d\mathbf{x}_{s+1} \dots d\mathbf{x}_N = 0. \quad (9)$$

I develop each term of (9) separately.

To introduce the s-particles distribution function on the first time-derivative term is quite trivial, just substitute the definition (8):

$$\frac{\partial}{\partial t} \int D_N d\mathbf{x}_{s+1} \dots d\mathbf{x}_N = \frac{1}{V^s} \frac{\partial f_s}{\partial t}. \quad (10)$$

To develop the second term I split the sum at the s^{th} term, so that f_s can be *extracted* smoothly.

$$\begin{aligned} \sum_{i=1}^N \int \frac{\partial D_N}{\partial \mathbf{q}_i} \cdot \frac{\mathbf{p}_i}{m} d\mathbf{x}_{s+1} \dots d\mathbf{x}_N &= \sum_{i=1}^s \frac{\mathbf{p}_i}{m} \cdot \frac{\partial}{\partial \mathbf{q}_i} \left(\frac{f_s}{V^s} \right) + \sum_{i=s+1}^N \int \frac{\partial D_N}{\partial \mathbf{q}_i} \cdot \frac{\mathbf{p}_i}{m} d\mathbf{x}_{s+1} \dots d\mathbf{x}_N \\ &= \sum_{i=1}^s \frac{\mathbf{p}_i}{m} \cdot \frac{\partial f_s}{\partial \mathbf{q}_i} \frac{1}{V^s} + \sum_{i=s+1}^N \frac{p_i^2}{2m} D_N|_{\partial\Omega} \end{aligned} \quad (11)$$

The second term is equal to zero since D_N vanishes at the boundaries, hence:

$$\sum_{i=1}^N \int \frac{\partial D_N}{\partial \mathbf{q}_i} \cdot \frac{\mathbf{p}_i}{m} d\mathbf{x}_{s+1} \dots d\mathbf{x}_N = \frac{1}{V^s} \sum_{i=1}^s \frac{\mathbf{p}_i}{m} \cdot \frac{\partial f_s}{\partial \mathbf{q}_i}. \quad (12)$$

To develop the third term, it is useful for compactness to adopt the formulation involving the electrostatic potential $\phi_{i,j}$:

$$\begin{aligned} e \sum_{i=1}^N \int \mathbf{E}_i \cdot \frac{\partial D_N}{\partial \mathbf{p}_i} d\mathbf{x}_{s+1} \dots d\mathbf{x}_N &= \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N \int \frac{\partial \phi_{i,j}}{\partial \mathbf{q}_i} \cdot \frac{\partial D_N}{\partial \mathbf{p}_i} d\mathbf{x}_{s+1} \dots d\mathbf{x}_N \\ &= \sum_{i=1}^s \sum_{\substack{j=1 \\ j \neq i}}^s \frac{\partial \phi_{i,j}}{\partial \mathbf{q}_i} \cdot \frac{\partial f_s}{\partial \mathbf{p}_i} \frac{1}{V^s} + \\ &\quad + \sum_{i=1}^s \sum_{j=s+1}^N \int \frac{\partial \phi_{i,j}}{\partial \mathbf{q}_i} \cdot \frac{\partial D_N}{\partial \mathbf{p}_i} d\mathbf{x}_{s+1} \dots d\mathbf{x}_N + \quad (13) \\ &\quad + \sum_{i=s+1}^N \sum_{j=1}^s \int \frac{\partial \phi_{i,j}}{\partial \mathbf{q}_i} \cdot \frac{\partial D_N}{\partial \mathbf{p}_i} d\mathbf{x}_{s+1} \dots d\mathbf{x}_N + \\ &\quad + \sum_{i=s+1}^N \sum_{\substack{j=s+1 \\ j \neq i}}^N \int \frac{\partial \phi_{i,j}}{\partial \mathbf{q}_i} \cdot \frac{\partial D_N}{\partial \mathbf{p}_i} d\mathbf{x}_{s+1} \dots d\mathbf{x}_N. \end{aligned}$$

Remark that $\int \frac{\partial D_N}{\partial \mathbf{p}_i} d\mathbf{p}_i = 0$ since D_N vanishes at the boundaries. As a consequence, the last two terms of the sum in (13) are equal to zero. On the other hand, to develop the second term on the r.h.s., imagine to fix $j = s + 1$.

$$\begin{aligned} \sum_{i=1}^s \int \frac{\partial \phi_{i,s+1}}{\partial \mathbf{q}_i} \cdot \frac{\partial D_N}{\partial \mathbf{p}_i} d\mathbf{x}_{s+1} \dots d\mathbf{x}_N &= \sum_{i=1}^s \int \frac{\partial \phi_{i,s+1}}{\partial \mathbf{q}_i} \cdot \frac{\partial}{\partial \mathbf{p}_i} \left(\int D_N d\mathbf{x}_{s+2} \dots d\mathbf{x}_N \right) d\mathbf{x}_{s+1} \\ &= \sum_{i=1}^s \int \frac{\partial \phi_{i,s+1}}{\partial \mathbf{q}_i} \cdot \frac{\partial}{\partial \mathbf{p}_i} \left(\frac{f_{s+1}}{V^{s+1}} \right) d\mathbf{x}_{s+1} \quad (14) \\ &= \frac{1}{V^{s+1}} \sum_{i=1}^s \int \frac{\partial \phi_{i,s+1}}{\partial \mathbf{q}_i} \cdot \frac{\partial f_{s+1}}{\partial \mathbf{p}_i} d\mathbf{x}_{s+1}. \end{aligned}$$

Now, by symmetry of D_N , the same result would be obtained by fixing " $j = s + 2$ ", " $j = s + 3$ " and so on up to " $j = N$ ". As a result, the second term in 13 can be written as the sum of $(N - s)$ terms equal to (14). Overall, (13) can be ultimately developed as:

$$e \sum_{i=1}^N \int \mathbf{E}_i \cdot \frac{\partial D_N}{\partial \mathbf{p}_i} d\mathbf{x}_{s+1} \dots d\mathbf{x}_N = \sum_{i=1}^s \sum_{\substack{j=1 \\ j \neq i}}^s \frac{\partial \phi_{i,j}}{\partial \mathbf{q}_i} \cdot \frac{\partial f_s}{\partial \mathbf{p}_i} \frac{1}{V^s} + \frac{(N-s)}{V^{s+1}} \sum_{i=1}^s \int \frac{\partial \phi_{i,s+1}}{\partial \mathbf{q}_i} \cdot \frac{\partial f_{s+1}}{\partial \mathbf{p}_i} d\mathbf{x}_{s+1}. \quad (15)$$

Combining these results, I finally obtain for (9) the following formulation:

$$\frac{1}{V^s} \frac{\partial f_s}{\partial t} + \frac{1}{V^s} \sum_{i=1}^s \frac{\mathbf{p}_i}{m} \cdot \frac{\partial f_s}{\partial \mathbf{q}_i} - \frac{1}{V^s} \sum_{i=1}^s \sum_{\substack{j=1 \\ j \neq i}}^s \frac{\partial \phi_{i,j}}{\partial \mathbf{q}_i} \cdot \frac{\partial f_s}{\partial \mathbf{p}_i} = \frac{(N-s)}{V^{s+1}} \sum_{i=1}^s \int \frac{\partial \phi_{i,s+1}}{\partial \mathbf{q}_i} \cdot \frac{\partial f_{s+1}}{\partial \mathbf{p}_i} d\mathbf{x}_{s+1} \quad (16)$$

$$\frac{\partial f_s}{\partial t} + \sum_{i=1}^s \frac{\mathbf{p}_i}{m} \cdot \frac{\partial f_s}{\partial \mathbf{q}_i} - \sum_{i=1}^s \sum_{\substack{j=1 \\ j \neq i}}^s \frac{\partial \phi_{i,j}}{\partial \mathbf{q}_i} \cdot \frac{\partial f_s}{\partial \mathbf{p}_i} = \frac{(N-s)}{V} \sum_{i=1}^s \int \frac{\partial \phi_{i,s+1}}{\partial \mathbf{q}_i} \cdot \frac{\partial f_{s+1}}{\partial \mathbf{p}_i} d\mathbf{x}_{s+1} \quad (17)$$

By taking the boundary to infinity, the particle density n_0 remains finite, thus:

$$N \rightarrow \infty, \quad V \rightarrow \infty$$

$$\frac{(N-s)}{V} \sim \frac{N}{V} = n_0.$$

In the end, the equation for f_s I was looking for is:

$$\frac{\partial f_s}{\partial t} + \sum_{i=1}^s \frac{\mathbf{p}_i}{m} \cdot \frac{\partial f_s}{\partial \mathbf{q}_i} - \sum_{i=1}^s \sum_{\substack{j=1 \\ j \neq i}}^s \frac{\partial \phi_{i,j}}{\partial \mathbf{q}_i} \cdot \frac{\partial f_s}{\partial \mathbf{p}_j} = n_0 \sum_{i=1}^s \int \frac{\partial \phi_{i,s+1}}{\partial \mathbf{q}_i} \cdot \frac{\partial f_{s+1}}{\partial \mathbf{p}_i} d\mathbf{x}_{s+1} \quad (18)$$

No approximation has been made to obtain this result and we see that the equation determining f_s contains f_{s+1} , and thus the equation for f_{s+1} contains f_{s+2} and so back to D_N . We do not appear to be any better off. Moreover, we now appeal to a physical argument to justify the use of simplifying approximations. This is why (18) constitutes a hierarchy of equations, that is indeed the so-called BBGKY hierarchy. To deal with this kind of hierarchy, some approximation must be made in order to close the system. In particular, in plasma physics the concept of Debye sphere allows to neglect interactions between electrons that are too far from each others.

The Debye sphere is defined as the sphere of radius λ_D around a particle. λ_D is the *Debye length*, defined² as:

$$\lambda_D^2 := \frac{\epsilon_0 k T}{n_0 e^2} \quad (19)$$

where ϵ_0 is the dielectric constant of vacuum, k is the Boltzmann's constant and T is the electron temperature. [$n_0 \lambda_D^3$ is often taken as the definition of plasma, a typical value for a hot laboratory plasma is $n_0 \sim 10^4 \text{ cm}^{-3}$, $T_e \sim 10^6 \text{ K}$ giving $n_0 \lambda_D^3 \sim 10^4$]

The physical meaning of the Debye length is found in the phenomenon of *Debye shielding*. Roughly speaking, each electron generates an electrostatic potential which attracts ions and repels all other electrons. As a consequence, around one electron the density of ions increases, while the electron density decreases. In fact, the ions around each electron build a sort of *shielding cloud* that tends to cancel the electron charge. As a result, at a given distance from the electron its potential is completely shielded by the surrounding ions. The Debye length (19) is such distance by definition.

Since the potential generated by a single electron has spherical symmetry, we can say that an electron can interact with other electrons only inside the Debye sphere around it. Indeed, long-range interactions between two electrons at distance above λ_D might be ignored: the electrostatic potential generated by one electron is negligible at the position of the other one (and vice-versa).

Remark that this is just an ideal interpretation of the Debye length, in a real plasma the particles are moving: each particle tries to gather its own shielding cloud, however they cannot be thoroughly successful ([3]). Nevertheless, for this treatment it is enough for justifying the assumption that long-range interactions ($|\mathbf{q}_1 - \mathbf{q}_2| > \lambda_D$) are negligible.

The Mayer cluster expansion is often used.

$$\begin{aligned} f_1(\mathbf{x}_1) &= f_1(\mathbf{x}) \\ f_2(\mathbf{x}_1, \mathbf{x}_2) &= f_1(\mathbf{x}_1)f_1(\mathbf{x}_2) + P(\mathbf{x}_1, \mathbf{x}_2) \\ f_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) &= f_1(\mathbf{x}_1)f_1(\mathbf{x}_2)f_1(\mathbf{x}_3) + \\ &\quad + f_1(\mathbf{x}_1)P(\mathbf{x}_2, \mathbf{x}_3) + f_1(\mathbf{x}_2)P(\mathbf{x}_1, \mathbf{x}_3) + f_1(\mathbf{x}_3)P(\mathbf{x}_1, \mathbf{x}_2) + \\ &\quad + T(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \\ &\dots \end{aligned} \quad (20)$$

where $P(\mathbf{x}_1, \mathbf{x}_2)$ gives a measure of the correlation between the two particles, $T(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ of the correlation between three particles and so on. Thanks to the *Debye shielding* it is possible to assume the contribution of the correlation terms in (20) (P, T, \dots) to be progressively smaller by increasing the

²In this case, the Debye lenght is defined as in [1] using the *International System* of units. So this definition is coherent with my system of unit choice.

number of interacting particles. In particular, it will be shown that the Vlasov equation is obtained as a zero-order approximation of (18), i.e. by neglecting every interaction even between two electrons. In view of the long range nature of the Coulomb interaction one might expect that $P(\mathbf{x}_1, \mathbf{x}_2)$ would be comparable in magnitude to $f_1(\mathbf{x}_1)f_1(\mathbf{x}_2)$, i.e. pairs of particles would be strongly correlated. However, the Coulomb interaction results in particles moving so as to preserve quasi-neutrality. Each particle attracts a screening charge cloud of the opposite sign so as to neutralize its own charge. The effect of this screening is that over almost the whole of phase space P/f_1f_1 is a small quantity of the order of g . This is called *plasma parameter* and is defined as:

$$g := \frac{1}{n_0 \lambda_D^3}. \quad (21)$$

The denominator of (21) is just the number of electrons in a cube which has side length λ_D .³ Small values of g allow indeed to neglect collisions among particles, making convenient the macroscopic treatment of the plasma.

Intuitively, one could think that g very large means that there are few particles in a *Debye sphere*, so the electrons are on average far apart from each other more than λ_D . This means that their motion is independent due to the Debye shielding and so a collision could happen. On the other hand, if g is very small it means that the electrons in the plasma are very close to each other with respect to λ_D , so they are able to *feel* the Coulomb repulsive force of other electrons. In the latter case, the electrons are less likely to collide and they have less freedom of motion.

Just to give an idea, in the limit case $g = 0$ the plasma is "collisionless" as the electrons cannot collide: the motion of every electron depends on the motion of every other electron and the whole population moves as one body. On the other hand, in the limit case $g = \infty$ electrons act as they have neutral charge and each one of them moves around independently.

We can now use this as a basis for simplifying equation (18). Assuming that three particle correlations are even weaker than two particle correlations, we can take $s = 1$ and write (18) as

$$\frac{\partial f_1}{\partial t} + \frac{\mathbf{p}_1}{m} \cdot \frac{\partial f_1}{\partial \mathbf{q}_1} = n_0 \int \frac{\partial \phi_{1,2}}{\partial \mathbf{q}_1} \cdot \frac{\partial}{\partial \mathbf{p}_1} f_1(\mathbf{x}_1) f_1(\mathbf{x}_2) d\mathbf{x}_2 + n_0 \int \frac{\partial \phi_{1,2}}{\partial \mathbf{q}_1} \cdot \frac{\partial}{\partial \mathbf{p}_1} P(\mathbf{x}_1, \mathbf{x}_2) d\mathbf{x}_2. \quad (22)$$

We do not write down the equation for $P(\mathbf{x}_1, \mathbf{x}_2)$ but since we neglect three-particle correlations equation (22) together with equation for $P(\mathbf{x}_1, \mathbf{x}_2)$ now form a closed system. However, even this system is too complicated for general use and we are forced to approximate further. First note the form of equation (22). The first term on the right hand side of the equation describes the uncorrelated motions and the second is due to particle correlations.

The effect of screening is such that two-particles correlations are very weak except over a very small region of phase space of volume $\sim g^2/n_0$. I have tried to prove this estimate. Let a be the spacing between two particles. Given the number density n_0 , such spacing would approximately scale like:

$$a \sim \left(\frac{V}{N}\right)^{\frac{1}{3}} = n_0^{-\frac{1}{3}}. \quad (23)$$

It follows that a measure of the relative density of particles with respect to λ_D , i.e. the average distance between two particles that are interacting is estimated by a^2/λ_D . Therefore, the volume in space in which two particles interactions are not negligible can be estimated by:

$$\left(\frac{a^2}{\lambda_D}\right)^3 \sim \left(\frac{1}{n_0^{\frac{2}{3}} \lambda_D}\right)^3 = \frac{1}{n_0^2 \lambda_D^3} = \frac{g}{n_0}. \quad (24)$$

Now I need to estimate the volume in which the two particles interactions are relevant in the velocity space. I can estimate the *side velocity* of such volume by multiplying a times some frequency.

Let me introduce the electron *plasma frequency* ω_p :

$$\omega_p^2 = \frac{n_0 e^2}{\epsilon_0 m}. \quad (25)$$

³Other definitions of g can be found in literature using instead the sphere of radius λ_D , the Debye sphere. It does not change much since the difference is only of a coefficient of $3/(4\pi)$.

Consider a two-population plasma with fixed ions and with electrons that can move with respect to the ions, but are highly rigid with respect to other electrons. If the two *grids* are displaced, an electric field is generated and the electrons are pulled back toward the ions. When the electrons overlap exactly with the ions, the net force is zero, but the electrons have gained some velocity, so another displacement is reached. The result is an harmonic oscillation of the electrons grid with respect to the ions grid. The electron plasma frequency ω_p is defined to be the frequency of such motion ([3]). Its product with the Debye length λ_D corresponds to the thermal speed of the plasma particles:

$$\lambda_D \omega_p = \left[\left(\frac{\epsilon_0 k T}{n_0 e^2} \right) \left(\frac{n_0 e^2}{\epsilon_0 m} \right) \right]^{\frac{1}{2}} = \left(\frac{k T}{m} \right)^{\frac{1}{2}} = v_T. \quad (26)$$

The plasma frequency is naturally adopted as the characteristic frequency for a plasma, I can estimate the volume in the velocity space as:

$$(a \omega_p)^3 \sim (v_T \frac{a}{\lambda_D})^3 = v_T^3 g \sim g \quad (27)$$

In the end, the volume in the phase space in which interactions are relevant scales indeed like: $g/n_0 \cdot g \sim g^2/n_0$.

In view of this we now aim to describe the system in terms of a 'smoothed-out' distribution which should be a good approximation over almost the whole of phase space. This smoothing procedure can be appreciated by returning to equation (18).

The terms of the equation (18) are dimensionally a frequency.⁴ As a consequence, it is natural to non-dimensionalize the equation by dividing both sides by the plasma frequency ω_p . If the equation is non-dimensionalized it will be found that the third term on the left hand side of the equation is of order $\sim g$. Again, I tried to justify this scaling. A single term of the sum within the third term of (18) in its non-dimensional form is of the kind:

$$\frac{1}{\omega_p} \frac{\partial \phi_{i,j}}{\partial \mathbf{q}_i} \cdot \frac{\partial f_s}{\partial \mathbf{p}_i}$$

I can approximately say that:

$$\frac{\partial f_s}{\partial \mathbf{p}_i} \sim \frac{1}{m v_T} = \frac{1}{m \lambda_D \omega_p} \quad (28)$$

as λ_D and ω_p are the reference length and frequency of this system and I am dealing here with the velocity gradient of the distribution function.

About the force:

$$\begin{aligned} \frac{\partial \phi_{i,j}}{\partial \mathbf{q}_i} &= \frac{\partial}{\partial \mathbf{q}_i} \left(\frac{e^2}{4\pi\epsilon_0 |\mathbf{q}_i - \mathbf{q}_j|} \right) \\ &= \frac{e^2}{4\pi\epsilon_0} \left(\frac{\mathbf{q}_i - \mathbf{q}_j}{|\mathbf{q}_i - \mathbf{q}_j|^3} \right) \\ &\sim \frac{e^2}{\epsilon_0} \frac{1}{\lambda_D^2}. \end{aligned} \quad (29)$$

In the end, I can deduce that the third term of (18) scales like:

$$\frac{1}{\omega_p} \frac{\partial \phi_{i,j}}{\partial \mathbf{q}_i} \cdot \frac{\partial f_s}{\partial \mathbf{p}_i} \sim \frac{1}{\omega_p} \frac{e^2}{\epsilon_0} \frac{1}{\lambda_D^2} \frac{1}{m \lambda_D \omega_p} = \frac{e^2}{m \epsilon_0 \omega_p^2} \frac{1}{\lambda_D^3} = \frac{1}{n_0 \lambda_D^3} = g. \quad (30)$$

which is exactly the estimate that I was looking for. Clearly this term does not appear in (22), but the fact that for higher values of s it is of the order $o(g)$ is a very interesting result. Indeed, this means

⁴Just look at the first, time derivative term: f_s is non-dimensional, so its derivative with respect to time is a frequency.

that for small values of g I can neglect the contribution of such term for whatever f_s equation. Neglect of this term allows the equation to be solved with the uncorrelated function:

$$f_s(\mathbf{x}_1, \dots, \mathbf{x}_s) = \prod_{i=1}^s f_1(\mathbf{x}_i). \quad (31)$$

This is possible under the approximation $g = 0$. Indeed, under this assumption there is physical ground to neglect every interaction among electrons of the plasma because they are relevant only over a phase-space volume that scales with g^2/n_0 .

This also makes sense mathematically speaking. In the textbook [5] (Section 2.3) I have found an explicit computation⁵ of $P(\mathbf{x}_1, \mathbf{x}_2)$ which gives as a result:

$$P = -\frac{g}{8\pi} \frac{\exp\{-|\mathbf{q}_2 - \mathbf{q}_1|/\lambda_D\}}{|\mathbf{q}_2 - \mathbf{q}_1|/\lambda_D} \quad (32)$$

which shows that $P(\mathbf{x}_1, \mathbf{x}_2) \sim g$. One could than find out that $T \sim g^2$ and, in general, that the Mayer cluster expansion of f_s (20) can be viewed as a series in the plasma parameter g . In fact, (31) is basically the Mayer cluster expansion after a zero-order truncation, i.e. obtained by neglecting every term that involves correlations ($P \sim g = 0, T \sim g^2 = 0, \dots$).

I notice that, in the context of kinetic description of a plasma, the assumption of zero plasma parameter corresponds to the more general molecular chaos assumption in kinetic theory of gases (*Stosszahlansatz*).

We also note that f_s is the distribution function of s-test particles. If we assume that the separation of these s-particles is large, then $|\mathbf{q}_i - \mathbf{q}_j|^{-1} \rightarrow 0 \forall i, j = 1, 2, \dots, s, i \neq j$ which again allow us to obtain the uncorrelated solution, equation (31).

The smoothing procedure is equivalent to allowing e/m and g to become very small ($\sim g$) while at the same time keeping e/m , kT/m and n_0e^2/m constant. Consider indeed the thermal velocity to be of the order of units $v_T = (kT/m)^{1/2} \sim 1$, while $g = 1/n_0(n_0e^2/\epsilon_0 kT)^{3/2}$ is small. Since also n_0e^2/m is constant, necessarily $1/n_0 \sim g$, thus $e^2/m \sim g$ as well. In the end, since e/m is kept constant, than $e \sim g$, so $m \sim g$ ([7]).

In the continuum limit $g = 0$ the masses are completely smoothed out and we have a continuous, six dimensional phase fluid. Instead of a singular distribution function consisting of a set of δ -functions we now have a smooth, continuously differentiable distribution function. In the limit $g = 0$, equation (22) reduces to:

$$\frac{\partial f_1}{\partial t} + \frac{\mathbf{p}_1}{m} \cdot \frac{\partial f_1}{\partial \mathbf{q}_1} - n_0 \int \frac{\partial \phi_{1,2}}{\partial \mathbf{q}_1} \cdot \frac{\partial}{\partial \mathbf{p}_1} f_1(\mathbf{x}_1) f_1(\mathbf{x}_2) d\mathbf{x}_2 = 0 \quad (33)$$

For semplicity of notation f_1 is simply replaced by f since there is no f_i involved for $i > 1$ thanks to the $g = 0$ approximation. Moreover, \mathbf{x} is not anymore a point in the phase space but it just replaces \mathbf{q} as notation for the position. The momentum \mathbf{p} as independent variable in the phase space is now replaced by the electron velocity \mathbf{v} . Therefore (33) written-down using the updated notation looks like

$$\frac{\partial f}{\partial t} + \mathbf{v}_1 \cdot \frac{\partial f}{\partial \mathbf{x}_1} - \frac{n_0}{m} \left[\int \frac{\partial \phi_{1,2}}{\partial \mathbf{x}_1} f(\mathbf{x}_2, \mathbf{v}_2) d\mathbf{x}_2 d\mathbf{v}_2 \right] \cdot \frac{\partial f}{\partial \mathbf{v}_1} = 0. \quad (34)$$

In contrast to a normal fluid the six-dimensional phase fluid described by equation (34) contains the thermal information relating to the distribution of velocities.

In the third term, it is possible to recognise the averaged electric field:

$$e\mathbf{E}(\mathbf{x}_1, t) = n_0 \int \frac{\partial \phi_{1,2}}{\partial \mathbf{x}_2} f(\mathbf{x}_2, \mathbf{v}_2, t) d\mathbf{x}_2 d\mathbf{v}_2. \quad (35)$$

The electric field acting on a single electron was already introduced by (6). In that case, such electric field was a microscopic field since it was applied to a single electron, in a discrete approach. Here, we

⁵The treatment starts from a Gibbs ensemble and assumes $P \ll T$. Moreover, two general charged particles are considered and not specifically two electrons, so this result holds also in a more general context. I arranged the result (32) to the case of two interacting electrons. I considered out of scope to include here the full derivation.

are dealing with a macroscopic electric field obtained as an averaged quantity using the distribution function as *weight*. In fact, the adoption of averaged quantities is one of the points of introducing the distribution function, this is a key concept in statistical mechanics. This macroscopic electric field is suitable for a continuous description of the plasma.

By writing-down the third term in terms of the electric field, in the end (34) becomes:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} - \frac{e}{m} \mathbf{E}(\mathbf{x}, t) \cdot \frac{\partial f}{\partial \mathbf{v}} = 0 \quad (36)$$

where we have now dropped the subscript 1 from (\mathbf{x}, \mathbf{v}) since this is now redundant.

Equation (36) is the required Vlasov equation which forms the basis of most of plasma physics. The Vlasov equation is coupled to Poisson's equation through the electric field \mathbf{E} .

$$\begin{cases} \frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} - \frac{e}{m} \mathbf{E}(\mathbf{x}, t) \cdot \frac{\partial f}{\partial \mathbf{v}} = 0 \\ \frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{E} = \frac{n_0 e}{\epsilon_0} \int f(\mathbf{x}, \mathbf{v}, t) d\mathbf{v}. \end{cases} \quad (37)$$

The electric field appearing in Vlasov's equation is the self-consistent field due to all the other electrons in the system moving in an uncorrelated way, i.e. uncorrelated to any individual particle but correlated to all particles by the self-consistent field. The existence of the self-consistent field is the reason why a plasma is so rich in its collective effects and is the solution to the problem posed at the outset - namely, how to take account of the fact that every particle could interact with every other particle. It may be viewed as an iterative solution to the Vlasov-Poisson system, i.e. assume a distribution f , calculate the resulting electric field \mathbf{E} and than determine the correction to f and so on until the electric field resulting from f produces the same distribution function.

Remark that the Vlasov equation (36) is indeed the Boltzmann equation for a one-population electron plasma under the effect of a self-consistent electric field \mathbf{E} . Indeed the electric field acts on an electron through the Coulomb force:

$$\mathbf{F}_C(\mathbf{x}, t) = -e \mathbf{E}(\mathbf{x}, t). \quad (38)$$

This explains also why it was useful to define the electric field in (6) and in (35) with a multiplicative coefficient equal to the electron charge: basically the microscopic Coulomb force was defined back than.

Using the definition (38) in (36) it becomes clear that we are dealing with a Boltzmann equation for a collisionless system:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \frac{\mathbf{F}_C(\mathbf{x}, t)}{m} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0. \quad (39)$$

2.1 Properties of the Vlasov Equation

In the previous part of section 2, the Vlasov equation (36) has been derived starting from the Liouville's equation (3). It was explained how the Lorentz force was not taken into account since the beginning as it depends on the particle momentum. It follows that (3) fails if the magnetic field is taken into account and an extra term would be needed in that case. For this reason only \mathbf{E} was included in the discussion.

However, it can be shown with a different procedure (and more calculations) that the Vlasov equation (36) is the exact same in presence of an electro-magnetic field. In that case the third term will involve also the contribution of the Lorentz force other than the Coulomb force:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \frac{\mathbf{F}_C(\mathbf{x}, t) + \mathbf{F}_L(\mathbf{x}, \mathbf{v}, t)}{m} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0. \quad (40)$$

The microscopic Lorentz force \mathbf{F}_L for a single electron has been already defined by (2). Here the self-consistent magnetic field $\mathbf{B} = \mathbf{B}(\mathbf{x}, t)$ is introduced. This is already a macroscopic vector field which is suitable for a continuous description of the plasma, similarly to the averaged electric field \mathbf{E} . The Lorentz force can be now defined as

$$\mathbf{F}_L(\mathbf{x}, \mathbf{v}, t) = -e \mathbf{v} \times \mathbf{B}(\mathbf{x}, t). \quad (41)$$

It follows that (40) can be written down as:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} - \frac{e}{m} [\mathbf{E}(\mathbf{x}, t) + \mathbf{v} \times \mathbf{B}(\mathbf{x}, t)] \cdot \frac{\partial f}{\partial \mathbf{v}} = 0. \quad (42)$$

The complete Vlasov-Maxwell system of equations⁶ is then

$$\begin{cases} \frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} - \frac{e}{m} [\mathbf{E}(\mathbf{x}, t) + \mathbf{v} \times \mathbf{B}(\mathbf{x}, t)] \cdot \frac{\partial f}{\partial \mathbf{v}} = 0 \\ \frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \\ \frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{B} = 0 \\ \frac{\partial}{\partial \mathbf{x}} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \frac{\partial}{\partial \mathbf{x}} \times \mathbf{B} = \mu_0 (\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}) \\ \rho(\mathbf{x}, t) = -e \int f(\mathbf{x}, \mathbf{v}, t) d\mathbf{v} \\ \mathbf{J}(\mathbf{x}, t) = -e \int \mathbf{v} f(\mathbf{x}, \mathbf{v}, t) d\mathbf{v}. \end{cases} \quad (43)$$

System (43) involves the Vlasov equation, the Maxwell equations for the electromagnetic field and two equations to close the system for the charge and current densities, which are *physical sources* for the electromagnetic field. In particular, ρ is the scalar field for the charge density and \mathbf{J} is the vector field for the current density. These two fields are also related by the following continuity equation:

$$\frac{\partial}{\partial t} \rho + \frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{J} = 0. \quad (44)$$

Paper [1] here includes in (43) only the Faraday law and the Ampère-Maxwell law. Also, it uses the magnetic field \mathbf{H} within the Faraday law without introducing it at all. Moreover, only the expression for \mathbf{J} is provided. To improve clarity, semplicity and completeness, I decided to write-down the full system of equation, expressing the Maxwell equations only in terms of the introduced fields \mathbf{E} and \mathbf{B} . The remaining part of this section is dedicated to some properties of the Vlasov equation.

Noting that the *particle orbit* corresponding to equation (42) is:

$$\begin{cases} \frac{d}{dt} \mathbf{x}(t) = \mathbf{v}(t) \\ \frac{d}{dt} \mathbf{v}(\mathbf{x}, t) = \frac{\mathbf{F}_C + \mathbf{F}_L}{m} = -\frac{e}{m} [\mathbf{E}(\mathbf{x}, t) + \mathbf{v} \times \mathbf{B}(\mathbf{x}, t)] \end{cases} \quad (45)$$

we can write the equation

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{x}} \cdot \frac{d\mathbf{x}}{dt} + \frac{\partial f}{\partial \mathbf{v}} \cdot \frac{d\mathbf{v}}{dt} = 0 \quad (46)$$

$$\frac{d}{dt} f(\mathbf{x}, \mathbf{v}, t) = 0 \quad (47)$$

thus enabling the equation to be integrated once we know the particle orbit (method of characteristics). The Vlasov equation thus appears as the analogue of Liouville's equation in the phase space of a single particle. As a general method of solution, equation (47) is not very helpful since the orbit is in the self-consistent electromagnetic field which is not known until f is determined. However, the method is very useful in small amplitude solutions when the equation can be linearized. The integration is then carried out routinely along the unperturbed or equilibrium orbit. Equilibrium solutions to equation (42) are also obtained as functions of the constants of the motion.

A fundamental property of Vlasov's equation is that it is time reversible, i.e. if all velocities were reversed and time was made to run backwards, all previous motions would be reproduced. In other words, the equation is invariant under the transformation:

$$\begin{cases} \mathbf{x}' = \mathbf{x}; \\ \mathbf{v}' = -\mathbf{v}; \\ t' = -t. \end{cases} \quad (48)$$

⁶Here, since both \mathbf{E} and \mathbf{B} are velocity-independent fields, I could have adopted the notation " ∇ " instead of " $\partial/\partial \mathbf{x}$ " without ambiguity for the Maxwell equations. However, I maintained the typical notation of statistical mechanics, that is necessary in the phase space, for the entire work.

Indeed, by writing down (42) in terms of the *transformed* independent variables I obtain:

$$\frac{\partial f}{\partial(-t')} - \mathbf{v}' \cdot \frac{\partial f}{\partial \mathbf{x}} - \frac{e}{m} [\mathbf{E}(\mathbf{x}, -t') - \mathbf{v}' \times \mathbf{B}(\mathbf{x}, -t')] \cdot \frac{\partial f}{\partial(-\mathbf{v}')} = 0 \quad (49)$$

$$-\frac{\partial f}{\partial t'} - \mathbf{v}' \cdot \frac{\partial f}{\partial \mathbf{x}} + \frac{e}{m} [\mathbf{E}(\mathbf{x}, -t') - \mathbf{v}' \times \mathbf{B}(\mathbf{x}, -t')] \cdot \frac{\partial f}{\partial \mathbf{v}'} = 0. \quad (50)$$

By performing a multiplication both sides times "−1" I obtain:

$$\frac{\partial f}{\partial t'} + \mathbf{v}' \cdot \frac{\partial f}{\partial \mathbf{x}} - \frac{e}{m} [\mathbf{E}(\mathbf{x}, -t') - \mathbf{v}' \times \mathbf{B}(\mathbf{x}, -t')] \cdot \frac{\partial f}{\partial \mathbf{v}'} = 0. \quad (51)$$

Now, let me develop the electric and magnetic field expressions with reversed time. Consider the Gauss law from (43):

$$\frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{E}(\mathbf{x}, -t') = \frac{\rho(\mathbf{x}, -t')}{\epsilon_0}. \quad (52)$$

Since $\rho(-t) = \rho(t)$, it follows that also $\mathbf{E}(\mathbf{x}, -t') = \mathbf{E}(\mathbf{x}, t')$. On the other hand, consider now the Faraday law:

$$\frac{\partial}{\partial \mathbf{x}} \times \mathbf{E}(\mathbf{x}, -t') = -\frac{\partial \mathbf{B}(\mathbf{x}, -t')}{\partial(-t')} \quad (53)$$

$$\frac{\partial}{\partial \mathbf{x}} \times \mathbf{E}(\mathbf{x}, t') = \frac{\partial \mathbf{B}(\mathbf{x}, -t')}{\partial t'}. \quad (54)$$

It follows from (54) that it must be: $\mathbf{B}(\mathbf{x}, -t') = -\mathbf{B}(\mathbf{x}, t')$.

By inserting these results for the reversed electric and magnetic fields into (51):

$$\frac{\partial f}{\partial t'} + \mathbf{v}' \cdot \frac{\partial f}{\partial \mathbf{x}} - \frac{e}{m} [\mathbf{E}(\mathbf{x}, t') + \mathbf{v}' \times \mathbf{B}(\mathbf{x}, t')] \cdot \frac{\partial f}{\partial \mathbf{v}'} = 0 \quad (55)$$

which is exactly the equation (36). This proves that the Vlasov equation is indeed time reversible. This is because the random binary collisions are not included in the equation. It is these collisions which introduce physical dissipation and irreversibility into the system. The interactions which are responsible for the self-consistent field cannot be interpreted as collisions in the usual sense. This is because the effect is produced by a large number of particles and is therefore a macroscopic effect. Since an average over a large number of particles is involved the process becomes certain rather than random and cannot cause an increase in the entropy of the system. This can be demonstrated explicitly by means of an H-theorem where

$$H(t) = \int f \log f d\mathbf{x} d\mathbf{v}. \quad (56)$$

Consider the Vlasov equation (42), by multiplying both sides by "1 + log f" and by integrating over both space and velocity I obtain:

$$\int \frac{\partial f}{\partial t} (1 + \log f) d\mathbf{x} d\mathbf{v} + \int \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} (1 + \log f) d\mathbf{x} d\mathbf{v} + \int \mathbf{a} \cdot \frac{\partial f}{\partial \mathbf{v}} (1 + \log f) d\mathbf{x} d\mathbf{v} = 0 \quad (57)$$

where I called for semplicity \mathbf{a} the acceleration of the particle, corresponding to the total Coulomb plus Lorentz force divided by the mass.

Now, let me remark that:

$$\begin{aligned} \frac{\partial}{\partial t} (f \log f) &= \log f \frac{\partial f}{\partial t} + f \frac{\partial}{\partial t} (\log f) \\ &= \log f \frac{\partial f}{\partial t} + f \frac{1}{f} \frac{\partial f}{\partial t} \\ &= (1 + \log f) \frac{\partial f}{\partial t}. \end{aligned} \quad (58)$$

By substituting (58) into the first term of (57) I obtain:

$$\begin{aligned}
 \int \frac{\partial f}{\partial t} (1 + \log f) d\mathbf{x} d\mathbf{v} &= \int \frac{\partial}{\partial t} (f \log f) d\mathbf{x} d\mathbf{v} \\
 &= \frac{\partial}{\partial t} \int (f \log f) d\mathbf{x} d\mathbf{v} \\
 &= \frac{\partial}{\partial t} H \\
 &= \frac{dH}{dt}
 \end{aligned} \tag{59}$$

since H is by definition a function only of the variable t .

Consider now the second term. The same reasoning of (58) can be made component-by-component for the spatial gradient. Hence, the second term can be written down as follows:

$$\begin{aligned}
 \int \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} (1 + \log f) d\mathbf{x} d\mathbf{v} &= \int \frac{\partial}{\partial \mathbf{x}} \cdot \left(\int f \log f \mathbf{v} d\mathbf{v} \right) d\mathbf{x} \\
 &= \int \frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{J} d\mathbf{x}
 \end{aligned} \tag{60}$$

The vector $\mathbf{J} = \mathbf{J}(\mathbf{x}, t)$ has been introduced. Since \mathbf{J} does not depend on velocity by definition, it is possible to apply the Gauss theorem for any space domain Ω

$$\int_{\Omega} \frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{J} d\mathbf{x} = \int_{\partial \Omega} \mathbf{J} \cdot \mathbf{n} d\sigma. \tag{61}$$

The distribution function f is null at infinity, so it is the vector \mathbf{J} . By pushing Ω to the limit, the flux integral of \mathbf{J} on the r.h.s. vanishes:

$$\int \frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{J} d\mathbf{x} = \int \mathbf{J} \cdot \mathbf{n} d\sigma = 0. \tag{62}$$

In the end, let me consider the third term. Although this term looks similar to the previous one, it is more tricky to deal with because the acceleration $\mathbf{a} = \mathbf{a}(\mathbf{x}, \mathbf{v}, t)$ depends on both space and velocity. Again, I can develop the integrand as in (58). I define with \mathcal{I} the integral over $d\mathbf{v}$:

$$\begin{aligned}
 \int \mathbf{a} \cdot \frac{\partial f}{\partial \mathbf{v}} (1 + \log f) d\mathbf{x} d\mathbf{v} &= \int \mathbf{a} \cdot \frac{\partial}{\partial \mathbf{v}} (f \log f) d\mathbf{v} d\mathbf{x} \\
 &= \int \mathcal{I} d\mathbf{x}.
 \end{aligned} \tag{63}$$

To develop \mathcal{I} , I perform an integration by part

$$\begin{aligned}
 \mathcal{I} &= \int \mathbf{a} \cdot \frac{\partial}{\partial \mathbf{v}} (f \log f) d\mathbf{v} \\
 &= \int \frac{\partial}{\partial \mathbf{v}} \cdot (f \log f \mathbf{a}) d\mathbf{v} - \int f \log f \frac{\partial}{\partial \mathbf{v}} \mathbf{a} d\mathbf{v} \\
 &= \mathcal{I}_1 - \frac{e}{m} \mathcal{I}_2
 \end{aligned} \tag{64}$$

with obvious meaning of notation.

To compute \mathcal{I}_1 I exploit again Gauss theorem. Similarely as it was for (62), I can say that the surface flux integral over the whole velocity space is null because f is vanishes at infinite velocities.

$$\begin{aligned}
 \mathcal{I}_1 &= \int \frac{\partial}{\partial \mathbf{v}} \cdot (f \log f \mathbf{a}) d\mathbf{v} \\
 &= \int f \log f \mathbf{a} \cdot \nu d\sigma = 0.
 \end{aligned} \tag{65}$$

On the other hand, for \mathcal{I}_2 I exploit the *circular shift* property of the triple product. Remark that \mathbf{E} and \mathbf{B} are independent on \mathbf{v}

$$\begin{aligned}\mathcal{I}_2 &= \int f \log f \frac{\partial}{\partial \mathbf{v}} \cdot (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \\ &= \int f \log f \frac{\partial}{\partial \mathbf{v}} \cdot (\mathbf{v} \times \mathbf{B}) \\ &= \int f \log f \mathbf{B} \cdot \left(\frac{\partial}{\partial \mathbf{v}} \times \mathbf{v} \right).\end{aligned}\quad (66)$$

I conclude that also $\mathcal{I}_2 = 0$ because the velocity curl of the velocity itself is trivially zero. In the end, I have shown that $\mathcal{I} = 0$.

By substituting every result into (57) I finally obtain the H-theorem for the Vlasov equation:

$$\frac{dH}{dt} + \int \mathbf{J} \cdot \mathbf{n} d\sigma + \int \mathcal{I} d\mathbf{x} = 0 \quad (67)$$

$$\frac{dH}{dt} = 0. \quad (68)$$

In the more general case of Boltzmann equation for a gas (homogeneous in space) and not specifically the Vlasov equation, the H-theorem states that

$$\frac{dH}{dt} \leq 0. \quad (69)$$

In this case the reversibility is not guaranteed since $\frac{dH}{dt}$ might be negative. Indeed by reversing time one would obtain:

$$\frac{dH}{d(-t)} = -\frac{dH}{dt} \geq 0 \quad (70)$$

which means that, if the transformation (48) is applied, the H-theorem would be violated. So, in general, the kinetic description of a gas allows reversibility only if (68) holds. This happens for instance when the distribution function f is a Maxwellian distribution. However, this is not necessarily the case in this treatment because in (42) collisions are ignored, so there is no collision operator involved. Referring to the case of electrostatic interactions any $f_{0a}(\mathbf{v})$ will be an equilibrium solution of equation (36)⁷ provided only:

$$\sum_a q_a \int f_{0a}(\mathbf{v}) d\mathbf{v} d\mathbf{x} = 0. \quad (71)$$

This relation corresponds to an electrostatic equilibrium condition for the plasma.

Under this assumption, the self-consistent electric field \mathbf{E} is null, making null the overall convective term of (36). Moreover, if f_{0a} depends only on velocity and it is uniform in space, its space gradient would be null as well. To sum up, in this case it holds:

$$\mathbf{v} \cdot \frac{\partial f_{0a}}{\partial \mathbf{x}} - \frac{e}{m} \mathbf{E} \cdot \frac{\partial f_{0a}}{\partial \mathbf{v}} = 0. \quad (72)$$

By substituting into the Vlasov equation, it simply follows that:

$$\frac{\partial f_{0a}}{\partial t} = 0. \quad (73)$$

In particular the Vlasov equation does not single out the Maxwell-Boltzmann distribution. The effect of binary collisions must be included to ensure that the distribution will relax to a Maxwell distribution.

⁷Here, the extension to the multi-population plasma is considered. "a" is the index over the plasma populations and " q_a " is the charge of the particles of the a -esim population.

3 The Effect of Binary Collisions

The Vlasov equation is often called the collisionless Boltzmann equation. In the justification of this equation with the aid of equation (22) we neglected the two particle correlation function $P(\mathbf{x}_1, \mathbf{x}_2)$ on the grounds of the effect of screening. However, as mentioned in the previous section it is the binary collisions which introduce dissipation into the system which in turn is responsible for classical transport phenomena. Let us now consider how the effect of binary collisions is incorporated into the kinetic theory. The discussion will be divided into two parts. In the first part we shall describe the simplest method of allowing for collisions through the introduction of a relaxation time. In the second part we shall discuss the more accurate Fokker-Planck model. This model applies specifically to a system, such as a plasma, where most of the scatterings are through small angles.

3.1 Relaxation Models

The simplest model of all arises through the introduction of a relaxation time τ , which in general will be velocity dependent. τ is the characteristic time that takes to an electron to suffer a collision and it is simply defined as:

$$\tau = \frac{1}{\nu} \quad (74)$$

where ν is the collision frequency, which expression reads:

$$\nu = n_0 v \sigma. \quad (75)$$

Here, σ is dimensionally a cross section. This means that in dimensional terms ν is the flux " $n_0 v$ " through such cross sectional area. Clearly this remark does not provide a proper physical interpretation since the cross section is not a *real* area of some actual surface. About physical meaning it is relevant to remark that ν scales like the number density n_0 and the particle velocity v : this is coherent since we expect more frequent collisions between particles in more *crowded* regions and when they have higher speed. A detailed derivation of the formula for ν , specifying what σ is, requires physical arguments that are beyond the scope of this work. I have found a simple mathematical derivation for ν in plasmas, yet not exactly rigorous, in section 1.6 of book [3]. The only result that I think it is interesting to mention here is:

$$\frac{\nu}{\omega_p} \sim g \quad (76)$$

where ω_p is the plasma frequency defined in (25) and g is the plasma parameter defined in (21). Estimate (76) shows that for sufficiently ionized plasmas ($\omega_p \gg \nu$) the collisionless assumption ($g \ll 1$) has physical ground and collisions can be either neglected or considered as a correction of the first collisionless analysis.

If we consider only changes to the distribution function due to binary collisions, than the relaxation model is:

$$(\frac{\partial f}{\partial t})_{\text{coll}} = \frac{f_0 - f(\mathbf{x}, \mathbf{v}, t)}{\tau(v)} \quad (77)$$

where f_0 is the equilibrium distribution function.

Such model is known more in general in Statistical Mechanics as the BGK model, from Bhatnagar, Gross and Krook. In particular, being the relaxation time velocity dependent, model (77) is a *modified* version of the simplest BGK, where the collision frequency is considered instead constant and uniform. Clearly, any small departure from the equilibrium distribution will decay exponentially on a timescale proportional to τ .

The adoption of model (77) introduces into the Vlasov equation a non-zero collision term:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} - \frac{e}{m} [\mathbf{E}(\mathbf{x}, t) + \mathbf{v} \times \mathbf{B}(\mathbf{x}, t)] \cdot \frac{\partial f}{\partial \mathbf{v}} = (\frac{\partial f}{\partial t})_{\text{coll}}. \quad (78)$$

Remark that this model is, obviously, also dimensionally coherent since (77) is a frequency such as every other term of the Vlasov equation.

This simple relaxation model, however, suffers from the defect that it does not conserve particle number. Indeed, by integrating the collision model (77) over the velocity I obtain:

$$\begin{aligned} \int \left(\frac{\partial f}{\partial t} \right)_{\text{coll}} d\mathbf{v} &= \int \frac{f_0}{\tau(v)} d\mathbf{v} - \int \frac{f}{\tau(v)} d\mathbf{v} \\ &= n_0 \int v \sigma(v) f_0 d\mathbf{v} - n_0 \int v \sigma(v) f d\mathbf{v} \neq 0. \end{aligned} \quad (79)$$

In fact, since the integrals of f and f_0 are weighted by $v \sigma(v)$, it is not guaranteed that their value is the same, so the particle number is not conserved. This means that charge is not conserved instantaneously by adopting the kinetic model (77), but only on the average over a cycle ([8]). This defect can be remedied by the Krook model which is:

$$\left(\frac{\partial f}{\partial t} \right)_{\text{coll}} = -\frac{f}{\tau} + \frac{n(\mathbf{x}, t)}{n_0} \frac{f_0}{\tau}. \quad (80)$$

This kinetic model has been introduced in [8], $n(\mathbf{x}, t) = \int f d\mathbf{v}$ is the fluctuating density and τ is indeed a suitable averaged collision time. Clearly, equation (80) does now conserve number density instantaneously. Indeed, having now a velocity independent τ I obtain:

$$\begin{aligned} \int \left(\frac{\partial f}{\partial t} \right)_{\text{coll}} d\mathbf{v} &= -\int \frac{f}{\tau} d\mathbf{v} + \int \frac{n(\mathbf{x}, t)}{n_0} \frac{f_0}{\tau} d\mathbf{v} \\ &= -\frac{1}{\tau} \int f d\mathbf{v} + \frac{\int f d\mathbf{v}}{n_0} \frac{1}{\tau} \int f_0 d\mathbf{v} \\ &= -\frac{1}{\tau} \int f d\mathbf{v} + \frac{\int f d\mathbf{v}}{n_0} \frac{1}{\tau} n_0 \\ &= -\frac{1}{\tau} \int f d\mathbf{v} + \frac{1}{\tau} \int f d\mathbf{v} = 0. \end{aligned} \quad (81)$$

The interpretation of equation (80) is that particles are absorbed at a rate proportional to f and re-emitted at a rate proportional to the local density with a Maxwell distribution. The particle distribution is therefore brought to rest as a whole with the result that although conserving particle number, equation (80) does not conserve momentum and energy. The gain or loss of total energy and momentum by collisions is provided by [8]:

$$\begin{aligned} \int m\mathbf{v} \left(\frac{\partial f}{\partial t} \right)_{\text{coll}} d\mathbf{v} &= -\frac{m}{\tau} \int \mathbf{v} f d\mathbf{v} + \frac{m}{\tau} \frac{n(\mathbf{x}, t)}{n_0} \int \mathbf{v} f_0 d\mathbf{v} \\ &= -\frac{m}{\tau} \int \mathbf{v} f d\mathbf{v} \neq 0; \end{aligned} \quad (82)$$

$$\begin{aligned} \int \frac{mv^2}{2} \left(\frac{\partial f}{\partial t} \right)_{\text{coll}} d\mathbf{v} &= -\frac{m}{2\tau} \int v^2 f d\mathbf{v} + \frac{m}{2\tau} \frac{n(\mathbf{x}, t)}{n_0} \int v^2 f_0 d\mathbf{v} \\ &= -\frac{m}{2\tau} \int v^2 f d\mathbf{v} + \frac{m}{2\tau} \frac{n(\mathbf{x}, t)}{n_0} \frac{3kT_0}{m} \\ &= \frac{m}{2\tau} \left\{ \frac{3kT_0}{mn_0} \int f d\mathbf{v} - \int v^2 f d\mathbf{v} \right\} \neq 0 \end{aligned} \quad (83)$$

with T_0 being the equilibrium temperature. This does not mean that this model is therefore of no value. It applies, for example, to electron neutral collisions where the momentum and energy would be lost from the electron fluid while the electron number would be conserved. Such a model is therefore appropriate to a weakly ionized plasma. Indeed, since an atom is so much more massive than an electron the electron would effectively be brought to rest.

The above collision model is linearly proportional to f which is appropriate to electron-neutral collisions. However, for a fully ionized plasma where electron-electron or ion-ion collisions will be proportional to f^2 . In addition, for these collisions momentum and energy must be conserved. Krook et al [8] have developed a simplified model to take account of such collisions as follows:

$$\left(\frac{\partial f}{\partial t} \right)_{\text{coll}} = -\frac{n(\mathbf{x}, t)}{n_0} \frac{f}{\tau} + \frac{n(\mathbf{x}, t)^2}{n_0} \frac{\Phi}{\tau} \quad (84)$$

where

$$\Phi = \left[\frac{m}{2\pi kT(\mathbf{x}, t)} \right]^{\frac{3}{2}} \exp \left\{ -\frac{m}{2\pi kT(\mathbf{x}, t)} [\mathbf{v} - \mathbf{q}(\mathbf{x}, t)]^2 \right\} \quad (85)$$

$$n(\mathbf{x}, t) = \int f d\mathbf{v} \quad (86)$$

$$\mathbf{q}(\mathbf{x}, t) = \frac{1}{n(\mathbf{x}, t)} \int \mathbf{v} f d\mathbf{v} \quad (87)$$

$$\frac{3kT}{m}(\mathbf{x}, t) = \frac{1}{n(\mathbf{x}, t)} \int (\mathbf{v} - \mathbf{q})^2 f d\mathbf{v} \quad (88)$$

and

$$\int \Phi d\mathbf{v} = 1 \quad (89)$$

where $\mathbf{q}(\mathbf{x}, t)$ and $T(\mathbf{x}, t)$ define the flow velocity and temperature at \mathbf{x} and t . The interpretation now is that the re-emitted particles (after the collision) at (\mathbf{x}, t) emerge with a Maxwellian distribution centered on the mean flow velocity $\mathbf{q}(\mathbf{x}, t)$ corresponding to the temperature $T(\mathbf{x}, t)$. It is easily verified that this model conserves particle number, momentum and energy and is therefore appropriate to describe like-particle collisions. I have tried to prove that the three quantities are conserved by adopting model (84). I started with the particle number.

$$\begin{aligned} \int \left(\frac{\partial f}{\partial t} \right)_{\text{coll}} d\mathbf{v} &= -\frac{n(\mathbf{x}, t)}{n_0 \tau} \int f d\mathbf{v} + \frac{n(\mathbf{x}, t)^2}{n_0 \tau} \int \Phi d\mathbf{v} \\ &= -\frac{n(\mathbf{x}, t)^2}{n_0 \tau} + \frac{n(\mathbf{x}, t)^2}{n_0 \tau} = 0. \end{aligned} \quad (90)$$

Consider now the conservation of momentum:

$$\begin{aligned} \int m\mathbf{v} \left(\frac{\partial f}{\partial t} \right)_{\text{coll}} d\mathbf{v} &= -m \frac{n(\mathbf{x}, t)}{n_0 \tau} \int \mathbf{v} f d\mathbf{v} + m \frac{n(\mathbf{x}, t)^2}{n_0 \tau} \int \mathbf{v} \Phi d\mathbf{v} \\ &= m \frac{n(\mathbf{x}, t)^2}{n_0 \tau} \left[\int \mathbf{v} \Phi d\mathbf{v} - \mathbf{q}(\mathbf{x}, t) \right] \\ &= m \frac{n(\mathbf{x}, t)^2}{n_0 \tau} \left[\int \mathbf{v} \Phi d\mathbf{v} - \mathbf{q}(\mathbf{x}, t) \int \Phi d\mathbf{v} \right] \\ &= m \frac{n(\mathbf{x}, t)^2}{n_0 \tau} \int (\mathbf{v} - \mathbf{q}) \Phi d\mathbf{v} = 0. \end{aligned} \quad (91)$$

Indeed Φ is a Maxwellian distribution centered in \mathbf{q} , so the integral above corresponds to the first momentum of a Maxwellian which is null.

To prove energy conservation, I consider the two terms of (84) separately.

$$\begin{aligned} \int v^2 f d\mathbf{v} &= \int (\mathbf{v} - \mathbf{q})^2 f d\mathbf{v} + \int 2\mathbf{q} \cdot \mathbf{v} f d\mathbf{v} - \int q^2 f d\mathbf{v} \\ &= \int (\mathbf{v} - \mathbf{q})^2 f d\mathbf{v} + 2\mathbf{q} \cdot (n(\mathbf{x}, t)\mathbf{q}) - q^2 n(\mathbf{x}, t) \\ &= \int (\mathbf{v} - \mathbf{q})^2 f d\mathbf{v} + q^2 n(\mathbf{x}, t). \end{aligned} \quad (92)$$

$$\begin{aligned} \int v^2 \Phi d\mathbf{v} &= \int (\mathbf{v} - \mathbf{q})^2 \Phi d\mathbf{v} + 2\mathbf{q} \cdot \int \mathbf{v} \Phi d\mathbf{v} - q^2 \int \Phi d\mathbf{v} \\ &= \frac{3kT(\mathbf{x}, t)}{m} + 2\mathbf{q} \cdot \int (\mathbf{v} - \mathbf{q}) \Phi d\mathbf{v} + 2\mathbf{q} \cdot \int \mathbf{q} \Phi d\mathbf{v} - q^2 \\ &= \frac{3kT(\mathbf{x}, t)}{m} + 2q^2 \int \Phi d\mathbf{v} - q^2 \\ &= \frac{1}{n(\mathbf{x}, t)} \int (\mathbf{v} - \mathbf{q})^2 f d\mathbf{v} + q^2. \end{aligned} \quad (93)$$

In other words:

$$\int v^2 \Phi d\mathbf{v} = \frac{1}{n(\mathbf{x}, t)} \int v^2 f d\mathbf{v}. \quad (94)$$

Therefore, I can easily show that model (84) conserves energy as well:

$$\int \frac{1}{2} m v^2 \left(\frac{\partial f}{\partial t} \right)_{\text{coll}} d\mathbf{v} = -\frac{m}{2\tau} \frac{n(\mathbf{x}, t)}{n_0} \int v^2 f d\mathbf{v} + \frac{m}{2\tau} \frac{n(\mathbf{x}, t)^2}{n_0} \int v^2 \Phi d\mathbf{v} = 0. \quad (95)$$

3.2 Fokker-Planck Model

The kinetic theory of gases is described by the Boltzmann equation. The collision integral in this equation describes large angle scattering due to close encounters. For Coulomb interactions in a plasma the slow fall off of the interaction causes the Boltzmann collision integral to diverge. This is an indication that distant encounters (small-angle scattering) are important. However, we know that the Coulomb interaction is screened so that the integral is cut off and does not diverge. Nevertheless, many small-angle collisions can accumulate to produce large-angle deflections and will therefore be important.

Paper [1] uses a physical argument to derive the Fokker-Planck collision operator. In this approach, we make direct use of the fact that changes in the particle velocity distribution are caused by small angle scatterings. Let $F(\mathbf{v}, \Delta\mathbf{v})$ be the probability that the particle velocity changes from \mathbf{v} to $\Delta\mathbf{v}$ in a time interval Δt due to collisions. The distribution at time t can therefore be expressed in terms of the distribution at the slightly earlier time $t - \Delta t$ as follows:

$$f(\mathbf{v}, t) = \int f(\mathbf{v} - \Delta\mathbf{v}, t - \Delta t) F(\mathbf{v} - \Delta\mathbf{v}, \Delta\mathbf{v}) d(\Delta\mathbf{v}). \quad (96)$$

The idea here is to expand the integrand in series up to the second order in velocity and first order in time. Due to the long-range nature of the Coulomb force most of the collisions yield small angle scatterings, so it is allowed to consider $\Delta\mathbf{v}$ and Δt as infinitesimal quantities to perform the expansion.

$$\begin{aligned} f(\mathbf{v}, t) &= \int \{ f(\mathbf{v}, t) F(\mathbf{v}, \Delta\mathbf{v}) + \\ &\quad - \Delta t F(\mathbf{v}, \Delta\mathbf{v}) \left(\frac{\partial f}{\partial t} \right)_{\text{coll}} + \\ &\quad - \Delta\mathbf{v} \cdot \frac{\partial}{\partial \mathbf{v}} [f(\mathbf{v}, t) F(\mathbf{v}, \Delta\mathbf{v})] + \sum_{i,j} \frac{1}{2} \Delta v_i^2 \Delta v_j^2 \frac{\partial^2}{\partial v_i \partial v_j} [f(\mathbf{v}, t) F(\mathbf{v}, \Delta\mathbf{v})] \} d(\Delta\mathbf{v}) \quad (97) \\ &= \{ f(\mathbf{v}, t) - \Delta t \left(\frac{\partial f}{\partial t} \right)_{\text{coll}} \} \int F(\mathbf{v}, \Delta\mathbf{v}) d(\Delta\mathbf{v}) + \\ &\quad + \int \{ -\Delta\mathbf{v} \cdot \frac{\partial}{\partial \mathbf{v}} [f F] + \sum_{i,j} \frac{1}{2} \Delta v_i^2 \Delta v_j^2 \frac{\partial^2}{\partial v_i \partial v_j} [f F] \} d(\Delta\mathbf{v}). \end{aligned}$$

Being $F(\mathbf{v}, \Delta\mathbf{v})$ a probability distribution function, it is normalized so:

$$\int F(\mathbf{v}, \Delta\mathbf{v}) d(\Delta\mathbf{v}) = 1. \quad (98)$$

By substituting (98) into (97), the following expression for the collision term is obtained:

$$\left(\frac{\partial f}{\partial t} \right)_{\text{coll}} = \frac{1}{\Delta t} \int \{ -\Delta\mathbf{v} \cdot \frac{\partial}{\partial \mathbf{v}} [f F] + \sum_{i,j} \frac{1}{2} \Delta v_i^2 \Delta v_j^2 \frac{\partial^2}{\partial v_i \partial v_j} [f F] \} d(\Delta\mathbf{v}). \quad (99)$$

We now write this as:

$$\left(\frac{\partial f}{\partial t} \right)_{\text{coll}} = -\frac{\partial}{\partial v_i} \left\{ \frac{\langle \Delta v_i \rangle}{\Delta t} f \right\} + \frac{1}{2} \frac{\partial^2}{\partial v_i \partial v_j} \left\{ \frac{\langle \Delta v_i \Delta v_j \rangle}{\Delta t} f \right\} \quad (100)$$

where $\langle \Delta v_i \rangle$ and $\langle \Delta v_i \Delta v_j \rangle$ are the average values of these quantities over the time interval Δt and are defined as:

$$\langle \Delta v_i \rangle = \int F(\mathbf{v}, \Delta \mathbf{v}) \Delta \mathbf{v} d(\Delta \mathbf{v}) \quad (101)$$

$$\langle \Delta v_i \Delta v_j \rangle = \int F(\mathbf{v}, \Delta \mathbf{v}) \Delta v_i \Delta v_j d(\Delta \mathbf{v}). \quad (102)$$

Remark that since (99) the Einstein notation has been adopted: to write summations in compact form the symbol " \sum " is omitted and a repeated index automatically denotes a summation.

Finally, we obtain:

$$\left(\frac{\partial f}{\partial t} \right)_{\text{coll}} = -\frac{\partial}{\partial v_i} (A_i f) + \frac{1}{2} \frac{\partial}{\partial v_i} (D_{ij} \frac{\partial f}{\partial v_j}) \quad (103)$$

where:

$$\begin{cases} A_i = \frac{\langle \Delta v_i \rangle}{\Delta t} - \frac{1}{2} \frac{\partial}{\partial v_j} \frac{\langle \Delta v_i \Delta v_j \rangle}{\Delta t} \\ D_{ij} = \frac{\langle \Delta v_i \Delta v_j \rangle}{\Delta t}. \end{cases} \quad (104)$$

Equation (103) is the Fokker-Planck equation. the first term has the character of a dynamical friction which decreases or increases the population of a group of particles depending on whether they are moving faster or slower than the corresponding equilibrium particles. The second term acts like a diffusion in velocity space and has the effect of spreading a velocity perturbation. The equilibrium Maxwell distribution results from a balance between these two terms.

We also note that the right hand side of the Fokker-Planck equation is the divergence in velocity space $-\frac{\partial s_i}{\partial v_i}$ of the vector

$$s_i = A_i f - \frac{1}{2} D_{ij} \frac{\partial f}{\partial v_j} \quad (105)$$

where s_i is the particle flux in velocity space. The Fokker-Planck equation is therefore an equation of continuity thus guaranteeing the conservation of the number of particles. In addition the quantities A_i and D_{ij} are connected by the equilibrium condition that the particle flux in velocity space must be zero.

In fact, a system in thermodynamic equilibrium satisfies $(\partial f / \partial t)_{\text{coll}} = 0$, which translates into $s_i = 0 \forall i$ in case of Fokker-Planck collision model. If the system is not at equilibrium the velocity of a group of particles may change or the velocity distribution may spread. The former phenomenon is related with unbalance due to the first term in (105) while the latter is due to the second, diffusion term ([5]).

In order to make practical calculations with the Fokker-Planck equation we need explicit forms for the quantities A_i and D_{ij} . To complete the discussion of the Fokker-Planck equation we will quote the result for these three terms.

The coefficients in the Fokker-Planck equation are obtained by considering the details of binary collisions of particles of species a with all other particles of the plasma. I found the full computations in [5] (Sections 6.2,6.3). It is a quite long and not trivial treatment, it refers to a multi-population, fully ionized plasma. The result is:

$$\left(\frac{\partial f}{\partial t} \right)_{\text{coll}} = - \sum_{a'} \frac{q_a^2 q_{a'}^2 \log \Lambda}{8\pi \epsilon_0^2 m_a} \frac{\partial}{\partial v_i} \int \left(\frac{\delta_{ij}}{|\mathbf{u}|} - \frac{u_i u_j}{|\mathbf{u}|^3} \right) \left\{ \frac{f_a}{m_{a'}} (\mathbf{v}) \frac{\partial f_{a'}}{\partial v'_j} (\mathbf{v}') - \frac{f_{a'}}{m_a} (\mathbf{v}') \frac{\partial f_a}{\partial v_j} (\mathbf{v}) \right\} d\mathbf{v} \quad (106)$$

where $\mathbf{u} = \mathbf{v} - \mathbf{v}'$, $(\frac{\delta_{ij}}{|\mathbf{u}|} - \frac{u_i u_j}{|\mathbf{u}|^3})$ comes from an average over scattering angles and $\log \Lambda$ is the factor which arises from cutting off the Coulomb interaction at small angles due to the effect of Debye screening. Note that the friction and diffusion coefficients depend on the distribution function with the result that the equation is nonlinear. The quadratic dependence on f for like particle collisions is shown explicitly.

The Fokker-Planck equation is often written in another form:

$$\left(\frac{\partial f}{\partial t} \right)_{\text{coll}} = - \sum_{a'} \frac{q_a^2 q_{a'}^2 \log \Lambda}{4\pi \epsilon_0^2 m_a^2} \frac{\partial}{\partial v_i} \left[f_a(\mathbf{v}) \frac{\partial}{\partial v_j} h(\mathbf{v}) - \frac{1}{2} \frac{\partial}{\partial v_j} (f_a(\mathbf{v}) \frac{\partial^2 g}{\partial v_i \partial v_j}) \right] \quad (107)$$

where $h(\mathbf{v})$ and $g(\mathbf{v})$ are the Rosenbluth potential

$$g(\mathbf{v}) = \int f_{a'}(\mathbf{v}') |\mathbf{u}| d\mathbf{v}' \quad (108)$$

$$h(\mathbf{v}) = \left(1 + \frac{m_a}{m_{a'}}\right) \int f_{a'}(\mathbf{v}') \frac{1}{|\mathbf{u}|} d\mathbf{v}'. \quad (109)$$

For many problems the distribution can be assumed to be close to thermal equilibrium so that the collision term can be linearized about a Maxwellian. Finally, when all scattering processes are included the Fokker-Planck equation satisfies conservation of number density, momentum and energy.

Here it has been considered the application of the Fokker-Planck equation to a fully ionized plasma, but in general (103) is valid for any system in which collisions produce small changes in the particles velocity ([5]).

In section 2 I remarked how the evolution equation for the one-particle distribution function (22) can be seen as an expansion in the plasma parameter g . In particular, the Vlasov equation is the zero-order truncation. An alternative derivation of the Fokker-Planck equation derives from the inclusion in the equation of terms up to order $\sim g$.

4 Fluid Models

In many situations a simpler description than the full kinetic model is desirable, either because the kinetic treatment would be too complicated or because the simpler model contains all the essential physics. In the final part of this lecture we will show how these simpler, fluid models may be obtained from the kinetic equation.

If the distribution function $f_a(\mathbf{x}, \mathbf{v}, t)$ is known, then the various physically significant quantities can be obtained as follows:

$$n_a(\mathbf{x}, t) = \int f_a(\mathbf{x}, \mathbf{v}, t) d\mathbf{v}. \quad (110)$$

Where the subscript "a" indicates the plasma population under consideration⁸.

Roughly speaking the idea of this section is to average kinetic model's quantities, which lie over the phase-space, over the velocity space in order to obtain a macroscopic, fluid description of the plasma in the physical space. Here, n_a is the number density of the a -esim plasma population in the physical space.

Let \mathbf{u}_a be the velocity of the population a defined as:

$$\mathbf{u}_a = \frac{1}{n_a} \int \mathbf{v} f_a d\mathbf{v}. \quad (111)$$

It follows that I can define, for every single particle of the population a , the random velocity ξ as the relative velocity of the particle with respect to the population velocity

$$\xi = \mathbf{v} - \mathbf{u}_a. \quad (112)$$

Notice that ξ represents a microscopic quantity, so if averaged over the whole velocity space it disappears:

$$\begin{aligned} \int \xi f_a d\mathbf{v} &= \int (\mathbf{v} - \mathbf{u}_a) f_a d\mathbf{v} \\ &= \int \mathbf{v} f_a d\mathbf{v} - \mathbf{u}_a \int f_a d\mathbf{v} \\ &= \int \mathbf{v} f_a d\mathbf{v} - n_a \mathbf{u}_a = 0. \end{aligned} \quad (113)$$

Let m_a be the particle mass of population a , which is clearly a constant value in both space and time. I can easily extract from (111) the momentum density $m_a n_a \mathbf{u}_a = m_a \int \mathbf{v} f_a d\mathbf{v}$ which is, component-by-component:

$$m_a n_a u_{a,i} = m_a \int v_i f_a d\mathbf{v}. \quad (114)$$

Now, I consider the flow of the i -th component of the momentum density (114) through direction j :

$$\begin{aligned} m_a \int v_i v_j f_a d\mathbf{v} &= m_a \int (u_{a,i} + \xi_i)(u_{a,j} + \xi_j) f_a d\mathbf{v} \\ &= m_a u_{a,i} u_{a,j} \int f d\mathbf{v} + m_a u_{a,i} \int \xi_j f_a d\mathbf{v} + m_a u_{a,j} \int \xi_i f_a d\mathbf{v} + m_a \int \xi_i \xi_j d\mathbf{v} \\ &= m_a n_a u_{a,i} u_{a,j} + m_a \int \xi_i \xi_j d\mathbf{v}. \end{aligned} \quad (115)$$

The second term of (115) has no macroscopic explanation. Since dimensionally it is a force, we define after this component the stress tensor for each population $\underline{\underline{P}}_a$

$$(\underline{\underline{P}}_a)_{ij} = m_a \int \xi_i \xi_j d\mathbf{v}. \quad (116)$$

⁸Actually paper [1] uses index "j" for the plasma populations, while for tensor indexes uses "n" and "m". However, in the previous sections it adopted "i" and "k". To improve clarity I simply adopted "a" for the plasma populations and usual "i" and "j" for indexes. This notation was maintained for the whole work in order to avoid ambiguity.

In vector form I can write down (115) as:

$$m_a \int \mathbf{v} \mathbf{v} f_a d\mathbf{v} = m_a n_a \mathbf{u}_a \mathbf{u}_a + \underline{\underline{P}}_a. \quad (117)$$

After the introduction of these quantities I am ready to proceed with the derivation of the momentum balance equation.

First, remark that from statistical mechanics it is well known that the collision operator of the Boltzmann equation Q verifies:

$$\int Q(f, f) \psi(\mathbf{v}) d\mathbf{v} = 0 \quad (118)$$

if $\psi(\mathbf{v}) = 1, v_1, v_2, v_3, v^2$ (which are the elementary collisional invariants) or any linear combination of them. Such functions $\psi(\mathbf{v})$ are called collisional invariants.

Since in this section I am basically averaging quantities over the velocity space, an idea for the derivation of the momentum balance equation would be to integrate over the velocity space the kinetic equation multiplied times $m_a \mathbf{v}$. Being every component of \mathbf{v} a collisional invariant, it is clear that the collision term would disappear. Hence, without loss of generality, I can directly start from the Vlasov equation (42), the collisionless case, to derive a momentum balance equation.

Before proceeding in this way, I first integrate the Vlasov equation (42) as it is over the velocity space. Remark indeed that also $\psi = 1$ is an (elementary) collisional invariant.

$$\int \frac{\partial f_a}{\partial t} d\mathbf{v} + \int \mathbf{v} \cdot \frac{\partial f_a}{\partial \mathbf{x}} d\mathbf{v} + \int \frac{q_a}{m_a} [\mathbf{E}(\mathbf{x}, t) + \mathbf{v} \times \mathbf{B}(\mathbf{x}, t)] \cdot \frac{\partial f_a}{\partial \mathbf{v}} d\mathbf{v} = 0. \quad (119)$$

The first two terms can be easily developed since integration and derivation are performed with respect to different variables:

$$\begin{aligned} \int \frac{\partial f_a}{\partial t} d\mathbf{v} + \int \mathbf{v} \cdot \frac{\partial f_a}{\partial \mathbf{x}} d\mathbf{v} &= \frac{\partial}{\partial t} \int f_a d\mathbf{v} + \frac{\partial}{\partial \mathbf{x}} \cdot \int \mathbf{v} f_a d\mathbf{v} \\ &= \frac{\partial}{\partial t} n(\mathbf{x}, t) + \frac{\partial}{\partial \mathbf{x}} \cdot (n_a \mathbf{u}_a). \end{aligned} \quad (120)$$

About the last integral, I can observe that it is null. The reasoning is the exact same one as for the proof of the H-Theorem (see equation (64)). The only difference with (64) is simply that here there is " f_a " in place of " $f \log f$ ", but since both vanish at infinity the procedure is still valid. I have derived from equation (119) a continuity equation for the plasma population density:

$$\frac{\partial}{\partial t} n_a + \frac{\partial}{\partial \mathbf{x}} \cdot (n_a \mathbf{u}_a) = 0. \quad (121)$$

Now I look forward to the momentum balance equation by integrating the Vlasov equation multiplied times " $m_a \mathbf{v}$ "

$$m_a \int \mathbf{v} \frac{\partial f_a}{\partial t} d\mathbf{v} + m_a \int \mathbf{v} \mathbf{v} \cdot \frac{\partial f_a}{\partial \mathbf{x}} d\mathbf{v} + q_a \int \mathbf{v} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_a}{\partial \mathbf{v}} d\mathbf{v} = 0. \quad (122)$$

Again I can treat the first two integrals as I did for the continuity equation:

$$m_a \int \mathbf{v} \frac{\partial f_a}{\partial t} d\mathbf{v} + m_a \int \mathbf{v} \mathbf{v} \cdot \frac{\partial f_a}{\partial \mathbf{x}} d\mathbf{v} = \frac{\partial}{\partial t} (m_a n_a \mathbf{u}_a) + \frac{\partial}{\partial \mathbf{x}} \cdot m_a \int \mathbf{v} \mathbf{v} f_a d\mathbf{v}. \quad (123)$$

I notice that the second term is the gradient of the momentum flux, which expression is given by (117). I develop it component-by-component by taking the derivative of (115) over the direction x_j :

$$\frac{\partial}{\partial x_j} m_a \int v_i v_j f_a d\mathbf{v} = m_a u_{a,i} \frac{\partial}{\partial x_j} (n_a u_{a,j}) + n_a u_{a,j} \frac{\partial}{\partial x_j} (m_a u_{a,i}) + \frac{\partial}{\partial x_j} (\underline{\underline{P}}_a)_{ij} \quad (124)$$

which in vector form reads

$$m_a \frac{\partial}{\partial \mathbf{x}} \int \mathbf{v} \mathbf{v} f_a d\mathbf{v} = m_a \mathbf{u}_a \frac{\partial}{\partial \mathbf{x}} \cdot (n_a \mathbf{u}_a) + n_a \mathbf{u}_a \frac{\partial}{\partial \mathbf{x}} \cdot (m_a \mathbf{u}_a) + \frac{\partial}{\partial \mathbf{x}} \cdot \underline{\underline{P}}_a. \quad (125)$$

Therefore, (123) reads:

$$\begin{aligned}
m_a \int \mathbf{v} \frac{\partial f_a}{\partial t} d\mathbf{v} + m_a \int \mathbf{v} \mathbf{v} \cdot \frac{\partial f_a}{\partial \mathbf{x}} d\mathbf{v} &= \\
&= m_a \frac{\partial}{\partial t} (n_a \mathbf{u}_a) + m_a \mathbf{u}_a \frac{\partial}{\partial \mathbf{x}} \cdot (n_a \mathbf{u}_a) + m_a n_a \mathbf{u}_a \frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{u}_a + \frac{\partial}{\partial \mathbf{x}} \cdot \underline{\underline{P}_a} \\
&= m_a n_a \frac{\partial \mathbf{u}_a}{\partial t} + m_a \mathbf{u}_a \frac{\partial n_a}{\partial t} + m_a \mathbf{u}_a \frac{\partial}{\partial \mathbf{x}} \cdot (n_a \mathbf{u}_a) + m_a n_a \mathbf{u}_a \frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{u}_a + \frac{\partial}{\partial \mathbf{x}} \cdot \underline{\underline{P}_a} \quad (126) \\
&= m_a n_a \frac{\partial \mathbf{u}_a}{\partial t} + m_a n_a (\mathbf{u}_a \cdot \frac{\partial}{\partial \mathbf{x}}) \mathbf{u}_a + m_a \mathbf{u}_a [\frac{\partial}{\partial t} n_a + \frac{\partial}{\partial \mathbf{x}} \cdot (n_a \mathbf{u}_a)] + \frac{\partial}{\partial \mathbf{x}} \cdot \underline{\underline{P}_a} \\
&= m_a n_a \frac{\partial \mathbf{u}_a}{\partial t} + m_a n_a (\mathbf{u}_a \cdot \frac{\partial}{\partial \mathbf{x}}) \mathbf{u}_a + \frac{\partial}{\partial \mathbf{x}} \cdot \underline{\underline{P}_a}
\end{aligned}$$

where the term with the brackets "[...]" vanishes thanks to the continuity equation (121).

Finally, I have to deal with the last integral of equation (122). I start with the electric field contribution, which I write down component-by-component and integrate by parts:

$$\begin{aligned}
\int v_i E_j \frac{\partial f_a}{\partial v_j} d\mathbf{v} &= - \int E_j f_a \frac{\partial v_i}{\partial v_j} d\mathbf{v} \\
&= - \int E_j f_a \delta_{ij} d\mathbf{v} \\
&= -E_i n_a
\end{aligned} \quad (127)$$

Similarly I can treat the magnetic field contribution. Remark that $(\partial/\partial v_j)(\mathbf{v} \times \mathbf{B})_j = 0$ by the *circular shift* property of the triple product.

$$\begin{aligned}
\int v_i (\mathbf{v} \times \mathbf{B})_j \frac{\partial f_a}{\partial v_j} d\mathbf{v} &= - \int f_a \frac{\partial}{\partial v_j} [v_i (\mathbf{v} \times \mathbf{B})_j] d\mathbf{v} \\
&= - \int f_a \frac{\partial v_i}{\partial v_j} (\mathbf{v} \times \mathbf{B})_j d\mathbf{v} - \int f_a \frac{\partial (\mathbf{v} \times \mathbf{B})_j}{\partial v_j} v_i d\mathbf{v} \\
&= - \int f_a \frac{\partial v_i}{\partial v_j} (\mathbf{v} \times \mathbf{B})_j d\mathbf{v} \\
&= - \int f_a \delta_{ij} \epsilon_{ijk} v_i B_k d\mathbf{v} \\
&= -\epsilon_{ijk} \left(\int f_a v_j d\mathbf{v} \right) B_k \\
&= -\epsilon_{ijk} u_{a,j} B_k \\
&= -(\mathbf{u}_a \times \mathbf{B})_i
\end{aligned} \quad (128)$$

where ϵ_{ijk} is the Levi-Civita object used to express the vector product between two field component-wise.

I put every result together (in vector form) to write down equation (122) as:

$$m_a n_a \frac{\partial \mathbf{u}_a}{\partial t} + m_a n_a (\mathbf{u}_a \cdot \frac{\partial}{\partial \mathbf{x}}) \mathbf{u}_a + \frac{\partial}{\partial \mathbf{x}} \cdot \underline{\underline{P}_a} - q_a n_a \mathbf{E} - q_a n_a (\mathbf{u}_a \times \mathbf{B}) = 0. \quad (129)$$

By moving the force terms to the right hand side and by dividing by $m_a n_a$, I obtain the momentum conservation equation in its usual form:

$$\frac{\partial}{\partial t} \mathbf{u}_a + (\mathbf{u}_a \cdot \frac{\partial}{\partial \mathbf{x}}) \mathbf{u}_a + \frac{1}{n_a m_a} \frac{\partial}{\partial \mathbf{x}} \cdot \underline{\underline{P}_a} = \frac{q_a}{m_a} (\mathbf{E} + \mathbf{u}_a \times \mathbf{B}). \quad (130)$$

Notice that the equation for each moment contains a term involving the next higher moment - the closure problem. Thus, the equation for n_a contains \mathbf{u}_a and the equation for \mathbf{u}_a contains $\underline{\underline{P}_a}$. As already mentioned there is no rigorous method of closure for this set of equations. Instead some physical justification is sought for terminating the hierarchy.

The next equation in the series is the energy equation obtained by taking the moment of $mv^2/2$. This is usually as far as one goes with the moment method and we shall not write down the general energy equation. Instead we shall briefly consider the various physical approximation which are used in a fluid description.

The simplest model is the COLD-plasma approximation, in which the term $\text{div}(\underline{\underline{P}}_a)$ is simply neglected thus closing the hierarchy of equations after the zeroth and first momenta. The plasma is thus described as two cold, interpenetrating fluids which may be expected to be a good approximation in a variety of wave propagation problems provided the phase velocity of the wave is much greater than the thermal velocities of the particles. The two fluid approximation is often extended to the case of two warm fluids in which the pressure tensor is now included but assumed to be diagonal, i.e. the pressure is isotropic. Thus

$$\underline{\underline{P}}_a = p_a \underline{\underline{I}}. \quad (131)$$

The final fluid approximation we would like to mention is the ONE-FLUID or magnetohydrodynamic (MHD) model. This is appropriate under the conditions where the electrons and ions move together maintaining charge neutrality. This occurs at low frequencies, i.e. $\omega \ll \Omega_a$, and long wavelengths $\lambda \ll v_{T,a}/\Omega_a$.

The cyclotron frequency (or gyrofrequency) Ω_a is the frequency at which a plasma particle rotates in its helical motion around the magnetic field lines. In fact, in a zero-order approximation plasma particles follow the magnetic field lines in their motion⁹. This fact is essential for several applications because the plasma can be confined by imposing closed magnetic field lines. However, the actual trajectory of the particle is a helix and Ω_a is the frequency of rotation around the guiding center *hosted* by \mathbf{B} . The ratio $v_{T,a}/\Omega_a$ is the Larmor radius and represents the radius of the helix.

Therefore, since λ and ω are taken as characteristic length and frequency for the plasma, the situation in which $\omega \ll \Omega_a$ and $\lambda \ll v_{T,a}/\Omega_a$ makes the magnetic field line alone a good approximation for the particles trajectory, as the helical motion around it is very fast and with a very small radius. Adding together the continuity equations for the electrons and ions we obtain the equation of continuity for mass flow:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \mathbf{x}} \cdot (\rho \mathbf{u}_0) = 0 \quad (132)$$

where

$$\rho = \sum_a n_a m_a \quad (133)$$

$$\mathbf{u}_0 = \frac{1}{\rho} \sum_a n_a m_a \mathbf{u}_a. \quad (134)$$

Adding the momentum equations gives the equation for the center of mass motion

$$\frac{\partial \mathbf{u}_0}{\partial t} + (\mathbf{u}_0 \cdot \frac{\partial}{\partial \mathbf{x}}) \mathbf{u}_0 = - \frac{\partial}{\partial \mathbf{x}} \cdot \underline{\underline{P}} + \mathbf{J} \times \mathbf{B} \quad (135)$$

where $\underline{\underline{P}} = \underline{\underline{P}}_i + \underline{\underline{P}}_e$ refers to the centre-of-mass velocity as is appropriate to a one-fluid description. The pressure term is usually assumed to be diagonal and \mathbf{J} is the electric current given by

$$\mathbf{J} = \sum_a n_a q_a \mathbf{u}_a. \quad (136)$$

Equation (135) is reminiscent of the Navier-Stokes equation of fluid mechanics without the viscous force but containing the body force $\mathbf{J} \times \mathbf{B}$ due to the fluid being magnetized. It is typically defined a *magnetic Reynolds number*, which is named this way because of the analogy with the Navier-Stokes equations.

⁹This does not mean that the particle always keeps the same direction. There are some cases in which the particle is mirrored and its trajectory is confined, resulting in the so-called *banana orbits*.

The one-fluid model is usually closed by making the adiabatic approximation which assumes a local Maxwellian so that:

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_0 \cdot \frac{\partial}{\partial \mathbf{x}} \right) \left(\frac{p}{\rho^\gamma} \right) = 0 \quad (137)$$

where γ is the ratio of specific heats. Equation (137) results from the energy equation by neglecting the heat flow and assuming perfect conductivity, i.e.

$$\mathbf{E} + \mathbf{u}_0 \times \mathbf{B} = 0. \quad (138)$$

In general, there are books and manuals just about the MHD model as it has important applications in several fields that involve plasma physics. The author here just provides a slim introduction to the MHD model, while the focus of [1] was more about the Vlasov equation and the Kinetic model. For this reason I considered beyond the scope of this work to go into much detail with the MHD equations.

A Appendix: Hamiltonian Mechanics

In this appendix it will be briefly shown how to derive Hamilton's equations starting from the Lagrangian of a mechanical system. Then, since they are used by the author of [1] in section 2, *Poisson brackets* and Liouville's theorem are introduced. The reference book for this appendix is [2].

The Hamiltonian formulation of the laws of mechanics introduces several advantages with respect to the Lagrangian formulation. Indeed, the first allows to change system of co-ordinates very easily. This agility makes convenient its adoption in the study of general problems using general co-ordinates and momenta. Moreover, the explicit dependence on momenta yields naturally the adoption of the phase-space, as it is done in kinetic problems (and also in [1]).

The *Hamilton's principle* states that every mechanical system is characterized by a definite function of space, velocity and time: $L(\mathbf{q}, \dot{\mathbf{q}}, t)$. Such function is called the *Lagrangian* of the system.

The motion of the system between two generic instants t_1 and t_2 always minimizes the *action* S :

$$S(\mathbf{q}, \dot{\mathbf{q}}) = \int_{t_1}^{t_2} L(\mathbf{q}, \dot{\mathbf{q}}, t) dt. \quad (139)$$

In particular, for a system of N particles, the Lagrangian is the difference between the kinetic energy and the potential energy of the system. In cartesian co-ordinates:

$$L = \sum_{a=1}^N \frac{1}{2} m_a v_a^2 - U(\mathbf{x}_1, \dots, \mathbf{x}_N) \quad (140)$$

where, for the a -esim particle, $v_a^2 = \dot{\mathbf{x}}_a \cdot \dot{\mathbf{x}}_a$.

Consider a general closed system with s degrees of freedom. By virtue of *homogeneity of time*, the Lagrangian does not depend explicitly on time. This property yields the definition of a conservation law of the kind:

$$\frac{d}{dt} E = 0 \quad (141)$$

where the conserved quantity is named *energy* of the system:

$$E := \sum_{i=1}^s \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L. \quad (142)$$

Another quantity of interest, which is itself conserved in case of *homogeneity in space*, is the *momentum* \mathbf{p}

$$\mathbf{p} := \sum_{i=1}^s \frac{\partial L}{\partial \dot{q}_i}. \quad (143)$$

Again, in the case of a closed system of N particles, using cartesian co-ordinates, these *integrals of the motions* are expressed in the form:

$$\begin{aligned} E &= \sum_{a=1}^N \frac{1}{2} m_a v_a^2 + U(\mathbf{x}_1, \dots, \mathbf{x}_N); \\ \mathbf{p} &= \sum_{a=1}^N m_a \mathbf{v}_a. \end{aligned} \quad (144)$$

The introduction of the *phase-space* makes the momentum \mathbf{p} an independent variable besides the position \mathbf{q} ¹⁰. Therefore, the mechanical system with s d.o.f is described in the $6s$ -dimensional phase-space. Time t is considered as a parameter.

¹⁰Arbitrarily, one could adopt the velocities as independent variables instead of momenta, as it is done in [6]. However, the difference does not really matter as it just involves a simple change of variables. Here, momenta are used to be coherent with the choice of [1]

In the Lagrangian definition, the dependence on the velocity of the particle $\dot{\mathbf{q}}$ was explicit, so it is basically effortless the formulation in phase-space:

$$L = L(\mathbf{q}, \mathbf{p}, t). \quad (145)$$

The energy of the system written in terms of the independent variables of the phase space takes the name of *Hamiltonian* of the system:

$$H(\mathbf{q}, \mathbf{p}, t) := \sum_{i=1}^s p_i \dot{q}_i - L(\mathbf{q}, \mathbf{p}, t) \quad (146)$$

The knowledge of this function in the phase-space allows to derive a simple system of $2s$ differential equations of motion for the mechanical system:

$$\begin{cases} \dot{q}_i = \frac{\partial H}{\partial p_i} & i = 1, \dots, s; \\ \dot{p}_i = -\frac{\partial H}{\partial q_i} & i = 1, \dots, s. \end{cases} \quad (147)$$

(147) are called *canonical equations*, or *Hamilton's equations*. Furthermore, one can easily show that:

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} \quad (148)$$

This means that, if the Hamiltonian does not depend explicitly on time, energy is conserved. Since the Hamiltonian (1) does not depend on explicitly time, from now on it will be considered $H = H(\mathbf{q}, \mathbf{p})$. The Hamiltonian formulation of a mechanical system makes easy to perform changes of variables. In fact, Hamiltonian equations are basically *immune* to a certain class of changes of variables. To introduce this class, the definition of *Poisson brackets* is needed.

Consider two functions of the co-ordinates $f = f(\mathbf{p}, \mathbf{q}, t)$ and $g = g(\mathbf{p}, \mathbf{q}, t)$. The Poisson brackets operator is indicated with $\{f, g\}$ ¹¹ and it is defined as follows:

$$\{f, g\} := \sum_{i=1}^s \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right). \quad (149)$$

Poisson brackets are equipped with several useful properties such as the *Poisson's theorem*, which states that if f, g are two integrals of the motion, than also $\{f, g\}$ is an integral of the motion.

Consider now a generic change of variables $Q_i = Q_i(q, p, t)$ and $P_i = P_i(q, p, t)$. Such transformation is said to be *canonical* if its application leaves the Hamilton's equation form unchanged, i.e.

$$\begin{cases} \dot{Q}_i = \frac{\partial H'}{\partial P_i} & i = 1, \dots, s; \\ \dot{P}_i = -\frac{\partial H'}{\partial Q_i} & i = 1, \dots, s, \end{cases} \quad (150)$$

where $H' = H'(\mathbf{P}, \mathbf{Q})$ is some Hamiltonian.

It is possible to prove that a change of variables is canonical if and only if $\{Q_i, Q_k\} = 0$, $\{P_i, P_k\} = 0$, $\{Q_i, P_k\} = \delta_{ik}$.¹²

To conclude this appendix, the *Liouville's theorem* is introduced. This result guarantees that the volume of a region in the phase-space is invariant with respect to a canonical transformation.

This theorem is crucial in statistical mechanics, where the transformation involved is the mapping induced by the motion of a system of N particles:

$$\mathbf{z} = (\mathbf{q}, \mathbf{p}) \mapsto \mathbf{Z} = \dot{\mathbf{z}} = (\mathbf{p}, \mathbf{F}).$$

where $\mathbf{F} = (\mathbf{F}_1, \dots, \mathbf{F}_N)$, with \mathbf{F}_i being the force acting on the i^{th} particle.

Eventually, this is a canonical transformation under the assumption

$$\text{div}(\mathbf{Z}) = 0.$$

¹¹[2] uses $[f, g]$ as notation, while paper [1] uses $\{f, g\}$. Here, as it has been done so far, the same notation of [1] is adopted.

¹²Here the Poisson brackets operator is applied with the respect to the *old* variables \mathbf{p}, \mathbf{q} .

This assumption is satisfied if and only if the force acting on each particle depends only on the position of the particle, and not on its momentum. Indeed, in the derivation of the Liouville's theorem, [6] explains that *we shall always consider velocity-independent forces*.

In particular, under this assumption, in the context of statistical mechanics the Liouville's theorem guarantees that the so-called *Liouville's equation* holds:

$$\frac{\partial P}{\partial t} + \sum_{i=1}^N \mathbf{p}_i \cdot \frac{\partial P}{\partial \mathbf{q}_i} + \sum_{i=1}^N \mathbf{q}_i \cdot \frac{\partial P}{\partial \mathbf{p}_i} = 0. \quad (151)$$

where P is the probability density function.

By exploiting the canonical equations (147) and the Poisson brackets (149), the Liouville equation (151) can be written down in compact form as follows:

$$\frac{\partial P}{\partial t} + \{P, H\} = 0. \quad (152)$$

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