

# Stochastic Simulations

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## Project 4

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### QMC integration of non-smooth functions: application to pricing exotic options

## 1 Introduction and background

In high-dimensional integration, the goal is to approximate a multiple integral over an  $m$ -dimensional ( $m \gg 1$ ) domain, e.g., the unit cube  $[0, 1]^m$ :

$$I_m(\phi) = \int_{[0,1]^m} f(x) dx = \int_0^1 \cdots \int_0^1 f(x_1, \dots, x_m) dx_1 \dots dx_m, \quad (1)$$

or, more generally, over  $\mathbb{R}^m$  with respect to some distribution  $\rho := \rho_1(x_1)\rho_2(x_2)\dots\rho_m(x_m)$ , i.e.,

$$I_m(\phi) = \int_{\mathbb{R}^m} f(x) \rho(x) dx = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \dots, x_m) \rho_1(x_1) \dots \rho_m(x_m) dx_1 \dots dx_m. \quad (2)$$

By now, quasi-Monte Carlo (QMC) has become a popular way to approximate the high-dimensional integration problem (1). Recall (c.f. lecture notes) that the approximation error of the quasi-Monte Carlo method can be controlled by the product of the star-discrepancy of the point set  $P_N = \{x_1, \dots, x_N\}$  and a term measuring the *smoothness* of the integrand. More precisely, the Koksma-Hlawka inequality states that the error can be bounded as

$$\left| \int_{[0,1]^m} f(x) dx - \frac{1}{N} \sum_{i=1}^N f(x_i) \right| \leq \underbrace{\left( \sum_{\mathbf{k} \subset \{1, \dots, s\}} \int_{[0,1]^{|\mathbf{k}|}} \left| \frac{\partial^{|\mathbf{k}|}}{\partial x_{\mathbf{k}}} f(x_{\mathbf{k}}, \mathbf{1}) \right| dx_{\mathbf{k}} \right)}_{:= V_{HK}} \mathcal{D}^*(P_N), \quad (3)$$

where  $x_{\mathbf{k}} = (x_{k_1}, x_{k_2}, \dots, x_{k_p})$ , if  $|\mathbf{k}| = p \leq s$ ,  $\mathcal{D}^*(P_N)$  is the so-called star-discrepancy of the point set  $P_N$  and  $V_{HK}$  is the total variation norm in the sense of Hardy and Krause. Notice that for (3) to be a meaningful error estimator,  $f$  needs to have integrable mixed first order derivatives.

In this mini-project we implement a method proposed in [1] for numerically computing integrals of the form (2) using QMC in the presence of “kinks” (i.e. discontinuities in the gradients) or “jumps” (i.e. discontinuities in the function) on the integrand  $f(x)$ , where those kinks and jumps are not aligned to the coordinate axis, hence the integrand  $f(x)$  fails to have bounded  $V_{HK}$ . Such functions commonly arise in financial mathematics, more concretely in option pricing, where the payoff function is often of the form  $f(x) = \max(0, \Theta(x) - k)$ , with

$\Theta : \mathbb{R}^m \rightarrow \mathbb{R}$  a smooth function for, say a call option, or  $f(x) = x\mathbb{I}_{\{\phi(x)>a\}}$ , for, e.g., binary digital options. In the first case,  $f$  may have a kink across the surface  $\Theta(x) = k$ , whereas in the second case  $f$  may have a jump across the surface  $\Phi(x) = a$ . More generally, let us consider integrands  $f : \mathbb{R}^m \mapsto \mathbb{R}$  of the form

$$f(x) = \theta(x)\mathbb{I}_{\{\phi(x)\}}, \quad (4)$$

for smooth functions  $\theta$  and  $\phi$ , and assume that,  $\exists j \in \{1, 2, \dots, m\}$  such that  $\frac{\partial \phi}{\partial x_j}(x) > 0 \forall x \in \mathbb{R}^m$  and that  $\phi(x) \rightarrow \pm\infty$  as  $x_j \rightarrow \pm\infty$ .

The method proposed in [1] is relatively simple: notice that we can write (2) as

$$I_m(\phi) = \int_{\mathbb{R}^m} f(x)\rho(x)dx = \int_{\mathbb{R}^{m-1}} \underbrace{\left( \int_{\mathbb{R}} f(x_j, x_{-j})\rho_j(x_j)dx_j \right)}_{:= p(x_{-j})} \rho_{-j}(x_{-j})dx_{-j}. \quad (5)$$

where we have written  $x = (x_j, x_{-j})$ , with  $x_{-j} = (x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_m)$ , and similarly for  $\rho_j$  and  $\rho_{-j}$ . Instead of evaluating  $\int_{\mathbb{R}^m} f(x)\rho(x)dx$  directly by a QMC formula, we could alternatively, first evaluate the inner integral for a set of given points  $x_{-j}$ , and then evaluate the outer integral in  $m - 1$  dimensions via, say MC or QMC. What is interesting about this approach is that the resulting inner function  $p(x_{-j})$  is smooth, i.e., it doesn't have a kink or a jump anymore. To see this, notice that for any fixed  $x_{-j} \in \mathbb{R}^{m-1}$  by the assumptions on the derivatives of  $\phi$ , the function  $x_j \mapsto f(x_j, x_{-j})$  has a jump at the (unique) point where  $\phi(x_j, x_{-j}) = 0$ . It follows from the implicit function theorem that for each  $x_{-j}$ , there exists a unique value  $x_j^* := \psi(x_{-j})$  for which  $\phi(x_j, x_{-j}) < 0$  if  $x_j < x_j^*$  and  $\phi(x_j, x_{-j}) > 0$  if  $x_j > x_j^*$ . Thus, given the chosen form of  $f(x)$ , we can write

$$p(x_{-j}) := \int_{\mathbb{R}} f(x_j, x_{-j})\rho_j(x_j)dx_j = \int_{\psi(x_{-j})}^{+\infty} \theta(x_j, x_{-j})\rho_j(x_j)dx_j, \quad (6)$$

with both  $\theta$  and  $\psi$  smooth, and

$$I_m(\phi) = \int_{\mathbb{R}^{m-1}} p(x_{-j})\rho_{-j}(x_{-j})dx_{-j}. \quad (7)$$

Notice that once  $\psi(x_{-j})$  has been found, the one dimensional integral (6) can be evaluated using, e.g., a Gauss quadrature and (7) can be recast into an  $(m - 1)$ -dimensional integral over the hypercube  $[0, 1]^{m-1}$  with, hopefully, a smooth integrand.

## 1.1 Application to option pricing

Consider the problem of pricing an option with maturity  $T > 0$  based on the stock price  $S$ , which is given as the solution to the stochastic differential equation

$$dS = rS dt + \sigma S dw_t, \quad S(0) = S_0.$$

Here  $S_0$  represents the initial value of the underlying asset,  $K$  is the so-called strike price,  $r$  is the interest rate,  $\sigma$  is the volatility and  $w_t$  denotes a standard one-dimensional Wiener process. One can show that  $S_t = S_0 \exp((r - \sigma^2/2)t + \sigma w_t)$ . It follows that  $S_t$  has a log-normal distribution for any  $t > 0$ . For  $m \in \mathbb{N}$ , let  $t_i = i\Delta t$  with  $\Delta t = T/m$  denote the

discrete observation times of the stock price  $S$  (e.g. daily at market closure). The problem of simulating asset prices can be reduced to the problem of simulating discretized Brownian motion paths taking values  $w = (w_{t_1}, \dots, w_{t_m})$ . Notice that  $w$  is normally distributed with covariance matrix having entries  $C_{i,j} = \min\{t_i, t_j\}$ ,  $i, j = 1, \dots, m$ . In particular, we will be interested in pricing Asian options with the following payoffs:

$$\begin{aligned}\Psi_1(w_{t_1}, \dots, w_{t_m}) &:= \left( \frac{1}{m} \sum_{i=1}^m S_{t_i}(w_{t_i}) - K \right)_+ = \left( \frac{1}{m} \sum_{i=1}^m S_{t_i}(w_{t_i}) - K \right) \mathbb{I}_{\left\{ \frac{1}{m} \sum_{i=1}^m S_{t_i}(w_{t_i}) - K \right\}}, \\ \Psi_2(w_{t_1}, \dots, w_{t_m}) &:= \mathbb{I}_{\left\{ \frac{1}{m} \sum_{i=1}^m S_{t_i}(w_{t_i}) - K \right\}}.\end{aligned}$$

Here,  $\Psi_1$  corresponds to the payoff of an Asian call option and  $\Psi_2$  corresponds to the payoff of a binary digital Asian option. Notice that the value  $V_i$ ,  $i = 1, 2$ , of such options is given by

$$V_i := e^{-rT} \mathbb{E}[\Psi_i(w_{t_1}, \dots, w_{t_m})] = \frac{e^{-rT}}{(2\pi)^{m/2} \sqrt{\det(C)}} \int_{\mathbb{R}^m} \Psi_i(w_{t_1}, \dots, w_{t_m}) e^{-\frac{1}{2} w^T C^{-1} w} dw, \quad (8)$$

Thus, the problem of pricing an option with  $m$  discrete time steps, can be treated as a problem of integration in  $m$  dimensions, however, with a discontinuous or only Lipschitz continuous function.

## 2 Goals of the project

Consider the task of estimating  $V_1, V_2$ , with parameters  $K = 100, S_0 = 100, r = 0.1, \sigma = 0.1, T = 1$ , and different discretization parameters, namely  $m = 32, 64, 128, 256, 512$ . To re-cast the integral (8) into a hypercube, consider rewriting it first in terms of independent standard normal random variables and then transforming them into uniform random variables in  $[0, 1]$ . For the former transformation, consider taking a Cholesky factorization of  $C$ , or a reparametrization into Gaussian increments  $\xi_i = \frac{w_{t_i} - w_{t_{i-1}}}{\sqrt{t_i - t_{i-1}}}$ , or using a Lévy-Ciesielski construction (see [2, §3.1]).

1. First estimate the value of the integral, as well as the error of the estimation, using a crude Monte Carlo and a randomized QMC, without the pre-integration trick, i.e., by generating (Q)MC sample  $(x_1, \dots, x_N)$  over the  $m$ -dimensional unit cube  $[0, 1]^m$ . Try increasing values of the sample sizes  $N = 2^7, \dots, 2^{13}$  and plot the estimated error versus  $N$ . Comment on the observed convergence rate.
2. Implement now the pre-integration trick. First, generate randomized (Q)MC points over  $[0, 1]^{m-1}$ , and decide on which direction  $x_j$  to perform the integration. Discuss your choice (or, alternatively, compare numerically different choices) depending also on the adopted re-parametrization of the Brownian path. Use the pre-integration approach to estimate  $V_i$  using both QMC and MC. Estimate your (Q)Monte Carlo error using sample sizes  $N = 2^7, \dots, 2^{13}$ . Plot the estimated error versus  $N$ . Compare your results to those obtained in point 1 in terms of error and computational cost.

3. Consider now  $K = 120$  and the goal of computing  $V_1$  and  $V_2$  with given relative accuracy  $10^{-2}$  in the mean squared sense. What sample size should be used for a crude MC estimator? Propose a variance reduction technique to improve the performance of the crude MC estimator. **Note:** You can change the value of  $K$  to suit your computational budget.
4. Can the variance reduction technique proposed in point 3 be used also for the QMC estimators (with and without the pre-integration trick)? If yes, how? Repeat point 3 for the QMC estimators. What do you observe?

## References

- [1] Andreas Griewank, Frances Y Kuo, Hernan Leövey, and Ian H Sloan. High dimensional integration of kinks and jumps—smoothing by preintegration. *Journal of Computational and Applied Mathematics*, 344:259–274, 2018.
- [2] René L. Schilling and Lothar Partzsch. *Brownian Motion*. De Gruyter, Berlin, Boston, 2012.