Optimization HW6

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Exer 1

a) **Do the case as it is and answer the business questions—you can do the model in Excel or Python or both.**

<u>Minimum cost solution:</u>
To solve the business question, we found the solution associated to the lowest possible cost while still meeting the demand. This solution can be found in the sheet named "optimal" in the attached Excel file. The model has 48 decision variables:

- monthly actual output (12x);
- monthly overtime workforce (12x);
- monthly number of hires (12x);
- monthly number of layoffs (12x)

Apart from the positivity constraints on all decision variables, we only have two constraints:

- actual production needs to be smaller than or equal to maximum possible production, which is given by (regular workforce + overtime workforce) * productivity per person
- demand is met, ie amount available for demand (inventory from previous month + production of this month) larger than or equal to demand

The minimum cost solution costs $6,227,280.00 which is considerably less than the solutions presented in the case; however, this solution is not necessarily the best solution. In May, the workforce needs to grow by almost 50%, which could be extremely complicated to put into practice in the real world. Moreover, as the demand slows down towards the end of the year and we assume that carrying inventory in the next period has no benefit but is costly, the model decides to fire a lot of workers in the final months of the year, which would probably affect the image of the company and damage union relations. Before moving into extensions of the model, we decided to gain a better understanding of the scenarios presented in the case. Our conclusions are presented in the next paragraphs.

<u>Scenario 1:</u>
In the first proposed scenario we want to have a constant workforce and enough production to meet the peak demand, while never utilizing overtime. To keep the workforce constant throughout the year, January was the only month when we were allowed to hire or layoff employees. Hence, in this scenario we had only 2 decision variables: number of hires in January and number of layoffs in January. The constraints

impose that demand must be met in every month. We solved the linear problem using OpenSolver and found that the final cost is $6,768,600.00, which agrees with the answer in the text of the case. This solutions leads to the greatest excess of inventory at the end of the year, but provides the best strategy for union/employee relations.

Scenario 2:

In the second proposed scenario we want to:

- have a constant workforce throughout the year;
- use overtime to meet demand in peaks months;
- restrict output in slow months so that we would never carry inventory.

In this case, we have 26 decision variables:

- monthly actual output (12x);
- monthly overtime workforce (12x);
- number of hires and layoffs in the month of January.

The constraints are such that demand is met in every month, the monthly inventory values must be equal to zero and the actual production is smaller than or equal to the total capacity (given by the sum number of regular workers and the number of overtime-months times the productivity per employee). while adjusting the overtime hours per month. We solved this linear problem using OpenSolver and we found that within the constraints of scenario 2, there exists a much better solution than the one analyzed in the text; this solution has a total cost of $6,559,500.00. The approach from this scenario puts moderate strain on union/employee relations, because it requires significant overtime at peak production intervals, though not as much strain as scenario 3. Additionally, this solution ends with no excess inventory heading into the next year.

Scenario 3:

In the third proposed scenario, we want to meet changes in demand by hiring and firing employees, without ever using overtime. We also want to make sure that we never carry a positive inventory. The solution presented in the case is such that the monthly workforce if computed by simply dividing the monthly predicted demand by the productivity per employee. However, we thought that using the idea from scenario 2 of restricting output in slow months (producing less than the maximum capacity in a month, instead of firing employees and then hiring other in the following month) would be beneficial. We therefore set up our model to have 36 decision variables:

- monthly actual output (12x);
- monthly number of hires (12x);
- monthly number of layoffs (12x)

The constraints are such that demand is met in every month, the monthly inventory values must be equal to zero and the actual production is smaller than or equal to the total capacity. After solving the linear program in OpenSolver, we found that our version of scenario 3 costs $6,477,000.00, which is just under $25,000 cheaper than the scenario 3 solution presented in the text of the case. This solution, while cheaper than the others, puts the greatest strain on union/employee relations, as constant hirings and firings are unanimously despised among workers.

b) Extend the analysis in the case to present some interesting results to the client.

We extended the model in the following way:

- We tried to make the model more applicable to real life. We started by setting constraints on the number of layoffs and hires, on the maximum overtime usage per month and the maximum inventory that could be carried to the next period. The limit on layoffs helped make the model more realistic, but did not fix the problem of large layoffs: layoffs still occurred, but were simply spread out over the course of several months, rather than all at once. Including a limit on hires as well forces the model to choose a solution that relies more heavily on overtime. Given the structure of the model, carrying a positive inventory into the following year is suboptimal, as it has no value but it is costly. However, this is an unrealistic assumption. When companies close their financial year, they value their end of year stock as part of their assets in their balance sheet. Hence, in the workbook 'value_eoy' we assigned a positive value to end of year inventory and then included this in the objective function. As a consequence, in the final months we increase production to make sure that we hit the maximum inventory constraint in December.
- The current model lasts for only 12 months, and does not take into consideration the effects on production for the following year. Therefore, at the end of the 12 month period, a lot of employees are laid off since there is no consideration for what is going to happen in month 13. The model suggests to almost double the workforce in May because of the demand spike in June. When the demand peak is over the model suggests to layoff almost 30% of the workforce since the demand decreases by about 30% as well. While the model is succeeding in its purpose to reduce cost, it affects the public image of the company, morale of the workforce, and union relations. To address this problem, we can assume demand continues into the next year by forecasting an additional 12 months, while only considering the cost associated with the first 12. We decided to replicate future demand for another 12 months assuming it remains the same as the previous year. However, given that the demand at the beginning of the year is very low, this did not help reduce the number of workers getting laid off. Anyways, this was a useful exercise as the company is not gonna terminate its operations in December of next year; better and updated predictions two years in the future would make this approach more meaningful.
- To tackle the huge layoffs, we further added a constraint on how many workers can be laid off at once as a percentage of the current workforce. We tried 5% and 10% of the workforce as the maximum number of new hires and layoffs. This helped reduce layoffs, but inevitably resulted in increased costs. The model chose a solution that relies more heavily on overtime and less on hiring/firing

employees, which is however not necessarily better for the company as this might have a negative effect on employees morale.

- In another attempt to tackle the issue of huge layoffs, we decided to introduce contractors in the workforce alongside full time employees. Since we now have contractors that we can hire on a monthly basis, we removed the possibility to ask workers to work overtime, in an effort to improve the morale of our full-time workforce. We put a 5% constraint on increasing and decreasing the full time employee headcount, and would have to spend less on hiring contractors compared to full time employees ($900 compared to $1,800). There is no cost to laying off contractors since the length of the contracts are already fixed beforehand. However, the monthly payment to contractors would be higher than the full time employees ($2,900 compared to $2,400) but lower than overtime. Running the model with all of these constraints, we end up with a cost of $6,251,260. This cost is lower than the variable workforce cost in Exhibit 3 (but this depends on our assumptions about costs associated with contractors). However, this model addresses the employee layoff problem as the model does not layoff more than 8 full time employees and there is no continuous layoffs. The contractors on the other hand, are laid off in huge numbers. But this is not a problem since the contracts can be agreed upon before hiring them.

Exer 2

a) **Build a model that determines what they should buy from each distributor including the minimum requirement of $15,000. Provide a summary write up of your approach to setting up the model (not the answer to this problem, but how you set up the optimization program).**

The decision variables in our model are, for each vendor:

- the number of units of each product purchased from that vendor
- a binary that indicates whether we decided to buy from that vendor or not

We set constraints on the total cubic feet, the maximum and minimum quantity per vegetable and minimum purchase value from each vendor.

The binary plays a crucial role in setting up the constraint on minimum purchase value from each vendor. To maintain the linearity of the problem, we used the 'big M' approach. For each vendor, the binary is constrained to be larger than or equal than the total number of units purchased from that vendor divided by a large number (we made sure that this number was larger than the maximum number of units you could buy from a single vendor, ie 18,000/1.25 = 14,400). Given this constraint, as soon as the model decides to buy even a single unit from a vendor, the binary for that vendor has to take value 1; on the contrary, the binary for a vendor is equal to 0 if and only if we are buying exactly 0 units from that vendor.

The binary is then multiplied by the minimum expenditure requirement and the sum of the value of vegetables purchased from each vendor is set to be larger than the product of the binary and the minimum expenditure requirement. Hence, the binary makes sure that if we buy at least 1 unit from a vendor, we also meet the minimum requirement constraint.

b) **Include a write-up of the results of the model and why it appeared to make the decisions that it did. That is, why did it recommend the suppliers it picked and the quantities purchased from each supplier. As part of your write up, you should at least note what happens when the purchasing requirement is $0 and $12,000.**

Given the data provided in the exercise, we know that the profit maximizing total cost is $35,381.50. This gives us an idea of approximately how much we will spend in all scenarios.

With a purchasing requirement of $15k, the optimal profit is $5,137.00 and we only purchase from vendor 2 and 3. Given what we said above total cost, it is not surprising that we are not buying from all 3 vendors, as that would imply a minimum cost of $45k, which is very far away from the optimal solution.
With this stringent constraint, we cannot buy all products at their minimum cost, which hurts our profits. We also observe that all products but spinach are bought either in the maximum or minimum quantity. This is because the model ranked products from most profitable to least profitable and bought as much as possible of the most desirables and as little as possible of the least desirables; spinach is the “middle” product, ie it’s more profitable than all products for which we buy the minimum quantity but less profitable than all products for which we buy the maximum. This is also proven by the fact that when we relax the constraint on volume by a single unit, the model buys a little bit more of spinach.

When the purchasing requirement is $0 for every vendor, we can reach the optimal solution and buy each vegetable from the vendor that sells it at the lowest unit cost. The optimal solution has a profit of $5,640.50.

When the purchasing requirement is $12k, the optimal profit is $5,433.08. Notice that here the constraint is “in between” what we saw in the first part and the lack of constraint we had with a purchasing requirement of $0; hence, the profit we obtain is in between the other two. In this case, we buy exactly $12k worth of vegetables from each vendor. This solution is interesting: while the total cost is higher in this case than when we had the $15k purchasing limit, by buying from all vendors, we can obtain a more profitable mixture of products and we can increase revenues by a factor that is larger than the increase in cost, ultimately obtaining higher profits. Moreover, this solutions opts to buy a series of product in a quantity that is not the maximum or the minimum possible.

As the purchasing requirement increases, the optimal profit decreases and it is harder to buy products from all three vendors. We tried setting a purchasing limit of $20k. As expected, profits were even smaller than in all previous cases. Moreover, the model decided to buy from a single vendor (vendor 1): this is not surprising, as buying from 2 producers would require a minimum expenditure of $40k, which is too high to compensate for the increase in revenue that comes from buying a better product mix.

Exer 3

a) **Add a third time period to our natural gas model we covered in class. Formulate and solve it in Excel first and then create and solve a Python version of the model.**

We assumed that gas bought in period 1 does not simply roll over to period 2, but also to period 3 -if unused. Thus, in our model, storage in year i = storage in year i-1 + buy to use in year i + buy to store in year i - demand in year i.

The decision variables are the amount that is “bought to use” and “bought to store”. Notice that given our definition of storage, the difference between “buy to use” and “buy to store” becomes blurry in all periods after the first one, as both “buy to use” and “buy to store” contribute to the storage. This is not a concern, as in real life a company cannot keep gas “bought to use” and “bough to store” separate and actually makes a single decision on how much gas to buy and then stores whatever is leftover after satisfying the demand of its customers. We could have formulated the model with a single decision variable “quantity to buy”, but we didn’t want to lose the nice split that this formulation gives at time 1.

The only constraint is that the quantity available for demand is larger than the actual demand. Notice that quantity available for demand in year i = storage in year i-1 + buy to use in year i.

We have setup a full probability matrix for the change of state from time 2 to time 3. This was inspired by the idea of a Markov transition matrix. This allows the model to be more flexible and have different likelihoods of seeing a specific type of weather in time 3 as a function of the weather observed in time 2.

b) **Test the model with the even split (33%), and then test it with a different mix of probabilities.**

With even splits of different scenarios (33% each), the expected cost is $2,302.78.
At time 1, we buy to store 150 units. At time 2:

- If the weather is normal, we take advantage of the low prices and buy 50 more units, to hedge against potentially higher prices at time 3;

- If the weather is cold, we don't buy any gas, as we already have 150 units that rolled over from the previous period. Given the extra cost of stocking gas, carrying a positive quantity of gas in the next period would be profit-maximising if and only if weather at time 3 was to be very cold. However, this does not happen often enough to push us to buy more gas than what we need for satisfying the contemporaneous demand;
- If the weather is very cold, we buy 30 units of gas, as we don't have enough gas to satisfy the current demand.

In period 3, we buy whatever is needed to meet the contemporaneous demand. In this model, "the world ends after 3 periods" so it doesn't make sense to buy more gas than what is needed to satisfy the current demand.

When we assign 5% to normal, 5% to cold, and 90% to very cold, the expected cost is $2,802.53.
At time 1, given our very negative expectations about future prices, we buy 280 units to store. At time 2:

- If the weather is normal, we have already stored enough gas to face the current demand (100) and any possible future demand (up to 180), so we don't buy any gas;
- If the weather is cold, we buy 20 units of gas; as we were mentioning in the previous analysis, carrying a positive quantity of gas in the next period would be profit-maximising if and only if weather at time 3 was to be very cold. However, now the probability of that happening are very high and we therefore decide to buy gas to store;
- If the weather is very cold, we don't buy any gas; after adjusting the price by adding the stocking cost, the prices in the next period can only be smaller than the current adjusted prices.

In period 4, we buy whatever is needed to meet the contemporaneous demand. Notice that in the Normal-Normal-Normal and Normal-Normal-Cold scenarios we end up with positive stocks of gas; this is due to the fact that these scenarios are extremely unlikely and the model decided to take the risk to over-buy, as the expected benefit from stocking a lot of units in period 1 in anticipation of very cold weather more than compensates.

We also tested the model with a different combination of probabilities. We assumed that in period 2, normal weather is by far the most likely outcome. However, transition probabilities from period 2 to period 3 show that weather is "negatively autocorrelated":

- If at time 2 we had normal weather, very cold is the most likely outcome;
- If at time 2 we had cold weather, all outcomes are equally probable;
- If at time 2 we had very cold weather, normal weather is the most likely outcome.

The results can be found in the Excel file, in the 'final_marko' sheet. Given that in period 2 we expect normal weather, we don't stock up in period 1. However, if we see normal weather in period 2, we then have a negative outlook on the future and we decide to stock up at that point.