## Homework 2 - Group 4

*Luca Colombo, Jill Fan, Rishabh Joshi*

Q1 (a, b, c)

We modeled the amount of ale and beer produced and the amount of corn, hops, and malt bought in different categories (also called tiers) as decision variables. The objective was to maximize the profit earned. The total amount of corn, hops, and malt were constrained to be less than the amount of each required to produce the ale and beer. The amount of corn, hops, and malt bought within each tier of price was also constrained to be less than the allowed limit. These constraints, along with the trivial non-negativity constraints on all decision variables, led to a maximum profit of $1,112.25 with 37 units of ale and 53 units of beer.

From the solution, we can see that we do not buy any corn in the third tier, but we buy hops and malt from all three tiers. Hops is the raw material that limits the production, as this is the only raw material for which we exhaust the maximum availability in all tiers.

An interesting thing we observed is that for each raw material the optimizer exhausts the availability in tier 1 before buying from tier 2 and exhausts the availability in tier 2 before buying from tier 3. This happens even though we did not specify explicit constraints that force the optimizer to follow this pattern. This is not surprising: the reason why this happen is that for all raw materials tier 1 is the cheapest, followed by tier 2 and 3; hence, in an effort to minimize cost, the optimizers choses to buy as much raw material from the first tier (the cheapest) before buying from the second tier. If the unit costs of the raw materials were decreasing in tiers, the solver would not behave as we would like it to.

Q1 (d)

These constraints can be enforced by modifying the limits of tiers 2 and 3 for all ingredients to be 0. This will ensure that the amount of corn, malt, and hops bought is less than 480, 160, and 1190 units respectively. The optimal quantities are the same as the ones for the original problem with 12 and 28 units of ale and beer produced respectively. The profit is $812, which is similar to what we found when we solved this problem during the first lecture ($800).

Q1 (e)

The optimal model chooses to buy 1350 units of corn and produce 90 units of beer and 0 units of ale. Given that corn is now relatively cheaper, it is not surprising to see the optimal solution to consist of beer only, as beer production is more corn-intensive than ale production.

The solver chooses only tier 3 corn: this is the issue we referred to in Q1(a,b,c). Ideally we would want the solver to first pick from tier 1, then tier 2 and finally tier 3, but this will not happen unless we find a way to explicitly introduce this as a constraint. This being said, since the price of the three tiers is constant, we could easily reframe this problem as having a single tier of corn. Moreover, as far as the total quantity of corn purchased does not change, we could think of a multitude of completely equivalent solutions: for example, 50 units from the first tier, 300 from the second, and 1000 from the third. The profit would always $1909.

Q1 (f)
In the original problem, we saw that the optimal production plan had 37 units of ale and 53 units of beer. In this problem, we set the minimum quantity of ale to be 40 units. The optimal solution is then 40 units of ale and 50 units of beer, for a profit of $1102.50. The total units of corn are 950, units of hops are 360, and units of malt are 2400.
The raw material that limits production is hops. Both ale and beer require 4 units of hops and given that the maximum amount of hops we can buy is 360, the production is limited to a total (beer + ale) of 360/4 = 90 units. There is no production plan of beer and ale that sums to 90 total units that requires more corn or malt than what we can buy, which proves our previous statement that hops is indeed the raw material limiting production.
In the original problem, we produced 37 units of ale and 53 units of beer (37+53=90). If we start from that solution and we want to produce an extra unit of ale, we would have to sacrifice one unit of beer in order to have enough hops to produce the extra unit of ale. The new solution would be 38 units of ale and 52 units of beer (38+52 = 90). However, this change in the production plan reduces profits by 3.25 dollars.
To produce 37 units of ale and 53 units of beer we need 980 units of corn and 2355 units of malt. This means that at the margin we are buying corn from tier 2 (at a price of $2) and malt from tier 3 (at a price of $1.55). Summing up, substituting 1 unit of beer with 1 unit of ale, would imply:

- No change in hops utilization
- An increase of 15 units in malt utilization, which is an extra cost of 15 * $1.55 = $23.25
- A decrease of 10 units in corn utilization, which is a saving of 10 * $2 = $20.

Hence, profits would change by 20 - 23.25 = - 3.25 dollars.
Going from a production plan of 37-53 to a production plan of 40-50 does not imply any change in the tiers from which we buy raw materials. Hence, the change in profit would be 3.25 * 3 = 9.75 dollars. As a matter of fact, profits changed from $1112.25 to $1102.5, which is exactly a difference of $9.75.
Given that each time we trade off one unit of beer for one unit of ale we hurt the total profits, the optimal solution with the new constraint is to make as few beer-ale tradeoffs as possible. For this reason, we produce the minimum required amount of ale (40 units) and use all remaining hops to produce as much beer as possible (50 units). Another way to think about this is that we are forcing the solver to move away from the optimal solution by adding a new constraint on the decision variables. The new solution will be the solution that meets the new constraints while moving as little as possible away from the original optimal point.

Q2
The objective is to minimize the cost of buying the bonds and the amount invested in savings while still being able to pay the pension funds outflows for all 14 years.
We have 4 decision variables: the number of each type of bonds to buy at the beginning of the first year (3 variables) and the amount to invest in the savings account at the beginning of the first year. At the end of any given year, the flow in of money is given by the sum of the coupons earned for each bond, the maturity value of the bonds (if a bond hits its maturity), and the

principal + interest earned from the savings account. The flow out of money consists of the payments made at the end of the year. The difference in flow in and flow out of money will be re-invested in the savings account.

We need to constrain the flow in of money to be greater than or equal to the flow out of money at the end of all years. We also impose non-negativity constraints on the decision variables.

The optimal solution of the problem shows that the difference between flow in and flow out at the end of 14 years is almost 0. This makes sense if we are trying to minimize the initial cost since there is no point in saving money after the end of the 14 years. The optimal plan is to invest 73.69 $000s on bond 1, 77.21 $000s on bond 2, $28.84 $000s on bond 3, and $9.38 $000s in the savings account. The total initial cost turns out to be 186.77 $000s.

Q3

All machines are constrained to run for less than 336 hours. To produce a minimum of 100 units of product A and 100 units of product B, machine 3 would be utilized for a minimum of $2 \times 100 + 1.5 \times 100 = 350$ hours. Since this is not possible, PuLP returns an infeasible solution. To make sure that this problem does not repeat in the future, we should always ensure whether there exists at least one solution (not necessarily the optimal one) to the given problem. One way to check this is to check whether the boundary conditions of the decision variables satisfy the other conditions. In this case, since the plant remains the same, we can ensure that the machines can be used to produce the market minimum amounts of products A and B specified by the VP. If not, we return a statement saying that the machines cannot meet the minimum demands.

A more general solution to this issue is to check whether at least one of the boundary points for the decision variables (defined by the minimum and maximum constraints on just the decision variables) satisfies all the constraints of the problem. If none of the boundary points satisfy the other constraints, then there would be no possible solution. This is because the linear conditions form a convex region and any point inside the convex region would be a linear combination of the boundary points.