Question 1

If X, Y, and Z are decision variables, which of the following relationships are valid in a linear program. Briefly explain why for each of your answers.

a. $X + Y = Z$

Can form a linear program.

b. $XY <= 100$

Can't form a linear program. In a linear program, the constraint should be the linear combination of decision variables (a set of terms multiplied by constants and then summed). However, in the equation above, two decision variables are being multiplied.

c. $3X + 2Y <= \sqrt{5}$

Can form a linear program.

d. $\sqrt{5}X + 2Y = 50$

Can form a linear program. The problem needs to be linear in the decision variables. Decision variables can be multiplied by any constant.

e. $\sqrt{5x} + 10Y = 100$

Can't form a linear program. Unlike the previous question, here we are taking the square root of the decision variable X, which make the problem non linear.

f. $X^2 + Y^2 >= 45$

Can't form a linear program. Squaring the decision variables makes this program non linear.

Question 2

a. Formulate and solve the "Two Products – Three Machine" problem from Class #1 as a linear program.

Decision Variables:
A (amount of product A), B (amount of product B)

Maximize:
Total Revenue by Profit - Plant Overhead Cost
470 x A + 420 x B - 50000

Subject to:
75 <= A <= 140
0 <= B <= 140
2 x A <= 336
2.5 x B <= 336
2 x A + 1.5 x B <= 336

Solution:
The optimal strategy to maximize the total revenue from this production plan is to produce 75 units of Product A and 124 units of Product B.
A = 75
B = 124

Profit = 37,330

Utilization of the three machines:
M1 = 150
M2 = 310
M3 = 336

b. Also, how would you set up this model so you could scale it? That is, how would you structure it if you had a lot of machines and a lot of products?

To ensure the scalability to large numbers of machines and products, we want to code the problem so that the data and the modelling process are completely independent. A good approach in Python would imply not hard-coding the values into the equations themselves, but writing the problem as a function of some parameters. This way, the data could be stored and retrieved from a database before running the optimization model.

Question 3

a. Your non-vegetarian friend wants to determine how many units of different kinds of meat he should eat to get the right amounts of vitamins A, C, B1, and B2 over the week. He also wants to minimize his cost. What should his diet be?

Decision Variables:
Units of Beef, Chicken, Fish, Ham, Meat Loaf, Turkey
B, C, F, H, M, T

Minimize:
The total cost to purchase food:
4.16 x B + 2.75 x C + 3.28 x F + 2.91 x H + 2.25 x M + 2.45 x T

Subject to:
Daily amounts of vitamins needed
A: 0.6 x B + 0.08 x C + 0.08 x F + 0.4 x H + 0.7 x M + 0.6 x T >= 1
C: 0.2 x B + 0.00 x C + 0.1 x F + 0.4 x H + 0.3 x M + 0.2 x T >= 1
B1: 0.1 x B + 0.2 x C + 0.15 x F + 0.35 x H + 0.15 x M + 0.15 x T >= 1
B2: 0.15 x B + 0.2 x C + 0.10 x F + 0.1 x H + 0.15 x M + 0.10 x T >= 1

Solution:

B = 0
C = 2.5
F = 0
H = 0
M = 3.3
T = 0

Cost = 14.38

Daily amount of vitamins
A = 253%
C = 100%
B1 = 100%
B2 = 100%

b. How would you modify the linear program to make it "better"?

From the optimal solution we can see that 2.5 units of chicken and 3.33 units of Meat Loaf are enough to satisfy the daily vitamin requirements at the minimum possible cost. Given the current objective function and constraints, the optimal weekly diet is simply given by repeating the optimal daily diet 7 times.
Improving the linear program depends on how we define “better”. One definition could be that a more diverse diet (which includes more variety in food consumption) is better. In such a case, we can impose constraints on the minimum weekly requirements of each product. This will ensure that the optimal solution includes other products instead of just chicken and meat loaf. The tradeoff with this is that the total cost will be higher.

Question 4

Solve the gas blending problem introduced in class. That is, determine how much of each type of gas you should buy and what you should use it for. Make sure you can explain the full model and especially the blending constraints.

Decision Variables:
How many barrels of raw gasoline type i to use to make fuel type j - Xij
X11, X12, X13, X21, X22, X23, X31, X32, X33, X41, X42, X43
How many barrels of raw gasoline type i to resell on the market - Yi
Y1, Y2, Y3, Y4

Notice: The data in the spreadsheet shows that all types of raw gas can be resold for a profit. Hence, in this specific formulation of the fuel mixing problem, it is always profit maximizing to buy as much raw gas as possible and exhaust the maximum quantity allowed: some of this gas will be mixed to make fuel and sold on the market; the leftover raw gas will be resold. This means that with the given costs and resell prices, the same solution will be

obtained if we do not include Yi's (the number of barrels of each raw gasoline type to resell on the market) into the set of decision variables. Then Yi is mechanically derived as
Yi = (maximum allowed quantity of raw gas i) - (Xi1 + Xi2 + Xi3)

Maximize:
Daily Profit = Total Fuel Revenue + Resale Revenue - Total Cost
45.15 x (X11 + X21 + X31 + X41) +
42.95 x (X12 + X22 + X32 + X42) +
40.99 x (X13 + X23 + X33 + X43) +
36.85 x (Y1 + Y2) + 36.95 x (Y3 + Y4) -
31.02 x (X11 + X12 + X13 + Y1) -
33.15 x (X21 + X22 + X23 + Y2) -
36.35 x (X31 + X32 + X33 + Y3) -
38.70 x (X41 + X42 + X43 + Y4)

Subject to:
Limits on availability of raw gasoline types
X11 + X12 + X13 + Y1 <= 4000
X21 + X22 + X23 + Y2 <= 5050
X31 + X32 + X33 + Y3 <= 7100
X41 + X42 + X43 + Y4 <= 4300
Non-negativity constraints apply to all decision variables in this problem

Minimum or Maximum demand for fuels
0 <= X11 + X21 + X31 + X41 <= 10000
0 <= X12 + X22 + X32 + X42 <= 999999
15000 <= X13 + X23 + X33 + X43 <= 999999

Minimum octane rating for fuels
"Non-Linear" Version -
(68 x X11 + 86 x X21 + 91 x X31 + 99 x X41) / (X11 + X21 + X31 + X41) >= 95
(68 x X12 + 86 x X22 + 91 x X32 + 99 x X42) / (X12 + X22 + X32 + X42) >= 90
(68 x X13 + 86 x X23 + 91 x X33 + 99 x X43) / (X13 + X23 + X33 + X43) >= 85
Linear Version -
27 x X11 + 9 x X21 + 4 x X31 - 4 x X41 <= 0
22 x X12 + 4 x X22 - 1 x X32 - 9 x X42 <= 0
17 x X13 - 1 x X23 - 6 x X33 - 14 x X43 <= 0

Solution:
X11 = 0
X12 = 0
X13 = 3457
X21 = 1510
X22 = 0
X23 = 3540
X31 = 0

X32 = 0
X33 = 7100
X41 = 3397
X42 = 0
X43 = 903

Y1 = 543
Y2 = 0
Y3 = 0
Y4 = 0

Profit = 140,431

Question 5

Solve the nurse scheduling problem introduced in class. You want to know the minimum number of nurses needed to cover the 24-hour period. You can assume that the next day is the same as this one.

The constraints are posed on four hour shifts and each nurse will complete a continuous eight hour shift. The decision variables for this problem are the number of nurses starting their shift at the beginning of each four hour shift. The four hour shifts are named A, B, C, D, E, and F respectively. Each nurse will be serving in two consecutive four hour shifts. The optimal solution to minimize the number of nurses is to assign 10 nurses to start at 4 AM, 12 nurses to start at 12 PM, and 4 nurses to start at 8 PM.

Decision Variables:
Number of nurses starting their 8-hour shift at each 4-hour slot
X1, X2, X3, X4, X5, X6

Minimize:
X1 + X2 + X3 + X4 + X5 + X6

Subject to:
X6 + X1 >= 4
X1 + X2 >= 8
X2 + X3 >= 10
X3 + X4 >= 7
X4 + X5 >= 12
X5 + X6 >= 4

Solution:
X1 = 0
X2 = 10
X3 = 0

X4 = 12
X5 = 0
X6 = 4
Number of Nurses = 26

There are multiple optimal solutions to this problem. Since they are all optimal, they all require exactly 26 nurses.

Question 6

Cutting Stock Problem. You need to cut 20 foot rolls of paper to meet your orders of shorter paper rolls.

The idea is to start by determining the different patterns that allow to cut a 20 ft roll into rolls of 5 ft, 7 ft, 9 ft. This can be done by setting up a linear optimization problem that minimizes the waste left from cutting each 20 ft roll, i.e., the amount of the original 20 ft roll left after cutting should be less than the length of the shortest roll we are interested in producing (5 ft in our case). For the final optimization problem, the decision variables are the number of times that each of the different cutting patterns is used. In other words, the decision variable Xj indicates how many 20 ft rolls are cut according to the j-th cutting pattern. The objective is to minimize the sum of these decision variables.

a. Identify the decision variables
We first need to find the possible patterns. Then we need to decide how many times to use each cutting pattern in order to minimize the number of 20 foot rolls used.

It is fair to assume that strictly dominated patterns would never be chosen. Hence, when determining the j patterns we know that no pattern will have a waste >= 5 (as you could get an extra 5 foot long roll at no cost)

The total length of all rolls cut from a 20 foot roll must therefore be in the range [16,20].
6 patterns meet this condition:

Pattern 1 - 5 + 5 + 5 + 5 (waste = 0)
Pattern 2 - 5 + 5 + 7 (waste = 3)
Pattern 3 - 5 + 5 + 9 (waste = 1)
Pattern 4 - 5 + 7 + 7 (waste = 1)
Pattern 5 - 7 + 9 (waste = 4)
Pattern 6 - 9 + 9 (waste = 2)

b. Formulate a linear program to solve the problem
Minimize:
Total Number of Rolls
X1 + X2 + X3 + X4 + X5 + X6

Interestingly, an equivalent solution was obtained when we tried to minimize the total waste and formulated the problem as
0 X1 + 3 X2 + X3 + X4 + 4 X5 + 2 X6
where the coefficients represent the waste associated to each cutting pattern.

Subject To:
4 x X1 + 2 x X2 + 2 x X3 + 1 x X4 + 0 x X5 + 0 x X6 >= 150
0 x X1 + 1 x X2 + 0 x X3 + 2 x X4 + 1 x X5 + 0 x X6 >= 200
0 x X1 + 0 x X2 + 1 x X3 + 0 x X4 + 1 x X5 + 2 x X6 >= 300
The constraints ensure that we produce at least 150 5 ft long rolls, 200 7 ft long rools and 300 9 ft long rolls.

c. Solve the linear program and determine the minimum number of 20-foot rolls you need.

Solution:
X1 = 12.5
X2 = 0
X3 = 0
X4 = 100
X5 = 0
X6 = 150

Since Xi has to be an integer, X1 = 13, and total number of rolls = 13 + 100 + 150 = 263.