Learning from Exchange Rates and Foreign Exchange Interventions*

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Abstract

We study the informational role of the exchange rate and its implications for the conduct of foreign exchange interventions (FXI) in a small open economy model with segmented financial markets and information frictions. Dispersed information renders the exchange rate a public signal on the fundamentals of the economy. The signal is noisy because the exchange rate respond to capital flows that are unrelated to fundamentals. In this environment, the macroeconomic effects of FXI depend not only on their volume but also on the transparency of their communication. If not observed, the volume of FXI can alter the information content of the exchange rate. If observed, the volume of FXI can provide information about fundamentals in addition to the exchange rate. We show that the optimal conduct of FXI depends on how expectations are formed. If expectations are rational, it is optimal to intervene publicly to provide more information. If expectations display an over-reaction bias, secret interventions aimed at reducing the information content of the exchange rate can be optimal. The model rationalizes a signaling channel of FXI as well as the opaqueness in many central banks' practices.

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1 Introduction

Open economies experience recurrent episodes of capital inflows resulting in large fluctuations of exchange rates and macro aggregates. Amidst these events, central banks regularly intervene in the foreign exchange (FX) market.[1] Although effectiveness and desirability of FX interventions are are still subject to debate, at least three observations point to an important role of public communication and information in FX markets. First, an increasing number of central banks reportedly believe that FX interventions work primarily by affecting market expectations (Patel and Cavallino, 2019). Second, policymakers' communication of FX interventions is characterized by low transparency as interventions are often not publicly announced or only published with some lag (Canales-Kriljenko, 2003; Patel and Cavallino, 2019). These two observations suggest that publicly observed interventions convey information to market participant, and that central banks may be reluctant to disclose such information. Third, exchange rates, as market prices, play an informational role by aggregating agents' knowledge and beliefs about economic fundamentals (Hayek, 1945; Grossman, 1976). By affecting exchange rate, interventions may thus be used to influence market's expectations about those fundamentals. While seemingly important in practice, these channels remain relatively unexplored in state-of-the-art macro models of FX interventions.

This paper develops a modern macroeconomic model to formalize the informational role of the exchange rate and investigate its implication for the conduct and communication of FX interventions. We show that FXI have an information channel which depends also on their communication transparency. Our take-home message is that the optimal conduct of FXI crucially depends on how expectations are formed. If expectations are rational, it is optimal to intervene publicly, disclosing the amount of FXI and the rule of the central bank. If expectations overreact to new information, the central bank faces a trade-off and secret interventions can be optimal. The model rationalizes a signaling channel of FXI as well as the opaqueness in many central banks practices.

Our dynamic small-open economy model has three distinctive features. First, in-

[1] While initially mostly utilized by emerging economies, in recent decades FX interventions have been increasingly prominent also in advanced economies such as Switzerland and Japan (Adler et al., 2021). For example, the Swiss National bank has spent 353 billion francs buying mainly dollars, euros and and yen since 2015, while the Bank of Japan intervened in September 2022 for the first time since the crisis of 1997-1998.

ternational asset markets are segmented, which implies that financial flows directly influence equilibrium exchange rates and interventions are effective, as in Gabaix and Maggiori (2015) and Fanelli and Straub (2021). Second, information is dispersed in that each agent has access to a different piece of private information about the economy's fundamentals. As a result, the exchange rate aggregates information and is used by agents to form expectations about future economic fundamentals, and thus to make consumption and investment decisions. This relaxation of the full-information assumption is the main departure from the previous literature that allows our model to speak to the information role of exchange rates and FXIs. Third, investors' expectation formation is subject to a cognitive distortion known as "over-extrapolation", which induces them to over-react to new information. As a result, agents learn from the exchange rate – a public signal – but they over-react to such information.

All three features received wide empirical support. First, recent work on currency demands provide evidence of segmentation consistent with our model (see evidence reviewed in Maggiori, 2022). Second, recent work also support the notion that exchange rates reflect, at least in part, available information about a country's future fundamentals. In particular, Chahrour et al. (2021) show that a large portion of exchange rate variation emanates from anticipated changes in future productivity, with a significant component of expectational noise.[2] Last, agents' expectations "over-reaction" is also well documented in survey expectations data. Bordalo et al. (2020a) documents that forecasters typically over-react to news about macroeconomic and financial variables, while Candian and De Leo (2021) show that expectations' over-reaction explains key properties of exchange rate dynamics.[3]

The model formalizes a novel informational role of exchange rate in macroeconomic allocation. The exchange rate aggregates all private information about the economy's fundamental, and therefore it represents a valuable source of information for agents. An equilibrium exchange rate appreciation, for example, induces agents to revise their expectations of future fundamentals upward, and increase their consumption, investment, and external borrowing. We call this channel the informational role of exchange rate in macro allocation, which operates above and beyond the traditional expenditure switching and wealth channels. The portion of exchange rate that reflects fluctuations

[2]See also Engel and West (2005) and Stavrakeva and Tang (2020). In our model of dispersed information, "noise-trader shocks" blurs the relationship between exchange rate and fundamentals and effectively act as noise in the public signal – the exchange rate – as in Bacchetta and Wincoop (2006).

[3]See also Angeletos et al. (2020).

in future fundamentals – the *informational content of the exchange rate* – depends on the relative volatility of underlying shocks and the economic structure through which they transmit to the exchange rate.

The informational channel of exchange rate transmits through agents' belief formation. It is thus central to ask whether agents use available information optimally when forming expectations. We show that expectations over-reaction to new information generates an independent inefficiency in the competitive equilibrium. More specifically, agents' over-reaction to the informational content of equilibrium exchange rate causes excessive volatility in macro-economic allocations. Ceteris paribus, the exchange rate is excessively volatile.

We articulate two main implications for FX intervention in an environment of imperfect information. First, in our model it matters whether FX intervention is conducted publicly or secretly. Second, FX interventions affect market expectations differently depending on whether the central bank conducts interventions following a rule or discretionarily. If not observed, the volume of FX intervention can alter the information content of the exchange rate. In fact, equilibrium exchange rate reflect interventions in the FX market, and agents factor that in when forming expectations. If the central bank intervenes discretionarily, secret intervention adds noise to exchange rate fluctuations and reduces its informational content. If the central bank follows a rule responding to the underlying shocks, secret intervention changes the stochastic properties of the exchange rate and thus its informational content. Whether interventions increase or reduce the informational content of the exchange rate depends upon the rule that the central bank follows. To the contrary, if the central bank intervention is publicly announced, the volume of FX intervention can become an additional public signal but only if the central bank follows a rule that depends on the underlying shocks. Therefore, public but discretionary interventions do not provide any additional information to private agents.

Suppose that the central bank intervenes to partly offset the effect of noise trading on the exchange rate – a specific form of "leaning against the wind." If interventions are secret but the rule is common knowledge, the informational content of the exchange rate increases, and agents attach a larger weight to it when forming expectations about fundamentals. Similarly, if the central bank intervenes secretly to offset the effect of the fundamental shock on the exchange rate, its correlation with the shock is lower and its informational content decreases. Instead, if both the rule and the volumes are publicly observed, FX intervention does not affect the information content of the exchange rate but inherits a "signaling channel" itself. The FX intervention becomes an additional public signal together with the exchange rate, which means it can only increase the amount of information available to agents.

We find that the optimal conduct of FXI crucially depends on how expectations are formed. In particular, it is not always welfare-maximizing for a central to increase information. If expectations are rational, providing information is welfare improving. As a result, it is optimal to intervene publicly and following a rule, i.e. disclosing the amount of the intervention as well as the central bank's reaction function. To the contrary, if expectations display an over-reaction bias, more information may not always be welfare improving, as agents use this information sub-optimally. In this case, the central bank faces a trade-off and secret interventions aimed at decreasing the informativeness of the exchange rate can be optimal.

The model therefore rationalizes a signaling channel of FXI as well as the opaqueness and secrecy in many central banks' practices (Sarno and Taylor, 2001; Patel and Cavallino, 2019) – two seemingly conflicting practices of central banks.

Relation to the literature This paper relates to several strands of literature in open-economy macroeconomics, international finance, and behavioral macroeconomics.

First, this paper belongs to the open-economy literature on exchange rate policy. The literature has focused on how different assumptions on currency pricing shape the optimal conduct of monetary policy in open economies. Prominent examples are Galí and Monacelli (2005), Benigno and Benigno (2003), Engel (2011), Devereux and Engel (2007), and Egorov and Mukhin (2020). These papers work in models with full information rational expectations, and where foreign exchange intervention is generally ineffective.

Second, our analysis speaks to recent work on the costs and benefits of FX interventions. In this literature, financial frictions in international capital markets are the main motive behind FX interventions (Gabaix and Maggiori, 2015; Ghosh et al., 2016; Cavallino, 2019; Fanelli and Straub, 2021; Amador et al., 2019; Itskhoki and Mukhin, 2022). Our paper argues that information frictions can be at least as important in understanding the conduct of FX interventions. At least three observations motivate our analysis. First, an increasing number of central banks reportedly believe that FX interventions work primarily by affecting market expectations (Patel and Cavallino, 2019). Second, policymakers' communication of FX interventions is characterized by low transparency as interventions are often not publicly announced or only published with some lag (Canales-Kriljenko, 2003; Patel and Cavallino, 2019). Third, exchange rates, as market prices, play an informational role by aggregating agents' knowledge and beliefs about economic fundamentals (Hayek, 1945; Grossman, 1976). All these observations can only be rationalized if information frictions are an important element in exchange rate determination and economic fluctuations.

Third, our model of dispersed information builds on the seminal work of Grossman (1976) as well as the more recent work of Bacchetta and Wincoop (2006). We apply some of their insights to a general equilibrium framework where the informational role of the exchange rate affects macroeconomic allocations. Besides, we incorporate possible departures from rational expectations and study foreign exchange intervention policy.[4]

Fourth, our analysis speaks to the literature on exchange rate policy under imperfect information. Kimbrough (1983, 1984) show that flexible exchange rate regimes allow agents to learn from the exchange rate, but only consider monetary policy. Vitale (1999, 2003) study the signaling role of FX intervention in a market micro-structure framework, where the central bank transparency is not about the size of the intervention but about its objective, i.e. the intervention rule. Fernholz (2015) also studies the implications of central bank transparency during foreign exchange interventions, but in a partial equilibrium setting where FX intervention affect fundamentals. Iovino and Sergeyev (2021) study the effects of central bank balance sheet policies in a model where people form expectations through an iterative level-k thinking process. Candian (2021) studies the benefits of central bank transparency in a two-country model with dispersed information among price-setting firms. We contribute to this literature by studying foreign exchange interventions in a unified general-equilbrium framework where financial markets are segmented, information is dispersed and there are possible departures from rational expectations (in the form of extrapolative expectations). We frame our normative analysis within a fully choice-theoretic environment as opposed to relying on *ad hoc* policy objective functions.

Finally, this project relates to the growing body of work on central bank communication on monetary policy or financial stability. Prominent examples are Angeletos and Sastry (2020), Chahrour (2014), Kohlhas (2020), Melosi (2017), Blinder et al. (2008), Born et al. (2014), Dávila and Walther (2022).

[4]A related literature initiated by Evans and Lyons (2002) focuses on the informational content of trades in foreign exchange markets, in the context of market microstructure.

2 Model

We consider a two-period real small-open economy model with a tradable sector and non-tradable sector extended to incorporate three features of interest. First, limited asset market participation gives rise to a finite elasticity of demand for foreign bonds and, therefore, a scope for foreign-exchange interventions. Second, the economy is affected by two aggregate shocks that are imperfectly observed by agents in the economy: productivity shocks and "noise" shocks to the demand for foreign bonds. Finally, agents observe the real exchange rate and learn from it.

2.1 Model setup

The small-open economy is populated by four types of agents: households, final-good producers, financiers, and a central bank. Households and financiers are located on a continuum of atomistic islands, $i \in [0, 1]$, as in Lucas (1972). Information is common within islands but heterogenous across islands. In particular, in each island, households and the financiers receive the same private noisy signal on next-period productivity of the small open-economy. Agents observe local output, prices, interest rates and the aggregate real exchange rate, which acts as a noisy public signal about next-period productivity. Time is discrete and indexed by $t = [0, 1]$. Foreign variables are denoted with a star symbol.

2.1.1 Households and goods markets

The preferences of the representative household of island i are described by the following utility function:

$$\frac{{C_0^i}^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_0 \left(\frac{{C_1^i}^{1-\sigma}}{1-\sigma} \right), \tag{1}$$

where C^i denotes consumption.

Households have an initial endowment of capital, $K_0^i = K_0 > 0$, which fully depreciated between periods and is used in the production of tradable goods:

$$Y_{T,0}^{i,H} = {K_0^i}^{\alpha}, \qquad Y_{T,1}^{i,H} = A_1 {K_1^i}^{\alpha}. \tag{2}$$

Above, A_1 represents stochastic period-1 productivity. In each period, the household also receives a constant endowment of the non-tradable good: $Y^i_{N,0} = Y^i_{N,1} = Y^i_N$. Consumption and period-1 capital are composites of tradable and non-tradable goods:

$$C^i_0 + K^i_1 = G(Y_N, Y^i_{T,0}), \qquad C^i_1 = G(Y_N, Y^i_{T,1}) \tag{3}$$

where $G(Y_N, Y_T) = \left[(1-\gamma)^{\frac{1}{\theta}} {Y_N}^{\frac{\theta-1}{\theta}} + \gamma^{\frac{1}{\theta}} {Y_T}^{\frac{\theta-1}{\theta}}\right]^{\frac{\theta}{\theta-1}}$ is homogenous of degree 1. The parameter θ denotes the elasticity of substitution between tradable and non-tradable goods in the production of final goods while γ is related to the share of tradable goods in the final composite good. In (3), $Y^i_{T,t}$ represents domestic absorption of the tradable good, which is the sum (difference) of production and imports from (exports to) the rest of the world $Y^i_{T,t} = Y^{i,H}_{T,t} + Y^{i,F}_{T,t}$. We assume that each island trades with the rest of the world but not with other islands to avoid full information revelation by inter-island interactions.

Since the aggregator G is homogenous of degree 1, we have, in equilibrium:

$$G(Y_{N,t}, Y^i_{T,t}) = P^i_{N,t} Y_{N,t} + \mathcal{Q}^i_t Y_{T,t}, \tag{4}$$

where $P^i_{N,t}$ and $\mathcal{Q}^i_t$ are the island-i prices of the non-tradable and tradable goods in units of the composite good, respectively.[5] Since we assume that the size of the home economy is negligible relative to that of the world economy, the latter corresponds to island-i real exchange rate, i.e., the price of the foreign consumption basket (the tradable good) relative to the consumption basket of island i.

The price of the tradable good relative to the non-tradable good is given, in equilibrium, by their marginal rate of transformation:

$$\frac{\mathcal{Q}^i_t}{P^i_{N,t}} = \frac{\partial G(Y_{N,t}, Y^i_{T,t})/\partial Y^i_{T,t}}{\partial G(Y_{N,t}, Y^i_{T,t})/\partial Y_{N,t}} = \left(\frac{\gamma}{1-\gamma}\frac{Y_{N,t}}{Y^i_{T,t}}\right)^{\frac{1}{\theta}}. \tag{5}$$

Combining this expression with (4) yields a relationship between the real exchange rate and the relative price of non-tradable goods:

$$1 = \left[(1-\gamma) {P^i_{N,t}}^{1-\theta} + \gamma {\mathcal{Q}^i_t}^{1-\theta}\right]^{\frac{1}{1-\theta}}. \tag{6}$$

[5] The relative price of the tradable good $\mathcal{Q}^i_t$ has a superscript i because it is affected by island specific factors. For example, higher demand of non-tradables in island i would increase the price the composite good in island i and thus reduce the relative price the tradable good.

As usual, when the price of the non-tradable goods increases, the real exchange rate appreciates (goes down, in our notation). Combining these last two equations we obtain the demand function for tradable goods:

$$Y_{T,t}^{i} = \chi \left[\mathcal{Q}_{t}^{i\,-(1-\theta)} - \gamma \right]^{\frac{\theta}{1-\theta}} Y_N, \tag{7}$$

with $\chi = \frac{\gamma}{(1-\gamma)^{\frac{1}{1-\theta}}}$. The household's budget constraints are:

$$\begin{aligned} C_0^i + K_1^i + \frac{B_1^i}{R_0^i} &= P_{N,0}^i Y_N + \mathcal{Q}_0^i Y_{T,0}^{i,H} + T_0^i, \\ C_1^i &= B_1^i + P_{N,1}^i Y_N + \mathcal{Q}_1^i Y_{T,1}^{i,H} + T_1^i. \end{aligned} \tag{8}$$

The date-0 budget constraint assumes no initial debt and states that the household's income from the sale of tradable and non-tradable goods as well as from government transfers, T_0^i, can be used to buy consumption goods, invest in physical capital, or save in a local bond denominated in domestic goods, B_1^i, whose interest rate is R_0^i. The date-1 budget constraint states that all the income of the household is used for consumption. Maximizing utility (1) subject to the budget constraints in (8) yields the following optimality conditions:

$$\beta R_0^i E_0^i \left[\left(\frac{C_1^i}{C_0^i} \right)^{-\sigma} \right] = 1 \tag{9}$$

$$\alpha \beta E_0^i \left[\left(\frac{C_1^i}{C_0^i} \right)^{-\sigma} \mathcal{Q}_1^i A_1 K_1^{i\,\alpha-1} \right] = 1 \tag{10}$$

Finally, using (3)-(4) in the budget constraints (8), island i households' budget constraints simplifies to:

$$\frac{B_1^i}{R_0^i} = \mathcal{Q}_0^i (Y_{T,0}^{H,i} - Y_{T,0}^i) + T_0^i, \qquad -B_1^i = \mathcal{Q}_1^i (Y_{T,1}^{H,i} - Y_{T,1}^i) + T_1^i. \tag{11}$$

where each island households leave no debt at the end of period 1.

2.1.2 Financial markets

Each island has a financial market where its inhabitants, i.e., the households, noise traders, and financiers, trade home and foreign currency bonds. The government instead operates in the financial market of every island. We now describe how each agent operates in these markets.

Households Following the literature (e.g., Gabaix and Maggiori, 2015, Fanelli and Straub, 2021, Itskhoki and Mukhin, 2021), we assume limited asset market participation. Specifically, we assume that the household cannot hold foreign bonds. This assumption captures the idea that it is difficult for many households in emerging markets to access international financial instruments without financial intermediation, especially when borrowing in foreign currency. Household's demand for home currency bonds in island i, B_1^i, is captured by the Euler equation (9).

Noise traders Noise traders in island i hold a zero-capital portfolio in home and foreign bonds denoted $(N_1^i, N_1^{i\star})$. A zero-capital portfolio implies:

$$\frac{N_1^i}{R_0^i} + \mathcal{Q}_0^i \frac{N_1^{i\star}}{R_0{}^\star} = 0,$$

where $\frac{N_1^{i\star}}{R_0{}^\star}$ is an exogenous liquidity demand shock for foreign "currency" (i.e., goods). Here $\frac{N_1^{i\star}}{R_0{}^\star} > 0$ means that noise traders short home-currency bonds to buy foreign-currency bonds.

Financiers We follow Fanelli and Straub (2021) and assume that there exists a continuum of risk-neutral financiers, labelled by $j \in [0, \infty)$, in each island i. Financiers also hold a zero-capital portfolio in home and foreign bonds denoted $(d_{j,1}^i, d_{j,1}^{i}{}^\star)$. Financier's investment decisions are subject to two important restrictions. First, each intermediary is subject to a net open position limit $D > 0$. Second, intermediaries face heterogeneous participation costs, as in Alvarez et al. (2009). In particular, each intermediary j active in the foreign bond market at time t is obliged to pay a participation cost of exactly j per unit of foreign currency invested.

Putting these ingredients together, intermediary j in island i optimally invests an amount $\frac{d_{j,1}^{i}{}^\star}{R_0^\star}$ in foreign bonds, solving

$$\max_{\frac{d_{j,1}^{i}{}^\star}{R_0^\star} \in [-D, D]} \frac{d_{j,1}^{i}{}^\star}{R_0^\star} E_0^i \left(\tilde{R}_1^{i}{}^\star \right) - j \left| \frac{d_{j,1}^{i}{}^\star}{R_0^\star} \right|,$$

where $\tilde{R}_1^{i\,\star}$ is the return on one foreign-currency unit holding expressed in foreign currency: $\tilde{R}_1^{i\,\star} \equiv R_0{}^\star - R_0^i \frac{\mathcal{Q}_0^i}{\mathcal{Q}_1^i}$. Intermediary j's expected cash flow conditional on investing is $D \left| E_0^i \left(\tilde{R}_1^{i\,\star} \right) \right|$ while participation costs are jD. Thus, investing is optimal for all intermediaries $j \in [0, \bar{j}]$, with the marginal active intermediary $\bar{j}$ given by $\bar{j} = \left| E_0^i \left(\tilde{R}_1^{i\,\star} \right) \right|$. The aggregate investment volume is then

$$\frac{D_1^{i\,\star}}{R_0^\star} = \bar{j} D \operatorname{sign} \left\{ E_0^i \left(\tilde{R}_1^{i\,\star} \right) \right\}.$$

Defining $\Gamma \equiv D^{-1}$ and substituting out $\bar{j}$, we obtain the total demand for foreign-currency bonds in island i, $D_1^{i\,\star} = \int d_{j,1}^{i\,\star} \mathrm{d}j$:

$$\frac{D_1^{i\,\star}}{R_0^\star} = \frac{1}{\Gamma} E_0^i \left(R_0{}^\star - R_0^i \frac{\mathcal{Q}_0^i}{\mathcal{Q}_1^i} \right). \tag{12}$$

The zero-capital portfolio of each financier implies, in terms of island aggregates:

$$\frac{D_1^i}{R_0^i} + \mathcal{Q}_0^i \frac{D_1^{i\,\star}}{R_0{}^\star} = 0. \tag{13}$$

Moreover, the income from the carry trade of the financiers in island i is:

$$\pi_1^{i,D\star} \equiv D_1^{i\,\star} + \frac{D_1^i}{\mathcal{Q}_1^i} = \cdots = \tilde{R}_1^{i\,\star} \frac{D_1^{i\,\star}}{R_0{}^\star}.$$

Equation (12) embodies that intermediaries' demand for foreign bonds has a finite (semi-)elasticity to the expected excess return. This equation is crucial to our analysis because it implies that changes in home bond demand, e.g., induced by FX interventions, can indeed affect the equilibrium exchange rate.

The critical parameter in (12) is the inverse demand elasticity Γ. If Γ is large, e.g., due to tight position limits D, intermediation is impeded. In equilibrium, this implies both small levels of $\frac{D_1^{i\,\star}}{R_0^\star}$ and a small sensitivity of $\frac{D_1^{i\,\star}}{R_0^\star}$ to the expected foreign-currency returns. In the extreme case where $\Gamma \to \infty$ intermediation is absent, $\frac{D_1^{i\,\star}}{R_0^\star} = 0$. By contrast, if Γ is small, e.g., due to relaxed position limits D, the equilibrium will feature both large $\frac{D_1^{i\,\star}}{R_0^\star}$ and a large sensitivity of $\frac{D_1^{i\,\star}}{R_0^\star}$ to the expected excess return. In the extreme case where $\Gamma \to 0$, bond demand adjusts so that $E_0^i \left(\tilde{R}_1^{i\,\star} \right) = 0$ and the elasticity is infinite. Henceforth, we assume $\Gamma \in (0, \infty)$.

Last, we assume that participation costs constitute transfers to households in the home island economy. Thus, no extra cost terms enter the household's budget constraint.

Central Bank/Government The economy-wide central bank holds a $(F_1^i, F_1^{i\star})$ portfolio for each island i. The value of the island-i portfolio is $\frac{F_1^i}{R_0^i} + \mathcal{Q}_0^i \frac{F_1^{i\star}}{R_0^{i\star}}$. We assume that the government finances its island-i operations with island-i transfers, T_0^i:

$$\begin{aligned} \frac{F_1^i}{R_0^i} + \mathcal{Q}_0^i \frac{F_1^{i\star}}{R_0{}^\star} &= -T_0^i, \\ 0 = F_1^i + \mathcal{Q}_1^i F_1^{i\star} + \tau \mathcal{Q}_1^i \pi_1^{i\star} &- T_1^i, \end{aligned} \tag{14}$$

where $\pi_1^{i\star}$ is the combined income from financial transactions of financiers and noise traders in island i, defined below.

Financial market clearing Home-currency bond positions of all four types of agents balance out in each island (in home-island composite goods):

$$B_1^i + N_1^i + D_1^i + F_1^i = 0. \tag{15}$$

The island-level net foreign asset position (NFA) in home-island composite goods is

$$\mathcal{Q}_0^i \frac{B_1^{i\star}}{R_0^\star} = \frac{B_1^i + F_1^i}{R_0^i} + \mathcal{Q}_0^i \frac{F_1^{i\star}}{R_0^\star}, \tag{16}$$

where $\frac{B_1^{i\star}}{R_0^\star}$ is the island's NFA position expressed in foreign goods. It can easily be shown that the NFA of the island equals the combined foreign-currency bond position in the island's financial market:

$$B_1^{i\star} = N_1^{i\star} + D_1^{i\star} + F_1^{i\star}. \tag{17}$$

Combining market clearing conditions (15)-(17), with the household and government budget constraints ((11) and (14)), the income from financial transactions of financiers and noise traders, and the production function for tradable goods (2), we obtain the consolidated island budget constraints:[6]

[6]The income from financial transactions of financiers and noise traders is:

$$\pi_1^{i\star} \equiv \pi_1^{i,D\star} + \pi_1^{i,N\star} = \tilde{R}_1^{i\star} \frac{D_1^{i\star} + N_1^{i\star}}{R_0^\star}$$

$$\frac{B_1^{i\star}}{R_0^\star} = \left(K_0^{i\,\alpha} - Y_{T,0}^i\right), \qquad B_1^{i\star} = -\left(A_1 K_1^{i\,\alpha} - Y_{T,1}^i\right) + (1-\tau)\tilde{R}_1^{i\star}\frac{B_1^{i\star} - F_1^{i\star}}{R_0^\star}. \tag{18}$$

2.2 Equilibrium characterization

2.2.1 Island-level equilibrium

We assume that households and financiers use the log-linearized model around a steady state with $A = 1, N^\star = 0$ when addressing their signal-extraction problem and we focus our attention on the equilibria that arise in the linearized economy. This assumption greatly simplifies the analysis because it allows for the use of the Kalman filter to characterize posterior beliefs analytically despite the presence of endogenous signals.

The log-linearized version of the household's optimality conditions (7), (9),(10) are:

$$\sigma(E_0^i c_1^i - c_0^i) = r_0^i, \tag{19}$$

$$(1-\alpha)k_1^i = E_0 q_1^i + E_0^i a_1 - r_0^i, \tag{20}$$

$$q_t^i = -\frac{1-\gamma}{\theta} y_{T,t}^i, \tag{21}$$

The log-linear optimality condition of financiers (12):

$$\tilde{\Gamma} d_1^{i\star} = E_0^i q_1^i - q_0^i - (r_0^i - r_0^\star) \tag{22}$$

where $d_1^{i\star} \equiv \frac{D_1^{i\star}}{Y_T}$ and $\tilde{\Gamma} \equiv \Gamma \cdot Y_T \cdot \beta^2$. The consolidated island budget constraints in (18) can be combined and loglinearized as:

$$\frac{1}{\beta} y_{T,0}^i = a_1 + \alpha k_1^i - y_{T,1}^i \tag{23}$$

where $b_1^{i\star} \equiv \frac{B_1^{i\star}}{Y_T}$. The final good aggregator in (3) yields:

$$(1-\phi)c_0^i + \phi k_1^i = \gamma y_{T,0}^i \qquad (1-\phi)c_1^i = \gamma y_{T,1}^i \tag{24}$$

where $\phi = \beta\alpha\gamma$. Finally, bond market clearing, (17) becomes:

$$b_1^{i\,\star} = d_1^{i\,\star} + n_1^{i\,\star} + f_1^{i\star} \tag{25}$$

where

$$b_1^{i\,\star} = -\frac{1}{\beta} y_{T,0}^{i} \tag{26}$$

2.2.2 Economy-wide exchange rate

In Appendix A.1 we derive the solution for the equilibrium aggregate real exchange rate as a function of shocks and expectations thereof:

$$q_0 = \frac{\Gamma\omega_1}{\Gamma\tilde{\theta}\omega_1 + \omega_3}(n_1{}^\star + f_1{}^\star) - \frac{\omega_2}{\Gamma\tilde{\theta}\omega_1 + \omega_3}\bar{E}_0 a_1 \tag{27}$$

where $\bar{E}_0$ is the average expectation across all islands and and $\omega_1 > 0, \omega_2 > 0, \omega_3 > 0$ are convolutions of parameters independent of Γ.

Consider the first term, that describes how a shock to the demand for foreign bond by noise trader leads to a exchange rate depreciation. First, a higher noise traders demand for foreign bond requires financiers to take an opposite position, meaning long on domestic bonds and short on foreign ones (25). In order to hold this currency risk, financiers require a compensation proportional to their limited brisk bearing capacity Γ (22). The real exchange rate depreciates today, so that its expected appreciation allow financiers to make profit on their long domestic bond position (the numerator of the first term). The movement in the exchange rate triggers a second round of general equilibrium effects. As the exchange rate depreciates today, the price of tradable goods become higher and agents decrease their import and increase borrowing, and therefore demand more domestic bond, (21) and (26). The position of households is opposite to the one of noise traders, and therefore attenuates the initial appreciation of the exchange rate (the denominator of the first term).

Consider now the second term, that describes how a shock to the expected future fundamental leads to a exchange rate appreciation. First, a higher future technology implies an higher supply of tradable relatively to non-tradable goods, which implies a lower relative price of tradables, meaning a future exchange rate appreciation (21). As for the UIP condition, the exchange rate today appreciates one-to-one with the future appreciation, as the financiers do not bear any additional risk at this point (22) (numerator of the second term). Similarly to before, a exchange rate appreciation triggers another round of general equilibrium effects. As the price of tradable goods become lower, households increase their import and increase borrowing, issuing domestic bonds to financiers. Correspondingly, financiers take a short position on foreign bonds and long on domestic bonds, they require an expected exchange rate appreciation (22), which leads to less appreciation today (the denominator of the first term).

2.2.3 Discussion of assumptions

Before we move on, let us discuss some of the assumptions that we made.

First, we have distributed agents along a continuum of islands that do not directly interact with one another. Allowing for interactions among all islands (for example, via inter-island trade) would completely reveal average expectations and, therefore, eliminate any marginal information role of aggregate public signals, may those be aggregate prices such as the exchange rate or quantities such as interventions.

Second, we have assumed that there is only one aggregate price that agents observe, namely the exchange rate, but two economic disturbances, productivity shocks and noise-trading shocks. This assumption ensures that agents cannot fully back out the aggregate state of the economy by simply observing the exchange rate. These first two assumptions parsimoniously capture the idea that economic agents, for various reasons, do not perfectly observe all the variables that are relevant to their decisions but that they use easily accessible information, such as exchange rates, to improve their inference about such variables.

Third, we have assumed that financial intermediaries are owned by the household. This assumption ensures that the profits and losses from carry trade activity do not represent a net benefit or cost to the small-open economy. The implications of FX interventions of "leakages" from carry trade if financial intermediaries were owned by foreigners has already been studied by Fanelli and Straub (2021). Instead, we focus, on the informational role of exchange rates and FX interventions.

Fourth, we have assumed that the small open economy can save in foreign bonds and physical capital. The presence of physical capital plays an important role in our model. The exchange rate, by affecting the relative demand for tradable and non-tradable goods, is a key determinant of the allocation of domestic income between domestic spending and external savings, as can be seen from (7)-(8). The breakdown of domestic spending between current consumption and capital investment depends on the expected marginal product of capital and thus on the expectation of future fundamentals, as embedded in (9)-(10). Absent capital, there is a one to one relationship between external saving and current consumption, and that relationship is entirely governed by the current exchange rate. Thus, a policymaker that is interested in affecting the path of consumption has no to reason to influence expectations if it can directly affect the exchange rate. The presence of capital ensures that, for a given level of the exchange rate, expectations are a concern for the policymaker because they determine the allocation of domestic spending between current consumption and investment, or (in part) future consumption.

2.3 First-best allocation

Consider an economy without any frictions, meaning with no intermediation friction $\Gamma = 0$ and full information $\bar{E}_0 a_1 = a_1$. As shown in Appendix A.2, The decentralized equilibrium of this economy attains the first best allocation. The first best equilibrium exchange rate is

$$q_0^{FB} = -\frac{\omega_2}{\omega_3} a_1. \tag{28}$$

The difference between the decentralized market equilibrium (27) and the first-best allocation is

$$q_0 - q_0^{FB} = \frac{\Gamma\omega_1}{\Gamma\tilde{\theta}\omega_1 + \omega_3} \underbrace{\left[(n_1{}^\star + f_1{}^\star) + \frac{\tilde{\theta}\omega_2}{\omega_3} a_1 \right]}_{\text{Intermediation wedge}} - \frac{\omega_2}{\Gamma\tilde{\theta}\omega_1 + \omega_3} \underbrace{\left(\bar{E}_0 a_1 - a_1 \right)}_{\text{Belief wedge}}. \tag{29}$$

The intermediation wedge represents the suboptimal exchange rate variation due to the limited risk bearing capacity of financial intermediaries ($\Gamma > 0$). The same term appears in the literature on models of FX interventions with intermediation frictions (Gabaix and Maggiori, 2015; Itskhoki and Mukhin, 2019, 2022). In this literature, FX intervention $f_1^\star$ is typically used to set this term to zero (unless there are other inefficiencies of the decentralized equilibrium). The belief wedge is due to frictions in belief formation. It stems from the fact that average beliefs may not coincide with the true value of fundamental technology. The following proposition highlights that a benevolent social planner will need to address both wedges to achieve the first-best allocation.

Proposition 1. *If $\bar{E}_0 a_1 - a_1 \neq 0$, then the first-best allocation cannot be achieved.*

Proof. See Appendix B. □

Proposition 1 means that engineering an intermediation wedge that exactly offset a non-zero belief wedge is not enough to attain the first best allocation of the overall macroeconomic equilibrium. Intuitively, a non-zero belief wedge affects the broad macro allocation, including investment decisions. As explained in Section 2.2.3, a first-best exchange rate only ensures that the allocation of domestic income between domestic spending and external savings is optimal. Nevertheless, if expectations of future technology are excessively optimistic, then the split of domestic spending between consumption and investment would be sub-optimal. For these reasons, the first-best allocation can only be attained when both the intermediation wedge and the belief wedge are simultaneously zero. In Section 3, we will examine whether the central bank can attain the first best equilibrium using FX interventions.

Next, we introduce the information structure of the small open economy and explore how different specifications of foreign exchange intervention influence equilibrium expectations and the above wedges.

2.4 Laissez-faire information structure

We now consider how expectations are formed and introduce two important assumptions: dispersed information and extrapolative expectations. In particular, we highlight the information role of the exchange rate and its equilibrium determination. In this section, we discuss the laissez-faire economy, that is the economy without FX interventions, $f_1^\star = 0$. Then, in Section 3 we introduce FX interventions. Under laissez-faire, the equilibrium exchange rate is:

$$q_0 = \frac{\Gamma\omega_1}{\Gamma\tilde{\theta}\omega_1 + \omega_3} {n_1}^\star - \frac{\omega_2}{\Gamma\tilde{\theta}\omega_1 + \omega_3}\bar{E}_0 a_1. \tag{30}$$

Dispersed information Households and financiers in each island $i \in [0, 1]$ can observe local fundamentals, prices, and quantities, in addition to a local signal about the future realization of the technology shock a_1:

$$v^i = a_1 + \epsilon^i, \qquad \epsilon^i \sim N(0, \beta_v^{-1}), \tag{31}$$

with $\int_i \epsilon^i di = 0$ and common prior $a_1 \sim N(0, \beta_a^{-1})$.

Agents can also observe the island-specific noise shock ${n_1^i}^\star$, that is ${n_1^i}^* = {n_1}^\star + \eta_n^i$ with ${n_1}^\star \sim N(0, \beta_n^{-1})$. Following Bacchetta and Wincoop (2006), we consider the limiting case where the variance of η_n^i approaches infinity, so that knowing island i' s noise traders shock does not provide any information about the average shock. This assumption is made for convenience, as a finite variance for η_n^i would provide additional information to the agents without changing the qualitative implications of the model.

Finally, while agents in each island cannot observe aggregate prices and quantities, they share the same currency and can therefore observe the aggregate real exchange rate q_0 given in (27). The aggregate exchange rate is in important endogenous signal because it carries information about the aggregate expectation of the common future technology shocks a_1.

Extrapolative expectations Consistent with growing empirical evidence (Bordalo et al., 2020b), we allow for the possibility that agents do not form beliefs completely rationally, but have an extrapolation bias that causes them to over-react to new information relative to the rational expectation benchmark. In particular, we assume

$$E_0^i a_1 = (1+\delta)(E_0^{i,RE} a_1) \tag{32}$$

where $E^{i,RE}$ is the rational expectation operator and the parameter $\delta \geq 0$ governs the degree of extrapolation. This setting can be viewed as a special case of the "diagnostic expectations" framework, with i.i.d. shocks and prior beliefs equal to zero (Bordalo et al., 2020a)

Because aggregate prices reflect average beliefs about fundamentals, agents in every island, when extracting information from the exchange rate, inherently need to forecast the forecast of agents in other islands. We thus need to specify how agents form these "higher-order beliefs." In this respect, we assume that, while agents are unaware of their extrapolation bias when forming beliefs, they know that the beliefs of all the other agents in the economy are biased. In other words, each agent thinks of themselves as rational and of every other agent as an extrapolator:

$$E_0^i[\bar{E}_0 a_1] = (1+\delta) E_0^i[\bar{E}_0^{RE} a_1] \tag{33}$$

The rationale behind this assumption is to preserve the tractability of the signal extraction problem from endogenous signals. Because we assume agents know about the behavioral bias of others, they understand how the average beliefs reflected in the aggregate price depend on the average signal and, therefore, on the fundamental. This means that agents are able to extract information from aggregate price correctly. Nevertheless, in line with the literature we assume that agents do not understand they are biased themselves, which leads them to use this information erroneously. If this was not the case and agents were unaware of the behavioral bias of others, they would misinterpret the mapping between aggregate beliefs and fundamental shock, which could potentially lead to either under- and over-reaction to it.

Finally, note that the special case of rational expectations corresponds to $\delta = 0$.

2.5 Learning from exchange rate

The equilibrium exchange rate solves the fixed point problem of clearing the bond market given expectations and determining expectations given market clearing (and the rest of the equilibrium) conditions. To solve for the equilibrium exchange rate in terms of the underlying structural shocks, we adopt the method of undetermined coefficient. That is, we conjecture an equilibrium exchange rate equation and then verify that it satisfies the equilibrium condition (27).

We conjecture that the equilibrium real exchange rate depends linearly on the (correct forecast of the) future fundamental a_1 and the noise trader shock $n_1^\star$

$$q_0 = \lambda_a a_1 + \lambda_b n_1^\star, \tag{34}$$

We now define the equilibrium under the laissez-faire information structure.

Definition 1 (Market equilibrium with laissez-faire)**.** *Given shocks realization* $\{a_1, n_1^{i\,\star}\}$ *and agents' prior and signals* $\{v^i, q_0\}_{i\in[0,1]}$*, a symmetric linear market equilibrium is defined as*

- *an allocation* $(\{c_0^i, c_1^i, k_1^i, y_{T,0}^i, y_{T,1}^i, b_1^{i\,\star}, d_1^{i\,\star}\}_{i\in[0,1]})$
- *a vector of local prices* $(\{q_0^i, r_0^i\}_{i\in[0,1]})$
- *A aggregate real exchange rate as a linear function of the states* $q_0 = \lambda_a a_1 + \lambda_b n_1^\star$

solving equations (19)-(26) *with expectations respecting* (32) *and* (33)*.*

The exchange rate depend on aggregate expectations about the fundamental, but it is itself an information source for agents when forming their beliefs. As a result, the relation between the exchange rate and the two shocks, λ_a, λ_b, is determined as he solution of a fixed point problem. In particular, one can rewrite 34 as

$$\frac{q_0}{\lambda_a} = a_1 + \frac{\lambda_b}{\lambda_a} n_1^{\star}. \tag{35}$$

In this formulation, $\frac{q_0}{\lambda_a}$ is an unbiased signal centered around the fundamental shock a_1 with a error variance of $\beta_q^{-1} \equiv \frac{\lambda_b^2}{\lambda_a^2}\beta_n^{-1}$.

To sum up, agent i has access to three sources of information: (i) the prior distribution of a_1; (ii) the private signal (31); (iii) the exchange rate (35). The rational individual posterior mean is the average of the signal weighted by their accuracy

$$E_0^{i,RE} a_1 = \frac{\beta_v v^i + \beta^q \frac{q_0}{\lambda_a}}{D}, \tag{36}$$

where $D \equiv \beta^v + \beta^q + \beta^a$ is the posterior belief accuracy. We can use (32) to compute the individual actual posterior belief and average across individual to get the average posterior belief $\bar{E} a_1 = \int^i E^i a_1 di$, using that $\int^i v^i di = a_1$. Substitute back in the exchange rate (30) to verify the conjectured exchange rate (34). The following proposition characterizes the unique equilibrium of the model economy.

Proposition 2. *Let $\Lambda \equiv \frac{\lambda_a}{\lambda_b}$. The symmetric linear market equilibrium is unique and the exchange rate is described by (34) with coefficients*

$$\begin{aligned} \lambda_a &= -\frac{\omega_2}{\Gamma\tilde{\theta}\omega_1 + \omega_3}(1+\delta)\frac{\beta_v + \Lambda^2\beta_n}{\beta_a + \beta_v + \Lambda^2\beta_n} \\ \lambda_b &= \frac{\Gamma\omega_1}{\Gamma\tilde{\theta}\omega_1 + \omega_3}\frac{\beta_v + \Lambda^2\beta_n}{\beta_v} \end{aligned} \tag{37}$$

where Λ^2 is unique and implicitly defined by

$$\Lambda^2 = \left(\frac{\omega_2}{\Gamma\omega_1}\right)^2 (1+\delta)^2 \frac{\beta_v^2}{(\beta_a + \beta_v + \Lambda^2\beta_n)^2} \tag{38}$$

while the explicit solution of Λ is reported in Appendix B.

Proof. See Appendix B. □

Proposition 2 describes how the equilibrium exchange rate depends on the two shocks and therefore how informative it is about the fundamental shock, $\beta_q \equiv \Lambda^2\beta_n$.

However, the information role of the exchange rate does not depend only on its own accuracy, but its relative accuracy with respect to the other signals. The higher is its relative accuracy compared to the other signals, the more weight agents assign to it when forming beliefs.

Definition 2 (Relative information content of exchange rate)**.** *Define the relative information content of the exchange rate as its relative accuracy as a signal about the fundamental shock a_1 compared to prior and private signal. That is, the Bayesian weight on public signal: $\mathcal{I}_R = \frac{\Lambda^2 \beta_n}{\beta_a + \beta_v + \Lambda^2 \beta_n}$.*

Let's now consider two limit cases highlighting the difference between exchange rate signal's absolute and relative accuracy. First, consider the case in which private signals do not carry any information. In this case, there is no information dispersion as all agents have the same incomplete information.

Corollary 1 (Incomplete information economy)**.** *In the case of perfectly inaccurate private signals, $\beta^v \to 0$, the exchange rate coefficients equal $\lambda_a = 0$ and $\lambda_b = \frac{\Gamma\omega_1}{\Gamma\tilde{\theta}\omega_1 + \omega_3}$. The relative information content of the exchange rate is nil, i.e. $\mathcal{I}_R = 0$ and the overall posterior accuracy is nil, i.e. $D = 0$.*

If agents have no private information, the exchange rate has no private information to aggregate and therefore it will also be uninformative. Both the absolute and relative accuracy of the exchange rate are nil, as the common prior is the only source of information.

Next, consider the case where agents receive perfectly informative signals, and thus they are perfectly informed.

Corollary 2 (Full Information economy)**.** *In the case of perfectly accurate private signals, $\beta_v \to \infty$, the exchange rate coefficients equal $\lambda_a = -\frac{\omega_2}{\Gamma\tilde{\theta}\omega_1 + \omega_3}(1+\delta)$ and $\lambda_b = \frac{\Gamma\omega_1}{\Gamma\tilde{\theta}\omega_1 + \omega_3}$. The relative information content of the exchange rate is nil, i.e. $\mathcal{I}_R = 0$, while the overall posterior accuracy is infinite, i.e. $D \to \infty$.*

Because the private signal is perfectly informative, the exchange rate – albeit a perfectly-revealing signal – does not provide additional information. As a result, its absolute informativeness is positive, but its relative informativeness is zero.

More generally, the information contribution of the exchange rate is to aggregate individual beliefs. Therefore if the information is commonly shared among agents, the exchange rate does not provide any additional information to agents. This happens both in the case where agents are fully informed (Corollary 2) and the case where the only information they have is their common prior (Corollary 1).

Informational role of the exchange rate Away from these two limiting cases, the exchange rate has an informational role in addition to the commonly explored expenditure switching and wealth channels. One can re-write the exchange rate as

$$q_0 = z\frac{\Gamma\omega_1}{\Gamma\omega_1 + \omega_3}n_1^\star - z\left[\frac{\Gamma\omega_1}{\Gamma\omega_1 + \omega_3}(1+\delta)\frac{\beta^v}{\beta_a + \beta_v + \Lambda^2\beta_n}\right]a_1 \tag{39}$$

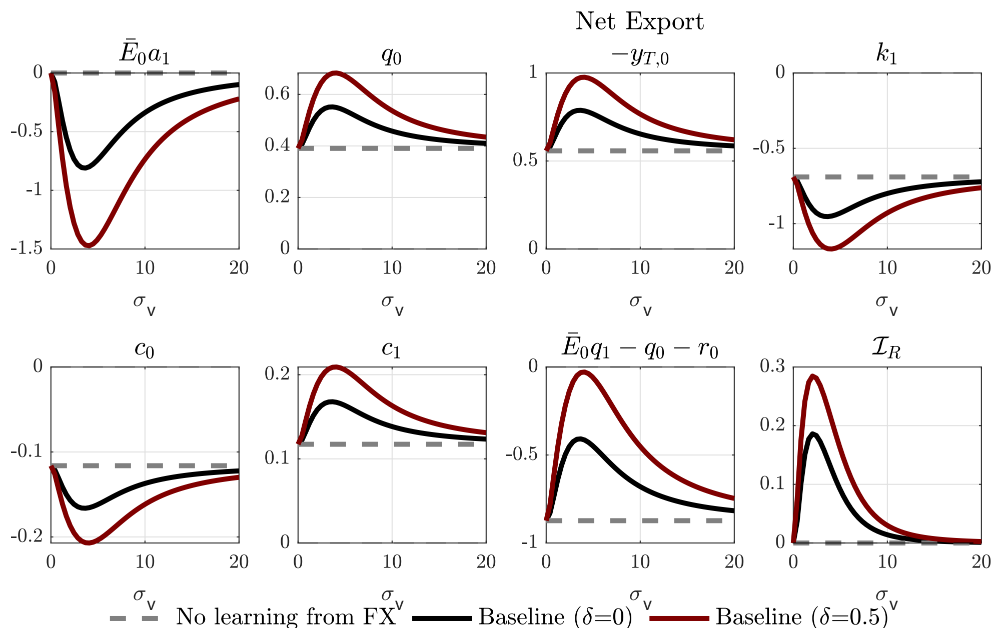
where $z \equiv 1 + \frac{\Lambda^2\beta_n}{\beta_v}$. In order to highlight the informational role of the exchange rate, consider an increase in noise-trader demand for foreign currency, $n_1^\star > 0$, with $a_1 = 0$.

Let's first consider a case where we assume agents do not use the exchange rate as a signal about fundamentals, i.e. $E_0^{i,RE}a_1 = \frac{\beta_v v^i}{\beta_v + \beta_a}$ (and therefore $\mathcal{I}_R = 0$, $z = 1$). In this case, $n_1^\star > 0$ causes a depreciation of the exchange rate through portfolio balance ($q_0 > 0$) and, in turn, results in a contraction in consumption and investment. In this case, the noise-trading shocks only operate through intermediation frictions and it impact on the exchange rate is the same as we would have in full information, $\frac{\Gamma\omega_1}{\Gamma\omega_1+\omega_3}$. The dashed lines in Figure 1 report the equilibrium responses of the model under different levels of noise in the private signal β_v^{-1}.

Let's now consider the baseline case in which agents learn from the exchange rate ($\mathcal{I}_R > 0$, $z > 1$). Agents in each island observe the exchange rate depreciation, but they do not know whether it is due to an expected reduction in future fundamental $a_1 < 0$ or to a non-fundamental noise-trading demand for foreign currency $n_1^\star > 0$. Thus, they confound, at least in part, the effect of the noise-trading shock on the exchange rate with the effect of lower future productivity and revise their beliefs about future fundamental downward, as described by (36). As a result, households reduce their consumption and investment decision. That is, they save to smooth consumption. This effect is analogous to a an exogenous news shock, but it is due to the endogenous response of exchange rate to the increase in foreign currency demand from noise traders.

This change in beliefs implies a second round of effects on the exchange rate. As agents expect lower fundamental tomorrow, they expect the future exchange rate to depreciate, which through he UIP condition implies a further depreciation today. In addition, as they increase saving the real interest rates decline, which further depreci-

Figure 1: Equilibrium responses to $n_1^\star = 1$ under different levels of private noise σ_v



Notes: This figure reports the equilibrium value of model variables for different levels of the noise in private signal, σ_v, under laissez faire. The rest of parameters are set as follows: $\beta = 0.99$, $\alpha = 0.3$, $\gamma = 0.3$, $\theta = 1$, $\sigma = 1$. The standard deviation of a_1 is $\sigma_a = 3$, while the standard deviation of $n_1^\star$ is $\sigma_n = 3$. We consider two values for the over-reaction parameter, $\delta = [0; 0.5]$. The "No learning from FX" scenario corresponds to a parametrization of $\sigma_v = \infty$.

ate the exchange rate today.[7] In summary, the rational confusion between noise and fundamental shock amplify the impact of the noise shocks.[8]

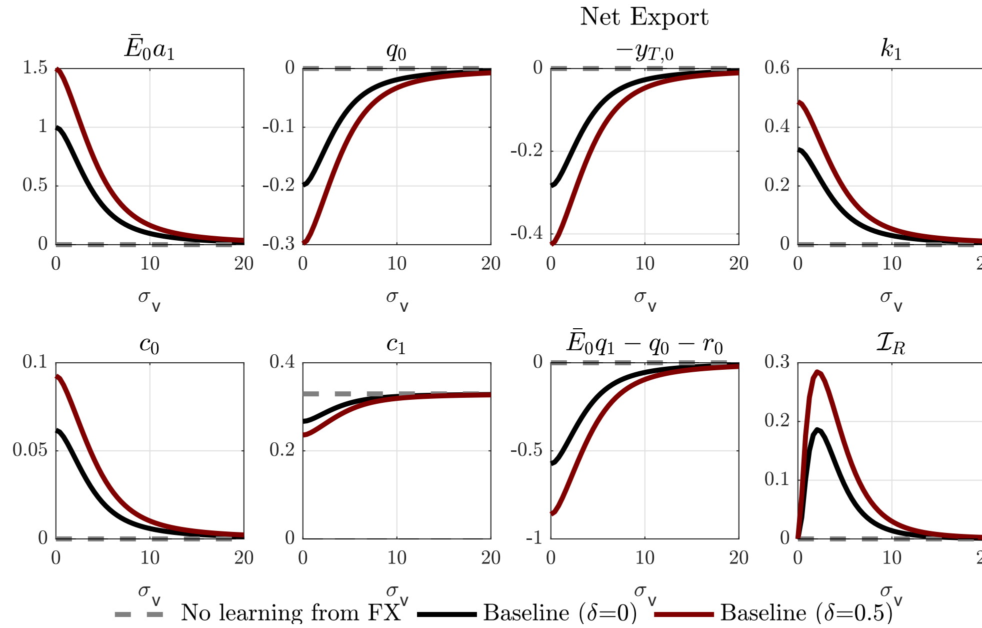
This effect is stronger the higher is the relative information content of the exchange rate $\mathcal{I}_R$, as agents would assign an higher weight to it in their beliefs formation. The solid lines in Figure 1 report the equilibrium responses of the model under different levels of noise in the private signal β_v^{-1}. As we consider cases closer to the common information economy characterized in Corollary 1 and 2, where β_v^{-1} approaches zero or infinity, the exchange rate ceases to be an informative public signal and it's information effect becomes zero.

The information channel of exchange rate described here does not rely on expectations over-reaction, and is operative even under rational expectations ($\delta = 0$). Yet, expectations over-reaction changes the quantitative role of the information channel, as

[7]The households' reduced demand for foreign currency (due to lower borrowing), instead, dampens the equilibrium amplification of the exchange rate depreciation (see second coefficient in eq. (27)).

[8]The amplification effect of noise shock due to rational confusion is highlighted also in Bacchetta and Wincoop (2006) in a more stylized setting. We analyze it in a a fully specified macroeconomic model.

Figure 2: Equilibrium responses to $a_1^\star = 1$ under different levels of private noise σ_v



Notes: This figure reports the equilibrium value of model variables for different levels of the noise in private signal, σ_v, under laissez faire. The rest of parameters are set as follows: $\beta = 0.99$, $\alpha = 0.3$, $\gamma = 0.3$, $\theta = 1$, $\sigma = 1$. The standard deviation of a_1 is $\sigma_a = 3$, while the standard deviation of $n_1^\star$ is $\sigma_n = 3$. We consider two values for the over-reaction parameter, $\delta = [0; 0.5]$. The "No learning from FX" scenario corresponds to a parametrization of $\sigma_v = \infty$.

depicted in Figure 1 under $\delta = 0.5$. Because of expectations over-reaction, agents not only confound the noise-trading shock for a fundamental one, but they over-react to this information. As a result, they assign a larger-than-rational informational role to the exchange rate and thus equilibrium variables exhibit an amplified response relative to the rational expectation case.

While we described the information channel conditional on a noise shock, it is in place every time agents use the exchange rate as a signal about future fundamentals. In Figure 2 we report the equilibrium responses of the model conditional on a unit increase in future productivity, $a_1 = 1$. The full-information economy response can be seen under $\sigma_v = 0$. In this case, agents perfectly foresee that productivity will increase and respond accordingly. They resort to external borrowing in order to increase current consumption and investment (and thus smooth consumption). When $\mathcal{I}_R > 0$ (and $\delta = 0$), agents' signal about future productivity is imprecise and they use the exchange rate to learn about it.

2.6 Belief wedge

The departure from the Full Information Rational Expectation Hypothesis introduces a wedge between the first best allocation and the decentralized equilibrium which we refer to as belief wedge, as discussed in Section 2.3. The belief wedge is proportional to the average forecast error, which depends on the two frictions on beliefs, meaning dispersed information and extrapolation. We now discuss how these two frictions affect the wedge. In particular, we study how the accuracy of the exchange rate signal $\beta_q \equiv \Lambda^2\beta_n$ affect this wedge. One can interpret this variation as due to a reduction in the variance of noise trading shocks (higher β_n) or an increase in Λ^2, for example due to a decrease in intermediation frictions Γ. In the next section we show that FX intervention can also alter the accuracy of the exchange rate as a signal, so the intuition developed here will apply also there.

Rational expectations Consider the case where agents have dispersed information but rational belief, i.e. $\delta = 0$. The average forecast error unconditional variance equals

$$var(\bar{E}_0^R a_1 - a_1) = \frac{1}{(\beta^v + \beta^q + \beta^a)} \frac{\beta^a + \beta^q}{(\beta^v + \beta^q + \beta^a)}. \tag{40}$$

Consider the two limits case. First, if the exchange rate signal is uninformative $\beta^q \to 0$, we are in the dispersed information case and the wedge is positive, $var(\bar{E}_0 a_1 - a_1) = \frac{\beta^a}{(\beta^v+\beta^a)^2}$. Second, if the exchange rate is perfectly accurate $\beta^q \to \infty$, we are in full information and the average error is zero, $var(\bar{E}_0 a_1 - a_1) = 0$. Average beliefs coincides with the fundamental and there is no belief wedge. It follows that with rational beliefs a more informative exchange rate signal lowers the belief wedge. However, this effect might be non-monotonic.[9]

[9] The effect of an increase in exchange rate accuracy on the belief wedge is non-monotonic. There are two margins through which the exchange rate accuracy Λ^2 affects the consensus forecast error variance. First, through the accuracy of the overall information available to each agent, which is represented by the first term in (40). An increase in public signal accuracy monotonically decreases the individual posterior uncertainty. Second, the aggregate externality due to the endogenous signal. An increase in public signal accuracy causes agents to put less weight on their private signal when forming beliefs, i.e. β_v/D, and more on public signal, β_q/D. Even if this is individually optimal, it is suboptimal from an aggregate point of view. Intuitively, the average private noise is zero in the aggregate, so if each agents updated one-to-on with their private signal, the average forecast would be perfectly accurate, even if the individual forecasts would no be. Higher weight on public signal, while individually optimal, means higher weight on the aggregate noise, which is not zero in the aggregate.

Proposition 3 (Belief wedge with rational expectations)**.** *With rational expectations $\delta = 0$, the unconditional variance of the consensus forecast error on fundamental* (40) *is zero with perfectly informative exchange rate signal, $\beta_q \to 0$.*

Proof. See Appendix B. □

In other words, if expectations are rational then in full information there is no belief wedge. Intuitively, if the belief wedge is only due to dispersed information, solving this friction offset completely the belief wedge.

Extrapolative beliefs Consider a more general case where agents have dispersed information but irrational beliefs, i.e. $\delta > 0$. The average forecast error unconditional variance equals

$$var(\bar{E}_0 a_1 - a_1) = \frac{[\delta(\beta^v + \beta^q) - \beta^a]^2}{(\beta^v + \beta^q + \beta^a)^2} \frac{1}{\beta^a} + \frac{(1+\delta)^2 \beta^q}{(\beta^v + \beta^q + \beta^a)^2}. \tag{41}$$

Consider the two limit cases. First, if the exchange rate signal is uninformative $\beta^q \to 0$, we are in dispersed information and the wedge is $var(\bar{\bar{E}}_0 a_1 - a_1) = \frac{(\delta\beta^v - \beta^a)^2 \frac{1}{\beta^a}}{(\beta^v + \beta^a)^2}$. Second, if the exchange rate is perfectly accurate $\beta^q \to \infty$, we are in full information but the consensus error variance is positive, $var(\bar{\bar{E}}_0 a_1 - a_1) = \delta^2 \frac{1}{\beta^a}$. Even if agents perfectly observe the fundamental, they overreact to this information due to their extrapolation bias. As a result, the wedge is positive even in full information. Whether the belief wedge is larger in the limit case of perfectly informative or uninformative exchange rate signal depends on the degree of extrapolation δ.

Proposition 4 (Belief wedge with extrapolative expectations)**.** *With extrapolative expectations $\delta > 0$, the unconditional variance of the consensus forecast error on fundamental* (41) *is minimized with perfectly informative exchange rate signal, $\beta_q \to \infty$, if $\delta < \frac{\beta_a}{\beta_a + 2\beta_v}$. Otherwise, it is minimized with perfectly uninformative exchange rate signal, $\beta_q \to 0$.*

Proof. See Appendix B. □

In other words, if the over-reaction in beliefs is stronger than the under-reaction caused by the dispersion of information, in full information the consensus forecast is less accurate with respect to the dispersed information case. As a result, lower information minimizes the belief wedge. A strong extrapolation bias may require the central bank to reduce instead of increasing information, in order to minimize the belief wedge. In the next section, we study how different communication strategy in FX intervention can alter the information content of the exchange rate.

3 Foreign Exchange Intervention

We now introduce the possibility for the central bank to intervene in the foreign exchange market by purchasing foreign-currency bond $f_1^\star$. We assume that FX interventions follow:

$$f_1^\star = \kappa_b n_1^\star + \kappa_a a_1 + \varepsilon_1^{f^\star}, \tag{42}$$

where $\kappa_b n_1^\star + \kappa_a a_1$ as the *rule-based* component of FXI, whereas $\varepsilon_1^{f^\star} \sim N\left(0, \beta_\varepsilon^{-1}\right)$ is the *discretionary* component of FXI.

The FX intervention has to be intermediated by the financiers in each island, analogously to the noise-trading demand. In this model, FX interventions are *effective*, i.e. can affect the exchange rate, because they alter the balance-sheet position of financiers. For example, a central bank's purchase of foreign bond $f_1^\star > 0$ requires financiers to take an opposite position (long on domestic bonds and short on foreign ones). As a result, financiers require a compensation, so the real exchange rate depreciates today to allow a premium on financiers position (27). Besides, in this model FX interventions may alter the information available to agents about future fundamentals, as we describe in details below.

We assume that the FXI in each island follows $f_1^{i\star} = f_1{}^\star + \eta_f^i$ with $\eta_f^i \sim N(0, \sigma_{\eta_f}^2)$ and $\int^i \eta_f^i = 0$. While the aggregate intervention always equal $f_1{}^\star$, the cross sectional variance of the intervention $\sigma_{\eta_f}^2$ determines the amount of information island i can extract about the aggregate size of the intervention from the local intervention $f_1^{i\star}$. We consider two limit cases. First, if $\sigma_{\eta_f}^2 \to 0$, agents can perfectly observe the FX intervention. We define this *public* FX intervention. Second, if $\sigma_\eta^2 \to \infty$, agents cannot infer the aggregate FX intervention. We define this case *secret* FX intervention.

In the next sections we explore sequentially discretionary and rule-based FX intervention, both under public and secret intervention, and highlight their different implications.

3.1 Discretionary FXI

Assume that $\kappa_a = \kappa_b = 0$ in (42), so that $f_1^\star = \varepsilon_1^{f^\star}$. While in practice it is unlikely that the central bank follows a completely random FX intervention, this case is useful to consider since many central banks do not currently conduct FX interventions according to a rule (Patel and Cavallino, 2019). Besides, this case is useful to build intuition and illustrates how FX intervention affect the information content of exchange rate. In this case, the equilibrium exchange rate is determined according to:

$$q_0 = \frac{\Gamma\omega_1}{\Gamma\tilde{\theta}\omega_1 + \omega_3}(n_1{}^\star + \varepsilon_1^{f^\star}) - \frac{\omega_2}{\Gamma\tilde{\theta}\omega_1 + \omega_3}\bar{E}_0 a_1. \tag{43}$$

Equation (43) shows that discretionary FX intervention represents an additional, exogenous shock to the foreign exchange market.

3.1.1 Public discretionary FXI

Let us consider first the case in which agents are able to observe the aggregate volume of FX intervention, $\varepsilon_1^{f^\star}$ (i.e., $\sigma^2_{\eta_f} \to 0$). Guess a linear solution for the exchange rate:

$$q_0 = \frac{\Gamma\omega_1}{\Gamma\tilde{\theta}\omega_1 + \omega_3} f_1^\star + \lambda_a a_1 + \lambda_b n_1^\star, \tag{44}$$

where $f_1^\star = \varepsilon_1^{f^\star}$. Since $f_1^\star$ is observed, we guess that the information problem does not change its relation with q_0. Define $\tilde{q}_0 \equiv q_0 - \frac{\Gamma\omega_1}{\Gamma\tilde{\theta}\omega_1+\omega_3} f_1^\star$. Agents use the exchange rate as signal

$$\frac{\tilde{q}_0}{\lambda_a} = a_1 + \frac{\lambda_b}{\lambda_a} n_1^\star, \tag{45}$$

with a error variance of $\beta_q^{-1} \equiv \frac{1}{\Lambda^2}\beta_n^{-1}$ with $\Lambda \equiv \frac{\lambda_a^2}{\lambda_b^2}$, the same as in the laissez-faire economy (35). Following the same solution method as in section 2.4 gives the same equilibrium λ_a and λ_b as in (37).

Proposition 5 (Public discretionary FXI)**.** *Suppose the central bank adopts a public discretionary FX intervention, i.e. $f_1^\star = \varepsilon_1^{f^\star}$ and $\sigma^2_{\eta_f} \to 0$. A more volatile FX intervention does not affect the relative information content of the exchange rate $\mathcal{I}_R$ nor the overall agents' posterior accuracy about fundamental D. The equilibrium exchange rate is given by (44) with the same λ_a and λ_b as in the laissez-faire equilibrium (37).*

Proof. See Appendix B. □

In other words, since the intervention is public, agents can partial out the intervention from the exchange rate when they solve their signal extraction problem. It follows that the intervention does not affect the information content of the exchange rate. Moreover, since the intervention is random, it only adds non-fundamental variation to the exchange rate.

3.1.2 Secret discretionary FXI

Consider now the case in which the central bank does not reveal the aggregate volume of discretionary FX intervention (i.e., $\sigma^2_{\eta_f} \to \infty$). Notice that the intervention $\varepsilon_1^{f^\star}$ and the noise shock $n_1^\star$ are both unobservable exogenous shock to the exchange rate (43). Guess a linear solution fo the exchange rate

$$q_0 = \lambda_a a_1 + \lambda_b (n_1^\star + \varepsilon^{f^\star}). \tag{46}$$

Agents use the exchange rate as signal

$$\frac{\tilde{q}_0}{\lambda_a} = a_1 + \frac{\lambda_b}{\lambda_a}(n_1^\star + \varepsilon^{f^\star}), \tag{47}$$

with a error variance of $\beta_q^{-1} \equiv \frac{1}{\Lambda^2}(\beta_n^{-1} + \beta_\varepsilon^{-1})$ with $\Lambda \equiv \frac{\lambda_a^2}{\lambda_b^2}$. Since the FX intervention is unobserved, it increases non-fundamental volatility to the exchange rate similarly to the liquidity demand from noise traders, and therefore decreases the information content of exchange rate $\mathcal{I}_R$.[10]

Proposition 6 (Secret discretionary FXI)**.** *Suppose the central bank adopts a secret discretionary FX intervention, i.e.* $f_1^\star = \varepsilon_1^{f^\star}$ *and* $\sigma^2_{\eta_f} \to \infty$. *A more volatile FX intervention decreases the relative information content of the exchange rate* $\mathcal{I}_R$ *and agents' posterior accuracy about fundamental* D. *The equilibrium exchange rate is given by* (46) *with* λ_a *and* λ_b *described in Apppendix* B.

Proof. See Appendix B. □

[10]In addition to directly increasing exchange rate non-fundamental volatility, higher FXI volatility also decreases the load of exchange rate on non-fundamental shock Λ^2. This second effect dampen the initial decrease in exchange rate informativeness $\mathcal{I}_R$, but it cannot reverse it. Intuitively, as the exchange rate becomes less accurate, agents put more weight on their own private signals. As a consequence, the exchange rate can now aggregate more private information and becomes therefore more accurate, attenuating the initial decline in accuracy.

Proposition 6 reveals that, when implemented secretly, FX intervention has an information effect. In particular, it alters agents' expectations of fundamentals by reducing the informativeness of exchange rate.

3.2 Rule-based FXI

We now explore the implications of rule-based FX interventions, i.e. when the central bank's volume of intervention is a function of the underlying shocks. Throughout, we assume that the central bank's reaction function is known to agents. Moreover, we assume that $\varepsilon_1^{f^\star} = 0$ so that equation (42) becomes

$$f_1^\star = \kappa_b n_1^\star + \kappa_a a_1 \tag{48}$$

That is, the central bank responds to the two shocks in the economy, noise trader demand shock $n_1^\star$ and fundamental shock a_1. The underlying assumption is that the central bank perfectly observes the shocks, and therefore its information set is different (and thus larger) than the agents in the economy. We discuss this assumption at the end of the section.

3.2.1 Public rule-based FXI

Consider first the case in which agents are able to observe the aggregate volume of FX intervention, $f_1^\star$ (i.e., $\sigma_{\eta_f}^2 \to 0$). Similarly to the case of discretionary public FXI, the intervention does not change the precision of the exchange rate as a public signal. To see that, guess a linear solution for the exchange rate

$$q_0 = \frac{\Gamma\omega_1}{\Gamma\tilde{\theta}\omega_1 + \omega_3} f_1^\star + \lambda_a a_1 + \lambda_b n_1^\star, \tag{49}$$

where $f_1^\star$ is given by equation (48). Define $\tilde{q}_0 \equiv q_0 - \frac{\Gamma\omega_1}{\Gamma\tilde{\theta}\omega_1+\omega_3} f_1^\star$. Agents use the exchange rate as signal

$$\frac{\tilde{q}_0}{\lambda_a} = a_1 + \frac{\lambda_b}{\lambda_a} n_1^\star, \tag{50}$$

with a error variance of $\beta_q^{-1} \equiv \frac{1}{\Lambda^2}\beta_n^{-1}$ with $\Lambda \equiv \frac{\lambda_a^2}{\lambda_b^2}$. Notice that the public signal (50) is the same as (45). Regardless of what the FX intervention $f_1^\star$ equals to, if it is observed it does not change the information content of the exchange rate.

However, while observing a discretionary FX intervention $f_1^\star = \varepsilon_1^{f^\star}$ does not convey any information per se, a rule-based FX intervention carries independent information about the shocks to which it responds. In particular, FX intervention becomes a public signal on the fundamental

$$\frac{f_1^\star}{\kappa_a} = a_1 + \frac{\kappa_b}{\kappa_a} n_1^\star. \tag{51}$$

Agents can now access two public signals, the exchange rate (50) and the FX intervention (51), which are two independent functions of the same two shocks a_1 and $n_1^\star$. As a result, agents are able to perfectly back out the fundamental and become perfectly informed.[11] In other words, with a transparent communication strategy by the central bank, the FX intervention has a *signaling* effect that increases agents information.

Proposition 7 (Public rule-based FXI)**.** *Suppose the central bank adopts a public rule-based FX intervention, i.e.* $f_1^\star = \kappa_b n_1^\star + \kappa_a a_1$ *and* $\sigma^2_{\eta_f} \to 0$. *The parameters* κ_b *and* κ_a *do not directly affect the accuracy of the exchange rate. However, the FX intervention and the exchange rate perfectly reveal the shocks* a_1 *and* $n_1^\star$, *so the economy is in full information. The relative information content of the exchange rate* $\mathcal{I}_R = 0$ *and the overall agents' posterior accuracy about fundamental* $D = \to \infty$. *The equilibrium exchange rate is given by* (44) *with the same* $\lambda_a = -\frac{\omega_2}{\Gamma\tilde{\theta}\omega_1+\omega_3}(1+\delta)$ *and* $\lambda_b = \frac{\Gamma\omega_1}{\Gamma\tilde{\theta}\omega_1+\omega_3}$.

Proof. See Appendix B. □

As the economy operates under full information, the exchange rate does not carry any additional information and thus it does not have any information channel. Since agents are fully informed, the exchange rate becomes

$$q_0 = \frac{\Gamma\omega_1}{\Gamma\tilde{\theta}\omega_1 + \omega_3}(1+\kappa_b){n_1}^\star - \frac{\omega_2(1+\delta) - \Gamma\omega_1\kappa_a}{\Gamma\tilde{\theta}\omega_1 + \omega_3} a_1. \tag{52}$$

When the future fundamental increases, $a_1 > 0$, in a laissez-faire economy (such as the one described by (30)) the exchange rate appreciates, $q_0 < 0$. The central bank can amplify this effect by selling foreign bonds, $\kappa_a < 0$, or dampen this effect (and even reverse it) by purchasing foreign bonds, $\kappa_a > 0$. If $\kappa_a = \frac{\omega_2(1+\delta)}{\Gamma\omega_1}$, the central bank can completely offset the effect of fundamental on the exchange rate. Similarly, the central bank can amplify noise shocks by purchasing foreign bond when noise traders do, $\kappa_b > 0$, or dampen them (and even reverse them) by taking the opposite position

[11] Consider the following linear combination of signals (50) and (51): $\left(\frac{f_1^\star}{\kappa_b} - \frac{\tilde{q}_0}{\lambda_b}\right) / \left(\frac{\kappa_a}{\kappa_b} - \frac{\lambda_a}{\lambda_b}\right)$. This signal would perfectly reveal the fundamental a_1.

$\kappa_b < 0$. If $\kappa_b = -1$, the central bank completely offset the noise shocks by taking a symmetrical position.

Discussion Note that the result that public rule-based intervention leads to full information is due to our assumption that the central bank is fully informed about fundamentals, and therefore observing the intervention, along with the exchange rate, reveals its perfect information. We make this assumption to simplify the information extraction problem and the exposition. If the central bank was not *perfectly* but more generally *differently* informed with respect to agents, then public intervention would still increase agents' information but only partially. Either way, the point is that transparent communication about FX intervention increases agents information about fundamentals by revealing central bank's own information.

3.2.2 Secret rule-based FXI

Finally, consider the case in which the central bank does not reveal the aggregate volume of FX intervention (i.e., $\sigma^2_{\eta_f} \to \infty$), but it still follows the rule described in (48). In this case, it is convenient to express the FX intervention rule (48) as a function of aggregate beliefs about the fundamental instead of in term of the fundamental itself.

$$f_1^\star = \tilde{\kappa}_b n_1^\star + \tilde{\kappa}_a \bar{E}_0 a_1, \tag{53}$$

Since we assume that the central bank is perfectly informed about noise shock and fundamental shocks, it can also observe the average expectation $\bar{E}_0 a_1$ which is a function of these two shocks. As a result, there is a one-to-one mapping between the FX rule in (48) and (53), with $\tilde{\kappa}_b$ and $\tilde{\kappa}_a$ as functions of κ_b and κ_a.[12]

Substitute (53) in the exchange rate (27) and get

$$q_0 = \frac{\Gamma\omega_1}{\Gamma\tilde{\theta}\omega_1 + \omega_3}(1 + \tilde{\kappa}_b) n_1{}^\star - \frac{\omega_2 - \Gamma\omega_1\tilde{\kappa}_a}{\Gamma\tilde{\theta}\omega_1 + \omega_3}\bar{E}_0 a_1. \tag{54}$$

The unobserved FX intervention changes the structural relation between the exchange rate and the two shocks, and therefore the information content of the exchange rate.

[12] In particular, $\tilde{\kappa}_a = \kappa_a \frac{\beta^v + \Lambda^2\beta^b}{\beta^a + \beta^v + \Lambda^2\beta^b}$ and $\tilde{\kappa}_b = \kappa_b + \kappa_a \frac{\Lambda\beta^b}{\beta^a + \beta^v + \Lambda^2\beta^b}$, where Λ is itself a function of $\tilde{\kappa}_a$ and $\tilde{\kappa}_b$ as explained in Proposition 8.

In order to solve the information problem, we guess

$$q_0 = \lambda_a a_1 + \lambda_b n_1^\star, \tag{55}$$

which is the same guess as in the laissez-faire economy (34). However, since the exchange rate (54) is different, the equilibrium λ_a, λ_b are different as well.

Proposition 8. *(Secret rule-based FXI) Suppose the central bank adopts a secret rule-based FX intervention, i.e.* $f_1^\star = \tilde{\kappa}_b n_1^\star + \tilde{\kappa}_a \bar{E}_0 a_1$ *and* $\sigma^2_{\eta_f} \to \infty$. *Then*

$$\begin{aligned} \lambda_a &= -\frac{\omega_2 - \Gamma\omega_1\tilde{\kappa}_a}{\Gamma\tilde{\theta}\omega_1 + \omega_3}(1+\delta)\frac{\beta_v + \Lambda^2\beta_n}{\beta_a + \beta_v + \Lambda^2\beta_n} \\ \lambda_b &= \frac{\Gamma\omega_1}{\Gamma\tilde{\theta}\omega_1 + \omega_3}(1+\tilde{\kappa}_b)\frac{\beta_v + \Lambda^2\beta_n}{\beta_v} \end{aligned} \tag{56}$$

where Λ^2 *is unique and implicitly defined by*

$$\Lambda^2 = \left(\frac{\omega_2 - \Gamma\omega_1\tilde{\kappa}_a}{\Gamma\omega_1(1+\tilde{\kappa}_b)}\right)^2 (1+\delta)^2 \frac{\beta_v^2}{(\beta_a + \beta_v + \Lambda^2\beta_n)^2} \tag{57}$$

while the explicit solution of Λ *is reported in Appendix B.*

Proof. See Appendix B. □

Similarly to the public rule-based intervention case in section 3.2.1, the FX intervention alter the stochastic properties of the exchange rate, and thus the structural relationship between exchange rate and the underlying shocks. Differently from the public rule-based case, however, FX interventions are not observed and therefore they alter the information content of exchange rate.

Corollary 3. *The exchange rate accuracy* $\beta^q \equiv \Lambda^2\beta^n$, *its relative information content* $\mathcal{I}_R$ *and the overall posterior accuracy* D *are proportional to* $(\omega_2 - \Gamma\omega_1\tilde{\kappa}_a)^2$, *the correlation between exchange rate and fundamentals, and inversely proportional to* $(1+\tilde{\kappa}_b)^2$, *the correlation between exchange rate and noise shocks.*

Intuitively, the more the exchange rate is correlated with the fundamental shock versus the noise shock, the more information it carries about it. The central bank can, through FX intervention, increase the equilibrium covariance between exchange rate and fundamental, increasing its information content and, as a result, the the overall amount of information in the economy. In a recent paper, Hassan et al. (2022) explore how exchange rate policy can influence the riskiness of that country's currency, by altering the stochastic properties of the exchange rate. In our paper, we also emphasize the ability of the central bank of affecting the macroeconomic allocation by altering the stochastic properties of the exchange rate, yet through a distinct, complementary channel: the informativeness of the exchange rate.

Thus, unlike public interventions, secret intervention allow a central bank to "manage" the informativeness of the exchange rate. In fact, they can even reduce the information content of the exchange rate relative to laissez-faire. In Section 3.4 we explore whether the central bank may find it desirable to intervene publicly or secretly.

3.3 FX Interventions and macroeconomic wedges

In this section, we turn to how FX interventions affect the macroeconomic equilibrium. In particular, we discuss how public and secret rule-based FX intervention impact the intermediation and the belief wedges described in Section 2.3. We highlight that providing information is welfare-improving in the rational expectation case, yet reducing information may be welfare-improving in the extrapolative beliefs case. We postpone a detailed characterization of the welfare-maximizing FX policy to Section 3.4.

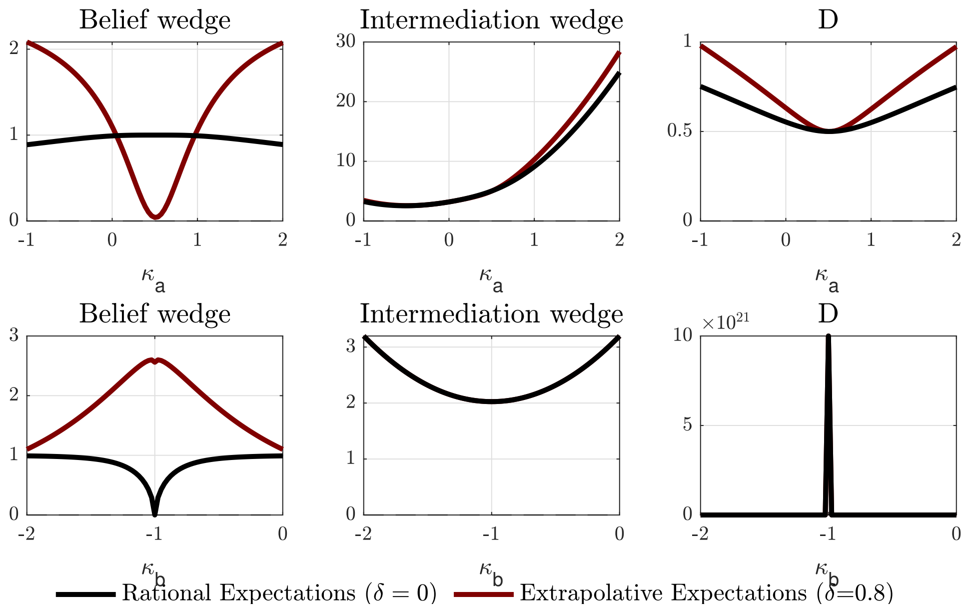
Rational expectations First, consider a public, rule-based intervention. A public intervention perfectly reveals the information of the central bank, which in this case is full information (Proposition 7). Moreover, with rational expectation and full information, the belief wedge is zero (Proposition 3). As a result, the central bank can use the FX intervention to close the only wedge left, the intermediation wedge. In particular, substituting for the FX intervention rule (48) in the exchange rate wedge (29)

$$q_0 - q_0^{FB} = \frac{\Gamma\omega_1}{\Gamma\tilde{\theta}\omega_1 + \omega_3}\left[(1+\kappa_b){n_1}^\star + \left(\frac{\tilde{\theta}\omega_2}{\omega_3} + \kappa_a\right)a_1\right]. \tag{58}$$

The central bank can eliminate the intermediation wedge, and therefore achieve the first best allocation, by setting $\kappa_b = -1$ and $\kappa_a = -\frac{\tilde{\theta}\omega_2}{\omega_3}$.

The same outcome can be achieved through a secret intervention, following the same reaction function. In fact, eliminating the intermediation wedge makes the exchange rate signal perfectly informative, and therefore the economy reaches full information

Figure 3: Secret FX intervention with different values of κ_a and κ_b



Notes: This figure reports the unconditional variance of intermediation wedge, $var((1+\kappa_b)n_1{}^{\star} + \left(\frac{\tilde{\theta}\omega_2}{\omega_3} + \kappa_a\right)a_1)$, belief wedge $var(\bar{E}_0 a_1 - a_1)$, and the posterior accuracy, $D = \beta_a + \beta_v + \beta_q$, for different levels of the FX intervention parameters, κ_a, κ_b. The rest of parameters are set as follows: $\beta = 0.99$, $\alpha = 0.3$, $\gamma = 0.3$, $\theta = 1$, $\sigma = 1$, $\kappa_b = 0$ in the first row, $\kappa_a = 0$ in the second row. The standard deviation of a_1 is $\sigma_a = 3$, while the standard deviation of $n_1^{\star}$ is $\sigma_n = 3$. We consider two values for the over-reaction parameter, $\delta = (0; 0.8)$.

endogenously. This is illustrated by the black line in Figure 3: minimizing the intermediation wedge leads to maximum posterior accuracy and zero belief wedge. To sum up, the central bank can achieve the first-best equilibrium by closing the intermediation gap with either secret or public FX intervention.

Extrapolative expectations Consider first the case with low extrapolation, $\delta < \frac{\beta_a}{\beta_a + 2\beta_v}$. In this case, the belief wedge is still minimized in full information, even if it is not zero (Proposition 4). Therefore, a public or secret intervention closing the intermediation wedge and providing full information would still be the best strategy for the central bank, similarly to the rational expectation equilibrium.

$$q_0 - q_0^{FB} = \frac{\Gamma\omega_1}{\Gamma\tilde{\theta}\omega_1 + \omega_3}\left[(1+\kappa_b)n_1{}^{\star} + \left(\frac{\tilde{\theta}\omega_2}{\omega_3} + \kappa_a\right)a_1\right] - \frac{\omega_2}{\Gamma\tilde{\theta}\omega_1 + \omega_3}\left(\bar{E}_0 a_1 - a_1\right) \quad (59)$$

Consider now the case of strong extrapolative expectations, $\delta > \frac{\beta_a}{\beta_a + 2\beta_v}$. In this case, the belief wedge is minimized with lower information (Proposition 4). As public intervention provides more information, it exacerbates the belief wedge – as agents overreact the information provided. As mentioned above, by utilizing secret intervention the central bank can manage the amount of information in the economy, that is both increase or decrease information. Here, the central bank faces a trade-off. In fact, closing the intermediation wedge by offsetting the noise traders $-1 < \kappa_b < 0$ would lower the noise in the exchange rate signal and therefore increase its informativeness (Corollary 3). This worsen the belief wedge, as agents overreact to their information. This is illustrated in the second row of Figure 3. On the other hand, the central bank can lower the information content of the exchange rate by demanding foreign bonds after a fundamental shock, dampening therefore its appreciation. The exchange rate becomes less correlated with the fundamental, and therefore less informative. However, this policy exacerbates the intermediation wedge, as it adds foreign bonds demand for the financiers to intermediate. This is illustrated by the first row of Figure 3. To sum up, the central bank can at the same time offset the noise traders and therefore lower the intermediation wedge with $-1 < \kappa_b < 0$ and decrease information and the belief wedge by offsetting the fundamental variation in the exchange rate, $0 > \tilde{\kappa}_a > \frac{\omega_2(1+\delta)}{\Gamma\omega_1}$.

3.4 Normative analysis of FXI

[To be written up]

4 Conclusions

We studied FX interventions in a macro model in which segmented financial markets and information frictions coexist. Both frictions generate wedges in aggregate consumption relative to its frictionless counterfactual, namely an intermediation wedge and a belief wedge. We formalized a novel informational role of exchange rate in macroeconomic allocation, as agents use the exchange rate to learn about future fundamentals and make consumption and investment decisions. FX interventions can contemporaneously influence the intermediation wedge, via the standard portfolio balance channel, and the belief wedge, both by altering the information content of the exchange rate and through signaling. We highlighted that the conduct (rule-based vs discretionary) and communication (public vs secret) are important in determining the effects of FX intervention. We then derived the conditions under which FX intervention can attain the first-best allocation. In this regard, an important take-away is that the expectation formation process, and whether it is rational, influences the central bank's ability to achieve the first best as well as the characteristics of the optimal intervention policy.

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Appendix

A Derivations

A.1 Derivations of equilibrium aggregate exchange rate

Consider the following set of island-level equilibrium equations: Demand for tradables at time 0 (cf. (21)):

$$q_0^i = -\frac{1-\gamma}{\theta} y_{T,0}^i \tag{A.1}$$

Demand for tradables at time 1 (cf. (21)):

$$q_1^i = -\frac{1-\gamma}{\theta} y_{T,1}^i \tag{A.2}$$

Modified UIP condition (cf. (22)):

$$-\frac{\tilde{\Gamma}}{\beta} y_{T,0}^i - \tilde{\Gamma} n_1^{i\star} - \tilde{\Gamma} f_1^{i\star} = E_0^i q_1^i - q_0^i - r_0^i \tag{A.3}$$

Combining the aggregate resource constraint and the Euler equation (cf. (19) and (24)):

$$r_0^i = \frac{\sigma\gamma}{1-\phi} E_0^i y_{T,1}^i - \frac{\sigma\gamma}{1-\phi} y_{T,0}^i + \frac{\sigma\phi}{(1-\phi)} k_1^i \tag{A.4}$$

Country budget constraint (cf. (23)):

$$\frac{1}{\beta} y_{T,0}^i = a_1 + \alpha k_1^i - y_{T,1}^i \tag{A.5}$$

Demand for capital (cf. (20)):

$$k_1^i = \frac{1}{1-\alpha} E_0^i q_1^i + \frac{1}{1-\alpha} E_0^i a_1 - \frac{1}{1-\alpha} r_0^i \tag{A.6}$$

The system of equations (A.1)-(A.6) comprises six endogenous variables ($y_{T,0}^i$, $y_{T,1}^i$, q_0^i, q_1^i, k_1^i, r_0^i) and 3 exogenous variables (a_1, $n_1^{i\star}$ and $f_1^{i\star}$).

Using this system we now derive solutions for the endogenous variables of the model as a function of exogenous variables and expectations thereof.

First, use (A.6) to eliminate k_1^i from (A.4) and (A.5):

$$r_0^i = \frac{\sigma\gamma}{1-\phi} E_0^i y_{T,1}^i - \frac{\sigma\gamma}{1-\phi} y_{T,0}^i + \frac{\sigma\phi}{(1-\phi)(1-\alpha)} E_0^i q_1^i + \frac{\sigma\phi}{(1-\phi)(1-\alpha)} E_0^i a_1 - \frac{\sigma\phi}{(1-\phi)(1-\alpha)} r_0^i \tag{A.7}$$

$$\frac{1}{\beta} y_{T,0}^i = a_1 + \frac{\alpha}{1-\alpha} E_0^i q_1^i + \frac{\alpha}{1-\alpha} E_0^i a_1 - \frac{\alpha}{1-\alpha} r_0^i - y_{T,1}^i \tag{A.8}$$

Second, use (22) to eliminate r_0^i from (A.7) and (A.8):

$$\begin{aligned} \frac{\sigma\phi\tilde{\Gamma} + (1-\phi)(1-\alpha)\tilde{\Gamma} + (1-\alpha)\beta\sigma\gamma}{(1-\phi)(1-\alpha)\beta} y_{T,0}^i + E_0^i q_1^i - \frac{\sigma\phi + (1-\phi)(1-\alpha)}{(1-\phi)(1-\alpha)} q_0^i = \\ \frac{\sigma\gamma}{1-\phi} E_0^i y_{T,1}^i + \frac{\sigma\phi}{(1-\phi)(1-\alpha)} E_0^i a_1 + \\ - \frac{\sigma\phi + (1-\phi)(1-\alpha)}{(1-\phi)(1-\alpha)} \tilde{\Gamma}(n_1^{i\star} + f_1^{i\star}) \end{aligned} \tag{A.9}$$

$$\frac{\alpha\tilde{\Gamma} + (1-\alpha)}{\alpha\beta} y_{T,0}^i + \tilde{\Gamma}(n_1^{i\star} + f_1^{i\star}) - q_0^i = \frac{1-\alpha}{\alpha} a_1 + E_0^i a_1 - \frac{1-\alpha}{\alpha} y_{T,1}^i \tag{A.10}$$

Third, use (A.2) to eliminate q_1^i from (A.9) and (A.10):

$$\begin{aligned} \frac{\sigma\phi\tilde{\Gamma} + (1-\phi)(1-\alpha)\tilde{\Gamma} + (1-\alpha)\beta\sigma\gamma}{(1-\phi)(1-\alpha)\beta} y_{T,0}^i - \frac{(1-\gamma)(1-\phi) + \theta\sigma\gamma}{(1-\phi)\theta} E_0^i y_{T,1}^i = \\ \frac{\sigma\phi + (1-\phi)(1-\alpha)}{(1-\phi)(1-\alpha)} q_0^i + \frac{\sigma\phi}{(1-\phi)(1-\alpha)} E_0^i a_1 + \\ - \frac{\sigma\phi + (1-\phi)(1-\alpha)}{(1-\phi)(1-\alpha)} \tilde{\Gamma}(n_1^{i\star} + f_1^{i\star}) \end{aligned} \tag{A.11}$$

$$y_{T,1}^i = -\frac{\alpha\tilde{\Gamma} + (1-\alpha)}{(1-\alpha)\beta} y_{T,0}^i - \frac{\alpha}{1-\alpha}\tilde{\Gamma}(n_1^{i\star} + f_1^{i\star}) + \frac{\alpha}{1-\alpha} q_0^i + a_1 + \frac{\alpha}{1-\alpha} E_0^i a_1 \tag{A.12}$$

Fourth, use (A.1) to eliminate $y_{T,0}^i$ from (A.11) and (A.12):

$$-\frac{\theta\sigma\phi\tilde{\Gamma}+(1-\phi)(1-\alpha)\theta\tilde{\Gamma}+(1-\alpha)\theta\beta\sigma\gamma+\beta(1-\gamma)\sigma\phi+\beta(1-\gamma)(1-\phi)(1-\alpha)}{(1-\phi)(1-\alpha)\beta(1-\gamma)}q_0^i = \frac{(1-\gamma)(1-\phi)+\theta\sigma\gamma}{(1-\phi)\theta}E_0^i y_{T,1}^i + \frac{\sigma\phi}{(1-\phi)(1-\alpha)}E_0^i a_1 + -\frac{\sigma\phi+(1-\phi)(1-\alpha)}{(1-\phi)(1-\alpha)}\tilde{\Gamma}(n_1^{i\star}+f_1^{i\star}) \tag{A.13}$$

$$y_{T,1}^i = \frac{\theta\alpha\tilde{\Gamma}+\theta(1-\alpha)+\alpha\beta(1-\gamma)}{(1-\alpha)\beta(1-\gamma)}q_0^i - \frac{\alpha}{1-\alpha}\tilde{\Gamma}(n_1^{i\star}+f_1^{i\star})+a_1+\frac{\alpha}{1-\alpha}E_0^i a_1 \tag{A.14}$$

Fifth, substitute (A.14) into (A.13) to obtain the island-real exchange rate:

$$q_0^i = \frac{\Gamma\omega_1}{\Gamma\tilde{\theta}\omega_1+\omega_3}(n_1^{i\star}+f_1^{i\star}) - \frac{\omega_2}{\Gamma\tilde{\theta}\omega_1+\omega_3}E_0^i a_1 \tag{A.15}$$

where $\tilde{\theta} \equiv \frac{\theta}{1-\gamma}$ and $\omega_1 > 0, \omega_2 > 0, \omega_3 > 0$ are all convolution of parameters:[13]

$$\omega_1 \equiv [\theta\sigma\alpha\gamma(1+\beta)+(1-\beta\alpha\gamma)(1-\alpha)\theta + (1-\gamma)(1-\beta\alpha\gamma)\alpha]$$

$$\omega_2 \equiv [(1-\gamma)(1-\beta\alpha\gamma)+\theta\sigma\gamma(1+\beta\alpha)]$$

$$\omega_3 \equiv \big((1+\beta)(1-\alpha)\theta^2\sigma\gamma+(1-\gamma)(1-\beta\alpha\gamma)\theta(1-\alpha)(1+\beta)\big)(\beta(1-\gamma))^{-1}+ +\theta\sigma\alpha\gamma(1+\beta)+(1-\beta\alpha\gamma)\alpha(1-\gamma)$$

Aggregating (A.15) across islands, we obtain the aggregate real exchange rate:

$$q_0 = \frac{\Gamma\omega_1}{\Gamma\tilde{\theta}\omega_1+\omega_3}(n_1{}^\star+f_1{}^\star) - \frac{\omega_2}{\Gamma\tilde{\theta}\omega_1+\omega_3}\bar{E}_0 a_1 \tag{A.16}$$

which is the equation (27) above.

[13] Recall that $\phi \equiv \beta\alpha\gamma$.

A.2 First-best allocation

[To be written up]

B Proofs

Proof of proposition 2. Consider the equilibrium exchange rate in case in case of full information, $\bar{E}_0 a_1 = a_1$ (but with intermedation friction, $\Gamma > 0$)

$$q_0^{FI} = \frac{\Gamma\omega_1}{\Gamma\tilde{\theta}\omega_1 + \omega_3}(n_1{}^\star + f_1{}^\star) - \frac{\omega_2}{\Gamma\tilde{\theta}\omega_1 + \omega_3}a_1 \tag{A.17}$$

Take the difference between (27) and A.17, and the difference between A.17 and 28. Sum them and get

$$q_0 - q_0^{FB} = \frac{1}{\Gamma\tilde{\theta}\omega_1 + \omega_3}\left(\Gamma\omega_1\left[(n_1{}^\star + f_1{}^\star) + \frac{\tilde{\theta}\omega_2}{\omega_3}a_1\right] - \omega_2(\bar{E}_0 a_1 - a_1)\right) \tag{A.18}$$

Substitute 21, 22, 25 and 26 in 20 to get

$$k_1 = \frac{1}{1-\alpha}q_0 + \frac{1}{1-\alpha}E_0 a_1 - \frac{\tilde{\Gamma}}{1-\alpha}\left(-\frac{1}{\beta}\frac{\theta}{1-\gamma}q_0 + (n_1^\star + f_1^\star)\right) \tag{A.19}$$

Consider the first best investment allocation, i.e. with $\Gamma = 0$ and $\bar{E}_0 a_1 = a_1$

$$k_1^{FB} = \frac{1}{1-\alpha}q_0^{FB} + \frac{1}{1-\alpha}a_1 \tag{A.20}$$

Take the difference and get

$$k_1 - k_1^{FB} = \frac{1}{1-\alpha}(q_0 - q_0^{FB}) + \frac{1}{1-\alpha}(E_0 a_1 - a_1) - \frac{\tilde{\Gamma}}{1-\alpha}\left(-\frac{1}{\beta}\frac{\theta}{1-\gamma}q_0 + (n_1^\star + f_1^\star)\right) \tag{A.21}$$

Using 27

$$
\begin{aligned}
k_1 - k_1^{FB} =& \frac{1}{1-\alpha}\left(\bar{E}_0 a_1 - a_1\right) + \frac{1}{1-\alpha}(q_0 - q_0^{FB}) + \\
&+ \frac{\Gamma}{1-\alpha}\left(\tilde{\theta}\frac{\Gamma\omega_1}{\Gamma\tilde{\theta}\omega_1 + \omega_4} - 1\right)(n_1^\star + f_1^\star) - \frac{\Gamma}{1-\alpha}\tilde{\theta}\frac{\omega_2}{\Gamma\tilde{\theta}\omega_1 + \omega_3}\bar{E}_0 a_1 \\
k_1 - k_1^{FB} =& \frac{1}{1-\alpha}\left(1 - \Gamma\,\tilde{\theta}\frac{\omega_2}{\Gamma\tilde{\theta}\omega_1 + \omega_3}\right)\left(\bar{E}_0 a_1 - a_1\right) + \frac{1}{1-\alpha}(q_0 - q_0^{FB}) + \\
&\frac{\Gamma}{1-\alpha}\left(\tilde{\theta}\frac{\Gamma\omega_1}{\Gamma\tilde{\theta}\omega_1 + \omega_4} - 1\right)(n_1^\star + f_1^\star) - \frac{\Gamma}{1-\alpha}\tilde{\theta}\frac{\omega_2}{\Gamma\tilde{\theta}\omega_1 + \omega_3} a_1 \\
k_1 - k_1^{FB} =& \frac{1}{1-\alpha}(q_0 - q_0^{FB}) + \\
&- \frac{1}{(1-\alpha)}\frac{1}{\Gamma\tilde{\theta}\omega_1 + \omega_3}\left(\Gamma\omega_3\left[(n_1^\star + f_1^\star) + \frac{\tilde{\theta}\omega_2}{\omega_3}a_1\right] - \left[\theta\Gamma(\omega_1 - \omega_2) + \omega_3\right]\left(\bar{E}_0 a_1 - a_1\right)\right)
\end{aligned}
\tag{A.22}
$$

Suppose the belief wedge $(\bar{E}_0 a_1 - a_1) \neq 0$. Then the exchange rate is optimal $q_0 = q_0^{FB}$ only if $\frac{\Gamma\omega_2}{\beta(1-\gamma)\omega_4}\left[\beta(1-\gamma)\omega_4(n_1^\star + f_1^\star) + \theta\omega_3 a_1\right] = \omega_3(\bar{E}_0 a_1 - a_1)$. However, in that case investment is not at optimum, $k_1 \neq k_1^{FB}$. Therefore, both capital and investment can't be simultaneously at optimum if $(\bar{E}_0 a_1 - a_1) \neq 0$..

□

Proof of proposition 2. From 36, average belief equals

$$
\int_0^1 E_0^i a_1 di = E_0 a_1 = (1+\delta)\frac{\beta_v a_1 + \beta_q \frac{q_0}{\lambda_a}}{D} \tag{A.23}
$$

Plug (A.23) in the solution for the exchange rate (??):

$$
q_0 = \left[1 + \frac{\omega_2}{\Gamma\tilde{\theta}\omega_1 + \omega_3}(1+\delta)\frac{\beta^q}{D\lambda_a}\right]^{-1}\left\{\frac{\Gamma\omega_1}{\Gamma\tilde{\theta}\omega_1 + \omega_3}n_1^\star - \left[\frac{\omega_2}{\Gamma\tilde{\theta}\omega_1 + \omega_3}(1+\delta)\frac{\beta^v}{D}\right]a_1\right\} \tag{A.24}
$$

To find the undetermined coefficients, set (A.24) equal to the guess (??). You get

$$
\begin{aligned}
-\frac{\omega_3}{\Gamma\omega_1 + \omega_4}(1+\delta)\frac{\beta^v}{D} &= \lambda_a\left[1 + \frac{\omega_3}{\omega_1}(1+\delta)\frac{\beta^q}{D\lambda_a}\right] \\
\lambda_a &= -\frac{\omega_3}{\Gamma\omega_1 + \omega_4}(1+\delta)\frac{\beta^v + \beta^q}{D}
\end{aligned}
\tag{A.25}
$$

and

$$\begin{aligned}\frac{\Gamma\omega_2}{\Gamma\omega_1+\omega_4} &= \lambda_b\left[1+\frac{\omega_3}{\omega_1}(1+\delta)\frac{\beta_q}{D\lambda_a}\right] \\ \lambda_b &= \frac{\Gamma\omega_2}{\Gamma\omega_1+\omega_4}\frac{\beta_v+\beta_q}{\beta_v}\end{aligned}\tag{A.26}$$

Take the ratio

$$\frac{\lambda_a}{\lambda_b} = -\frac{\omega_3}{\Gamma\omega_2}(1+\delta)\frac{\beta_v}{D}\tag{A.27}$$

Define $\Lambda \equiv \frac{\lambda_a}{\lambda_b}$. Then:

$$\begin{aligned}\Lambda &= -\frac{\omega_3}{\Gamma\omega_2}(1+\delta)\frac{\beta_v}{\beta_v+\beta_a+\Lambda^2\beta_n} \\ \Lambda^3+\left(\frac{\beta_v}{\beta_n}+\frac{\beta_a}{\beta_n}\right)\Lambda &+ \frac{\omega_3}{\Gamma\omega_2}(1+\delta)\frac{\beta_v}{\beta_n} = 0\end{aligned}\tag{A.28}$$

Define $\rho_1 \equiv \frac{(\beta_v+\beta_a)}{\beta_n}$ and $\rho_2 \equiv \frac{\omega_3}{\omega_2}(1+\delta)\frac{\beta_v}{\beta_n}$. Thus, rewrite (A.28) as:

$$\Lambda^3+\rho_1\Lambda+\rho_2 = 0\tag{A.29}$$

Cubics of this form are said to be "depressed." Cardano's formula states the following. If

1. the cubic equation is of the form in (A.29)
2. ρ_1 and ρ_2 are real numbers
3. $\frac{\rho_2^2}{4}+\frac{\rho_1^3}{27} > 0$ (which is satisfied in our context for any real value of $\frac{\omega_3}{\Gamma\omega_2}$)

Then, equation (A.29) has:

(i) the real root:

$$\sqrt[3]{-\frac{\rho_2}{2}+\sqrt{\frac{\rho_2^2}{4}+\frac{\rho_1^3}{27}}}+\sqrt[3]{-\frac{\rho_2}{2}-\sqrt{\frac{\rho_2^2}{4}+\frac{\rho_1^3}{27}}}\tag{A.30}$$

(ii) and two other roots that are non-real complex conjugate numbers.

□

Proof of Proposition. 3 Take the limit of (40),

$$\lim_{\beta_q \to \infty} var(\bar{E}_0^R a_1 - a_1) = 0$$
$$\lim_{\beta_q \to 0} var(\bar{E}_0^R a_1 - a_1) = \frac{\beta^a}{(\beta^v + \beta^a)^2} \tag{A.31}$$

Moreover,

$$\frac{\partial var(\bar{E}_0^R a_1 - a_1)}{\partial \beta_q} = \frac{\beta_v - (\beta_a + \beta_q)}{\beta_v + \beta_a + \beta_q} \tag{A.32}$$

Therefore, $\frac{\partial var(\bar{E}_0^R a_1 - a_1)}{\partial \beta_q} > 0$ as long as $\beta_q < \beta_v - \beta_a$, and $\frac{\partial var(\bar{E}_0^R a_1 - a_1)}{\partial \beta_q} < 0$ otherwise. As a result, $var(\bar{E}_0^R a_1 - a_1)$ is at a global minimum when $\beta_q \to \infty$. □

Proof of Proposition. 4 Take the limit of (41),

$$\lim_{\beta_q \to \infty} var(\bar{E}_0 a_1 - a_1) = \delta^2 \frac{1}{\beta^a}$$
$$\lim_{\beta_q \to 0} var(\bar{E}_0 a_1 - a_1) = \frac{(\delta\beta^v - \beta^a)^2 \frac{1}{\beta^a}}{(\beta^v + \beta^a)^2} \tag{A.33}$$

As a result, $\lim_{\beta_q \to \infty} var(\bar{E}_0 a_1 - a_1) < \lim_{\beta_q \to 0} var(\bar{E}_0 a_1 - a_1)$ if $\delta < \frac{\beta_a}{\beta_a + 2\beta_v}$.

Moreover,

$$\frac{\partial var(\bar{E}_0 a_1 - a_1)}{\partial \beta_q} = \frac{1}{(\beta^v + \beta^q + \beta^a)^3}(1+\delta)\left(2[\delta(\beta^v + \beta^q) - \beta^a] - (1+\delta)(\beta^q - (\beta^a + \beta^v))\right) \tag{A.34}$$

Therefore $\frac{\partial var(\bar{E}_0 a_1 - a_1)}{\partial \beta_q} > 0$ as long as $\beta^q < \frac{1+3\delta}{1-\delta}\beta^v - \beta^a$, then $\frac{\partial var(\bar{E}_0 a_1 - a_1)}{\partial \beta_q} < 0$.

As a result, if $\delta < \frac{\beta_a}{\beta_a + 2\beta_v}$, the $\lim_{\beta_q \to \infty} var(\bar{E}_0 a_1 - a_1)$ is the global minimum. If $\delta > \frac{\beta_a}{\beta_a + 2\beta_v}$, the $\lim_{\beta_q \to 0} var(\bar{E}_0 a_1 - a_1)$ is the global minimum □

Proof of Proposition. 5 [To be written up] □

Proof of Proposition. 6 [To be written up] □