Problem Set Measure Theory

OSELab2019
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I'm extremely sorry for not being able to finish more of this problem set. I tried, got help, then failed, (in loop) and decided at some point to focus on the other topics of this week.

Problem 1.3

Following the definitions we can see that: $\mathcal{G}_1$ is not a (σ-)algebra as $\mathcal{G}_1$ is not closed under complements. $\mathcal{G}_2$ is an algebra but not a σ-algebra. $\mathcal{G}_3$ is an algebra and a σ-algebra.

Problem 1.7

The power set is the 'largest' possible σ-algebra because it includes all possible subsets of X.
As $\mathcal{A}$ is an algebra, we know that $\varnothing \in \mathcal{A}$. For the second requirement to hold it must be true that:$\varnothing^c = X - \varnothing^c = X \in \mathcal{A}$. Thus, by fullfilling the definition of an algebra, we know that $\{\varnothing, X\} \subset \mathcal{A}$, which is the 'smallest' possible σ-algebra.

Problem 1.10

To proof that the intersection is also a σ-algebra we have to proof the three conditions that need to be met.

1) $\emptyset \in \cap_\alpha S_\alpha$:
As $\{\mathcal{S}_\alpha\}$ is a family of σ-algebras, by definition, $\varnothing \in \mathcal{S}_\alpha$ holds for each σ-algebra in the family. Thus, also for the intersection.

2) Let $A \in \cap_\alpha S_\alpha \ \forall \ \alpha$. We know by definition: $A \in S_\alpha \ \forall \ \alpha$. As S_α is a σ-algebra, it must hold that $A^c \in S_\alpha \ \forall \ \alpha$. Hence, this also holds for the family of σ-algebras: $A^c \in \cap_\alpha \ S_\alpha \ \forall \ \alpha \implies \cap_\alpha S_\alpha$ is closed under complements and finite unions.

3) ...

Problem 1.22

Monotone:
As A is a subset of B and by the definition of a measure: $\mu(B) = \mu(B \cap A^c) + \mu(A)$. As the measure is set to be non-negative, this implies that $\mu(A) \leq \mu(B) = \lambda(A) \leq \mu(B)$

Countably subadditive:
...

Problem 1.23

i) λ is defined such that $\lambda(A) = \mu(A \cap B)$. If $A = 0 \implies \lambda(\emptyset) = \mu(\emptyset \cap B) = \mu(\emptyset)$
ii) By the definition of a measure assuming that: $\{A_i\}_{i=1}^\infty \subset \mathcal{S}$ s.t. $A_i \cap A_j = \emptyset \forall i \neq j$

holds. Then $\mu(\cup_{i=1}^{\infty} A_i) = \mu((\cup_{i=1}^{\infty} A_i) \cap B)$ Since the A_i's are disjoint, we can re-write: $\mu(\cup_{i=1}^{\infty}(A_i \cap B)) = \sum_{i=1}^{\infty} \mu(A_i \cap B)$

Problem 1.26 Let $A = \cap_{n=1}^{\infty} A_n$ and $B = \cup_{n=1}^{\infty} B_n$. This is an increasing sequence $\implies \{B_n\}_{n=1}^{\infty}$

Hence:

$$\mu(A_1 - A) = \mu(B) = \mu(\cup_{n=1}^{\infty} B_n) = \lim_{n\to\infty} \mu(B_n) = \mu(A_1) - \lim_{n\to\infty} \mu(A_n))$$

Problem 2.10

B can be written as: $B = (B \cap E) \cup (B \cap E^c)$. By countable subadditivity $\mu^*(B) \leq \mu^*(B \cap E) + \mu^*(B \cap E^c)$ and $\mu^*(B) \geq \mu^*(B \cap E) + \mu^*(B \cap E^c)$. Hence, $\mu^*(B) = \mu^*(B \cap E) + \mu^*(B \cap E^c)$ holds.

Problem 2.14

Problem 3.1

Let $A = (a_1, a_2, \dots a_n)$ be a countable set. Let $a \in R$, then $a \subset [a - \epsilon, a + \epsilon]$, then $\lambda(a) \leq \lambda(a - \epsilon, a + \epsilon) = 2\epsilon$ for all $\epsilon > 0$. Hence, $\lambda(a) = 0$ and as $A = (a_1, a_2, \dots a_n)$ is a countable set, we know that: $\lambda(A) = 0$

Problem 3.7

To show:

$$\{x \in X : f(x) < a\}$$

can be replaced with any of the following:

$$\begin{gathered} \{x \in X : f(x) \leq a\} \\ \{x \in X : f(x) > a\} \\ \{x \in X : f(x) \geq a\} \end{gathered}$$

We know that: $\{x \in X : f(x) < a\} \in \mathcal{M}$ and closed under complements. $\implies f^{-1}([a, \infty)) = (f^{-1}(-\infty, a))^c$ hence, $f^{-1}([a, \infty)) \in \mathcal{M}$. Therefore, we know that: $\{x \in X : f(x) < a\} = \{x \in X : f(x) \leq a\}$.

Next, we use that $f^{-1}((a, \infty)) = \cap_{n=1}^{\infty} f^{-1}\left(\left(a - \frac{1}{n}, \infty\right)\right)$. As $\mathcal{M}$ is closed under countable intersection, we have $f^{-1}(a, \infty) \in \mathcal{M}$. Hence, we know that: $\{x \in X : f(x) \leq a\} = \{x \in X : f(x) > a\}$

Analogue procedure for the last one.