Dynamic Programming

July 8, 2019

1 Asset Market Equilibrium

[1]:
```
#Luca Maria Schüpbach
```

Households: h={1,2} with endowments $e^1 = (1, 1, 2, 1, 2)$ and $e^2 = (1, 3, 1, 3, 1)$; States: $s = 1, 2, 3, 4$; Assets: A^1 and A^2 with Payoffs: $(1, 1, 1, 1)$ and $(1, 1, 1.5.1.5)$ respectively.

The Agents' utility has the following form:

$$\max_{\theta_1,\theta_2} U(c^z) = v(c_0) + \frac{1}{4}\sum_{i=1}^{4} v(c_i)$$

where $v(c) = \dfrac{c^{1-\gamma}}{1-\gamma}$

For all states in s, agents maximize their utility and choose over the amount of assets to hold: (θ_1^h, θ_2^h)

$$\max_{\theta_1^h,\theta_2^h} U(c^h) = v(c_0^h) + E[v(c_s^h)]$$

s.t.

$$c_0^h = e_0^h - q_1 * \theta_1^h - q_2 * \theta_2^h$$
$$c_s^h = e_s^h + A_s^1 * \theta_1^h + A_s^2 * \theta_2^h$$

The first order conditions for this problem look like the following:

$$-q_1 v'(c_0^1) + E[v'(c_s^1)A_s^1)] = 0$$
$$-q_1 v'(c_0^2) + E[v'(c_s^2)A_s^1)] = 0$$
$$-q_2 v'(c_0^1) + E[v'(c_s^1)A_s^2)] = 0$$
$$-q_2 v'(c_0^2) + E[v'(c_s^2)A_s^2)] = 0$$

As households trade with eachother, the market clears whenever the aggregate holdings are zero for each asset.

$$\theta_1^1 + \theta_1^2 = 0$$
$$\theta_2^1 + \theta_2^2 = 0$$

[2]:
```
import numpy as np
from scipy.optimize import fsolve

def AME(x):
    gam=2
```

```
    the_1_1=x[0]
    the_1_2=x[1]
    the_2_1=x[2]
    the_2_2=x[3]
    q_1=x[4]
    q_2=x[5]
    f=np.zeros(6)
    c_0_1=1-q_1*the_1_1-q_2*the_2_1
    c_0_2=1-q_1*the_1_2-q_2*the_2_2
    c_1_1=1+the_1_1+the_2_1
    c_1_2=3+the_1_2+the_2_2
    c_2_1=2+the_1_1+the_2_1
    c_2_2=1+the_1_2+the_2_2
    c_3_1=1+the_1_1+1.5*the_2_1
    c_3_2=3+the_1_2+1.5*the_2_2
    c_4_1=2+the_1_1+1.5*the_2_1
    c_4_2=1+the_1_2+1.5*the_2_2
    f[0]=-q_1*c_0_1**(-gam)+0.
 ↪25*(c_1_1**(-gam)+c_2_1**(-gam)+c_3_1**(-gam)+c_4_1**(-gam))
    f[1]=-q_1*c_0_2**(-gam)+0.
 ↪25*(c_1_2**(-gam)+c_2_2**(-gam)+c_3_2**(-gam)+c_4_2**(-gam))
    f[2]=-q_2*c_0_1**(-gam)+0.25*(c_1_1**(-gam)+c_2_1**(-gam)+1.
 ↪5*c_3_1**(-gam)+1.5*c_4_1**(-gam))
    f[3]=-q_2*c_0_2**(-gam)+0.25*(c_1_2**(-gam)+c_2_2**(-gam)+1.
 ↪5*c_3_2**(-gam)+1.5*c_4_2**(-gam))
    f[4]=the_1_1+the_1_2
    f[5]=the_2_1+the_2_2
    return f
```

[2]:
```
fsolve(AME, [0.1, -0.1, 0.1, -0.1, 1, 1])
```

[2]:
```
array([ 1.95406655e-02, -1.95406655e-02,  1.18668498e-11, -1.18668498e-11,
        5.89777656e-01,  7.37222070e-01])
```

As expected, we see that for each Asset one agent is long and the other one is shor in an asset. (Side note: the values of the holdings in each asset for each agent greatly depend on the initial guess in the fsolve command). By increasing the risk aversion coefficient gamma, the intuition is that Asset 2, the risky asset will be less desired. However, increasing gamma does not show consistent change in the holdings.

2 Ramsey 1

[23]:
```
#50 points
```

[19]:
```
N = 50 # number of grid-points for the capital grid
k_low = 0.1 # lower bound for the capital grid
k_high = 10 # upper bound for capital grid
k_grid = np.linspace(k_low, k_high, N).reshape(1, N) #grid for capital
```

```
# Now we initialize the value function, I like to initialize it to zero.
V_init = np.zeros((2, N))
beta = 0.9
```

[21]:
```
def u(c):
    """
    input:
    c: consumption

    output:
    utility received from consumption
    """
    return np.log(c)

#Value function update for a given state:

def actionvalue_allchoices(k_index, V_old):
    """
    input:
    k_index: index so that k_grid[index] corresponds to value of capital this␣
 ↪period (state)
    V_old: approximation to the value function. V_old[i] approximates␣
 ↪V(k_grid[i]).

    output:
    action_value: value of all possible state-action pairs.
    """
    k = k_grid[0,k_index]

    action_value = np.zeros_like(V_old) #(2,N)
    c = np.zeros_like(V_old)
    c[0,:] = 0.9*k**0.3 + 0.3 * k - k_grid # consumption implied by policy␣
 ↪k_next in state k
    c[1,:] = 1.1*k**0.3 + 0.9 * k - k_grid

    action_value[c <= 0] = -999999 # set value to -HUGE for negative␣
 ↪consumption
    action_value[c > 0] = u(c[c > 0])

    EV_old=V_old.mean(axis=0).reshape(1,N)

    action_value=action_value + beta * EV_old #(2,N)

    return action_value

def vf_update(i, V_old):
```

```python
    """
    input:
    i: index corresponding to the entry of the value-function vector which is␣
 ↪updated
    V_old: value function vector from the previous iteration

    output:
    Vi_new: updated value for the value function vector at entry i.
    """
    Vi_new = actionvalue_allchoices(i, V_old).max(axis=1)

    return Vi_new

#one update iteration:
def vf_update_iteration(V_old):
    """
    input:
    V_old: array with current approximation of the value function

    output:
    V_new: updated approximation of the value function
    """
    V_new = np.zeros_like(V_old)
    for ii in range(V_new.shape[1]):
        V_new[:,ii] = vf_update(ii, V_old)
    return V_new
```

[22]:
```python
from matplotlib import pyplot as plt

difference_list = []
threshold = 1e-10
max_iterations = 10000
plot_interval = 50

V = V_init.copy()

for iteration in range(max_iterations):
    print('Iteration: {}'.format(iteration + 1))

    V_new = vf_update_iteration(V)
    difference = np.max(np.abs(V_new-V))

    difference_list.append(difference)

    V = V_new.copy()

    if difference < threshold:
```

```
        print('Converged after iteration {}'.format(iteration + 1))

        plt.figure()
        plt.plot(k_grid[0,:], V[0,:], label='s=1')
        plt.plot(k_grid[0,:], V[1,:], label='s=2')
        plt.xlabel('k')
        plt.ylabel('V(k)')
        plt.title('Value function after convergence')
        plt.show();
        break

    if iteration%plot_interval == 25:

        plt.figure()
        plt.plot(np.arange(1, iteration+2), np.log10(np.array(difference_list)))
        plt.xlabel('Iteration')
        plt.ylabel('Maximum update [log10]')
        plt.show();
```

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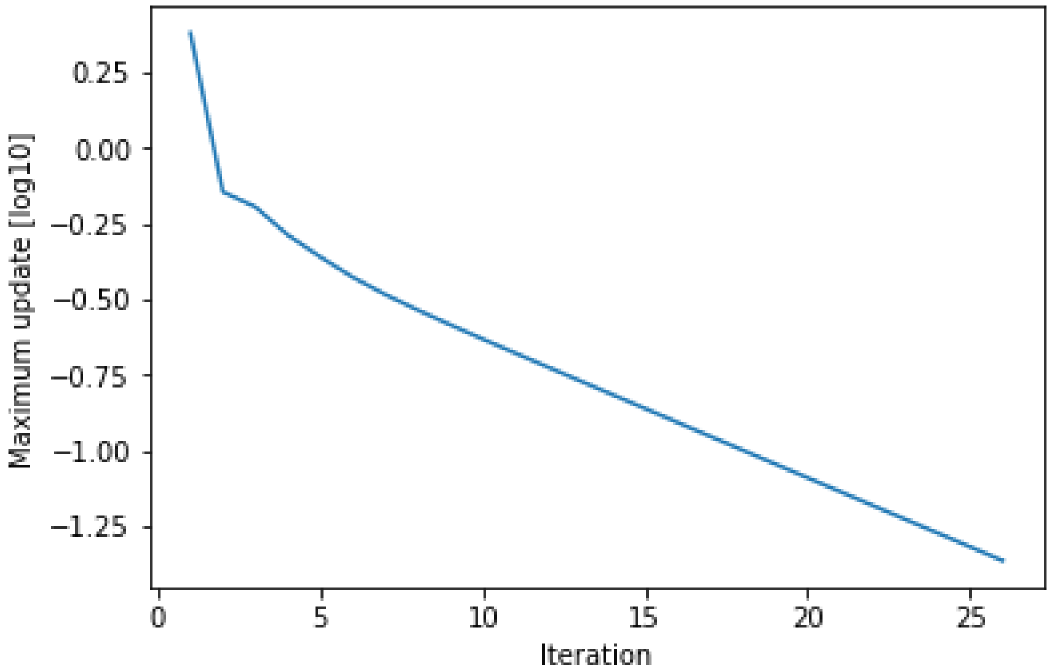


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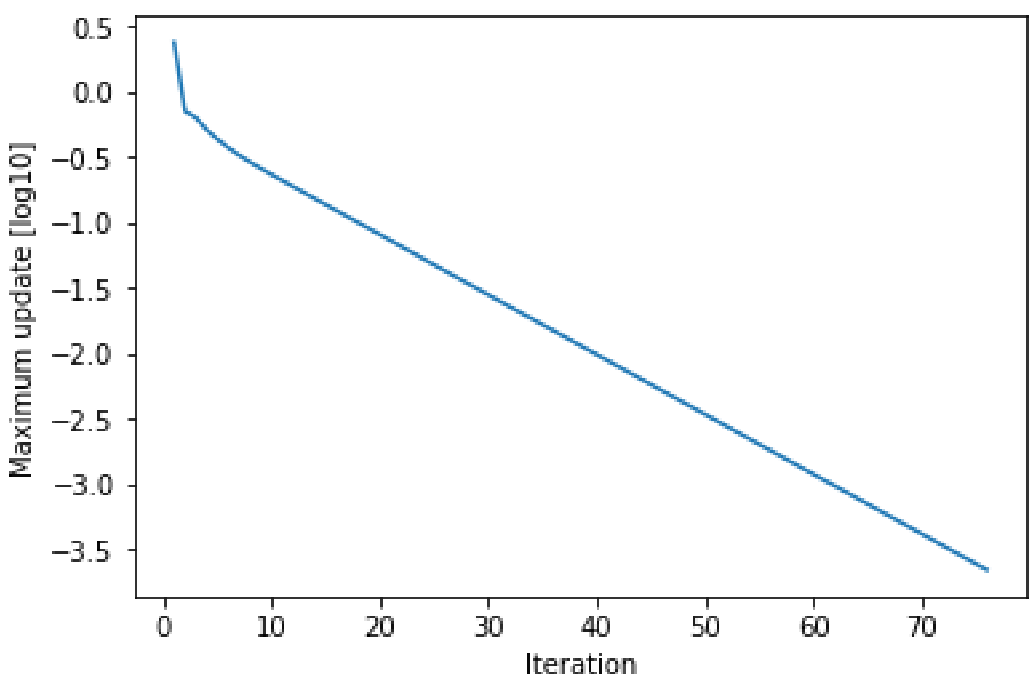

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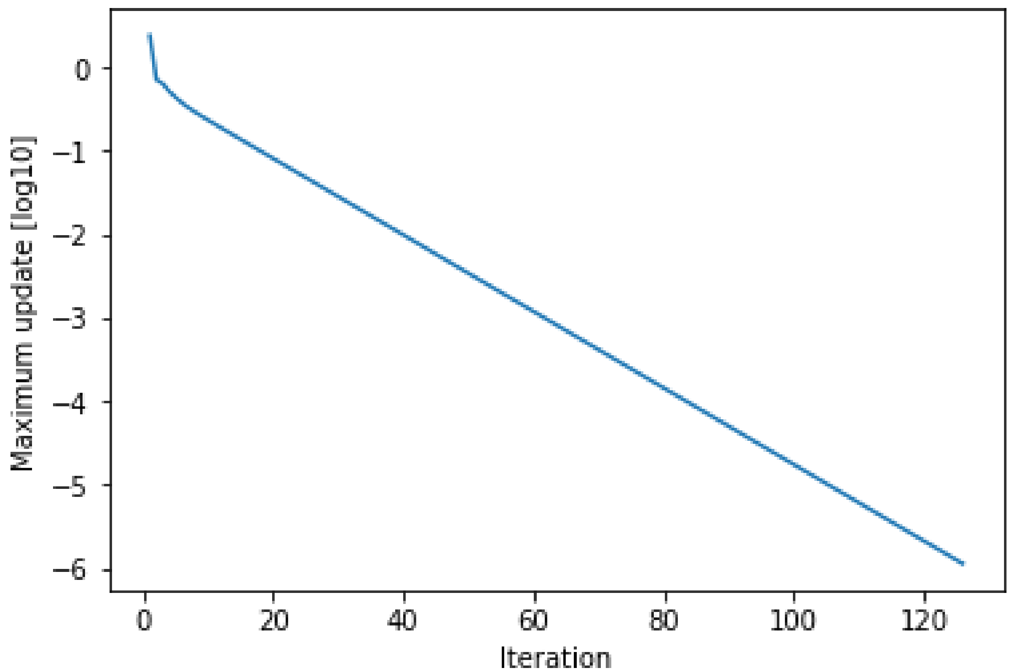


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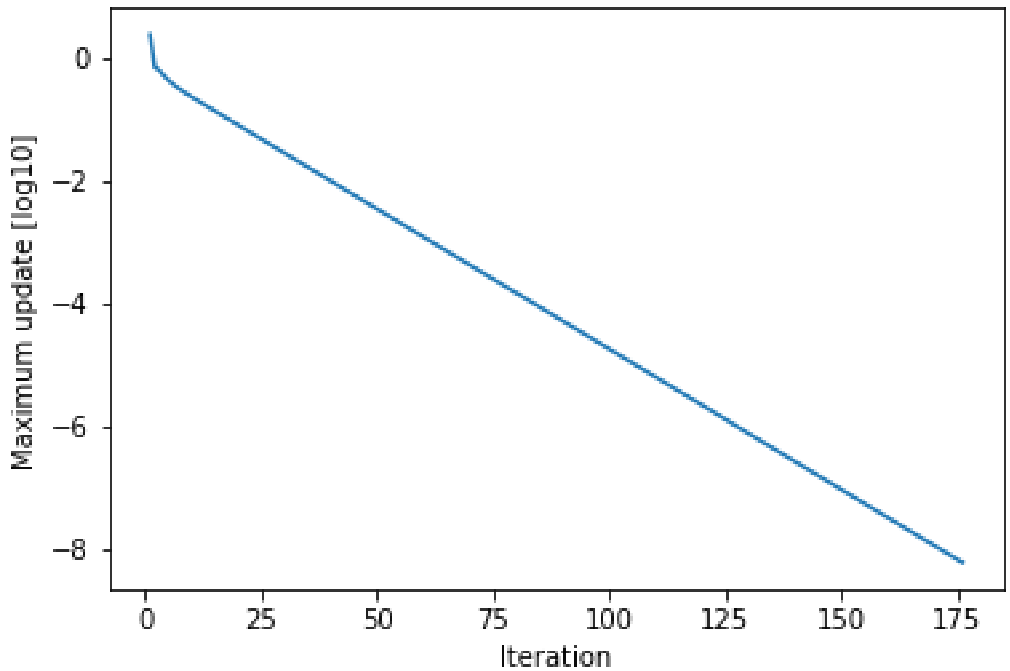


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Converged after iteration 215
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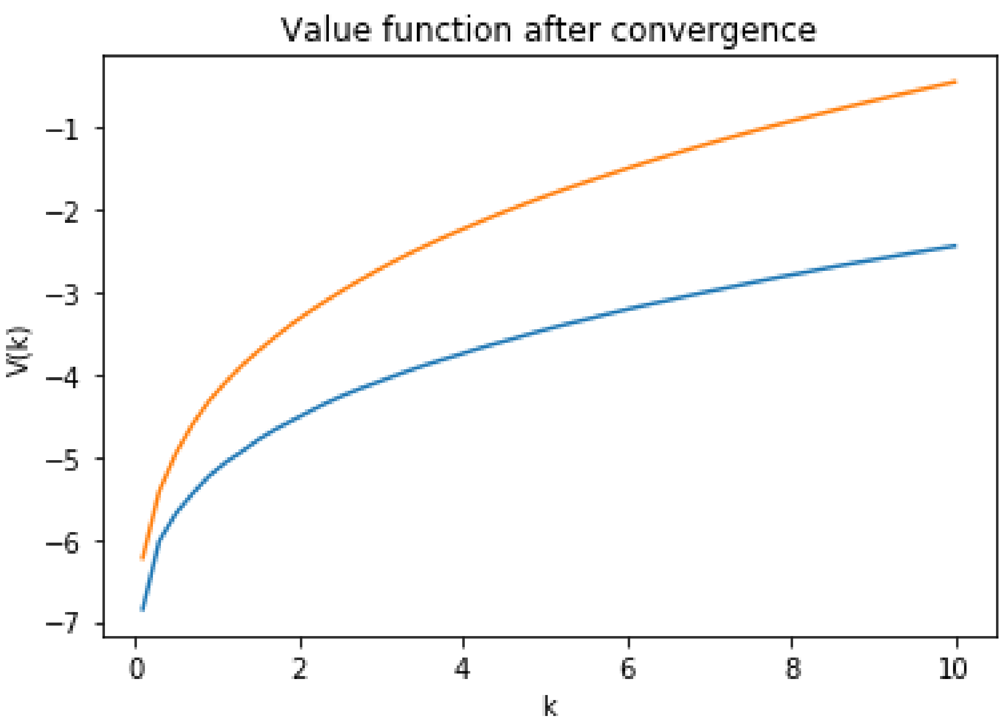

[23]:
```
# N=500
```

[24]:
```
N = 500 # number of grid-points for the capital grid
k_low = 0.1 # lower bound for the capital grid
k_high = 10 # upper bound for capital grid
k_grid = np.linspace(k_low, k_high, N).reshape(1, N) #grid for capital

# Now we initialize the value function, I like to initialize it to zero.
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```python
V_init = np.zeros((2, N))
```

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```python
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    V_new = vf_update_iteration(V)
    difference = np.max(np.abs(V_new-V))

    difference_list.append(difference)

    V = V_new.copy()

    if difference < threshold:
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        plt.figure()
        plt.plot(k_grid[0,:], V[0,:], label='s=1')
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        plt.title('Value function after convergence')
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        break

    if iteration%plot_interval == 25:

        plt.figure()
        plt.plot(np.arange(1, iteration+2), np.log10(np.array(difference_list)))
        plt.xlabel('Iteration')
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        plt.show();
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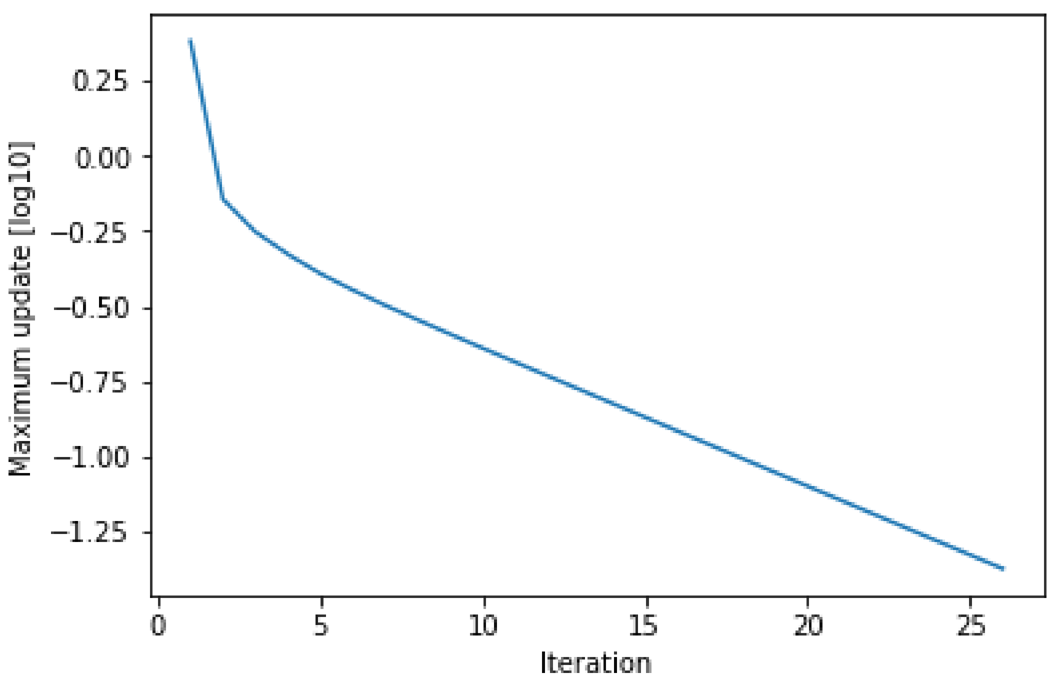


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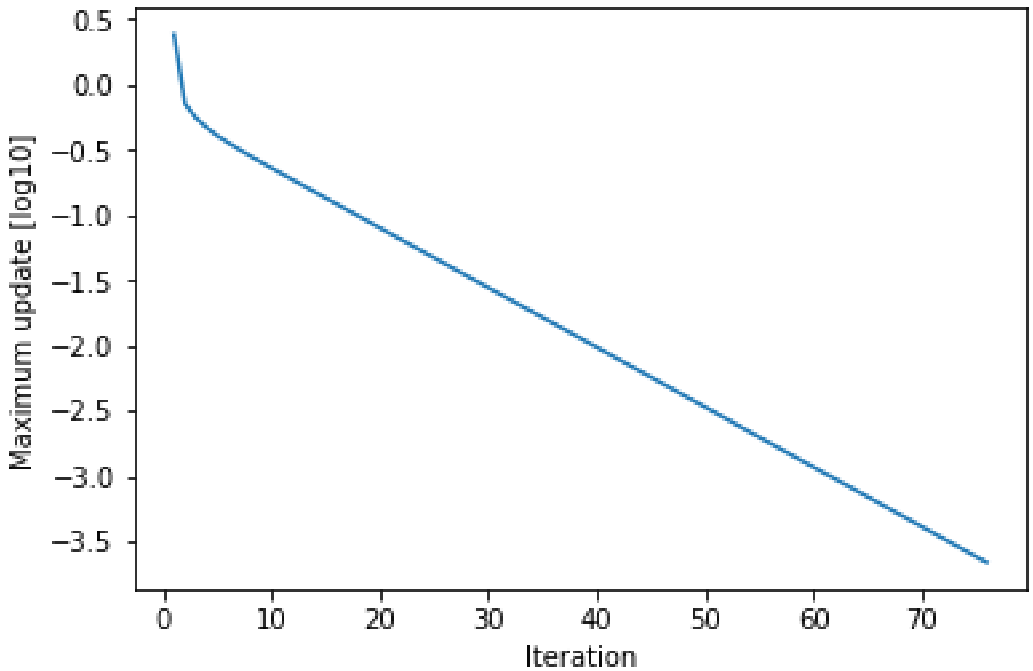


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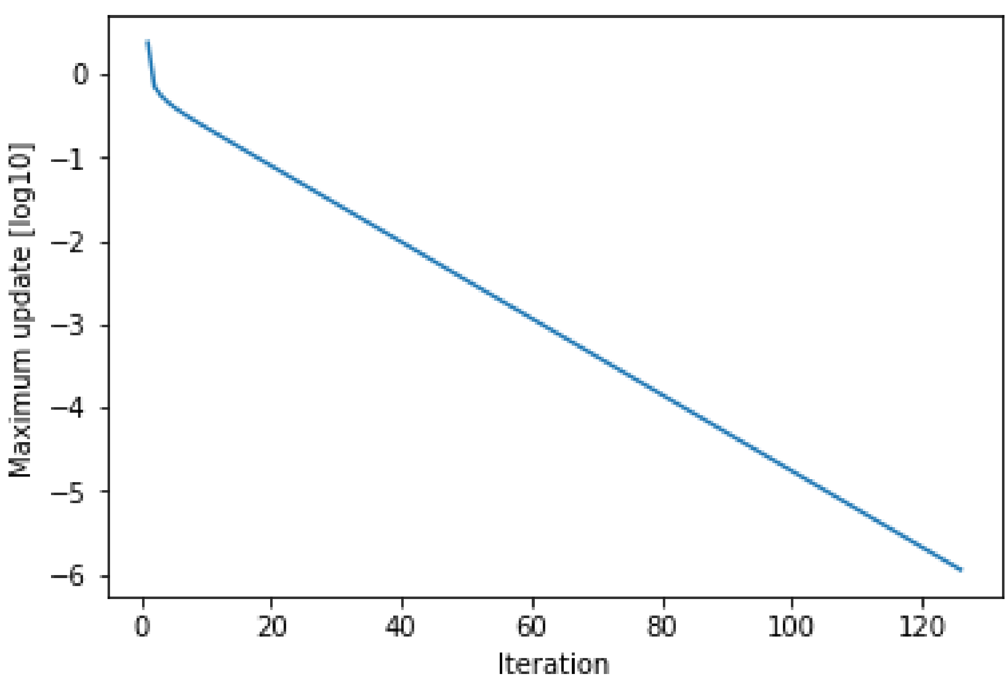

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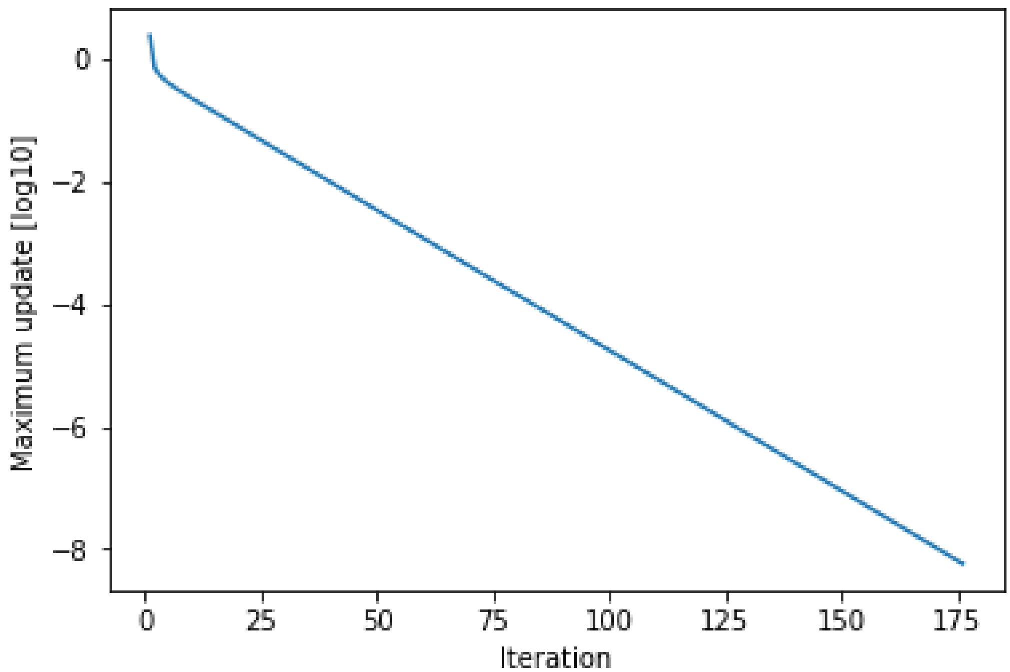


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Converged after iteration 215
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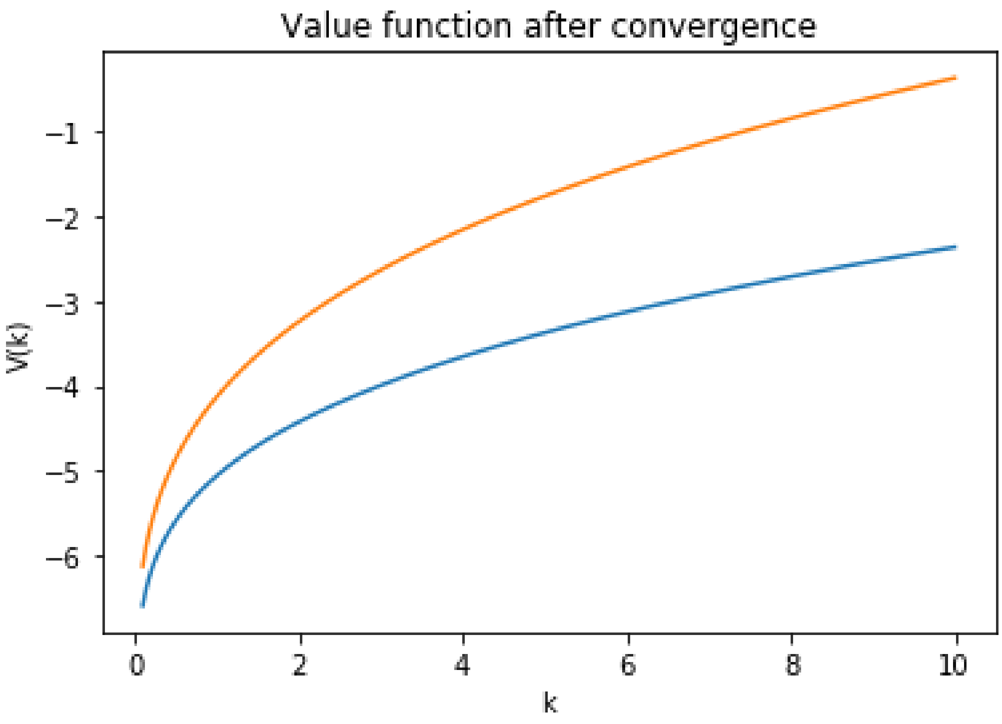

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