Problem Set: DSGE

OSELab 2019
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Exercise 1 Consumers in the Brock and Mirman model do not face a trade-off between labor and leisure.We can solve the model by guessing an initial policy function:

$$K_{t+1} = Ae^{z_t}K_t^\alpha \tag{1}$$

$$K_{t+2} = Ae^{\rho z_t}[Ae^{z_t}K_t^\alpha]^\alpha \tag{2}$$

Pluging in (1) and (2), the Euler equation becomes:

$$\frac{1}{e^{z_t}K_t^\alpha - Ae^{z_t}K_t^\alpha} = \frac{\beta\alpha e^{\rho z_t}(Ae^{z_t}K_t^\alpha)^{\alpha-1}}{e^{\rho z_t}(Ae^{\rho z_t}K_t^\alpha)^\alpha - Ae^{\rho z_t}(Ae^{z_t}K_t^\alpha)^\alpha} = \frac{\beta\alpha e^{\rho z_t}(Ae^{z_t}K_t^\alpha)^{\alpha-1}}{e^{\rho z_t}(Ae^{z_t}K_t^\alpha)^\alpha(1-A)}$$

$$\iff \frac{Ae^{z_t}K_t^\alpha}{e^{z_t}K_t^\alpha(1-A)} = \frac{\alpha\beta}{(1-A)}$$

$$\iff A = \alpha\beta$$

Exercise 2

Functional forms:

$$u(c_t, l_t) = \ln(c_t) + \alpha\ln(1-l_t)$$
$$F(K_t, l_t, z_t) = e^{z_t}K_t^\alpha L_t^{1-\alpha}$$

Characterizing equations:

$$c_t = (1-\tau)[w_t l_t + (r_t - \delta)k_t] + k_t + T_t - k_{t+1} \tag{3}$$

$$\frac{1}{c_t} = \beta E\{\frac{1}{c_{t+1}}[(r_{t+1} - \delta)(1-\tau) + 1]\} \tag{4}$$

$$\frac{\alpha}{1-l_t} = \frac{1}{c_t}w_t(1-\tau) \tag{5}$$

$$r_t = \alpha e^{z_t}k_t^{\alpha-1}l_t^{1-\alpha} \tag{6}$$

$$w_t = (1-\alpha)e^{z_t}k_t^\alpha l_t^{-\alpha} \tag{7}$$

$$\tau(w_t l_t + (r_t - \delta)k_t) = T_t \tag{8}$$

$$z_t = (1-\rho_t)\overline{z} + \rho_t z_{t-1} + \epsilon_t^z \tag{9}$$

We cannot use the same trick as in exercise 1. The reason is that leisure now enters the utility function and the consumer faces the labor-leisure trade-off. The expectation in the Euler Equation would no longer disappear and hence, we could not solve it with the same approach as before anymore.

Exercise 3

Functional forms:

$$u(c_t, l_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma} + \alpha ln(1 - l_t)$$

$$F(K_t, l_{t,z_t}) = e^{z_t} K_t^{\alpha} l_t^{1-\alpha}$$

Characterizing equations:

$$c_t = (1 - \tau)[w_t l_t + (r_t - \delta)k_t] + k_t + T_t - k_{t+1} \tag{10}$$

$$c_t^{-\gamma} = \beta E\{c_{t+1}{}^{-\gamma}[(r_{t+1} - \delta)(1 - \tau) + 1]\} \tag{11}$$

$$\frac{\alpha}{1 - l_t} = c_t^{-\gamma} w_t (1 - \tau) \tag{12}$$

$$r_t = \alpha e^{z_t} k_t^{\alpha-1} l_t^{1-\alpha} \tag{13}$$

$$w_t = (1 - \alpha) e^{z_t} k_t^{\alpha} l_t^{-\alpha} \tag{14}$$

$$\tau(w_t l_t + (r_t - \delta)k_t) = T_t \tag{15}$$

$$z_t = (1 - \rho_t)\overline{z} + \rho_t z_{t-1} + \epsilon_t^z \tag{16}$$

Exercise 4

Functional forms:

$$u\left(c_t, l_t\right) = \frac{c_t^{1-\gamma} - 1}{1-\gamma} + a\frac{\left(1 - \ell_t\right)^{1-\xi} - 1}{1-\xi}$$

$$F\left(K_t, L_t, z_t\right) = e^{z_t}\left[\alpha K_t^{\eta} + (1-\alpha) L_t^{\eta}\right]^{\frac{1}{\eta}}$$

Characterizing equations:

$$c_t = (1 - \tau)\left[w_t l_t + \left(r_t - \delta\right) k_t\right] + k_t + T_t - k_{t+1} \tag{17}$$

$$c_t^{-\gamma} = \beta E\left\{c_{t+1}^{-\gamma}\left[\left(r_{t+1} - \delta\right)\left(1 - \tau\right) + 1\right]\right\} \tag{18}$$

$$a\left(1 - l_t\right)^{-\xi} = c_t^{-\gamma} w_t (1 - \tau) \tag{19}$$

$$r_t = e^{z_t}\alpha k_t^{\eta-1}\left(\alpha k_t^{\eta} + (1-\alpha)l_t^{\eta}\right)^{1/\eta-1} \tag{20}$$

$$w_t = e^{z_t}(1-\alpha)l_t^{\eta-1}\left(\alpha k_t^{\eta} + (1-\alpha)l_t^{n}\right)^{1/\eta-1} \tag{21}$$

$$\tau\left[w_t l_t + \left(r_t - \delta\right)k_t\right] = T_t \tag{22}$$

$$z_t = \left(1-\rho_z\right)\overline{z} + \rho_z z_{t-1} + \epsilon_t^z \tag{23}$$

Exercise 5

From the exercise:

$$u\left(c_t\right) = \frac{c_t^{1-\gamma}-1}{1-\gamma}$$

$$F\left(K_t, L_t, z_t\right) = K_t^{\alpha}\left(L_t e^{z_t}\right)^{1-\alpha}$$

Characterizing equations:

$$c_t = \left(1-\tau\right)\left[w_t l_t + \left(r_t-\delta\right)k_t\right] + k_t + T_t - k_{t+1} \tag{24}$$

$$c_t^{-\gamma} = \beta E\left\{c_{t+1}^{-\gamma}\left[\left(r_{t+1}-\delta\right)\left(1-\tau\right)+1\right]\right\} \tag{25}$$

$$u_{l_t}(c_t, l_t) = c_t^{-\gamma} w_t (1-\tau) \tag{26}$$

$$r_t = \alpha k_t^{\alpha-1}\left(l_t e^{z_t}\right)^{1-\alpha} \tag{27}$$

$$w_t = (1-\alpha)k_t^{\alpha} l_t^{-\alpha} e^{(1-\alpha)z_t} \tag{28}$$

$$\tau\left[w_t l_t + \left(r_t - \delta\right)k_t\right] = T_t \tag{29}$$

$$z_t = \left(1-\rho_z\right)\overline{z} + \rho_z z_{t-1} + \epsilon_t^z \tag{30}$$

Steady State equations: (using $\overline{z} = 0$ and l_t=1)

$$\overline{c} = (1-\tau)[\overline{w} + (\overline{r}-\delta)\overline{k}] + \overline{T} \tag{31}$$

$$\overline{c}^{-\gamma} = \beta\left(\overline{c}^{-\gamma}(\overline{r}-\delta)(1-\tau)+1\right)] \tag{32}$$

$$u_{\overline{l}}(\overline{c}, \tau) = \overline{c}^{-\gamma}\overline{w}(1-\tau) \tag{33}$$

$$\overline{r} = \alpha\overline{k}^{\alpha-1} \tag{34}$$

$$\overline{w} = (1-\alpha)\overline{k}^{\alpha} \tag{35}$$

$$\tau[\overline{w} + (\overline{r} - \delta)\overline{k}] = \overline{T} \tag{36}$$

Plug (36) into (31):

$$\overline{c} = \overline{w} + (\overline{r} - \delta)\overline{k} \tag{37}$$

re-write (32):

$$1 = \beta(\overline{r} - \delta)(1-\tau) + \beta$$

$$1 - \beta = \beta\overline{r}(1-\tau) - \beta\delta(1-\tau)$$

$$\overline{r} = \frac{1-\beta}{\beta(1-\tau)} + \delta \tag{38}$$

Plug $\overline{r}$ into (38):

$$\overline{k} = \left(\frac{\overline{r}}{\alpha}\right)^{\frac{1}{\alpha-1}} = \left(\frac{1-\beta}{\alpha\beta(1-\tau)} + \delta\right)^{\frac{1}{\alpha-1}} \tag{39}$$

Investment as defined in the readings:

$$I = k_{t+1} - (1-\delta)k_t$$

In the steady state: $k_{t+1} = k_t = \overline{k}$ Hence,

$$I = \delta\overline{k}$$

Output:

$$\overline{y} = \overline{k}^{\alpha}$$

Plugging in the given parameters into the steady state value of $\overline{k} = 7.2875$, I $=$ 0.7288, $\overline{y} = 2.2133$. Solving this problem numerically in python yields to a solution that is extremely close to what I derived analytically (please see .ipyn file for numerical solution).

Exercise 6

Functional forms:

$$u\left(c_t, l_t\right) = \frac{c_t^{1-\gamma} - 1}{1\gamma} + a\frac{\left(1-\ell_t\right)^{1-\xi} - 1}{1-\xi}$$

$$F\left(K_t, L_t, z_t\right) = K_t^{a}\left(L_t e^{z_t}\right)^{1-\alpha}$$

Characterizing equations:

$$c_t = (1-\tau)\left[w_t l_t + (r_t - \delta)\, k_t\right] + k_t + T_t - k_{t+1} \tag{40}$$

$$c_t^{-\gamma} = \beta E\left\{c_{t+1}^{-\gamma}\left[(r_{t+1} - \delta)(1-\tau) + 1\right]\right\} \tag{41}$$

$$a\,(1-l_t)^{-\xi} = c_t^{-\gamma} w_t (1-\tau) \tag{42}$$

$$r_t = \alpha k_t^{\alpha-1}\left(l_t e^{z_t}\right)^{1-\alpha} \tag{43}$$

$$w_t = (1-\alpha) k_t^{\alpha} l_t^{-\alpha} e^{(1-\alpha) z_t} \tag{44}$$

$$\tau\left[w_t l_t + (r_t - \delta)\, k_t\right] = T_t \tag{45}$$

$$z_t = (1-\rho_z)\,\overline{z} + \rho_z z_{t-1} + \epsilon_t^z \tag{46}$$

Steady State equations:

$$\bar{c} = (1-\tau)[\overline{w}\bar{l} + (\overline{r} - \delta)\overline{k}] + \overline{T} \tag{47}$$

$$1 = \beta[(\overline{r} - \delta)(1-\tau) + 1] \tag{48}$$

$$a(1-\bar{l})^{-\xi} = \overline{c}^{-\gamma}\overline{w}(1-\tau) \tag{49}$$

$$\overline{r} = \alpha\overline{k}^{\alpha-1}\overline{l}^{-\alpha} \tag{50}$$

$$\overline{w} = (1-\alpha)\overline{k}^{\alpha}\overline{l}^{-\alpha} \tag{51}$$

$$\tau[\overline{w}\bar{l} + (\overline{r} - \delta)\overline{k}] = \overline{T} \tag{52}$$

(please see .ipyn file for solution)

Exercise 7 (please see .ipyn file for solution)

Problem Set: Linear

Exercise 1

To find the analytical values of F,G,H,L,M, N to the Brock and Mirman model in use Uhlig's notation I start from the following notation from the lecture slides:

$$E_t\left\{\beta\frac{\alpha e^{z_{t+1}}K_{t+1}^{\alpha-1}\left(e^{z_t}K_t^{\alpha}-K_{t+1}\right)}{e^{z_{t+1}}K_{t+1}^{\alpha}-K_{t+2}}\right\}=1 \tag{53}$$

Take FOCs (to be consistent with the lecture notes, but i think we're missing Beta in the lecture notes):

$$F=\beta\frac{\alpha\overline{K}^{\alpha-1}}{\overline{K}^{\alpha}-\overline{K}}$$
$$G=-\beta\frac{\alpha\overline{K}^{\alpha-1}\left(\alpha+\overline{K}^{\alpha-1}\right)}{\overline{K}^{\alpha}-\overline{K}}$$
$$H=\beta\frac{\alpha^2\overline{K}^{2(\alpha-1)}}{\overline{K}^{\alpha}-\overline{K}}$$
$$L=-\beta\frac{\alpha\overline{K}^{2\alpha-1}}{\overline{K}^{\alpha}-\overline{K}}$$
$$M=\beta\frac{\alpha^2\overline{K}^{2(\alpha-1)}}{\overline{K}^{\alpha}-\overline{K}}$$

Further:

$$N=\rho$$

(please see .ipyn file for solution)

Exercise 2 (please see .ipyn file for solution)

Exercise 3

Given:

$$E_t\left\{F\tilde{X}_{t+1}+G\tilde{X}_t+H\tilde{X}_{t-1}+L\tilde{Z}_{t+1}+M\tilde{Z}_t\right\}=0 \tag{54}$$

$$\tilde{Z}_t=N\tilde{Z}_{t-1}+\varepsilon_t \tag{55}$$

$$\tilde{X}_t=P\tilde{X}_{t-1}+Q\tilde{Z}_t \tag{56}$$

The goal is to express equation (1) in terms of $\tilde{X}_{t-1}$ and $\tilde{Z}_t$. Successively plugging (2) and (3) into (1):

$$E_t\left\{F(P\tilde{X}_t+Q\tilde{Z}_{t+1})+G(P\tilde{X}_{t-1}+Q\tilde{Z}_t)+H\tilde{X}_{t-1}+L(N\tilde{Z}_{t-1}+\varepsilon_t)+M\tilde{Z}_t\right\}=0$$

$$E_t\left\{F(P(P\tilde{X}_{t-1}+Q\tilde{Z}_t)+Q(\tilde{Z}_t+\varepsilon_{t+1}))+G(P\tilde{X}_{t-1}+Q\tilde{Z}_t)+H\tilde{X}_{t-1}+L(N\tilde{Z}_{t-1}+\varepsilon_t)+M\tilde{Z}_t\right\}=0$$

$$E_t[\varepsilon_t=0]\ \forall t$$

Hence, we can re-write:

$$F(P(P\tilde{X}_{t-1}+Q\tilde{Z}_t)+Q(\tilde{Z}_t+\varepsilon_{t+1}))+G(P\tilde{X}_{t-1}+Q\tilde{Z}_t)+H\tilde{X}_{t-1}+L(N\tilde{Z}_{t-1}+\varepsilon_t)+M\tilde{Z}_t=0$$

which is equivalent to:

$$[(FP+G)P+H]\tilde{X}_{t-1}+[(FQ+L)N+(FP+G)Q+M]\tilde{Z}_t=0$$

Exercise 4 (please see .ipyn file for solution)
Exercise 5 (please see .ipyn file for solution)
Exercise 6 (please see .ipyn file for solution)
Exercise 7 (please see .ipyn file for solution)
Exercise 8 (will be done in .ipyn if time allows)
Exercise 9 (please see .ipyn file for solution)

Problem Set: Pertubation

Exercise 1

Starting from equation (5) from the readings:

$$\begin{aligned}&F_{xx}\{x(u),u\}x_u(u)x_u(u)+F_{xu}\{x(u),u\}x_u(u)\\&+F_x\{x(u),u\}x_{uu}(u)+F_{xu}\{x(u),u\}x_u(u)\\&+F_{uu}\{x(u),u\}=0\end{aligned}\tag{57}$$

Taking FOC wrt u again:

$$\begin{aligned}&F_{xxx}\{x(u),u\}x_u(u)^3+3*(F_{xxu}\{x(u),u\}x_u(u)^2\\&+F_{uux}\{x(u),u\}x_u(u)+(F_{xu}\{x(u),u\}x_{uu}(u)\\&+F_{xx}\{x(u),u\}x_u(u)x_{uu}(u))+F_x\{x(u),u\}x_{uuu}(u)\\&+F_{uuu}\{x(u),u\}=0\end{aligned}\tag{58}$$

re-arrange yields to (choosing shorter notation for space reasons):

$$x_{uuu}=-\frac{F_{xxx}x_u^3+3\left(F_{xxu}x_u^2+F_{uux}x_u+F_{xu}x_{uu}+F_{xx}x_ux_{uu}\right)+F_{uuu}}{F_x}\tag{59}$$