Dynamic Programming

July 8, 2019

1 Asset Market Equilibrium

Households: h={1,2} with endowments $e^1 = (1,1,2,1,2)$ and $e^2 = (1,3,1,3,1)$; States: $s = 1,2,3,4$; Assets: A^1 and A^2 with Payoffs: $(1,1,1,1)$ and $(1,1,1.5.1.5)$ respectively.

The Agents' utility has the following form:

$$\max_{\theta_1,\theta_2} U(c^z) = v(c_0) + \frac{1}{4}\sum_{i=1}^{4} v(c_i)$$

where $v(c) = \dfrac{c^{1-\gamma}}{1-\gamma}$

For all states in s, agents maximize their utility and choose over the amount of assets to hold: (θ_1^h,θ_2^h)

$$\max_{\theta_1^h,\theta_2^h} U(c^h) = v(c_0^h) + E[v(c_s^h)]$$

s.t.

$$c_0^h = e_0^h - q_1 * \theta_1^h - q_2 * \theta_2^h$$
$$c_s^h = e_s^h + A_s^1 * \theta_1^h + A_s^2 * \theta_2^h$$

The first order conditions for this problem look like the following:

$$-q_1 v'(c_0^1) + E[v'(c_s^1)A_s^1)] = 0$$
$$-q_1 v'(c_0^2) + E[v'(c_s^2)A_s^1)] = 0$$
$$-q_2 v'(c_0^1) + E[v'(c_s^1)A_s^2)] = 0$$
$$-q_2 v'(c_0^2) + E[v'(c_s^2)A_s^2)] = 0$$

As households trade with eachother, the market clears whenever the aggregate holdings are zero for each asset.

$$\theta_1^1 + \theta_1^2 = 0$$
$$\theta_2^1 + \theta_2^2 = 0$$

```
In [2]: import numpy as np
        from scipy.optimize import fsolve

        def AME(x):
            gam=2
            the_1_1=x[0]
            the_1_2=x[1]
```

```
        the_2_1=x[2]
        the_2_2=x[3]
        q_1=x[4]
        q_2=x[5]
        f=np.zeros(6)
        c_0_1=1-q_1*the_1_1-q_2*the_2_1
        c_0_2=1-q_1*the_1_2-q_2*the_2_2
        c_1_1=1+the_1_1+the_2_1
        c_1_2=3+the_1_2+the_2_2
        c_2_1=2+the_1_1+the_2_1
        c_2_2=1+the_1_2+the_2_2
        c_3_1=1+the_1_1+1.5*the_2_1
        c_3_2=3+the_1_2+1.5*the_2_2
        c_4_1=2+the_1_1+1.5*the_2_1
        c_4_2=1+the_1_2+1.5*the_2_2
        f[0]=-q_1*c_0_1**(-gam)+0.25*(c_1_1**(-gam)+c_2_1**(-gam)+c_3_1**(-gam)+c_4_1**(-ga
        f[1]=-q_1*c_0_2**(-gam)+0.25*(c_1_2**(-gam)+c_2_2**(-gam)+c_3_2**(-gam)+c_4_2**(-ga
        f[2]=-q_2*c_0_1**(-gam)+0.25*(c_1_1**(-gam)+c_2_1**(-gam)+1.5*c_3_1**(-gam)+1.5*c_
        f[3]=-q_2*c_0_2**(-gam)+0.25*(c_1_2**(-gam)+c_2_2**(-gam)+1.5*c_3_2**(-gam)+1.5*c_
        f[4]=the_1_1+the_1_2
        f[5]=the_2_1+the_2_2
        return f
```

```
In [2]: fsolve(AME, [0.1, -0.1, 0.1, -0.1, 1, 1])
```

```
Out[2]: array([ 1.95406655e-02, -1.95406655e-02,  1.18668498e-11, -1.18668498e-11,
                5.89777656e-01,  7.37222070e-01])
```

As expected, we see that for each asset one agent is long and the other one is short in an asset. (Side note: the values of the holdings in each asset for each agent greatly depend on the initial guess in the fsolve command). By increasing the risk aversion coefficient gamma, the intuition is that safer assets are more desired than risky ones. However, increasing gamma does not show consistent change in the holdings.

2 Tic Tac Toe

```
In [3]: import copy

        class XO:
            def opp_sign(self, sign):
                return 'O' if sign == 'X' else 'X'

        class Board:
            def __init__(self, size):
                self.s = size
                self.q = size*size
                self.empty = [i for i in range(self.q)]
                self.grid = ['.'] * self.q
```

```
    def ins(self, move, sign):
        self.grid[move] = sign
        self.empty = [i for i in self.empty if i != move]
    def is_full(self):
        return not len(self.empty)
    def get_col(self, col):
        return [self.grid[i] for i in range(col-1, self.q, self.s)]
    def get_row(self, row):
        return self.grid[(row-1)*self.s:row*self.s]
    def get_diag1(self):
        return [self.grid[i] for i in range(0, self.q, self.s+1)]
    def get_diag2(self):
        return [self.grid[i] for i in range(self.s-1, self.q, self.s-1)][:-1]
    def __str__(self):
        return '\n'.join([' '.join(map(str,self.grid[i:i+self.s]))
                for i in range(0, self.q, self.s)]) + '\n'

class Tree:
    def find_best_move(self,board,depth,sign):

        if (board.empty==[]): return None

        best_move=-(2**(board.s**2))
        m=board.empty[0]
        for move in board.empty:
            b=copy.deepcopy(board)
            b.ins(move,sign)
            if (self.is_win(b,sign) or self.is_win(b,xo.opp_sign(sign))):
                return move
            curr_move=self.minimax(b,depth,False,xo.opp_sign(sign))
            if (curr_move > best_move):
                best_move = curr_move
                m=move
        return m

    def minimax(self,board,depth,myTurn,sign):

        if (self.is_win(board,xo.opp_sign(sign))):
            if myTurn:
                return -(board.s**2+1) + depth
            else:
                return (board.s**2+1) - depth

        elif (board.is_full()):
            return 0

        if (myTurn):
            bestVal=-(2**700)
```

```
            for move in board.empty: #empty - the empty squares at the board
                b = copy.deepcopy(board)
                b.ins(move, sign)
                value=self.minimax(b,depth+1,not myTurn, xo.opp_sign(sign))
                #xo.opp_sign(sign) - if function for the opposite sign: x=>o and o=>x
                bestVal = max([bestVal,value])

        else:
            bestVal = (2**700)
            for move in board.empty:
                b = copy.deepcopy(board)
                b.ins(move, sign)
                value = self.minimax(b, depth + 1, not myTurn, xo.opp_sign(sign))

                bestVal = min([bestVal, value])

        return bestVal

    def is_win(self,board, sign):

        temp=board.s
        wins = []
        for i in range(1, temp + 1):
            wins.append(board.get_col(i))
            wins.append(board.get_row(i))
        wins.append(board.get_diag1())
        wins.append(board.get_diag2())

        for i in wins:
            if (self.is_same(i, sign)):
                return True
        return False

    def is_same(self, l, sign):
        for i in l:
            if (i != sign):
                return False
        return True

xo = XO()
board = Board(3)
tree = Tree()
sign = 'O'
human = False
while not board.is_full() and not tree.is_win(board, sign):
    sign = xo.opp_sign(sign)
    human = not human
    if human:
```

```python
            move = input('Play {} (0-8):'.format(sign))
        else:
            print('on it...')
            move = tree.find_best_move(board, 0, sign)
        board.ins(int(move), sign)
        print (board)
```

```
Play X (0-8): 2


. . X
. . .
. . .

on it...
. . X
. O .
. . .



Play X (0-8): 6


. . X
. O .
X . .

on it...
. O X
. O .
X . .



Play X (0-8): 7


. O X
. O .
X X .

on it...
. O X
. O .
X X O
```

```
Play X (0-8): 0


X O X
. O .
X X O

on it...
X O X
O O .
X X O



Play X (0-8): 5


X O X
O O X
X X O
```

3 Ramsey 1

In [23]:
```
#50 points
```

In [19]:
```
N = 50 # number of grid-points for the capital grid
k_low = 0.1 # lower bound for the capital grid
k_high = 10 # upper bound for capital grid
k_grid = np.linspace(k_low, k_high, N).reshape(1, N) #grid for capital

# Now we initialize the value function, I like to initialize it to zero.
V_init = np.zeros((2, N))
beta = 0.9
```

In [21]:
```
def u(c):
    """
    input:
    c: consumption

    output:
    utility received from consumption
    """
    return np.log(c)

#Value function update for a given state:
```

```python
def actionvalue_allchoices(k_index, V_old):
    """
    input:
    k_index: index so that k_grid[index] corresponds to value of capital this period
    V_old: approximation to the value function. V_old[i] approximates V(k_grid[i]).

    output:
    action_value: value of all possible state-action pairs.
    """
    k = k_grid[0,k_index]

    action_value = np.zeros_like(V_old) #(2,N)
    c = np.zeros_like(V_old)
    c[0,:] = 0.9*k**0.3 + 0.3 * k - k_grid # consumption implied by policy k_next in
    c[1,:] = 1.1*k**0.3 + 0.9 * k - k_grid

    action_value[c <= 0] = -999999 # set value to -HUGE for negative consumption
    action_value[c > 0] = u(c[c > 0])

    EV_old=V_old.mean(axis=0).reshape(1,N)

    action_value=action_value + beta * EV_old #(2,N)

    return action_value

def vf_update(i, V_old):
    """
    input:
    i: index corresponding to the entry of the value-function vector which is updated
    V_old: value function vector from the previous iteration

    output:
    Vi_new: updated value for the value function vector at entry i.
    """
    Vi_new = actionvalue_allchoices(i, V_old).max(axis=1)

    return Vi_new

#one update iteration:
def vf_update_iteration(V_old):
    """
    input:
    V_old: array with current approximation of the value function

    output:
    V_new: updated approximation of the value function
    """
    V_new = np.zeros_like(V_old)
```

```
        for ii in range(V_new.shape[1]):
            V_new[:,ii] = vf_update(ii, V_old)
        return V_new
```

In [22]:
```
from matplotlib import pyplot as plt

difference_list = []
threshold = 1e-10
max_iterations = 10000
plot_interval = 50

V = V_init.copy()

for iteration in range(max_iterations):
    print('Iteration: {}'.format(iteration + 1))

    V_new = vf_update_iteration(V)
    difference = np.max(np.abs(V_new-V))

    difference_list.append(difference)

    V = V_new.copy()

    if difference < threshold:
        print('Converged after iteration {}'.format(iteration + 1))

        plt.figure()
        plt.plot(k_grid[0,:], V[0,:], label='s=1')
        plt.plot(k_grid[0,:], V[1,:], label='s=2')
        plt.xlabel('k')
        plt.ylabel('V(k)')
        plt.title('Value function after convergence')
        plt.show();
        break

    if iteration%plot_interval == 25:

        plt.figure()
        plt.plot(np.arange(1, iteration+2), np.log10(np.array(difference_list)))
        plt.xlabel('Iteration')
        plt.ylabel('Maximum update [log10]')
        plt.show();
```

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Iteration: 1
Iteration: 2
Iteration: 3
Iteration: 4
Iteration: 5
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Iteration: 6
Iteration: 7
Iteration: 8
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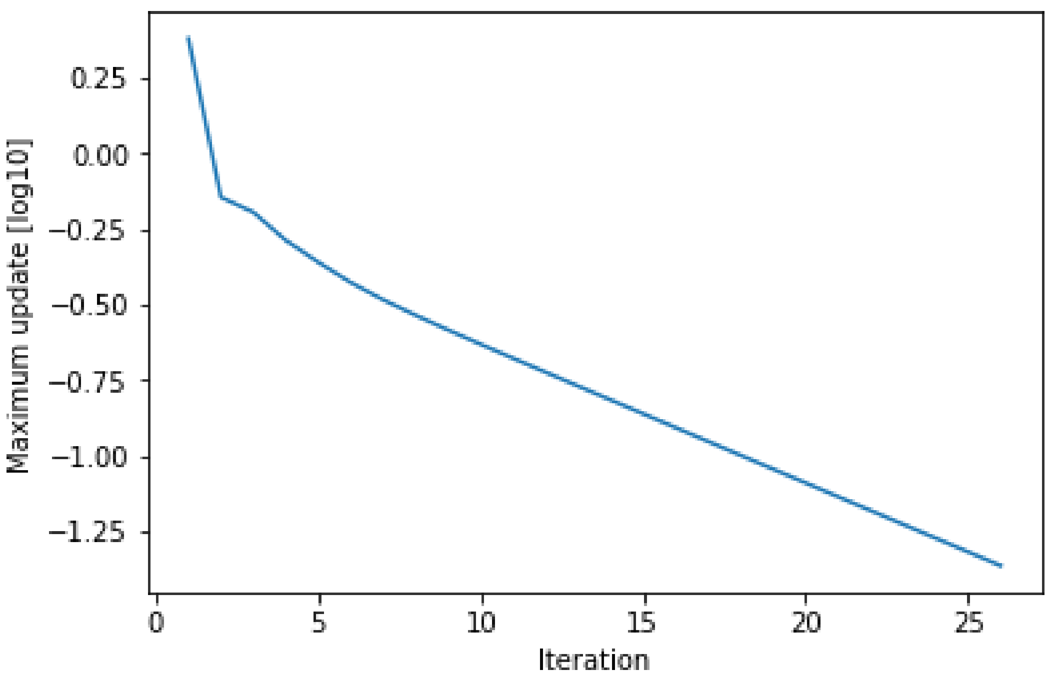
Iteration: 26
```

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Iteration: 28
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Iteration: 29
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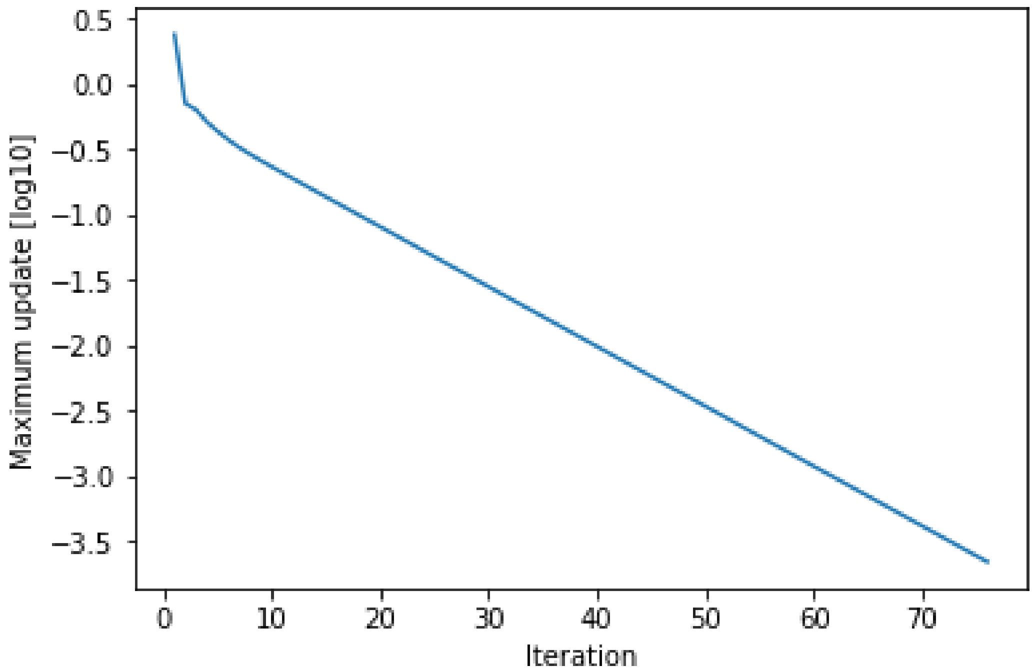


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Iteration: 126
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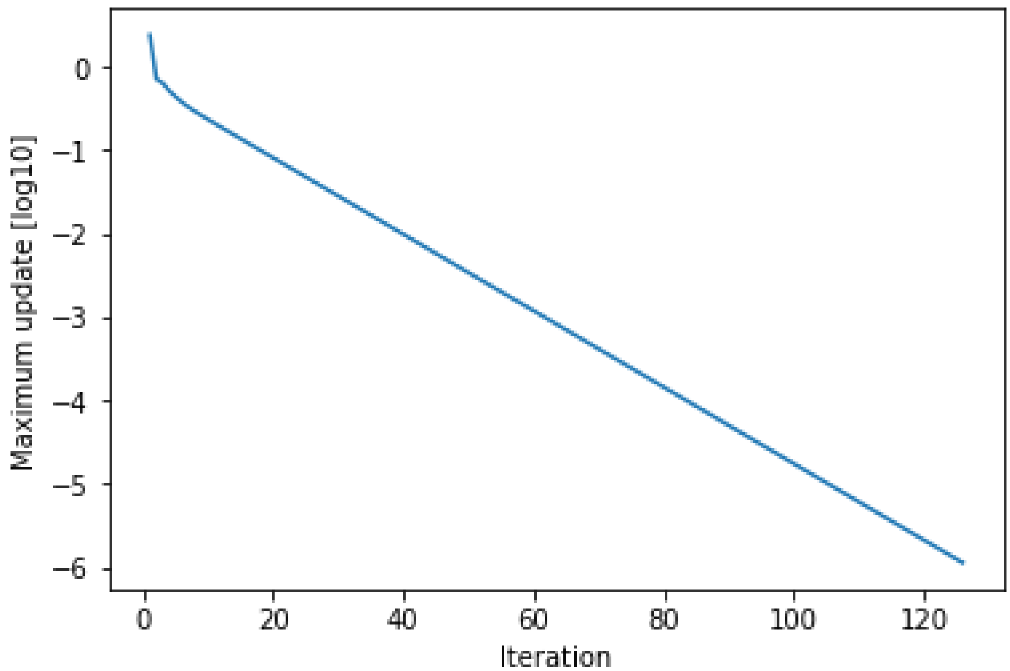


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Iteration: 127
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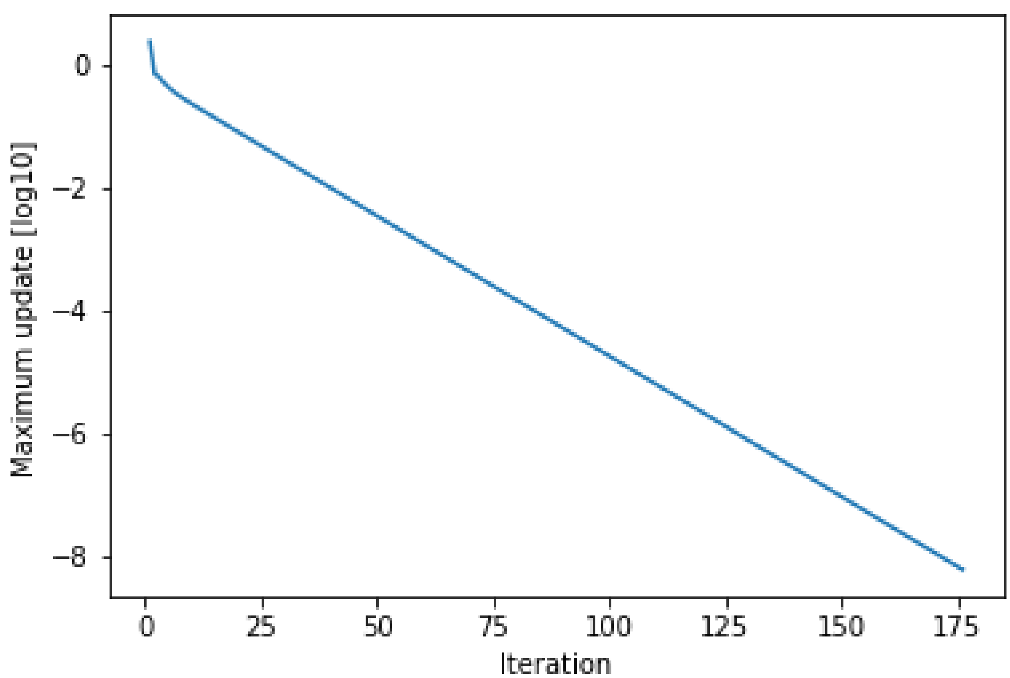

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Converged after iteration 215
```

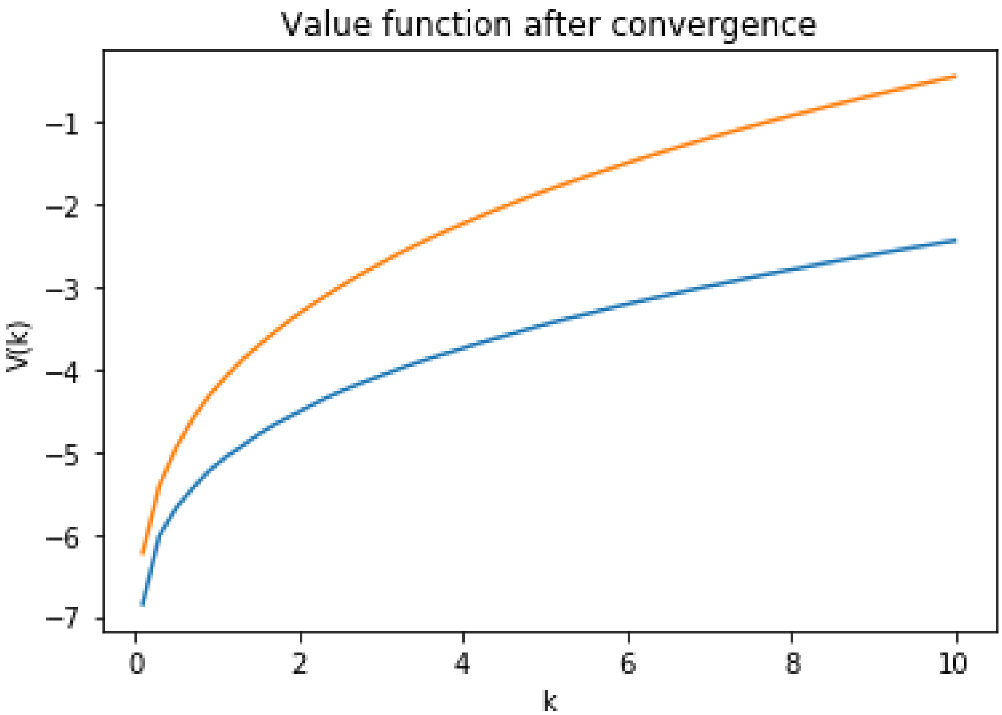


```
In [23]: # N=500

In [24]: N = 500 # number of grid-points for the capital grid
         k_low = 0.1 # lower bound for the capital grid
         k_high = 10 # upper bound for capital grid
         k_grid = np.linspace(k_low, k_high, N).reshape(1, N) #grid for capital

         # Now we initialize the value function, I like to initialize it to zero.
         V_init = np.zeros((2, N))

In [25]: from matplotlib import pyplot as plt

         difference_list = []
         threshold = 1e-10
         max_iterations = 10000
         plot_interval = 50

         V = V_init.copy()

         for iteration in range(max_iterations):
             print('Iteration: {}'.format(iteration + 1))

             V_new = vf_update_iteration(V)
             difference = np.max(np.abs(V_new-V))
```

```python
            difference_list.append(difference)

            V = V_new.copy()

            if difference < threshold:
                print('Converged after iteration {}'.format(iteration + 1))

                plt.figure()
                plt.plot(k_grid[0,:], V[0,:], label='s=1')
                plt.plot(k_grid[0,:], V[1,:], label='s=2')
                plt.xlabel('k')
                plt.ylabel('V(k)')
                plt.title('Value function after convergence')
                plt.show();
                break

            if iteration%plot_interval == 25:

                plt.figure()
                plt.plot(np.arange(1, iteration+2), np.log10(np.array(difference_list)))
                plt.xlabel('Iteration')
                plt.ylabel('Maximum update [log10]')
                plt.show();
```

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Iteration: 1
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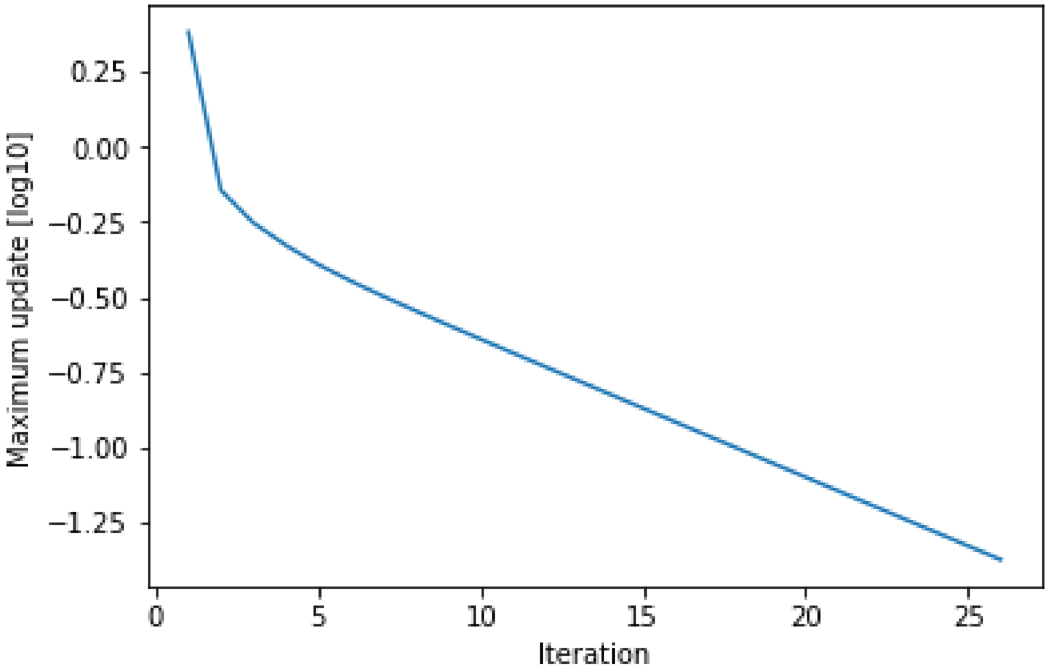


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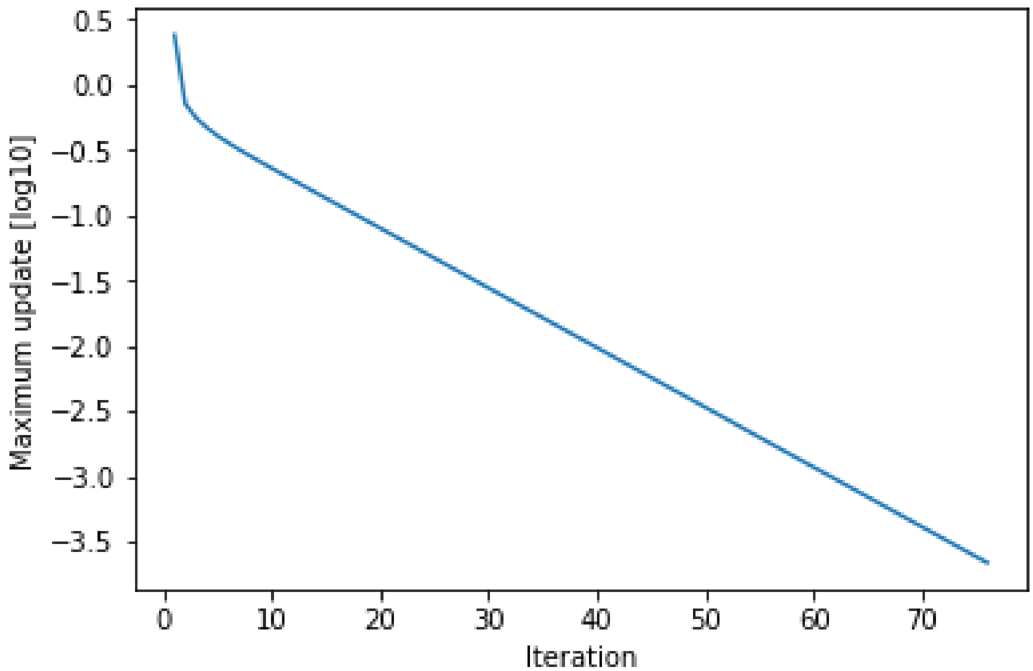


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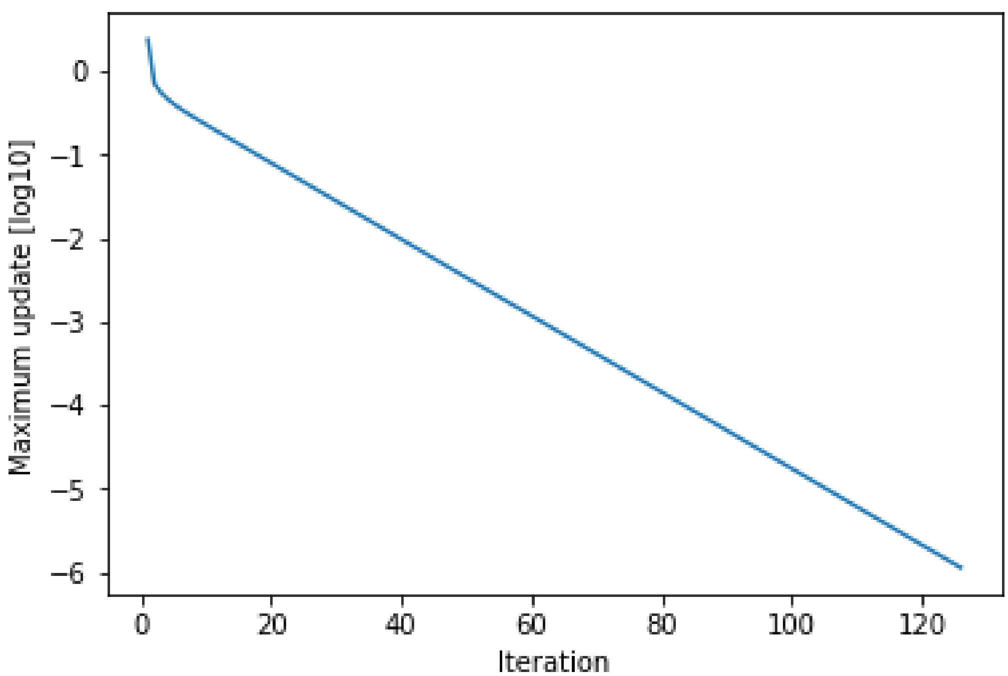

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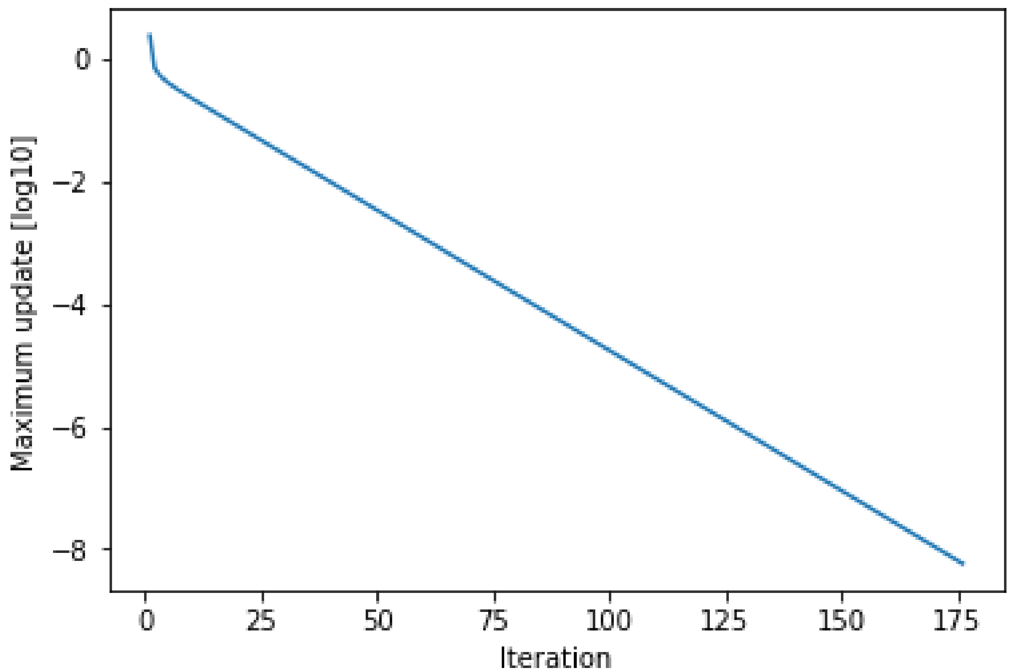


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Iteration: 200
Iteration: 201
Iteration: 202
Iteration: 203
Iteration: 204
Iteration: 205
Iteration: 206
Iteration: 207
Iteration: 208
Iteration: 209
Iteration: 210
Iteration: 211
Iteration: 212
Iteration: 213
Iteration: 214
Iteration: 215
Converged after iteration 215
```