On the Learning Parity with Noise Problem

Luca Melis

Università degli Studi di Firenze

Århus Universitet

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Advisors:
Prof. Alessandro Piva
Prof. Fabrizio Argenti

Co-advisors:
Dr. Claudio Orlandi
Prof. Ivan Damgård

Scenario

Cryptography schemes

- address the security of communication across an insecure medium
- are usually based only on complexity assumptions (standard model)

Near Future:

- Problem: What if someone constructs large quantum computers?
- Cryptography world may fall apart:
  1. cryptographic assumptions broken by efficient quantum algorithms
  2. proof of security becomes invalid

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Post-Quantum cryptography

Schemes that are believed to resist classical & quantum computers

- **Hash-based cryptography**
- **Code-based cryptography**
- **Lattice-based cryptography**

Our contribution

- We investigate about the Learning Parity with Noise (LPN) problem
- We propose a Threshold Public-Key Encryption scheme based on LPN
- We propose a Commitment protocol based on LPN

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Learning Parity with Noise Problem LPN

- Dimension ℓ (security parameter), $q \gg \ell$, $\tau \in \left(0, \frac{1}{2}\right]$
- **Search**: find $\boldsymbol{s} \in \mathbb{Z}_2^\ell$ given "noisy random inner products"

Errors $e_i \leftarrow \mathrm{Ber}_\tau$, i.e. $\Pr(e_i = 1) = \tau$

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$$\vdots$$

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- **Decision**: <u>distinguish</u> (a_i, b_i) from uniform (a_i, b_i)
- *decisional and search* LPN are "*polynomially equivalent*"

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Learning Parity with Noise Problem LPN

Hardness of LPN

Breaking the search LPN problem takes

- $2^{\Theta(\ell/\log \ell)}$ having the same number of samples q
- $2^{\Theta(\ell/\log\log \ell)}$ having $q = poly(\ell)$ samples
- $2^{\Theta(\ell)}$ having $q = \Theta(\ell)$ samples

where ℓ is the security parameter

Interesting features

- **Efficiency** $\Rightarrow$ suitable for limited computing power devices (e.g. RFID).
- **quantum algorithm resistance**

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Threshold Public-Key Encryption schemes

Public-key cryptography

- the ability of decrypting or signing is restricted to the owner of the secret key.
- $\Rightarrow$ only one person has all the power

Threshold Public-Key Encryption schemes

Solution: Threshold PKE

- Shares trust among a group of users, such that *enough* of them, the *threshold*, is needed to sign or decrypt
- The secret key is split into shares and each share is given to a group of users.

Our contribution

A Threshold Public-Key Encryption scheme which is:

- based on LPN
- secure in the *Semi-honest* model

Alekhnovich PKE scheme

Key Generation

The receiver R chooses

- a secret key $\boldsymbol{s} \xleftarrow{R} \mathbb{Z}_2^{\ell}$
- $\boldsymbol{A} \xleftarrow{R} \mathbb{Z}_2^{q\times\ell}$ and the error $\boldsymbol{e} \leftarrow \mathrm{Ber}_{\tau}^{q}$, where $\tau \in \Theta(\frac{1}{\sqrt{\ell}})$ and computes the pk as $(\boldsymbol{A}, \boldsymbol{b} = \boldsymbol{A}\boldsymbol{s} \oplus \boldsymbol{e})$

Encryption of a message bit $m \in \mathbb{Z}_2$

Sender S | Receiver R

choose a vector $\boldsymbol{f} \leftarrow \mathrm{Ber}_{\tau}^{q}$

compute $\boldsymbol{u} = \boldsymbol{f} \cdot \boldsymbol{A}$

$c = \langle \boldsymbol{f}, \boldsymbol{b} \rangle \oplus m \quad \xrightarrow{(\boldsymbol{u}, c)}$

Decryption

The receiver R computes $d = c \oplus \langle \boldsymbol{s}, \boldsymbol{u} \rangle = \cdots = \langle \boldsymbol{f}, \boldsymbol{e} \rangle \oplus m$

ThPKE: Protocol phases

- **Key Generation**
- **Key Assembly**
- **Encryption**
- **Partial Decryption**
- **Finish Decryption**

ThPKE: Protocol phases

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Key Generation

- All the receivers share a matrix $\boldsymbol{A} \xleftarrow{R} \mathbb{Z}_2^{q \times \ell}$
- Each receiver $\mathtt{R_i}$ indipendently choose a secret key $\boldsymbol{s_i} \xleftarrow{R} \mathbb{Z}_2^{\ell}$ and an error $\boldsymbol{e_i} \leftarrow \mathrm{Ber}_\tau^q$
- the public key for $\mathtt{R_i}$ is the pair $(\boldsymbol{A}, \boldsymbol{b_i} = \boldsymbol{A}\boldsymbol{s_i} \oplus \boldsymbol{e_i})$

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Key Assembly

The combined public key is the pair $(\boldsymbol{A}, \boldsymbol{b})$, where

$$\boldsymbol{b} = \bigoplus_{i \in I} \boldsymbol{b}_i$$

and I is the users subset

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Sender S

Receivers $\mathtt{R_i}, \mathtt{R_j}$

$$(\boldsymbol{C_1}, \boldsymbol{c_2}) \leftarrow \texttt{ThLPN.Enc}(m, \boldsymbol{b})$$

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$$(\boldsymbol{C_1}, \boldsymbol{c_2}) \leftarrow \mathtt{ThLPN.Enc}(m, \boldsymbol{b})$$

Encryption function (Alekhnovich scheme)

$$\boldsymbol{C_1} = \boldsymbol{F} \cdot \boldsymbol{A}, \ \boldsymbol{c_2} = \boldsymbol{F} \cdot \boldsymbol{b} \oplus \underbrace{\begin{bmatrix} 1 \\ \dots \\ 1 \end{bmatrix}}_{q} \cdot m \qquad \text{where } \boldsymbol{F} := \begin{bmatrix} \boldsymbol{f_1} \\ \dots \\ \boldsymbol{f_q} \end{bmatrix}, \ \boldsymbol{f_i} \leftarrow \mathrm{Ber}_{\tau}^{q}$$

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Receiver $\mathtt{R_i}$

Receiver $\mathtt{R_j}$

$$\boldsymbol{d_i} \leftarrow \mathtt{ThLPN.Pdec}(\boldsymbol{C_1}, \boldsymbol{c_2}, \boldsymbol{s_i})$$

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Partial decryption function (Alekhnovich scheme)

$$\boldsymbol{d_i} = \boldsymbol{C_1} \cdot \boldsymbol{s_i} \oplus \boldsymbol{\nu_i}$$

where $\boldsymbol{\nu_i} \leftarrow \mathrm{Ber}_\sigma^q$

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Finish decryption

- Each receiver indipendently computes the vector

$$\boldsymbol{d} = \boldsymbol{c_2} \bigoplus_{i \in I} (\boldsymbol{d}_i) = \boldsymbol{F} \cdot \boldsymbol{e} \oplus \underbrace{\begin{bmatrix} 1 \\ \dots \\ 1 \end{bmatrix}}_{q} \cdot m \bigoplus_{i \in I} (\boldsymbol{\nu}_i)\,.$$

- the bit in the vector $\boldsymbol{d}$ that is in majority is separately chosen by each receiver as the plaintext m

Protocol Security Analysis

Semi-honest model

A semi-honest party:

1. Follows the protocol properly
2. Keeps a record of all its intermediate computations

Security

- **Encryption:** from the Alekhnovich's scheme security
- **Decryption:** from the LPN hardness assumption, as each $\mathtt{R}_i$ is generating LPN samples

$$\boldsymbol{d_i} = \boldsymbol{C_1} \cdot \boldsymbol{s_i} \oplus \boldsymbol{\nu_i}$$

Protocol Security Analysis

Relaxed Semi-honest model

- Semi-honest model not so realistic (*replay attacks* may occur)
- Problem: if the same message is encrypted multiple times then it is possible to recover information about the secret key from the ciphertexts

Proposed solutions

1. implement the receivers as stateful machines (not good in resource-constrained devices)
2. make use of pseudorandom functions (i.e. deterministic algorithms that simulate truly random functions, given a “seed”)

Commitment Protocols

Commitment protocol

- can be thought as the digital analogue of a sealed envelope
- **Commit:** the sender `S` commit to a message m and the receiver `R` does not learn any information about m (hiding property)
- **Open:** `S` can choose to open the commitment and reveal the content m, but no other value (binding property)

Our contribution

We presented a Commitment protocol

- based on the commitment protocol by *Jain et al*
- based on Exact-LPN problem (where $\mathrm{wt}(e) = \lceil \tau \cdot \ell \rceil$)
- does not need a trusted third party

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The commitment protocol by *Jain et al*

Setup Phase

In order to commit a message $\boldsymbol{m} \in \mathbb{Z}_2^k$ where $k \in \Theta(\ell + v)$
We state $\boldsymbol{A} = [\boldsymbol{A'} \| \boldsymbol{A''}] \in \mathbb{Z}_2^{k \times (\ell+v)}$ as the common reference string (CRS).
Finally, we set $w = \lfloor \tau k \rceil$.

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Opening phase

define $\boldsymbol{d} = (\boldsymbol{m'}, \boldsymbol{r'})$ $\xrightarrow{\quad \boldsymbol{d} \quad}$

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Setup Phase

In order to commit a message $\boldsymbol{m} \in \mathbb{Z}_2^k$ where $k \in \Theta(\ell + v)$
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Commitment phase

Sender $\underline{\mathtt{S}}$ — Receiver $\underline{\mathtt{R}}$

chooses $\boldsymbol{r} \xleftarrow{R} \mathbb{Z}_2^{\ell}$, $\boldsymbol{e} \in \mathbb{Z}_2^k$ s.t. $\boldsymbol{wt}(\boldsymbol{e}) = w$
computes $\boldsymbol{c} = \boldsymbol{A}(\boldsymbol{r} \| \boldsymbol{m}) \oplus \boldsymbol{e}$ $\xrightarrow{\boldsymbol{c}}$

Opening phase

define $\boldsymbol{d} = (\boldsymbol{m'}, \boldsymbol{r'})$ $\xrightarrow{\boldsymbol{d}}$

computes $\boldsymbol{e'} = \boldsymbol{c} \oplus \boldsymbol{A}\,(\boldsymbol{r'} \| \boldsymbol{m'})$

$\xleftarrow{Yes,\, No}$ accepts iff $\boldsymbol{wt}(\boldsymbol{e'}) = w$

Proposed commitment protocol

Problem

We need a trusted third party for the common matrix $\boldsymbol{A} = [\boldsymbol{A}' \| \boldsymbol{A}'']$

Solution:

The Commitment and Opening phases are the same as in the original scheme

Proposed commitment protocol

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We need a trusted third party for the common matrix $\boldsymbol{A} = [\boldsymbol{A}' \| \boldsymbol{A}'']$

Solution:

Setup phase

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chooses $\boldsymbol{A}' \xleftarrow{R} \mathbb{Z}_2^{k\times \ell}$	$\xrightarrow{\boldsymbol{A}'}$	

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Proposed commitment protocol

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We need a trusted third party for the common matrix $\boldsymbol{A} = [\boldsymbol{A'} \| \boldsymbol{A''}]$

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	$\xleftarrow{\quad \boldsymbol{A''} \quad}$	chooses $\boldsymbol{A''} \xleftarrow{R} \mathbb{Z}_2^{k\times v}$

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We need a trusted third party for the common matrix $\boldsymbol{A} = [\boldsymbol{A'} \| \boldsymbol{A''}]$

Solution:

Setup phase

$$\begin{array}{ccc}
\text{Sender } \underline{\mathtt{S}} & & \text{Receiver } \underline{\mathtt{R}} \\
\text{chooses } \boldsymbol{A'} \xleftarrow{R} \mathbb{Z}_2^{k \times \ell} & \xrightarrow{\quad \boldsymbol{A'} \quad} & \\
 & \xleftarrow{\quad \boldsymbol{A''} \quad} & \text{chooses } \boldsymbol{A''} \xleftarrow{R} \mathbb{Z}_2^{k \times v}
\end{array}$$

The Commitment and Opening phases are the same as in the original scheme

The commitment protocol: a LPN-based variant

LPN variant

- security directly based on the standard LPN problem
- **Commit phase:** we set $w' = 2 \cdot \lfloor \tau k \rceil$ and we choose $\boldsymbol{e}$ such that $\boldsymbol{wt}(\boldsymbol{e}) \leq w'$

Choice of parameters

According to

Levieil, Éric and Fouque, Pierre-Alain
An Improved LPN Algorithm
Springer Berlin Heidelberg, 2006

we choose $\ell = 768$ and noise rate $\tau = \frac{1}{8} \Rightarrow 2^{90}$ bytes of memory to solve LPN

Theorem

Our commitment scheme is statistically binding and computationally hiding

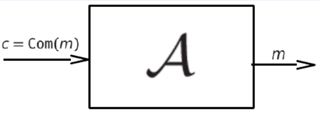
Proof

Statistically binding

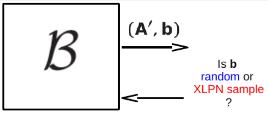
even if S is computationally unbounded she cannot cheat with probability greater than 2^{-k}

Computationally hiding

- proof for reduction (single bit message)
- we assume that $\mathcal{A}$ is able to break the commitment scheme

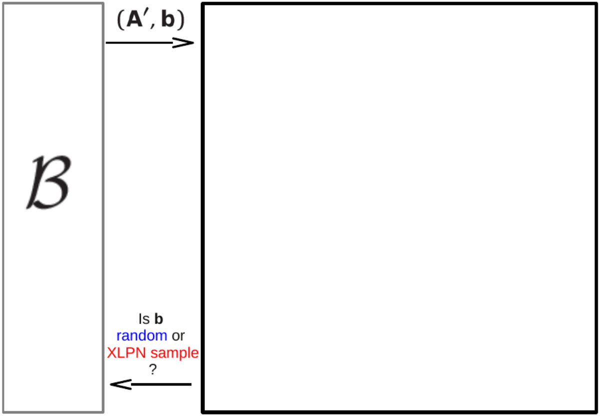


- Let $\mathcal{B}$ an oracle

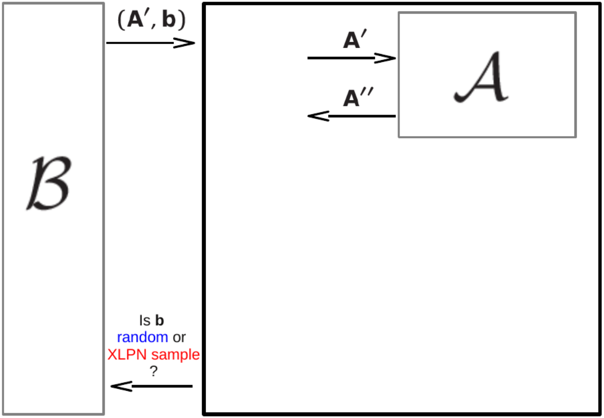


where $\boldsymbol{b} = \begin{cases} \text{random} & w.p.\ 1/2 \\ \boldsymbol{A's} \oplus \boldsymbol{e} & w.p.\ 1/2 \end{cases}$

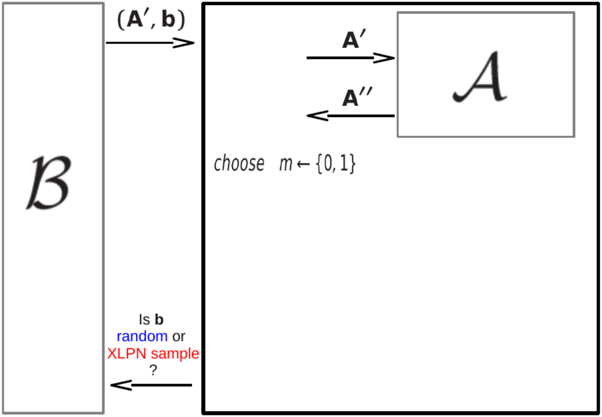
Proof



Proof

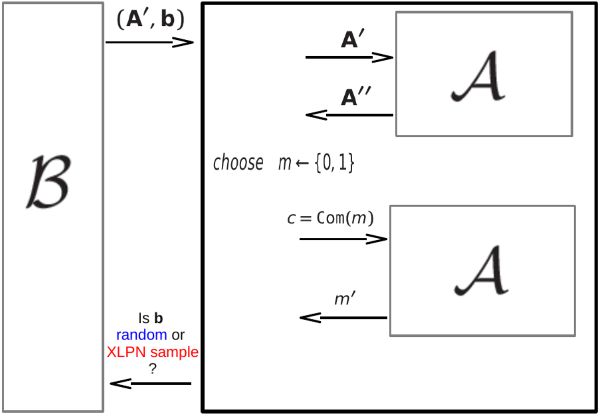

$(\mathbf{A}', \mathbf{b})$
$\mathbf{A}'$
$\mathbf{A}''$
$\mathcal{A}$
$\mathcal{B}$
Is **b**
random or
XLPN sample
?

Proof



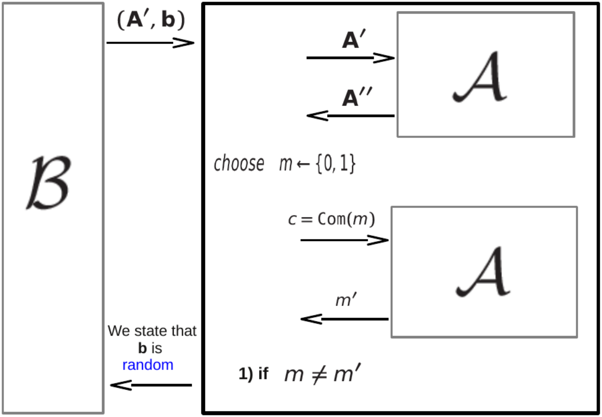
$(\mathbf{A}', \mathbf{b})$
$\mathbf{A}'$
$\mathbf{A}''$
$\mathcal{A}$
$\mathcal{B}$
choose $m \leftarrow \{0,1\}$
Is $\mathbf{b}$
random or
XLPN sample
?

Proof



(A′, b)
A′
A′′
$\mathcal{A}$
$\mathcal{B}$
choose m ← {0, 1}
c = Com(m)
$\mathcal{A}$
m′
Is b
random or
XLPN sample
?

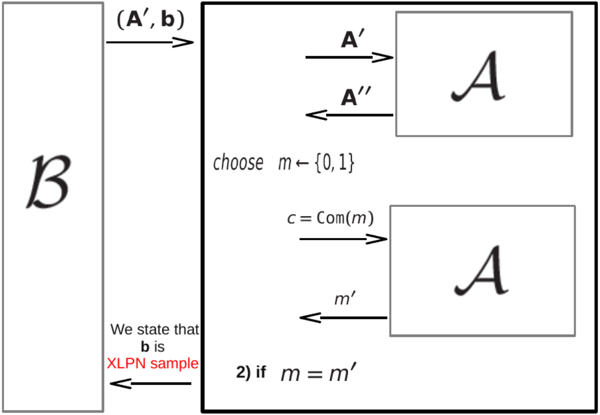
Proof



case 1)

$\boldsymbol{b}$ is random $\Rightarrow$ $\boldsymbol{c} = \boldsymbol{b} \oplus \boldsymbol{A}''m$ is a onetime-pad encryption
$\Rightarrow \mathcal{A}$ guesses w.p. $\frac{1}{2}$

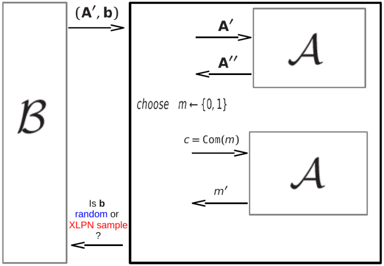
Proof



case 2)

$\boldsymbol{b}$ is a Exact-LPN sample $\Rightarrow$ $\boldsymbol{c}$ is a well formed commitment
$\Rightarrow$ $\mathcal{A}$ guesses w.p. 1 (by hypothesis)

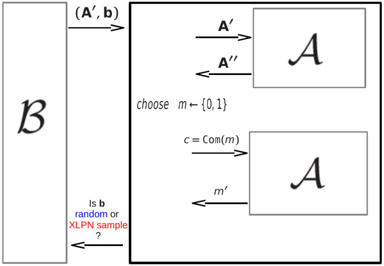
Proof



case 1) and 2)

Let E = the reduction breaks the Exact-LPN problem,

Proof

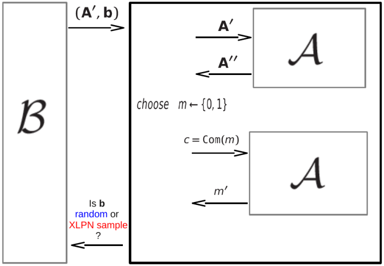


case 1) and 2)

Let E = the reduction breaks the Exact-LPN problem,

$$\Pr(E) = \Pr\left(E \middle|\, \boldsymbol{b} = \boldsymbol{A}'\boldsymbol{s} \oplus \boldsymbol{e}\right) \cdot \Pr\left(\boldsymbol{b} = \boldsymbol{A}'\boldsymbol{s} \oplus \boldsymbol{e}\right) + \Pr\left(E \middle|\, \boldsymbol{b} \text{ is random}\right) \cdot \Pr\left(\boldsymbol{b} \text{ is random}\right)$$

Proof



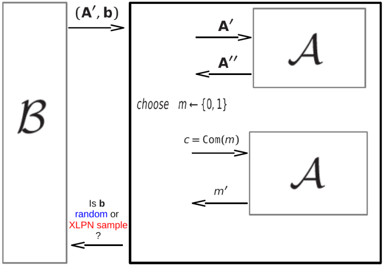
case 1) and 2)

Let E = the reduction breaks the Exact-LPN problem,

$$\Pr(E) = \Pr\left(E \middle| \, \boldsymbol{b} = \boldsymbol{A's} \oplus \boldsymbol{e}\right) \cdot \Pr\left(\boldsymbol{b} = \boldsymbol{A's} \oplus \boldsymbol{e}\right) + \Pr\left(E \middle| \, \boldsymbol{b} \text{ is random}\right) \cdot \Pr\left(\boldsymbol{b} \text{ is random}\right)$$

$$= 1 \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$$

Proof



case 1) and 2)

Let E = the reduction breaks the Exact-LPN problem,

$$\Pr(E) = \Pr\left(E \middle|\, \boldsymbol{b} = \boldsymbol{A'}\boldsymbol{s} \oplus \boldsymbol{e}\right) \cdot \Pr\left(\boldsymbol{b} = \boldsymbol{A'}\boldsymbol{s} \oplus \boldsymbol{e}\right) + \Pr\left(E \middle|\, \boldsymbol{b} \text{ is random}\right) \cdot \Pr\left(\boldsymbol{b} \text{ is random}\right)$$
$$= 1 \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$$

Exact-LPN hardness $\Rightarrow$ Hiding commitment

Conclusions and Open Problems

Our contribution

- study the security of our Threshold Public-Key Encryption scheme in the *malicious model*
- find statistically hiding commitments
- find efficient statistically binding commitments

LPN open problems

- relation between standard LPN and some variants
- Does LPN with noise rate τ imply anything about LPN with $\tau' < \tau$? Is there a threshold?
- how to get some basic primitives from standard LPN?