On the Learning Parity with Noise Problem

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Scenario

Cryptography schemes

- address the security of communication across an insecure medium
- are usually based only on complexity assumptions (standard model)

Near Future:

- Problem: What if someone constructs large quantum computers?
- Cryptography world may fall apart:
    1. cryptographic assumptions broken by efficient quantum algorithms
    2. proof of security becomes invalid

Post-Quantum cryptography

Schemes that are believed to resist classical & quantum computers

- **Hash-based cryptography**
- **Code-based cryptography**
- **Lattice-based cryptography**

Our contribution

- We investigate about the Learning Parity with Noise (LPN) problem
- We propose a Threshold Public-Key Encryption scheme based on LPN
- We propose a Commitment protocol based on LPN

Learning Parity with Noise Problem LPN

- Dimension ℓ (security parameter), $q \gg \ell$, $\tau \in \left(0, \frac{1}{2}\right]$
- **Search**: find $\boldsymbol{s} \in \mathbb{Z}_2^\ell$ given "noisy random inner products"

Errors $e_i \leftarrow \mathrm{Ber}_\tau$, i.e. $\Pr(e_i = 1) = \tau$

- **Decision**: distinguish $(\boldsymbol{a_i}, \boldsymbol{b_i})$ from uniform $(\boldsymbol{a_i}, b_i)$
- *decisional* and *search* LPN are *"polinomially equivalent"*

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$$\boldsymbol{a_1} \xleftarrow{R} \mathbb{Z}_2^\ell \quad , \quad b_1 =< \boldsymbol{a_1} \, , \, \boldsymbol{s} > \oplus e_1$$

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$$\vdots$$

$$\boldsymbol{a_q} \xleftarrow{R} \mathbb{Z}_2^\ell \quad , \quad b_q =< \boldsymbol{a_q} , \boldsymbol{s} > \oplus e_q$$

Errors $e_i \leftarrow \mathrm{Ber}_\tau$, i.e. $\Pr(e_i = 1) = \tau$

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- **Search**: find $\boldsymbol{s} \in \mathbb{Z}_2^\ell$ given "noisy random inner products"

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{a_1} \\ \vdots \\ \boldsymbol{a_q} \end{pmatrix}, \boldsymbol{b} = \boldsymbol{A} \cdot \boldsymbol{s} \oplus \boldsymbol{e}$$

Errors $e_i \leftarrow \mathrm{Ber}_\tau$, i.e. $\Pr(e_i = 1) = \tau$

- **Decision**: distinguish $(\boldsymbol{a_i}, \boldsymbol{b_i})$ from uniform $(\boldsymbol{a_i}, b_i)$
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Learning Parity with Noise Problem LPN

Hardness of LPN

Breaking the search LPN problem takes

- $2^{\Theta(\ell/\log \ell)}$ having the same number of samples q
- $2^{\Theta(\ell/\log\log \ell)}$ having $q = poly(\ell)$ samples
- $2^{\Theta(\ell)}$ having $q = \Theta(\ell)$ samples

where ℓ is the security parameter

Interesting features

- **Efficiency** $\Rightarrow$ suitable for limited computing power devices (e.g. RFID).
- **quantum algorithm resistance**

Threshold Public-Key Encryption schemes

Public-key cryptography

- the ability of decrypting or signing is restricted to the owner of the secret key.
- $\Rightarrow$ only one person has all the power

Threshold Public-Key Encryption schemes

Solution: Threshold PKE

- Shares trust among a group of users, such that *enough* of them, the *threshold*, is needed to sign or decrypt
- The secret key is split into shares and each share is given to a group of users.

Our contribution

A Threshold Public-Key Encryption scheme which is:

- based on LPN
- secure in the *Semi-honest* model

Alekhnovich PKE scheme

Key Generation

The sender S chooses

- a secret key $\boldsymbol{s} \xleftarrow{R} \mathbb{Z}_2^{\ell}$
- $\boldsymbol{A} \xleftarrow{R} \mathbb{Z}_2^{q \times \ell}$ and the error $\boldsymbol{e} \leftarrow \mathrm{Ber}_{\tau}^{q}$, where $\tau \in \Theta(\frac{1}{\sqrt{\ell}})$ and computes the pk as $(\boldsymbol{A}, \boldsymbol{b} = \boldsymbol{A}\boldsymbol{s} \oplus \boldsymbol{e})$

Encryption of a message bit $m \in \mathbb{Z}_2$

Sender S		Receiver R
choose a vector $\boldsymbol{f} \leftarrow \mathrm{Ber}_{\tau}^{q}$		
compute $\boldsymbol{u} = \boldsymbol{f} \cdot \boldsymbol{A}$		
$c = \langle \boldsymbol{f}, \boldsymbol{b} \rangle \oplus m$	$\xrightarrow{(\boldsymbol{u}, c)}$	

Decryption

The receiver R computes $d = c \oplus \langle \boldsymbol{s}, \boldsymbol{u} \rangle = \cdots = \langle \boldsymbol{f}, \boldsymbol{e} \rangle \oplus m$

Protocol phases

- **Key Generation**
- Key Assembly
- Encryption
- Partial Decryption
- Finish Decryption

Key Generation

- All the receivers share a matrix $\boldsymbol{A} \xleftarrow{R} \mathbb{Z}_2^{q \times \ell}$
- Each receiver $\mathtt{R_i}$ indipendently choose a secret key $\boldsymbol{s_i} \xleftarrow{R} \mathbb{Z}_2^{\ell}$ and an error $\boldsymbol{e_i} \leftarrow \mathrm{Ber}_{\tau}^{q}$
- the public key for $\mathtt{R_i}$ is the pair $(\boldsymbol{A}, \boldsymbol{b_i} = \boldsymbol{A}\boldsymbol{s_i} \oplus \boldsymbol{e_i})$

Protocol phases

- Key Generation
- **Key Assembly**
- Encryption
- Partial Decryption
- Finish Decryption

Key Assembly

The combined public key is the pair $(\boldsymbol{A}, \boldsymbol{b})$, where

$$\boldsymbol{b} = \bigoplus_{i \in I} \boldsymbol{b_i}$$

and I is the users subset

Protocol phases

- Key Generation
- Key Assembly
- **Encryption**
- Partial Decryption
- Finish Decryption

Sender S

Receivers $\mathtt{R_i}, \mathtt{R_j}$

$$(\boldsymbol{C_1}, \boldsymbol{c_2}) \leftarrow \mathtt{ThLPN.Enc}(m, \boldsymbol{b})$$

Protocol phases

- Key Generation
- Key Assembly
- **Encryption**
- Partial Decryption
- Finish Decryption

Sender $\underline{\mathtt{S}}$ Receivers $\underline{\mathtt{R_i}, \mathtt{R_j}}$

$$(\boldsymbol{C_1}, \boldsymbol{c_2}) \leftarrow \mathtt{ThLPN.Enc}(m, \boldsymbol{b})$$

Encryption function (Alekhnovich scheme)

$$\boldsymbol{C_1} = \boldsymbol{F} \cdot \boldsymbol{A},\ \boldsymbol{c_2} = \boldsymbol{F} \cdot \boldsymbol{b} \oplus \underbrace{\begin{bmatrix} 1 \\ \dots \\ 1 \end{bmatrix}}_{q} \cdot m \qquad \text{where } \boldsymbol{F} := \begin{bmatrix} \boldsymbol{f_1} \\ \dots \\ \boldsymbol{f_q} \end{bmatrix}, \boldsymbol{f_i} \leftarrow \mathrm{Ber}_\tau^q$$

Protocol phases

- Key Generation
- Key Assembly
- **Encryption**
- Partial Decryption
- Finish Decryption

Sender $\underline{\mathtt{S}}$ Receivers $\underline{\mathtt{R_i}, \mathtt{R_j}}$

$$(\boldsymbol{C_1}, \boldsymbol{c_2}) \leftarrow \mathtt{ThLPN.Enc}(m, \boldsymbol{b}) \quad \xrightarrow{(\boldsymbol{C_1}, \boldsymbol{c_2})}$$

Encryption function (Alekhnovich scheme)

$$\boldsymbol{C_1} = \boldsymbol{F} \cdot \boldsymbol{A}, \ \boldsymbol{c_2} = \boldsymbol{F} \cdot \boldsymbol{b} \oplus \underbrace{\begin{bmatrix} 1 \\ \dots \\ 1 \end{bmatrix}}_{q} \cdot m \qquad \text{where } \boldsymbol{F} := \begin{bmatrix} \boldsymbol{f_1} \\ \dots \\ \boldsymbol{f_q} \end{bmatrix}, \boldsymbol{f_i} \leftarrow \mathrm{Ber}_\tau^q$$

Protocol phases

- Key Generation
- Key Assembly
- Encryption
- **Partial Decryption**
- Finish Decryption

Receiver $\mathtt{R_i}$

Receiver $\mathtt{R_j}$

$$d_i \leftarrow \mathtt{ThLPN.Pdec}(C_1, c_2, s_i)$$

Protocol phases

- Key Generation
- Key Assembly
- Encryption
- **Partial Decryption**
- Finish Decryption

Receiver $\mathtt{R_i}$

Receiver $\mathtt{R_j}$

$$\boldsymbol{d_i} \leftarrow \mathtt{ThLPN.Pdec}(\boldsymbol{C_1}, \boldsymbol{c_2}, \boldsymbol{s_i})$$

Partial decryption function (Alekhnovich scheme)

$$\boldsymbol{d_i} = \boldsymbol{C_1} \cdot \boldsymbol{s_i} \oplus \boldsymbol{\nu_i}$$

where $\boldsymbol{\nu_i} \leftarrow \mathrm{Ber}_\sigma^q$

Protocol phases

- Key Generation
- Key Assembly
- Encryption
- **Partial Decryption**
- Finish Decryption

Receiver $\underline{\mathtt{R_i}}$ Receiver $\underline{\mathtt{R_j}}$

$$\boldsymbol{d_i} \leftarrow \mathtt{ThLPN.Pdec}(\boldsymbol{C_1}, \boldsymbol{c_2}, \boldsymbol{s_i}) \quad \xrightarrow{\quad \boldsymbol{d_i} \quad}$$

Partial decryption function (Alekhnovich scheme)

$$\boldsymbol{d_i} = \boldsymbol{C_1} \cdot \boldsymbol{s_i} \oplus \boldsymbol{\nu_i}$$

where $\boldsymbol{\nu_i} \leftarrow \mathrm{Ber}_\sigma^q$

Protocol phases

- Key Generation
- Key Assembly
- Encryption
- **Partial Decryption**
- Finish Decryption

Receiver $\mathtt{R_i}$ | Receiver $\mathtt{R_j}$

$\boldsymbol{d_i} \leftarrow \mathtt{ThLPN.Pdec}(\boldsymbol{C_1}, \boldsymbol{c_2}, \boldsymbol{s_i}) \xrightarrow{\boldsymbol{d_i}}$

$\boldsymbol{d_j} \leftarrow \mathtt{ThLPN.Pdec}(\boldsymbol{C_1}, \boldsymbol{c_2}, \boldsymbol{s_j})$

Partial decryption function (Alekhnovich scheme)

$$\boldsymbol{d_i} = \boldsymbol{C_1} \cdot \boldsymbol{s_i} \oplus \boldsymbol{\nu_i}$$

where $\boldsymbol{\nu_i} \leftarrow \mathrm{Ber}_\sigma^q$

Protocol phases

- Key Generation
- Key Assembly
- Encryption
- **Partial Decryption**
- Finish Decryption

Receiver $\underline{\mathtt{R_i}}$ Receiver $\underline{\mathtt{R_j}}$

$\boldsymbol{d_i} \leftarrow \mathtt{ThLPN.Pdec}(\boldsymbol{C_1}, \boldsymbol{c_2}, \boldsymbol{s_i})$ $\xrightarrow{\boldsymbol{d_i}}$

$\xleftarrow{\boldsymbol{d_j}}$ $\boldsymbol{d_j} \leftarrow \mathtt{ThLPN.Pdec}(\boldsymbol{C_1}, \boldsymbol{c_2}, \boldsymbol{s_j})$

Partial decryption function (Alekhnovich scheme)

$$\boldsymbol{d_i} = \boldsymbol{C_1} \cdot \boldsymbol{s_i} \oplus \boldsymbol{\nu_i}$$

where $\boldsymbol{\nu_i} \leftarrow \mathrm{Ber}_\sigma^q$

Protocol phases

- Key Generation
- Key Assembly
- Encryption
- Partial Decryption
- **Finish Decryption**

Finish decryption

- Each receiver indipendently computes the vector

$$\boldsymbol{d} = \boldsymbol{c_2} \bigoplus_{i \in I} (\boldsymbol{d}_i) = \boldsymbol{F} \cdot \boldsymbol{e} \oplus \underbrace{\begin{bmatrix} 1 \\ \dots \\ 1 \end{bmatrix}}_{q} \cdot m \bigoplus_{i \in I} (\boldsymbol{\nu}_i) \, .$$

- the bit in the vector $\boldsymbol{d}$ that is in majority is separately chosen by each receiver as the plaintext m

Protocol Security Analysis

Semi-honest model

A semi-honest party:

1. Follows the protocol properly
2. Keeps a record of all its intermediate computations

Security

- **Encryption:** from the Alekhnovich's scheme security
- **Decryption:** from the LPN hardness assumption, as each $\mathtt{R}_i$ is generating LPN samples

$$\boldsymbol{d_i} = \boldsymbol{C_1} \cdot \boldsymbol{s_i} \oplus \boldsymbol{\nu_i}$$

Protocol Security Analysis

Relaxed Semi-honest model

- Semi-honest model not so realistic (*replay attacks* may occur)
- Problem: if the same message is encrypted multiple times then it is possible to recover information about the secret key from the ciphertexts

Proposed solutions

1. implement the receivers as *stateful* machines (not good in resource-constrained devices)
2. make use of pseudorandom functions (i.e. deterministic algorithms that simulate truly random functions, given a "seed")

Commitment Protocols

Commitment protocol

- can be thought as the digital analogue of a sealed envelope
- **Commit:** the sender `S` commit to a message m and the receiver `R` does not learn any information about m (hiding property)
- **Open:** `S` can choose to open the commitment and reveal the content m, but no other value (binding property)

Our contribution

We presented a Commitment protocol

- based on the commitment protocol by *Jain et al*
- based on Exact-LPN problem (where $\mathbf{wt}(\boldsymbol{e}) = \lfloor \tau \cdot \ell \rceil$)
- not in a *common reference string (CRS)* model

The commitment protocol by *Jain et al*

Setup Phase

In order to commit a message $\boldsymbol{m} \in \mathbb{Z}_2^k$ where $k \in \Theta(\ell + v)$
We let $\boldsymbol{A'} \xleftarrow{R} \mathbb{Z}_2^{k\times\ell}$ and $\boldsymbol{A''} \xleftarrow{R} \mathbb{Z}_2^{k\times v}$. We state $\boldsymbol{A} = [\boldsymbol{A'} \| \boldsymbol{A''}] \in \mathbb{Z}_2^{k\times(\ell+v)}$ as the common reference string (CRS). Finally, we set $w = \lfloor \tau k \rceil$.

Commitment phase

Sender $\underline{\mathtt{S}}$		Receiver $\underline{\mathtt{R}}$
chooses $\boldsymbol{r} \xleftarrow{R} \mathbb{Z}_2^{\ell}$, $\boldsymbol{e} \in \mathbb{Z}_2^k$ s.t. $\boldsymbol{wt}(\boldsymbol{e}) = w$		
computes $\boldsymbol{c} = \boldsymbol{A}(\boldsymbol{r} \| \boldsymbol{m}) \oplus \boldsymbol{e}$	$\xrightarrow{\boldsymbol{c}}$	

Opening phase

computes $\boldsymbol{d} = (\boldsymbol{m'}, \boldsymbol{r'})$	$\xrightarrow{\boldsymbol{d}}$	
		computes $\boldsymbol{e'} = \boldsymbol{c} \oplus \boldsymbol{A}\,(\boldsymbol{r'} \| \boldsymbol{m'})$
	$\xleftarrow{Yes,\,No}$	accepts iff $\boldsymbol{wt}(\boldsymbol{e'}) = w$

Proposed commitment protocol

Problem

We need a trusted third party for the common matrix $\boldsymbol{A} = [\boldsymbol{A'} \| \boldsymbol{A''}]$

Solution:

Setup phase

$$\begin{array}{ccc} \text{Sender } \underline{\mathtt{S}} & & \text{Receiver } \underline{\mathtt{R}} \\ \text{chooses } \boldsymbol{A'} \xleftarrow{R} \mathbb{Z}_2^{k \times \ell} & \xrightarrow{\quad \boldsymbol{A'} \quad} & \\ & \xleftarrow{\quad \boldsymbol{A''} \quad} & \text{chooses } \boldsymbol{A''} \xleftarrow{R} \mathbb{Z}_2^{k \times v} \end{array}$$

The Commitment and Opening phases are the same as in the original scheme

The commitment protocol: a LPN-based variant

LPN variant

- security directly based on the standard LPN problem
- **Commit phase:** we set $w' = 2 \cdot \lfloor \tau k \rceil$ and we choose $\boldsymbol{e}$ such that $\boldsymbol{wt}(\boldsymbol{e}) \leq w'$

Choice of parameters

According to

Levieil, Éric and Fouque, Pierre-Alain
An Improved LPN Algorithm
Springer Berlin Heidelberg, 2006

we choose $\ell = 768$ and noise rate $\tau = \frac{1}{8} \Rightarrow 2^{90}$ bytes of memory to solve LPN

Theorem

Our commitment scheme is statistically binding and computationally hiding

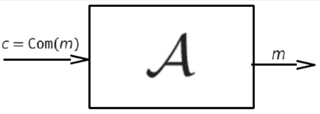
Proof

Statistically binding

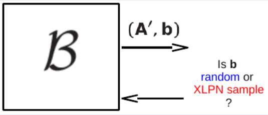
even if S is computationally unbounded she cannot cheat with probability greater than 2^{-k}

Computationally hiding

- proof for reduction (single bit message)
- we assume that $\mathcal{A}$ is able to break the commitment scheme

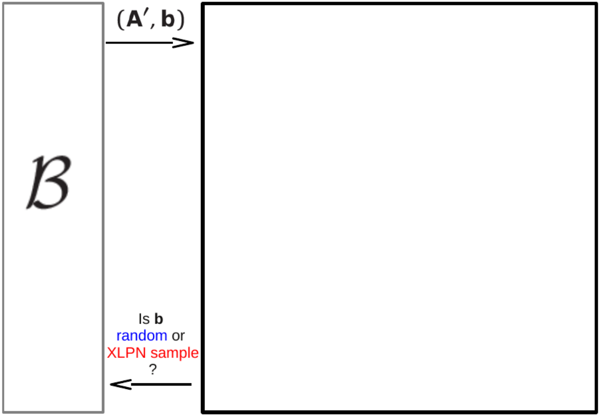


- Let $\mathcal{B}$ an oracle

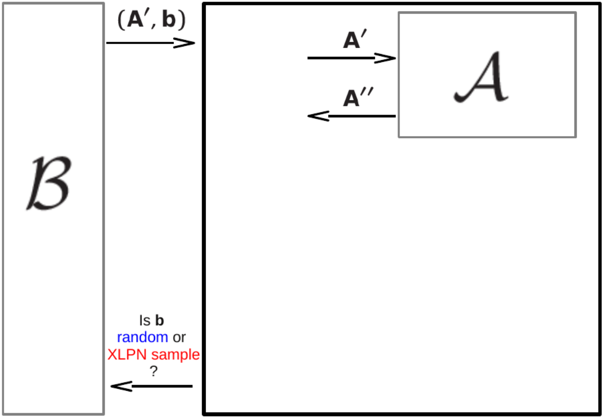


where $\boldsymbol{b} = \begin{cases} \text{random} & w.p.\ 1/2 \\ \boldsymbol{A's} \oplus \boldsymbol{e} & w.p.\ 1/2 \end{cases}$

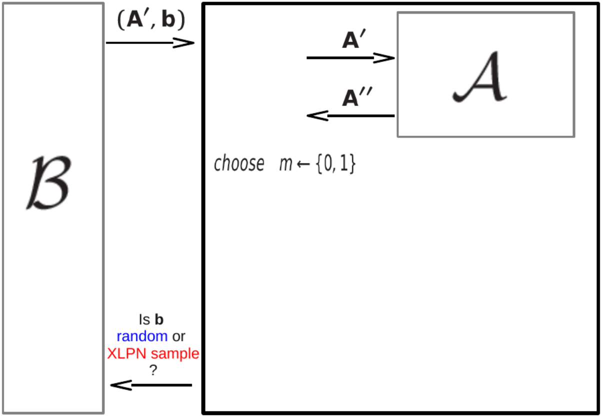
Proof



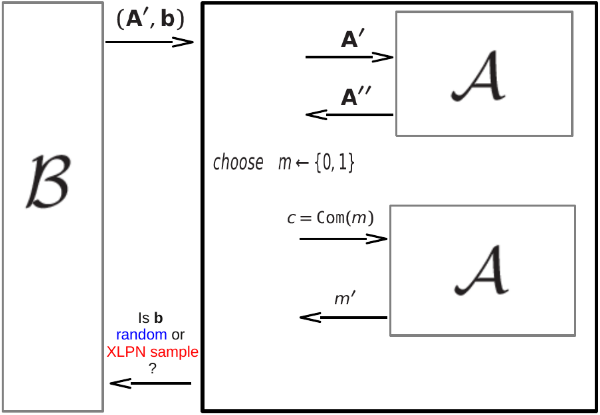
Proof



Proof

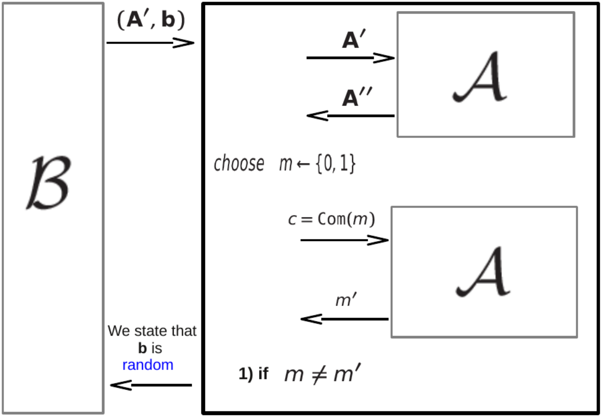

$(\mathbf{A}', \mathbf{b})$
$\mathbf{A}'$
$\mathbf{A}''$
$\mathcal{A}$
$\mathcal{B}$
choose $m \leftarrow \{0, 1\}$
Is **b**
random or
XLPN sample
?

Proof



$(\mathbf{A}', \mathbf{b})$
$\mathbf{A}'$
$\mathbf{A}''$
$\mathcal{A}$
$\mathcal{B}$
choose $m \leftarrow \{0,1\}$
$c = \mathrm{Com}(m)$
$\mathcal{A}$
m'
Is $\mathbf{b}$
random or
XLPN sample
?

Proof

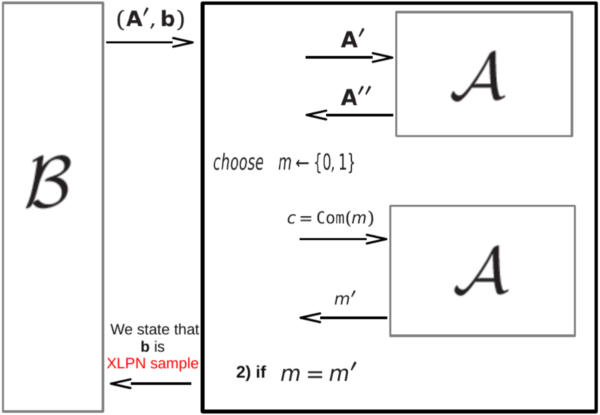


case 1)

$\boldsymbol{b}$ is random $\Rightarrow \boldsymbol{c} = \boldsymbol{b} \oplus \boldsymbol{A''}m$ is a onetime-pad encryption

$\Rightarrow \mathcal{A}$ guesses w.p. $\frac{1}{2}$

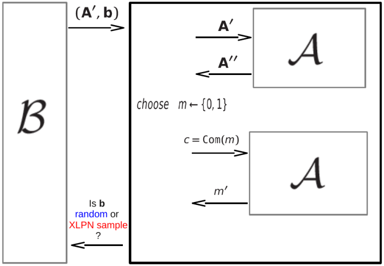
Proof



case 2)

$\boldsymbol{b}$ is a Exact-LPN sample $\Rightarrow$ $\boldsymbol{c}$ is a well formed commitment $\Rightarrow \mathcal{A}$ guesses w.p. 1 (by hypothesis)

Proof



case 1) and 2)

Let E = the reduction breaks the Exact-LPN problem,

$$\Pr(E) = \Pr\left(E \middle| \, \boldsymbol{b} = \boldsymbol{A'}\boldsymbol{s} \oplus \boldsymbol{e}\right) \cdot \Pr\left(\boldsymbol{b} = \boldsymbol{A'}\boldsymbol{s} \oplus \boldsymbol{e}\right) + \Pr\left(E \middle| \, \boldsymbol{b} \text{ is random}\right) \cdot \Pr\left(\boldsymbol{b} \text{ is random}\right)$$

$$= 1 \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4} \gg 2^{-k}$$

Exact-LPN hardness $\Rightarrow$ Hiding commitment

Conclusions and Open Problems

Our contribution

- study the security of our Threshold Public-Key Encryption scheme in the *malicious model*
- find statistically hiding commitments
- find efficient statistically binding commitments

LPN open problems

- relation between standard LPN and some variants
- LPN with noise rate τ imply anything about LPN with $\tau' < \tau$? Is there a threshold?
- how to get some basic primitives from standard LPN?