# Assignment 1 - Computer Vision (EEN020)

Luca Modica

November 13, 2023

# 1 Points in Homogeneous Coordinates.

## Theoretical exercise 1

The 2D Cartesian coordinates of the first 3 given points are the following:

$$x_1^{'} = \begin{pmatrix} 4/2 \\ -16/2 \\ 2/2 \end{pmatrix} = \begin{pmatrix} 2 \\ -8 \\ 1 \end{pmatrix},$$

$$x_2^{'} = \begin{pmatrix} -3/(-1) \\ 7/(-1) \\ (-1)/(-1) \end{pmatrix} = \begin{pmatrix} 3 \\ -7 \\ 1 \end{pmatrix},$$

$$x_3^{'} = \begin{pmatrix} 9\lambda/6\lambda \\ -3\lambda/6\lambda \\ 6\lambda/6\lambda \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}, \lambda \neq 0.$$

Now let's consider the point

$$x_4 = \begin{pmatrix} -6 \\ 3 \\ 0 \end{pmatrix}.$$

Since $x_3 = 0$, the related point in 2D Cartesian coordinates

$$x_4^{'} = \begin{pmatrix} -6/0 \\ 3/0 \\ 0/0 \end{pmatrix}$$

would represent a 2D point that tends to infinity. In particular, the interpretation will be a line that will go in the direction $\begin{pmatrix} -6 \\ 3 \end{pmatrix}$ in the 2D Cartesian.

For the above interpretation, the point

$$x_5 = \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix},$$

corresponding to a point that will go indefinitely in the direction $\begin{pmatrix} -4 \\ 3 \end{pmatrix}$, is not the same point as $x_5$.

## Computer exercise 1

Main reference matlab file: `comp_ex1.m`.

Resulting plots after applying the function `pflat` to the points in `x2D` and `x3D`:

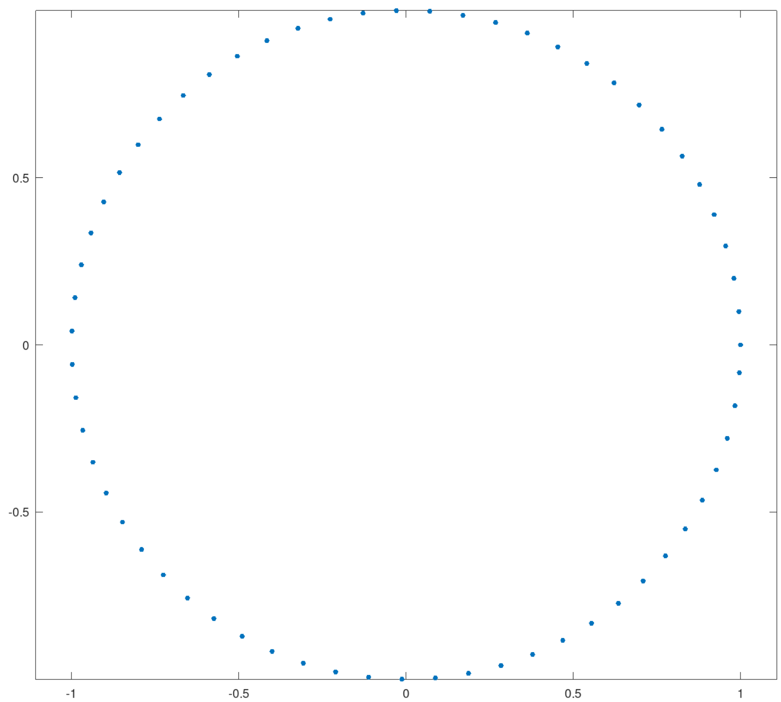


Figure 1: Points in `x2D` after applying the function `pflat`.

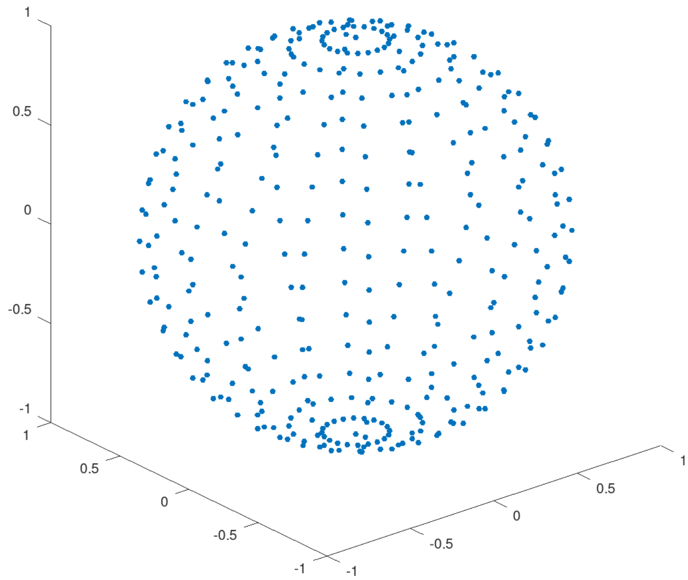


Figure 2: Points in `x3D` after applying the function `pflat`.

## 2 Lines.

### Theoretical exercise 2

Given the following lines:

$$l_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, l_2 = \begin{pmatrix} 6 \\ 3 \\ 1 \end{pmatrix},$$

the homogeneous coordinates of their intersection (in $\mathbb{P}^2$) can be found by computing the cross product between the 2 lines vector. In other words:

$$l_1 \times l_2 = \begin{pmatrix} (1 \cdot 1) - (1 \cdot 3) \\ (1 \cdot 6) - (-1 \cdot 1) \\ (-1 \cdot 3) - (1 \cdot 6) \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \\ -9 \end{pmatrix}$$

Normalizing the point we will have $\begin{pmatrix} 2/9 \\ -7/9 \\ 1 \end{pmatrix}$.

An alternative way to find the cross product is to solve the related null-space of the related matrix:

$$\text{inters} = \text{null}\left(\begin{pmatrix} -1 & 1 & 1 \\ 6 & 3 & 1 \end{pmatrix}\right) = \begin{pmatrix} 431/2500 \\ -151/250, \\ 437/562 \end{pmatrix}.$$

The normalized value will also give the same point inters $= \begin{pmatrix} 2/9 \\ -7/9 \\ 1 \end{pmatrix}$. Moreover, in both cases the corresponding point in $\mathbb{R}^2$ is $\begin{pmatrix} 2/9 \\ -7/9 \end{pmatrix}$.

Now, let's consider the following lines:

$$l_3 = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}, l_4 = \begin{pmatrix} 5 \\ 0 \\ 4 \end{pmatrix},$$

which represent 2 vertical parallel lines. Computing the cross product, their intersection in $\mathbb{P}^2$ will be:

$$l_1 \times l_2 = \begin{pmatrix} (0 \cdot 4) - (1 \cdot 0) \\ (1 \cdot 5) - (-3 \cdot 4) \\ (-3 \cdot 0) - (0 \cdot 5) \end{pmatrix} = \begin{pmatrix} 0 \\ 17 \\ 0 \end{pmatrix}.$$

In $\mathbb{R}^2$, the obtained intersection point $\begin{pmatrix} 0 \\ 17 \\ 0 \end{pmatrix}$ will tend to infinity, mainly for 2 reasons.

- As mentioned before, $l_3$ and $l_4$ represent 2 vertical parallel lines. Thus, for any scalar $\lambda \neq 0$, the intersection $(0, \lambda, 0)^T \in [(0, 1, 0)^T]$ will be a point to infinity in $\mathbb{R}^2$ since:

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \lim_{\epsilon \to \pm 0} \begin{pmatrix} 0 \\ 1 \\ \epsilon \end{pmatrix} \equiv \lim_{\epsilon \to \pm 0} \begin{pmatrix} 0/\epsilon \\ 1/\epsilon \end{pmatrix}$$

- The null space of the matrix related to $l_3$ and $l_4$

$$M = \begin{pmatrix} -3 & 0 & 4 \\ 5 & 0 & 4 \end{pmatrix}$$

has no solution. As a consequence, there is no unique defined intersection point between the 2 lines.

To end, let's consider the following points:

$$x_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, x_2 = \begin{pmatrix} 6 \\ 3 \end{pmatrix}.$$

In order to compute the line that goes through those points, we first consider the related point in $\mathbb{P}^2$ of $x_1$ and $x_2$, respectively $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 3 \\ 1 \end{pmatrix}$. Then, referencing the calculation above, we notice that line vector corresponds to the intersection between $l_1$ and $l_2$. That is

$$\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 6 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \\ -9 \end{pmatrix} := \text{line passing through } x_1 \text{ and } x_2.$$

## Theoretical exercise 3

The intersection point (in homogeneous coordinates) of $l_1$ and $l_2$ (from the theoretical exercise 2) is in the null space of the matrix

$$M = \begin{pmatrix} 6 & 3 & 1 \\ -1 & 1 & 1 \end{pmatrix},$$

since it still corresponds to related matrix of the 2 mentioned lines. To prove even further, we can see that the dot product between the matrix $M$ and the intersection point itself will result in the 0 vector:

$$\begin{pmatrix} 6 & 3 & 1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 7 \\ -9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Moreover, the related intersection point in $\mathbb{P}^2$ $\begin{pmatrix} 2/9 \\ -7/9 \\ 1 \end{pmatrix}$ represent the only point in the null space of $M$. This is because 2 lines $l_1, l_2 \in \mathbb{P}^2$, with $l_1 \neq l_2$, have a unique intersection in $\mathbb{P}^2$.

## Computer exercise 2

Main reference matlab file: `comp_ex2.m`.

In the plot below, the following elements will be displayed:

- the image,
- the pairs of image points $p_1$, $p_2$ and $p_3$, contained in `compEx2.mat`,
- the computed lines (using the `rital` function) for each pairs points,
- the intersection between the second and the third line (obtained from the pairs $p_2$ and $p_3$)). The point in the image will be highlighted in red.

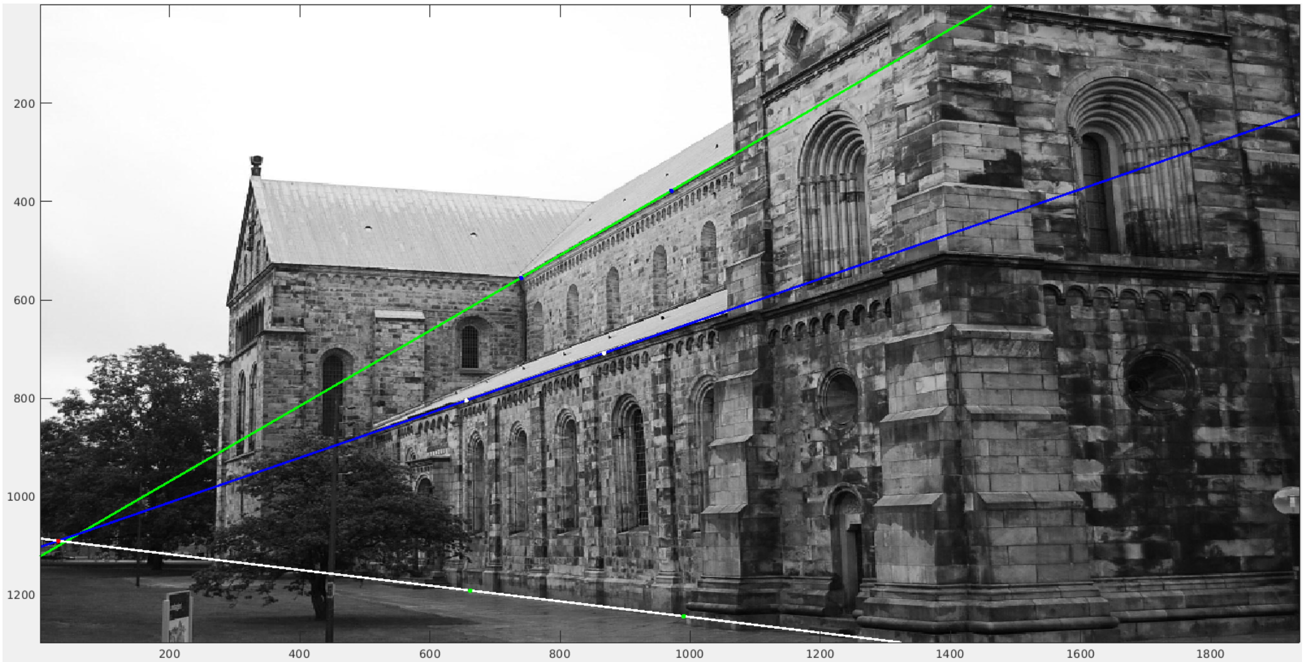


Figure 3: Plot related to the computer exercise 2, with all the elements mentioned above.

As we can see from the plot above, the 3 lines appears to be parallel in 3D. However, despite the perception given from the plot, by checking the component-wise ratios of each line component for each pair of lines those does not appear to be constant: this means that the lines are not parallel, since a non-constant component ratio means that that the lines does not point to the same direction.

This observation followed by the calculations to find the intersection between the second line and the third line, involving the null space of the 2 mentioned lines. Note that the result that will be reported is already normalized.

$$\text{null}\left(\begin{pmatrix} l_2 \\ l_3 \end{pmatrix}\right) \approx \begin{pmatrix} 28.31 \\ 1,089.77 \\ 1 \end{pmatrix}.$$

Thus, the intersection point in $\mathbb{R}^2$ is $\begin{pmatrix} 28.31 \\ 1,089.77 \end{pmatrix}$.

To conclude, the distance between the computed intersection point above and the first line will be reported. The distance, given the intersection point $x = (x_1, x_2)^T$ and the line $l_1 = (a_1, b_1, c_1)$, is computed as follows:

$$d = \frac{|a_1 x_1 + b_1 x_2 + c_1|}{\sqrt{a_1^2 + b_1^2}} = 8.2695.$$

The distance found is close to 0, for 2 main reasons.

- Considering the scale of the plotted image, which are in the order order of thousand, a distance of 8.2695 can considered close to 0.
- By computing the null space (that is, their intersection) related to the other pair of lines ($l_1$ and $l_2$, $l_1$ and $l_3$), we can notice that the point are really close to each other (note that also in this

case the results are already normalized):

$$\text{null}\left(\begin{pmatrix} l_1 \\ l_2 \end{pmatrix}\right) \approx \begin{pmatrix} 61.76 \\ 1,074.55 \\ 1 \end{pmatrix},$$

$$\text{null}\left(\begin{pmatrix} l_1 \\ l_3 \end{pmatrix}\right) \approx \begin{pmatrix} 39.56 \\ 1,091.56 \\ 1 \end{pmatrix}.$$

Where the points are, respectively: $\begin{pmatrix} 61.76 \\ 1,074.55 \end{pmatrix}$ and $\begin{pmatrix} 39.56 \\ 1,091.56 \end{pmatrix}$.

# 3 Projective Transformations.

## Theoretical exercise 4

Let:

$$H = \begin{pmatrix} 4 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}, x_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, x_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

The transformations are computed as follows:

$$y_1 \sim Hx_1 = \begin{pmatrix} \lambda_1 \\ \lambda_1 \\ \lambda_1 \end{pmatrix},$$

$$y_2 \sim Hx_2 = \begin{pmatrix} \lambda_2 \\ 2\lambda_2 \\ 0 \end{pmatrix},$$

where $\lambda_1, \lambda_2 \neq 0$ are 2 scalar such that $Hx_1 = \lambda_1 y_1$ and $Hx_2 = \lambda_2 y_2$.

Now we will compute 2 lines that contains $x_1 x_2$ and $y_1 y_2$, respectively, by using the cross product operation. Note that the resulted lines are normalized.

$$l_1 = x_1 \times x_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix},$$

$$l_2 = y_1 \times y_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}.$$

Finally, we also compute:

$$(H^{-1})^T l_1 = \begin{pmatrix} -1 \\ 1/2 \\ 1/2 \end{pmatrix}.$$

This value, compared to $l_2$, represents a line pointing in the same direction since those are linearly dependent (and so, parallel). This is also confirmed by the fact that, if we normalize the components of $(H^{-1})^T l_1$, we obtain $l_2$.

## Theoretical exercise 5

Let's consider the line $l_1$ and the point $x$, such that $x \in l_1$ (so, $l_1^T x = 0$) by the implication of the statement we want to prove. Let's also consider a projective transformation matrix $H$ and a point $y = Hx \in y \sim Hx$.

By assumption, considering a another line $l_2$:

$$y \in l_2 \rightarrow l_2^T y = 0 \rightarrow l_2^T Hx = 0.$$

From the following system of equations:

$$\begin{cases} l_1^T x = 0 \\ l_2^T Hx = 0 \end{cases} \rightarrow l_2^T = l_1^T H^{-1}.$$

Lastly, consider $l_1^T x = 0 \rightarrow l_1^T H^{-1} H x = 0$. Considering the value of $l_2$ and then $y$, by substitution we can derive:

$$l_2^T H x = 0 \rightarrow l_2^T y = 0 \rightarrow y \in l_2, \forall y \in y \sim Hx.$$

This concludes the proof, for which every projective transformation $H$ preserves lines.

■

## Theoretical exercise 6

### (a)

$H_1$, $H_2$ and $H_3$ are projective transformations, since those the only matrices that are invertible and has rank $= 3$.

### (b)

$H_1$ represent an affine transformation, since its top left sub-matrix is invertible and the last row is conforming with the form: $\begin{bmatrix} 0 & 0 & 0 & ... & 1 \end{bmatrix}$.

### (c)

None of the above matrices are an euclidean transformation, since they are not orthogonal or they don't preverve the the lengths between points up to a scale factor.

### (d)

None of the above matrices are an euclidean transformation, since they are not orthogonal or they don't preverve the the lengths between points.

### (e)

The euclidean transformations are the types that preserve distances, thus lengths between points. In our case, none of the transformations are euclidean transformations.

### (f)

The projective, affine, similarity and euclidean transformations are the types that map lines to lines. In our case, the transformations are $H_1$, $H_2$ and $H_3$.

### (g)

The affine, similarity and euclidean transformations are the types that map parallel lines to parallel lines. In our case, none of the transformations are euclidean transformations. In our case, the transformation is $H_1$.

# 4 The Pinhole Camera.

## Theoretical exercise 7

The projections of $X_1, X_2$ and $X_3$ in homogeneous coordinates are computed as follows (note that the end results will be normalized):

$$X_1\text{proj} = PX_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/0 \\ 2/0 \\ 0/0 \end{pmatrix},$$

$$X_2\text{proj} = PX_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ 1 \end{pmatrix},$$

$$X_3\text{proj} = PX_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1/2 \\ 1 \end{pmatrix}.$$

As we can notice, the projection of $X_1$ represents a 2D infinite point in $\mathbb{P}^2$ in the direction $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Moreover, It's not visible by the camera matrix $P$ for his third coordinate value.

For the camera center, C is computed using the null-space of the matrix P; in other words, the camera center is the 3D-coordinate point $x$ for which: $Px = 0$.

$$C = \text{null}(P) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} C_0 \\ 1 \end{pmatrix} \in \mathbb{P}^3.$$

Where the camera center is the related 3D point $C_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^3$.

The principal axes, instead, is represented by the first 3 components of the third row of the camera matrix:

$$P_{3,1:3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

## Computer exercise 3

Main reference matlab file: `comp_ex3.m`.

Values of the cameras centers in Cartesian coordinates and normalized:

$$\text{null}(P1) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} C_1 \\ 1 \end{pmatrix} \in \mathbb{P}^3, C_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$C2 = \text{null}(P1) \approx \begin{pmatrix} 6.63 \\ 14.84 \\ -15.07 \\ 1 \end{pmatrix} = \begin{pmatrix} C_2 \\ 1 \end{pmatrix} \in \mathbb{P}^3, C_2 = \begin{pmatrix} 6.63 \\ 14.84 \\ -15.07 \end{pmatrix}.$$

Principal axes normalized to length one:

$$\text{axes1} = P1_{3,1:3} \approx \begin{pmatrix} 3.74 \\ 11.30 \\ 1 \end{pmatrix}$$

$$\text{axes2} = P2_{3,1:3} \approx \begin{pmatrix} 0.03 \\ 0.36 \\ 1 \end{pmatrix}$$

Plots created:

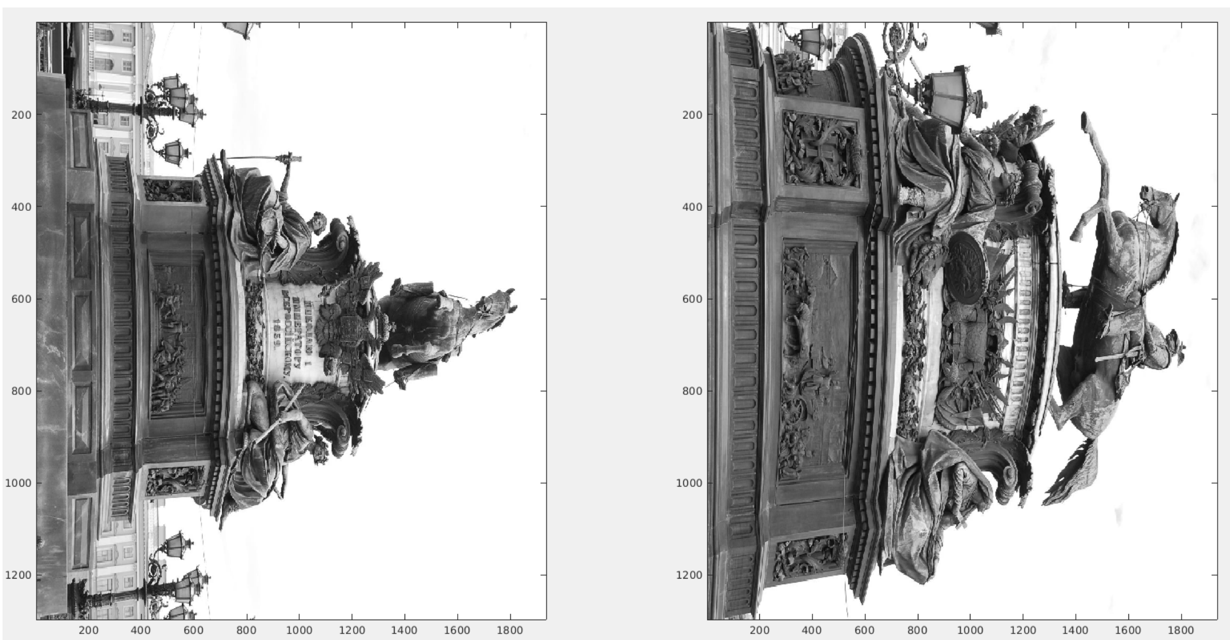


Figure 4: Reference images related to the exercise.

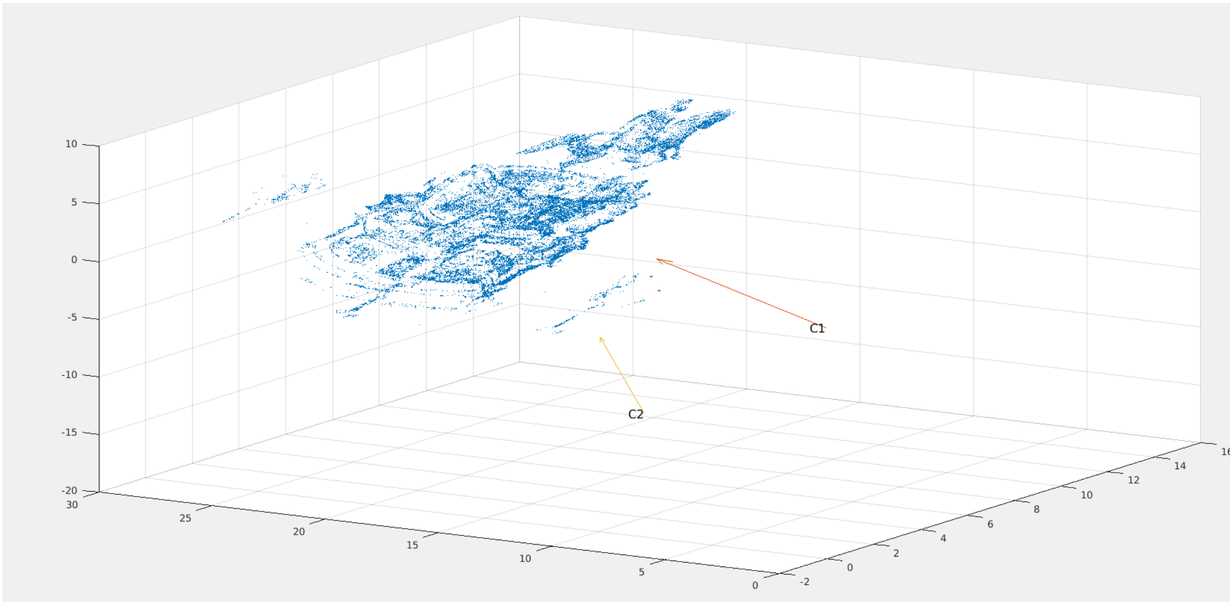


Figure 5: 3D plot with the 2 camera matrices (scaled by 1 and 6, respectively), and the point model of the statue $U$.

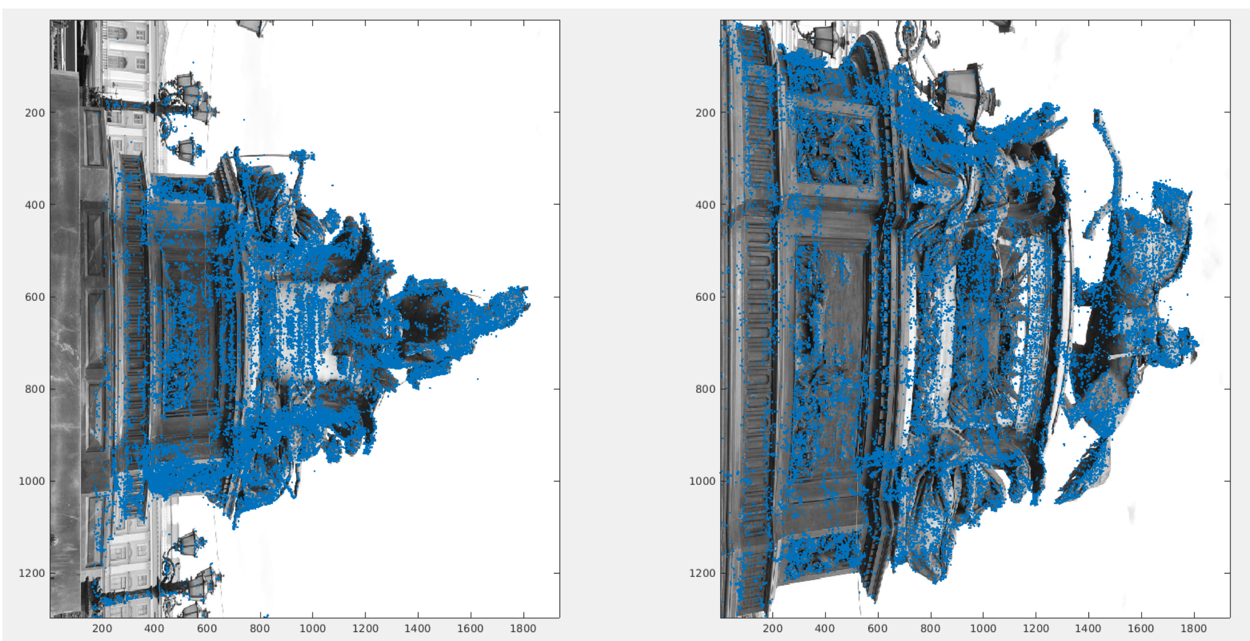

Figure 6: Plot of the projection of the points in $U$ into the cameras $P1$ and $P2$, on the initial images. As we can notice, the result of the projections seems reasonable, since they align with the actual image points.

## Theoretical exercise 8

To perform the following verification:

$$U \sim \begin{pmatrix} x \\ s \end{pmatrix},$$

we can compute the projection of $U(s) = \left(x^T, s\right)^T$ using $P_1 = \left[I|0\right]$.

$$P_1 U(s) = \left[I|0\right] \begin{pmatrix} x \\ s \end{pmatrix} = (I \times x + 0 \times s)^T = x \in \mathbb{P}^2.$$

The result is still the 2D point $x$, which suggest that the scalar $s \in \mathbb{R}$ doesn't play a role in the projection. In this case, the object $U(s)$ can be considered as the viewing ray for the image point $x$ of camera $P$, where $s$ is projective depth / parameter to move on the line/ray.

Related to the interpretation, It is not possible to determine s using only information from $P_1$, since the camera matrix does not provide direct information about the depth or scaling of the 3D points.

Assume now that $U$ belongs to to plane $\Pi = \left(\pi, 1\right)^T$ ($\pi \in \mathbb{R}^3$). To compute a scalar $s$ such that $\Pi^T U(s) = 0$, we can solve the following system of equations:

$$\begin{cases} \Pi^T U = 0 \\ \Pi^T U(s) = 0 \end{cases} \quad \rightarrow \Pi^T U = \Pi^T \begin{pmatrix} x \\ s \end{pmatrix} \rightarrow \Pi^T U = \pi^T x + s \rightarrow s = \Pi^T U - \pi^T x$$

Since $\Pi^T U = 0$, $s = -\pi^T x$.

In the last point we will verify that, if $x \sim P_1 U$, $x \sim P_2$ and $\Pi^T U = 0$ then the homography

$$H = (R - t\pi^T),$$

where $P_2 = [R|t]$, maps $x$ to $y$ ($y = Hx$). Developing our assumption, we obtain:

$$y = Hx = (R - y\pi^T)x = Rx - t\pi^T x.$$

Since we know that the 3D point $U$ belongs to the 3D plane $\Pi$ ($\Pi^T U = 0$), for $s = -\pi^T x$ we also have $\Pi^T U(s) = 0$. Then, we can write:

$$y = P_2 U = P_2 U(s) = [R|t] \begin{pmatrix} x \\ -\pi^T x \end{pmatrix} = Rx - t\pi^T x.$$

Also recalling that if $x \sim P_1 U$ then $U \sim \begin{pmatrix} x \\ s \end{pmatrix}$ (for any $s$ points of the form $U(s)$ always projects to x), the above equation confirm that the homography $H$ maps $x$ into $y$.

## Computer exercise 4

Main reference matlab file: `comp_ex4.m`.

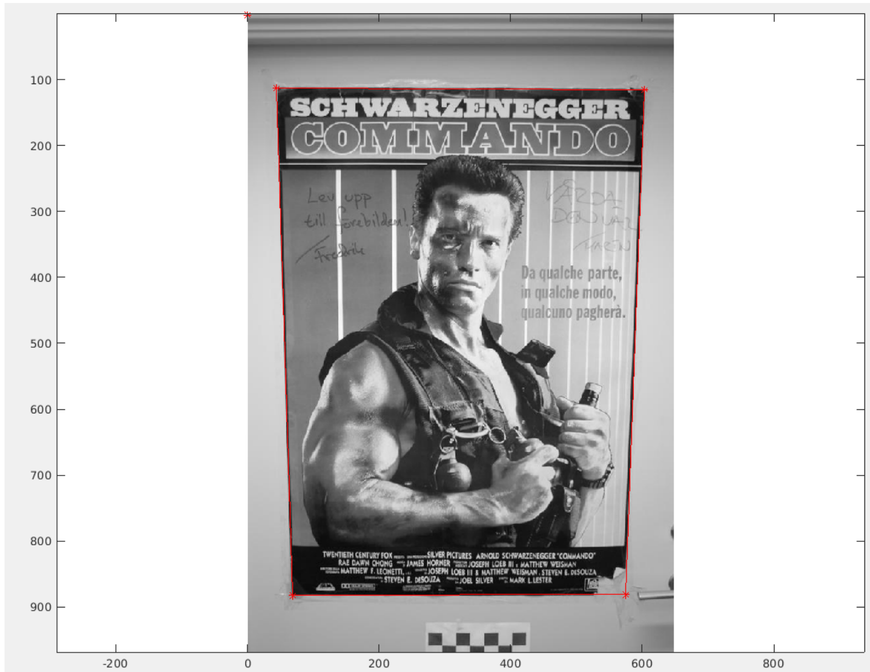


Figure 7: Initial plot of the image and the corner points. The origin of the image coordinate system is located in (0, 0).

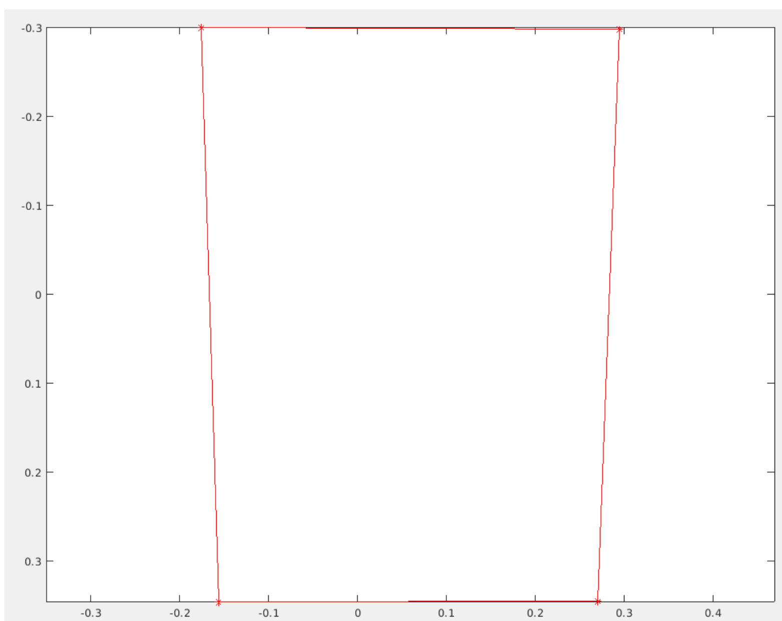


Figure 8: Plot of the normalized corner points, to ensure that we have calibrated cameras. In this case, the origin of the image coordinate system is located at (-0.17, -0.3).

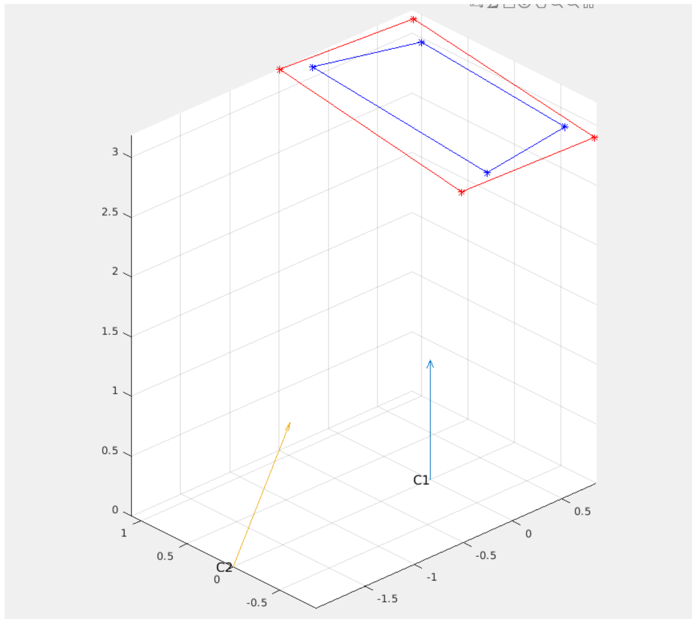


Figure 9: 3D-plot of the original camera (with center in C1) and the computed one (with center C2), with both the projected original corner points and the ones obtained from the homography $H$ (obtained with the rotation matrix of the second camera matrix) on the plane $v$. The result seems reasonable, since each camera matrix point to the related projected corner points.

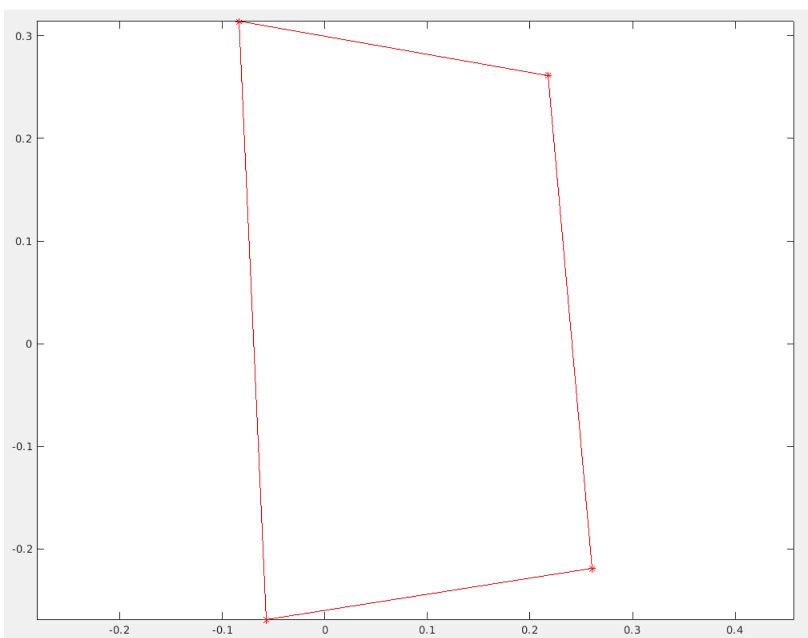

Figure 10: Plot of the corner points tranformed with the homography obtained from the rotation matrix of the second camera. The result, especially related to the previous figure, is the expected one in case a camera would be moved according to the provided $R$.

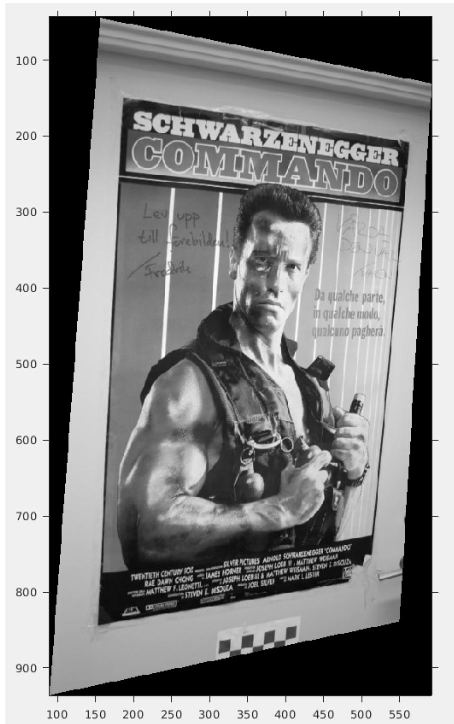


Figure 11: Plot with of the image, after by tranformed by the homography.