Assignment 3 - Computer Vision (EEN020)

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1 The Fundamental Matrix

Theoretical Exercise 1

From the cameras P_2, we can derive the following matrices:

$$[T]_\times = [(P_2)_{:4}]_\times \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ -2 & 2 & 0 \end{pmatrix}, \quad R = (P_2)_{33} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

With the information above, the fundamental matrix F can be computed as follows:

$$F = [T]_\times R = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ -2 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ -2 & 0 & 0 \end{pmatrix}.$$

Suppose now a point $x = (0, 2)$ is the projection of a 3D-point X into P_1. the epipolar line in the second image generated from x can computed in the following way:

$$l = Fx = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix}.$$

The epipolar line l can be used to check if a point could be a projection of the same point X in P_2, by checking if the dot-product and the line itself is equal to zero. This check was done in the following points:

- $y_1 = (1, 0), y_1^T l = 2 \rightarrow$ not a projection of X in P_2,
- $y_2 = (3, 2), y_2^T l = 2 \rightarrow$ not a projection of X in P_2,
- $y_3 = (1, 2), y_3^T l = 0 \rightarrow$ projection of X in P_2.

Theoretical Exercise 2

The epipoles by projecting the camera centers are computed as follows:

$$e_1 = P_1 C_2 = \begin{bmatrix} I & 0 \end{bmatrix} \text{null}(P_2) = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix},$$

$$e_2 = P_2 C_1 = \begin{pmatrix} 0 & 1 & 1 & 2 \\ 3 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{pmatrix} \text{null}(P_1) = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}.$$

To evaluate the fundamental matrix F, instead, the same formula as in the previous theoretical exercise can be used:

$$F = [T]_\times R = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -2 \\ -1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 3 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & -6 \\ 6 & 3 & -1 \end{pmatrix}.$$

Where its determinant is $det(F) = 0$. The fundamental matrix was then used to perform the following checks:

$$e_2^T F = \begin{pmatrix} 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & -6 \\ 6 & 3 & -1 \end{pmatrix} = 0,$$

$$F e_1 = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & -6 \\ 6 & 3 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = 0.$$

Theoretical Exercise 3

To compute the epipoles, we first need the camera centers:

$$C_1 = \text{null}(P_1) = \textbf{null}(\begin{bmatrix} I & 0 \end{bmatrix}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

$$C_2 = \text{null}(P_2) = \textbf{null}(\begin{bmatrix} A & t \end{bmatrix}),$$

$$\begin{bmatrix} A & t \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{pmatrix} = 0 \rightarrow Ax + t = 0 \rightarrow x = A^{-1}t \rightarrow C_2 = \begin{pmatrix} -A^{-1}t \\ 1 \end{pmatrix}.$$

The epipoles by projecting the camera centers are then computed as follows:

$$e_1 = P_1 C_2 = \begin{bmatrix} I & 0 \end{bmatrix} \begin{pmatrix} -A^{-1}t \\ 1 \end{pmatrix} = -A^{-1}t,$$

$$e_2 = P_2 C_1 = \begin{bmatrix} A & t \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = t.$$

Considering the fundamental matrix $F = [t]_\times A$, the next step is to check the epipoles contraints: $e_2^T F = 0$ and $F e_1 = 0$.

$$e_2^T F = 0 \rightarrow t^T [t]_\times A = 0 \rightarrow \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} A = 0 \quad \blacksquare,$$

$$F e_1 = 0 \rightarrow [t]_\times A(-A^{-1}t) = 0 \rightarrow [t]_\times A A^{-1}(-t) = 0 \rightarrow [t]_\times (-t) = 0 \quad \blacksquare.$$

To conclude, given the above results, the fundamental matrix F need to have the determinant 0. This because otherwise F would be a full rank matrix, implying that the epipolar lines corresponding to a pair of points will not intersect at a common point (the epipole); moreover, the same epipole will not satisfy the epipolar constraints above. Also for this reason, the rank of the fundamental matrix has to be 2.

Theoretical Exercise 4

Consider

$$\tilde{x_1} = N_1 x_1 \quad \text{and} \quad \tilde{x_2} = N_2 x_2.$$

If $\tilde{F}$ fulfills $\tilde{x_2}^T \tilde{F} \tilde{x_1} = 0$, the fundamental matrix F that fulfills $x_2^T F x_1 = 0$ can be derived as follows:

$$\tilde{x_2}^T \tilde{F} \tilde{x_1} = 0 \rightarrow (N_1 x_1)^T \tilde{F} N_2 x_2 = 0 \rightarrow x_2^T N_2^T \tilde{F} N_1 x_1 = 0 \rightarrow F = N_2^T \tilde{F} N_1,$$

where the last step was performed in order to fulfill the requirement for F.

Computer Exercise 1

The complete solution is reported in the MATLAB file `comp_ex1.m`.

Part 1

As first checks, after performing the 8-point algorithm to estimate F we checked the following values:

- minimum singular value $= 0$,
- $||Mv|| = 0.049642$,

resulting in small values as expected.

Another check performed, after enforcing the determinant of the fundamental matrix to 0, is related to the epipolar constraints. The plot below shows that the constraints is roughly fulfilled.

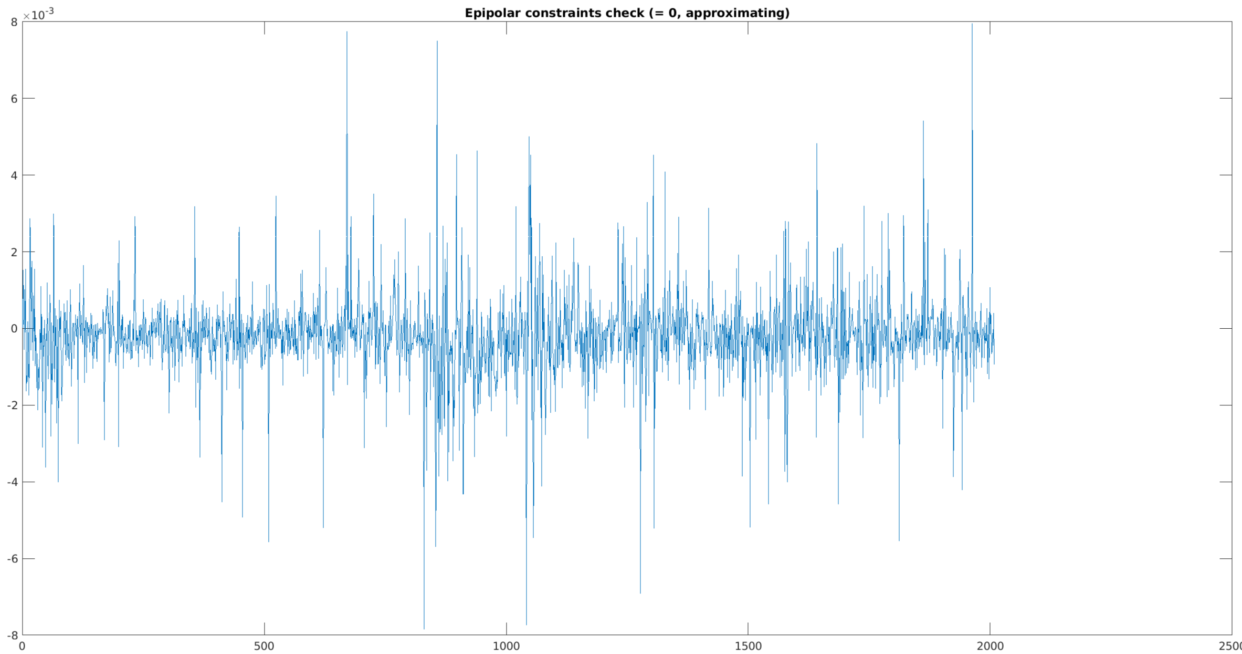

Figure 1: Values of the epipolar constraints $\tilde{x_2^T}\tilde{F}\tilde{x_1} = 0$. As we can see the values are small and close to 0 roughly satisfying the mentioned constraints.

Fundamental matrix F after the un-normalization (and with $F(3,3) = 0$):

$$F = \begin{pmatrix} -3.3901e^{-8} & -3.7201e^{-6} & 0.00577 \\ 4.6674e^{-6} & 2.8936e^{-7} & -0.0267 \\ -0.0072 & 0.0263 & 1 \end{pmatrix}.$$

The same above matrix was then used to compute the epipolar lines for a sample of the image points of the second figure and the epipolar errors, represented by an histogram.

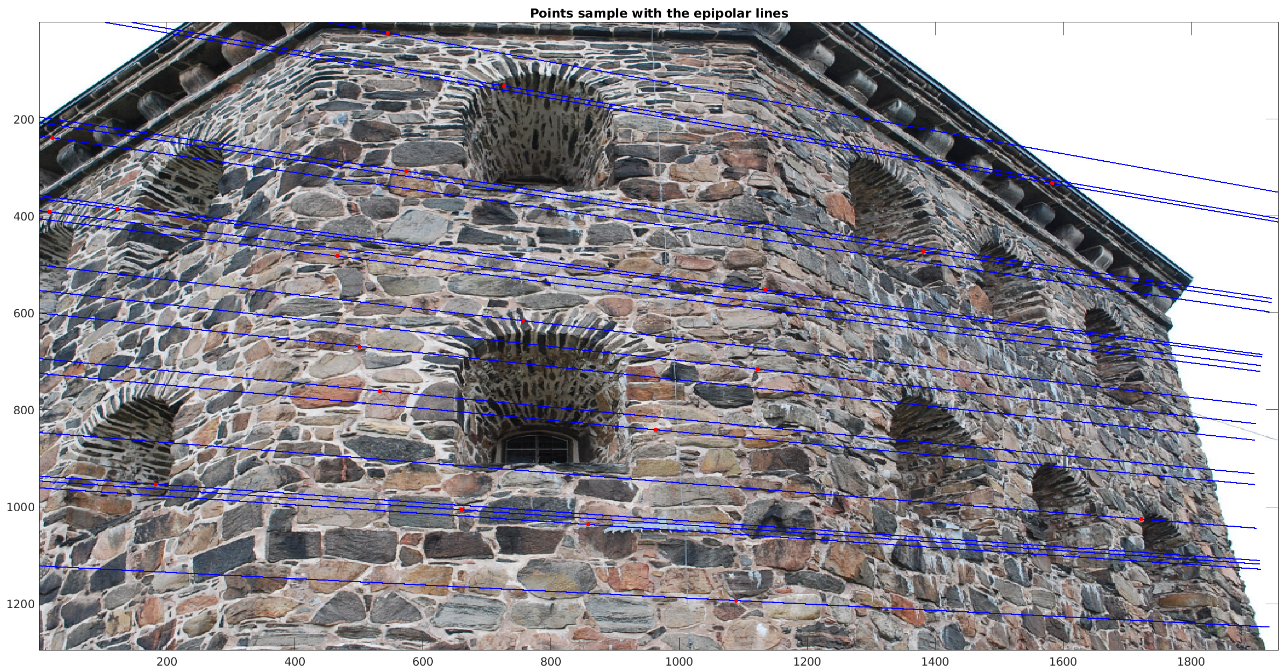

Figure 2: Points of the sample and the related epipolar lines on the image figure. As we can notice each point is close to the correspoinding epipolar line.

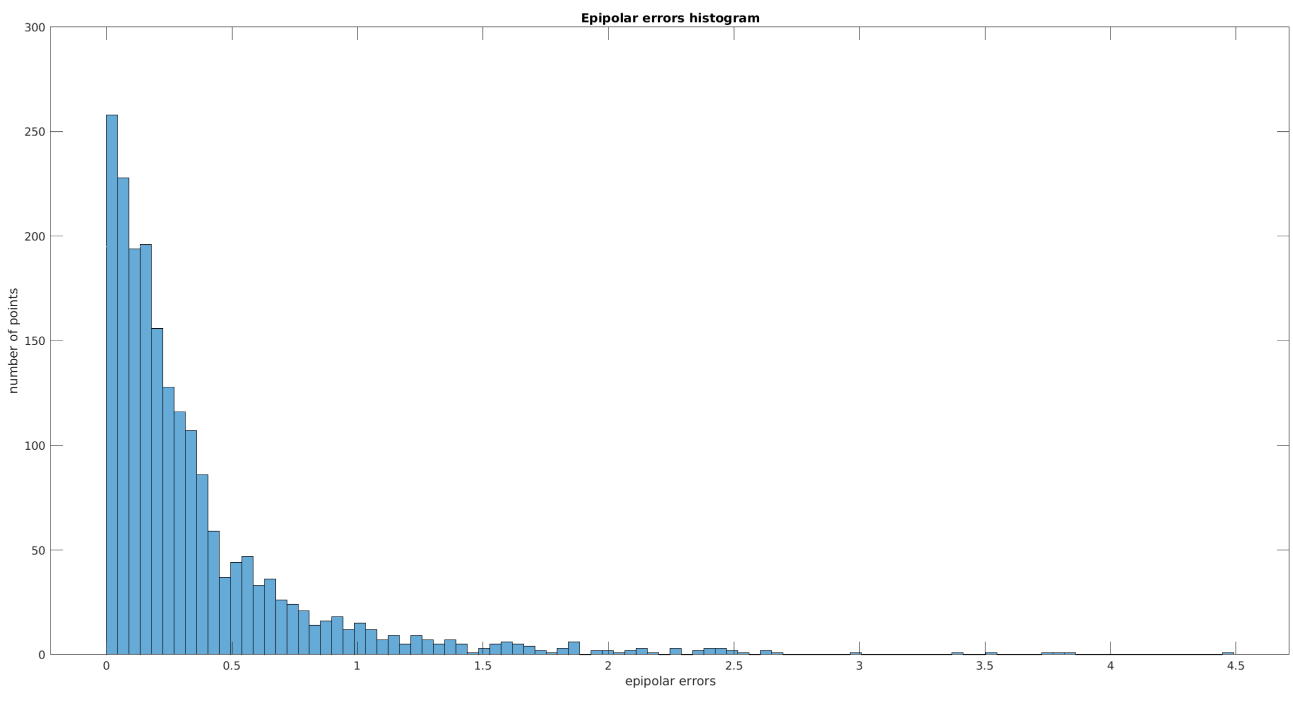

Figure 3: Histogram of the computed epipolar errors. The mean distance between the points and lines (that is, the epipolar errors mean) is 0.36123.

Part 2

In the second part, the experiment in Part 1 was repeated but without normalization (that is, with $N_1 = N_2 = I$). In this case, the mean distance (or epipolar errors mean) increases to 0.48784.

Theoretical Exercise 5

To check the required constraints, we first need to compute the epipolar line e_2 and then the camera P_2:

$$e_2 = null(F^T) = \begin{pmatrix} -0.7071 \\ 0 \\ 0.7071 \end{pmatrix},$$
$$P_2 = \begin{bmatrix} [e_2]_\times F & e_2 \end{bmatrix} = \begin{pmatrix} -1.4142 & 0 & -2.8284 & -0.7071 \\ 0 & 1.4142 & 1.4142 & 0 \\ -1.4142 & 0 & -2.8284 & 0.7071 \end{pmatrix}.$$

After the above computations, we can now verify that the projections of the 2 3D points X_1 and X_2 fulfill the epipolar constraints, for both cameras.

For the projections in P_1 (x_1) and P_2 (x_2) of the point X_1 we have:

$$x_2^T F x_1 = 0 \rightarrow \begin{pmatrix} -3.5355 & 5.6569 & -2.1213 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 4 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} = 0. \quad \blacksquare$$

For the projections in P_1 (x_1) and P_2 (x_2) of the point X_2 we have:

$$x_2^T F x_1 = 0 \rightarrow \begin{pmatrix} 0.7071 & 2.8284 & 2.1213 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 4 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = 0. \quad \blacksquare$$

After the epipolar constraints checks, also the camera center of P_2 was computed (the results will be reported in homogeneus coordinates):

$$C_2 = \text{null}(P_2) \approx \begin{pmatrix} -1.0524e^{16} \\ -5.2621e^{15} \\ 5.2621e^{15} \\ 1 \end{pmatrix}.$$

2 The Essential Matrix

Theoretical Exercise 6

To show that the eigenvalues of $[t]_\times^T[t]_\times$ are squared singular values (that is, the values in the diagonal matrix S from the Singular Value Decomposition of $[t]_\times^T[t]_\times$), we can derive the following:

$$\begin{aligned}
&[t]_\times^T[t]_\times = (USV^T)^T(USV^T)\\
&= VS^TU^TUSV^T\\
&= VS^TISV^T\\
&= V(S^TS)V^T.
\end{aligned}$$

Note that $U^TU = I$ since U is an orthogonal matrix. From linear algebra theory, we can state that the values of S (thus, the singular values) are the square roots of $[t]_\times^T[t]_\times$. In other words, the eigenvalues of $[t]_\times^T[t]_\times$ are the squared singular values.

Then, assuming a vector w we can derive the following:

$$\begin{aligned}
&-t \times (t \times w) = -t \times ([t]_\times w)\\
&= [-t]_\times[t]_\times w\\
&= [t]_\times^T[t]_\times w.
\end{aligned}$$

With the derivation above, since the eigenvector is a vector v such that:

$$Av = \lambda v, A \text{ is a matrix}, \lambda \text{ any eigenvalue of } A,$$

w can be considered as an eigenvector as well.

Next, considering the formula from "Linjär ar Algebra" by Sparr (on page 96) and setting $u = -t$ and $v = t$, we can write:

$$-t \times (t \times w) = (-t \cdot w)t - (-t \cdot t)w;$$

where, by also setting $w = t$, we will obtain:

$$(-t \cdot t)t - (-t \cdot t)t = 0.$$

From the result above, we can say that $w = t$ represents an eigenvector of $[t]_\times^T[t]_\times$ with eigenvalue 0.

Moreover, when instead w is perpendicular to t (that is, $t \cdot w = 0$), by referencing to the last result above we can derive:

$$\begin{aligned}
&(-t \cdot w)t - (-t \cdot t)w\\
&= -(-t \cdot t)w\\
&= ||v||^2 w.
\end{aligned}$$

The result obtained confirm that any w that is perpendicular to t is an eigenvector with eigenvalue $||v||^2$. Furthermore, also through software verification (by computing the eigenvectors with MATLAB code), we can also affirm that all the eigenvectors found (all the vector w perpendicular to t and t itself), are the only eigenvectors of $[t]_\times^T[t]_\times$.

For the next result, recalling:

- the first result of the exercise;

- the eigenvalues of $[t]_\times^T[t]_\times$ are $||t||^2, ||t||^2$ and 0
- since $[t]_\times$ is a normal matrix, we can have the following Singular Value Decomposition with the form $[t]_\times = UDU^T$;

we can affirm that that the eigenvalues of $[t]_\times$ are instead $||t||, ||t||$ and 0.

Computer Exercise 2

The complete solution is reported in the MATLAB file `comp_ex2.m`.

Essential matrix E, obtained using the 8-point algorithm with the matrix F and then scaled such that the non-zero singular values are 1:

$$E = \begin{pmatrix} 0.0032 & 0.3619 & -0.1357 \\ -0.4507 & 0.0282 & 0.8808 \\ 0.1701 & -0.9176 & -3.5980e^{-4} \end{pmatrix}.$$

As the previous exercise also the epipolar constraints were checked, noticing that they are roughly fulfilled.

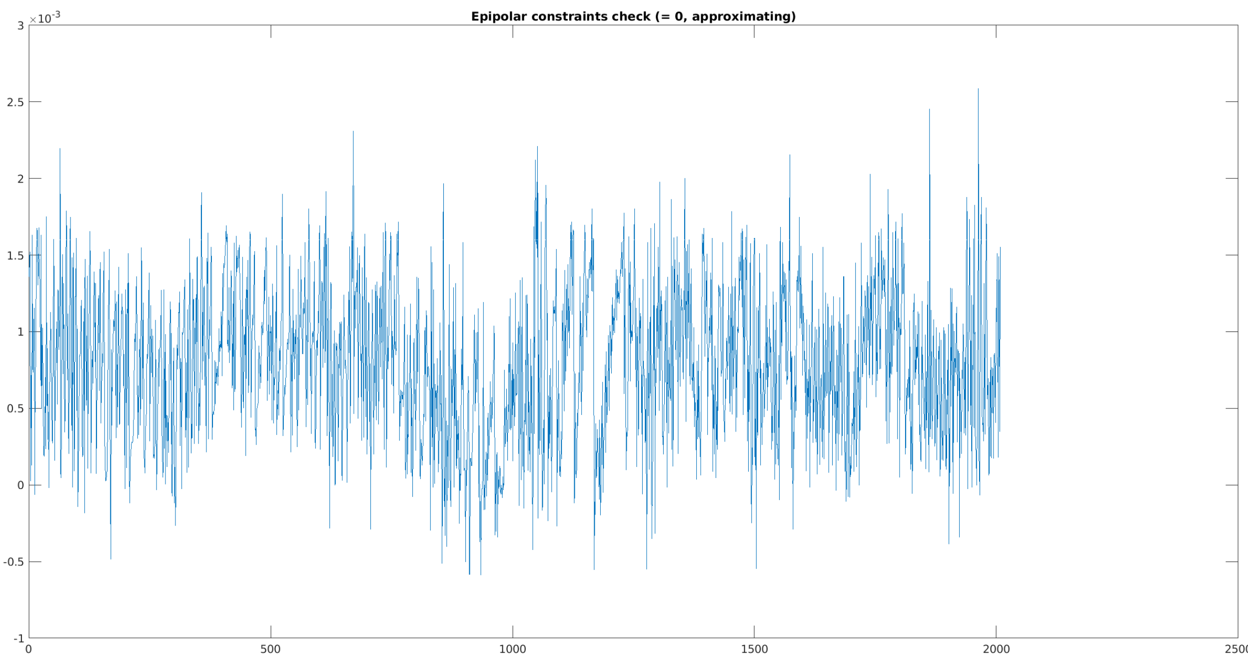

Figure 4: Values of the epipolar constraints $\tilde{x_2}^T E \tilde{x}_1 = 0$. As we can see the values are small and close to 0 roughly satisfying the mentioned constraints.

The essential matrix was then used to retrieve the fundamental one. F was then used to compute the epipolar lines for a sample of the image points of the second figure and the epipolar errors, represented by an histogram.

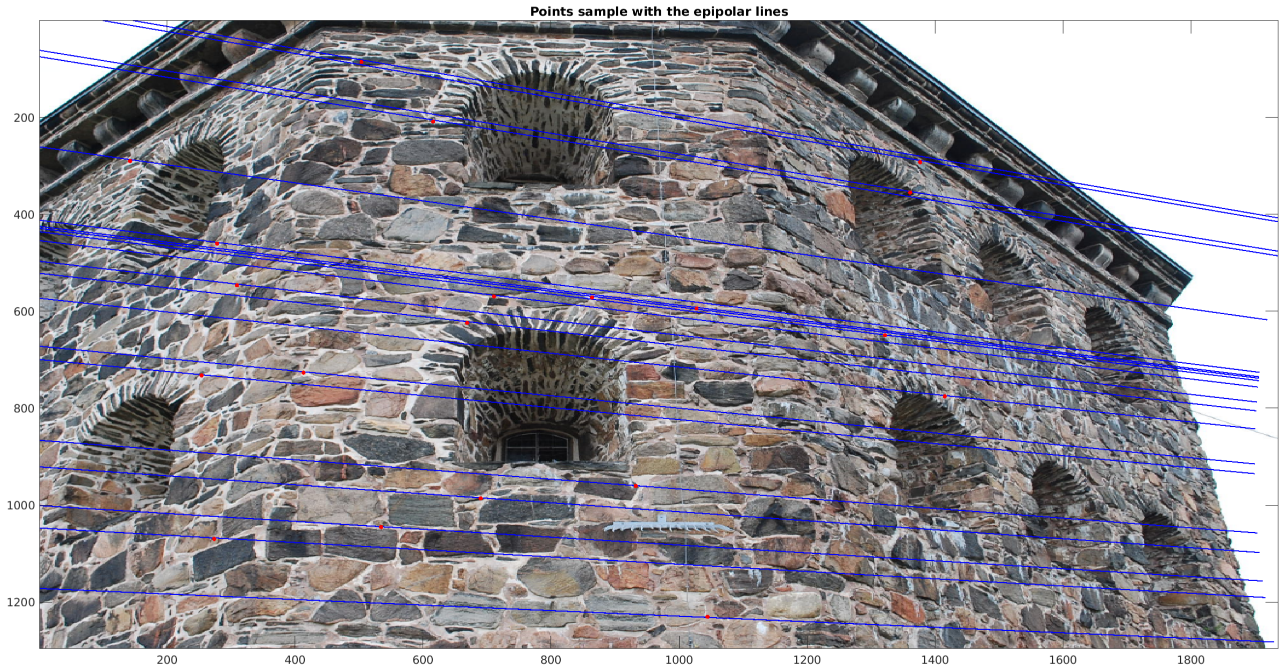

Figure 5: Points of the sample and the related epipolar lines (computed with the matrix F converted from E) on the image figure. As we can notice each point is close to the corresponding epipolar line.

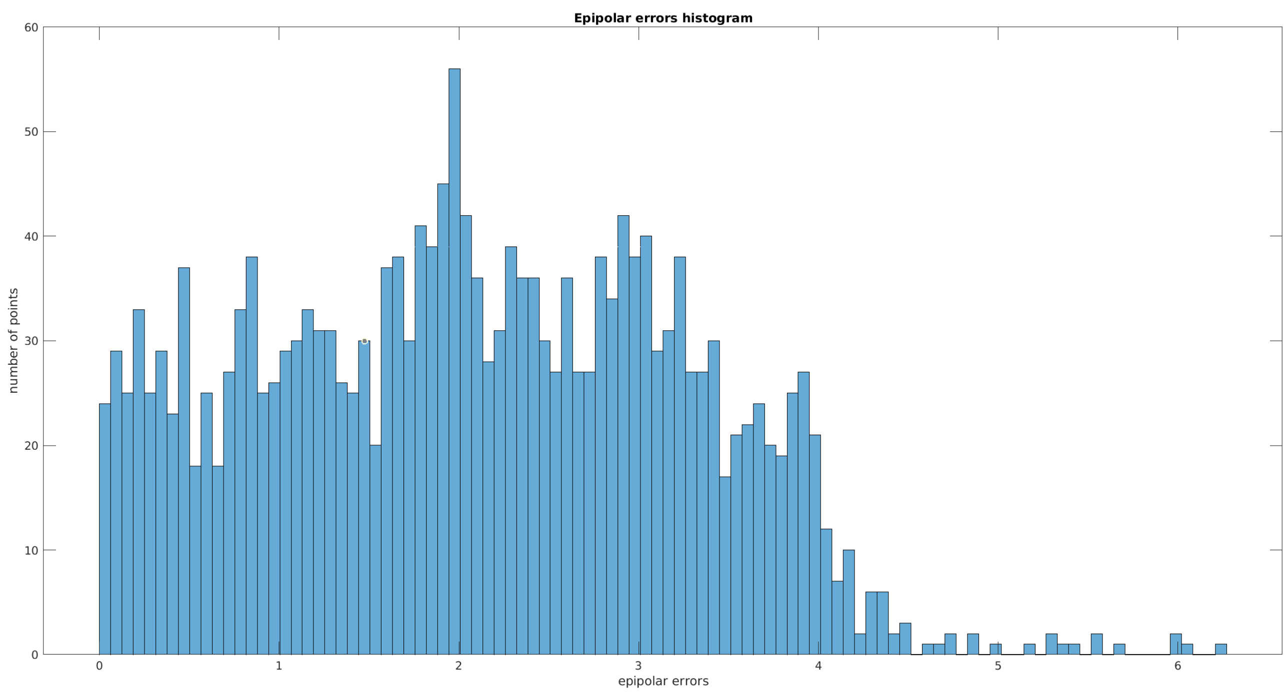

Figure 6: Histogram of the computed epipolar errors, with mean distance between the points and lines (that is, the epipolar errors mean) of 2.0838. Comparing with the results obtained in the computer exercise 1, we can notice a much bigger mean distance, thus a much bigger error.

Computer Exercise 3

In this computer exercise, the essential matrix E obtained from the previous one was used to extrapolate the 4 cameras (also using the twisted pair assumption). The following plots will use the camera where the highest number of points are in front of the cameras (P_1 and one the camera found). Moreover, note also that the point found are derived from the triangulated 3D points, for each camera (in this case, the points used will be the one associated with the chosen camera).

Figure 7: Image points and the projected points from the selected camera (image1).

Figure 8: image points and the projected points from the selected camera (image2)".

The actual image points are in red, while the triangulated and then projected 3D points are represented by blue circles. In both cases, the error looks small for the alignment between the points of the images and the projection of the points retrieved from the computed camera.

Below, instead, the triangulated 3D points are used to visualize the 3D reconstruction of the building:

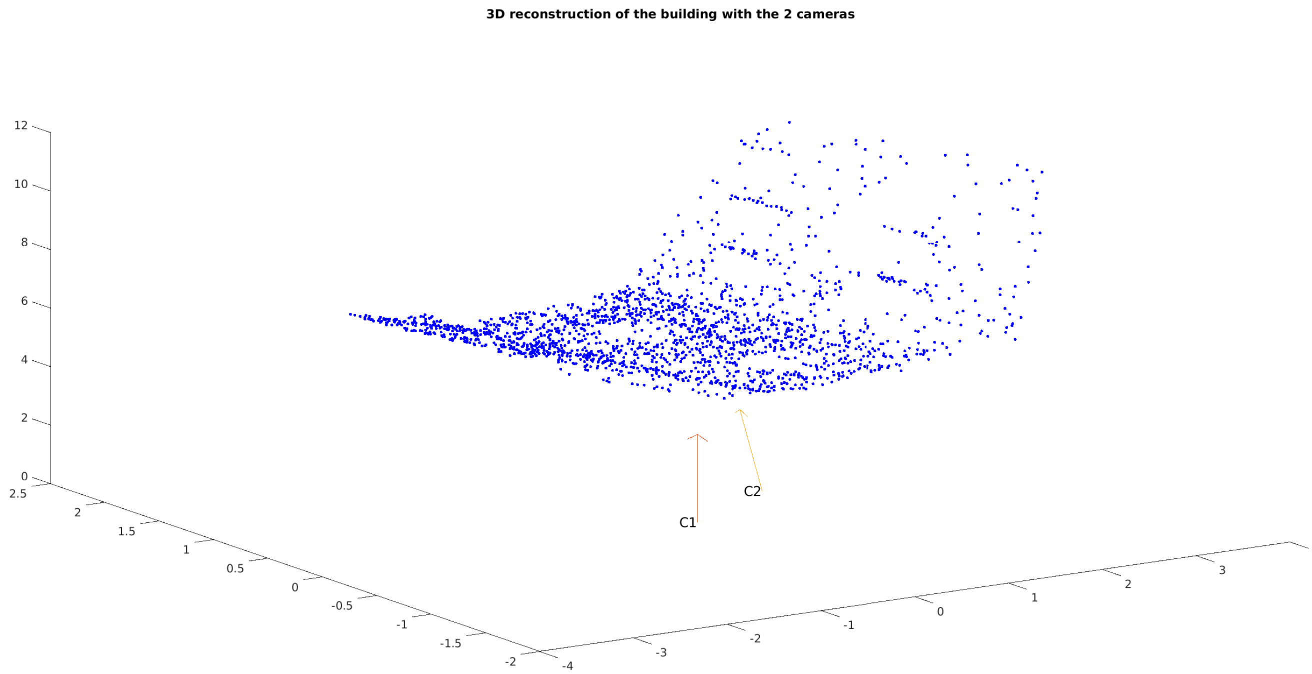

Figure 9: 3D reconstruction of the building with the 2 cameras. the 3D reconstruction looks like It should be expected, reflecting the building in the 2D images and with the 2 cameras pointing in the corrected directions.