Obligatory assignment 2, MVE550, autumn 2022

Petter Mostad

November 16, 2023

1. Consider a discrete-time Markov chain with possible states $1, 2, 3$. The transition matrix P is unknown, but we have observed the first 12 steps of a realization of the chain:

$$1, 2, 3, 2, 3, 1, 2, 1, 3, 2, 1, 3.$$

Let P_1, P_2, and P_3, be the rows of P. We assume independent Dirichlet priors for these, with all pseudo-counts equal to 1.

(a) Given the data, describe the posterior distributions for P_1, P_2, and P_3, and thus for P. Compute the expectation of the posterior for P.

(b) Write an R function `simulate1` that simulates a continuation of the chain above into a chain of total length 500, as follows: Sequentially, for each of the added steps, one should simulate a new state for the chain, using the expected value of the posterior for P when the data is all observed and all simulated values so far.

The result of the function `simulate1` should be the expected value of P when the data is all observed and all simulated values.

(c) Use `simulate1` to produce a histogram of 1000 values of the expected value of P_{23} given the observed and simulated data. (You might try the function `replicate`).

(d) Adapt `simulate1` to `simulate2` so that when you simulate a new state for the chain, you use the expected value of P based only on the actual 12 data values. However, the final result should be the expectation based on all actual and simulated data. Make a new histogram as in (c). Describe and explain any differences between the histograms in (c) and (d).

(e) Adapt `simulate1` to `simulate3` by starting with sampling[1] a P from the posterior found in (a). Then you use this sampled value every time you simulate new states in the chain. Make a histogram as in (c) and (d). Describe similarities and differences between the three histograms, and explain these theoretically.

[1] Hint: Consider the R function `rdirichlet`. To access the function you first need to download the R package "LearnBayes" by using `install.packages(LearnBayes)` and then activate it using `library(LearnBayes)`.

(f) It turns out that contextual information implies that the Markov chain can never directly repeat the same state, it will always change to a different state. Describe the adapted prior that incorporates this information.

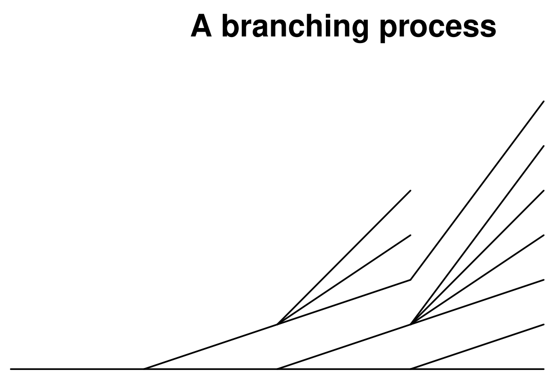

Figure 1: The 5 first steps of the Branching process for question 1

2. Assume a branching process has been observed for $Z_0, \ldots, Z_4$, and it looks like Figure 1. Assume the offspring distribution is Poisson with expectation λ. Assume that we use the improper prior $\pi(\lambda) \propto_\lambda 1/\lambda$ for λ. Compute the probability of extinction as follows:

   (a) Using the information about the offspring distribution appearing in Figure 1, compute the posterior distribution for λ.

   (b) Consider the continuation of the branching process pictured in Figure 1 into generations Z_5, Z_6, etc. Implement in R a function that takes as input a value for λ and outputs the probability that this branching process will become extinct[2].

   (c) Now compute the probability of extinction for the branching process taking the uncertainty in λ into account: Write down an integral representing this probability in terms of the results from (a) and (b). Then compute the integral using numerical integration.

   (d) Use simulation to check your result in (c): Simulate from the posterior found in (a), and combine with the code found in (b).

[2]Hint: Consider the R function `optimize`

(e) What is the maximum likelihood[3] estimate for λ? What is the probability of extinction for the branching process if you use this estimate for λ in our computation?

[3] The *likelihood* is the function of λ which for each λ gives the probability of the observations, as used in (a). The maximum likelihood estimate is found by maximizing this function.