Obligatory assignment 1, MVE550, autumn 2023

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November 1, 2023

NOTE: In the tasks below, you may use R functions for density/mass functions, cumulative distribution functions, quantile functions, and random data generation functions. Use the help function in R to learn more.

1. The probability density function of a Weibull random variable with shape parameter $\kappa > 0$ and scale parameter $\lambda > 0$ is given by

$$\pi(x \mid \kappa, \lambda) = \frac{\kappa}{\lambda}\left(\frac{x}{\lambda}\right)^{\kappa-1} \mathtt{exp}\left(-\frac{x^\kappa}{\lambda^\kappa}\right), \qquad x > 0$$

   Assume that the failure times of a specific electrical component in a device come from a Weibull distribution with known shape parameter $\kappa = 2$ and unknown scale parameter $\lambda > 0$.

   Now letting $\theta = \lambda^2$, the likelihood is

$$\pi(x \mid \theta) = \frac{2x}{\theta}\mathtt{exp}\left(-\frac{x^2}{\theta}\right), \qquad x > 0 \tag{1}$$

   Finally, we have observed the following failure times

$$\mathbf{x} = (0.66, 2.30, 1.98, 1.49, 0.62)$$

   (a) Determine $C(\alpha, \beta)$ so that the function

$$\pi(\theta \mid \alpha, \beta) = C(\alpha, \beta)\frac{1}{\theta^{\alpha+1}}\mathtt{exp}\left(-\frac{\beta}{\theta}\right), \quad \theta > 0$$

   is a probability density function for any $\alpha, \beta > 0$.

   (b) Guess at a conjugate family of priors to the likelihood in Equation (1). Prove that this family is conjugate, and compute the formula for the posterior density for θ given any prior density from the family.

   (c) Choose a prior from your conjugate class and plot the posterior for θ given your data.

   (d) Compute the formula for the posterior predictive distribution of the failure time for the 6th component. Use the formula you derived to calculate $\mathrm{P}(1 < X_6 < 2 \mid \mathbf{x})$, for example using numerical integration.

(e) Assuming a `Unif`$(0, 5)$ distribution as your prior, compute and plot the posterior for θ using discretization.

(f) Using the new prior, implement a function that computes the posterior predictive distribution of the failure time for the 6th component, $\pi(X_6 \mid \mathbf{x})$, using numerical integration. Now using your function calculate $\mathrm{P}(1 < X_6 < 2 \mid \mathbf{x})$, for example using numerical integration.

(g) Simulate a sample of size 100000 from the predictive distribution of the failure time for the 6th component, using the posterior found in (e). Create a histogram of the simulated times and calculate the probability $\mathrm{P}(1 < X_6 < 2 \mid \mathbf{x})$.

2. The board for a modified Snakes and Ladder game is shown in Figure 1. A counter starts at square 1 and the game ends when the counter reaches square 9 (e.g. if the counter is at square 7 and the result of the next throw is either 2, 3 or 4, then the game ends). If your counter lands at the bottom of a ladder, you move up to the top of the ladder and if your counter lands on the head of the snake, you must slide down to the bottom of the snake. For each round a fair 4-sided dice is thrown and the player moves the counter the corresponding number of steps. Find formulas for the answers to the questions below. You may use R to compute the actual values.

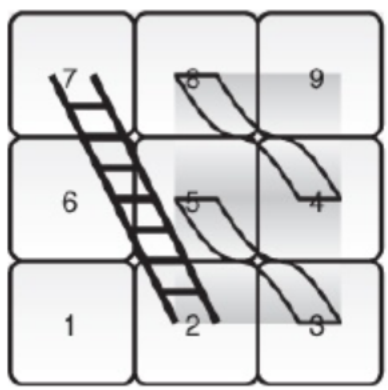

Figure 1: Snakes and Ladder game board

(a) Find the expected length of the game.

(b) What is the probability that the counter will land on square 6 before the end of the game?

(c) Assume that the counter is on square 6. Find the probability that the counter will land on square 3 before finishing the game.

(d) Verify the answers you obtained in questions (a), (b) and (c) using simulation.