PS 3 Networks MT22

Let A denote the adjacency matrix of an unweighted, undirected network (no double edges and self loops.

a) Where in the complex plane does the spectrum lie?

The matrix A is symmetric $\rightarrow$ spectrum is real

Quick proof:

$$Ax = \lambda x \Rightarrow \lambda^* x^T = (Ax)^T$$

$$\lambda^* x^T = (Ax)^T = x^T A^T \overset{\text{bcs of symmetry}}{=} x^T A \Rightarrow \lambda^* x^T x = x^T A x = \lambda x^T x \Rightarrow \lambda^* = \lambda$$

Can we say more? Yes! $\rightarrow$ Perron-Frobenius Theorem

can be extended to non-negative matrices

Let A be a $n \times n$ positive matrix. There is a positive real number r called the Perron root or Perron-Frobenius eigenvalue or leading eigenvalue s.t. r is an eig. value of A and all the other e.v. λ are smaller than r, $|\lambda| < r$.

$$\Rightarrow \rho(A) \in (-r, r]$$

b) Give an upper bound for $\rho(A)$ in terms of the vertex edges

The Perron-Frobenius theorem also sets boundaries for r,

$$\min_i \sum_j a_{ij} \leq r \leq \max_i \sum_j a_{ij} \Rightarrow \rho(A) \leq \max_i \{d_i\}$$

c) Where is the spectrum of $(I - \alpha A)$?

- The eigenvectors $\{v_i\}$ of A are also e.v. of $I - \alpha A$

$$\Rightarrow (I - \alpha A) v_i = I v_i - \alpha A v_i = (1 - \alpha \lambda_i) v_i$$

$$\Rightarrow \text{spectrum of } I - \alpha A \text{ is in } [1 - \alpha r, 1 + \alpha r)$$

d)

For Katz centrality to converge, $I - \alpha A$ must be invertible.

$\Rightarrow 1 - \alpha r > 0 \implies \alpha r < 1 \Rightarrow r < \frac{1}{\alpha}$. In terms of the degree dist, we have $\alpha < \frac{1}{\max_i \{d_i\}} < \frac{1}{r}$.

<u>Ex 2</u>

The Jordan Normal form of A will be in the form

$$A = Z \operatorname{diag}(\lambda_1, \dots, \lambda_5) Z^T \qquad (\text{NB. for our choice of } Z \text{ we have } Z^{-1} = Z^T)$$

it is easy to see that in this case Z is the matrix where the columns are the eigenvectors of A.

You saw in the ~~class~~ course notes that a matrix function $f(A)$ can be defined as

$$f(A) = Z f(J) Z^{-1}, \quad \text{where } J = \operatorname{diag}(\lambda_1, \dots, \lambda_5)$$

$\Rightarrow$

$$\exp(I - \alpha A) = Z \operatorname{diag}\left(\exp(1 - \alpha\lambda_1), \dots, \exp(1 - \alpha\lambda_5)\right) Z^T$$

$$\log(I - \alpha A) = Z \operatorname{diag}\left(\log(1 - \alpha\lambda_1), \dots, \log(1 - \alpha\lambda_5)\right) Z^T$$

<u>Note to self</u>

In class: show what is the intuition: change of basis + taylor expansion + change back

The principal power is defined as $A^s = e^{s \log A}$

$$\Rightarrow (I - \alpha A)^{\frac{1}{\sqrt{2}}} = Z \operatorname{diag}\left[\exp\left(\frac{1}{\sqrt{2}} \log(1 - \alpha\lambda_1)\right), \dots, \exp\left(\frac{1}{\sqrt{2}} \log(1 - \alpha\lambda_5)\right)\right]$$

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Community ↗ edited by Paris:

→ Radicchi et al. : set of nodes for which the internal degree is larger than external degree, i.e., denser cliques

→ Modularity : # // larger than random

The authors define Q as:

$$Q = \sum_{s=1}^{M} \left[\frac{l_s}{L} - \left(\frac{d_s}{2L}\right)^2 \right]$$

L = # links in the network

d_s = sum of degrees for nodes inside s

l_s = # links in module s

In the modularity max framework a subgraph s is a module iff

$$\frac{l_s}{L} - \left(\frac{d_s}{2L}\right)^2 > 0$$

Let's write the number of links going out of s as

$$l_s^{out} = a \cdot l_s \Rightarrow d_s = 2 l_s + l_s^{out} = (a+2) l_s$$

$$\Rightarrow \frac{l_s}{L} - \left[\frac{(a+2) l_s}{2L}\right]^2 > 0 \Rightarrow l_s < \frac{4L}{(a+2)^2}$$

Now if $a = 0$ (no out link) we get $l_s < L$ (always true) so no problem. But if ~~we set~~ $a > 0$ then we have an upper limit to the number of internal links that S must have to be a module → odd! The definition of community depends on the size of the network

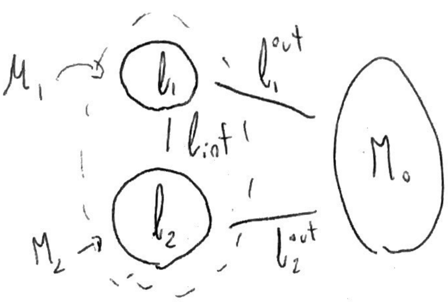
If $a < 2 \rightarrow 2 l_s > l_s^{out} \Rightarrow d_s^{in} > d_s^{out}$ = community by Radicchi ("weak definition")

For $a < 2$

$$\frac{L}{4} < \frac{4L}{(a+2)^2} < L \Rightarrow \text{assume } l_s < \frac{L}{4} \text{ to meet these conditions}$$

What does this have to do with the resolution limit?

Imagine a network



Let $l_1^{out} = b_1 l_1$, $l_2^{out} = b_2 l_2$, $l^{int} = a_1 l_1 = a_2 l_2$

Condition $l_s < \frac{L}{4}$ implies $l_1 < \frac{L}{4}$, $l_2 < \frac{L}{4}$ (M_1, M_2 modules)

Condition $d_s^{out} < d_s^{in}$ implies $a_1 + b_1 \leq 2$, $a_2 + b_2 \leq 2$

Now let's take two partitions of the network:

P_1: M_1 and M_2 are separated

P_2: M_1 and M_2 are the same module

$$Q_1 = Q_0 + \frac{l_1}{L} - \left[\frac{(a_1 + b_1 + 2) l_1}{2L}\right]^2 + \frac{l_2}{L} - \left[\frac{(a_2 + b_2 + 2) l_2}{2L}\right]^2;$$

$$Q_2 = Q_0 + \frac{l_1 + l_2 + a_1 l_1}{L} - \left[\frac{(2a_1 + b_1 + 2) l_1 + (b_2 + 2) l_2}{2L}\right]^2.$$

$$\Rightarrow \Delta Q = Q_2 - Q_1 = \left[2 L a_1 l_1 - (a_1 + b_1 + 2)(a_2 + b_2 + 2) l_1 l_2\right] / 2L^2$$

Modules are resolved iff $\Delta Q < 0$

$$\Rightarrow l_2 > \frac{2 L a_1}{(a_1 + b_1 + 2)(a_2 + b_2 + 2)}$$

But wait! This depends on L! By making L grow in M_0 the two communities might not be resolved anymore.

Limitation: From our result we deduce that it is not possible to exclude that modules of virtually any size may be clusters of modules, although the problem is more likely to occour for modules with a number of external links $\sim \sqrt{2L}$ or smaller.