PS 1 - Networks CS.4

1.

a) The expected value $E[x]$ of a variable x is

$$E[x] = \sum_K K\, p(x=K).$$

For Bernoulli:

$$E[x] = 1 \cdot p + 0 \cdot (1-p) = p.$$

Higher powers:

$$E[x^K] = 1^K \cdot p + 0^K (1-p) = p$$

$$\Rightarrow Var[x] = E[x^2] - E^2[x] = p - p^2 = p(1-p)$$

3.

a) The minimal example is

$$\circ \longrightarrow \circ$$

In a RW, walkers are trapped in the "sink" or "absorbing" nodes. The density in all the other nodes goes to zero.

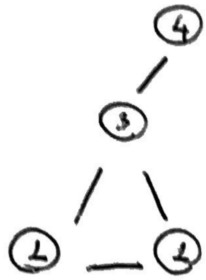
b) Undirected networks have symmetric adjacency matrices $\rightarrow$ their eigenvalues are real. This is not the case for dir. networks.

4.

local : $c_i = \dfrac{\# \nabla_i}{\# V_i} = \dfrac{\text{\# of couples of } i\text{'s friends that are friends themselves}}{\text{\# possible couples of friends of } i}$

average : $\langle c_i \rangle = \dfrac{1}{N} \sum_{i=1}^{N} c_i$

global : $\dfrac{\text{\# closed triplets}}{\text{\# } \cancel{\text{closed}} \text{ possible triplets}}$



$c_1 = \dfrac{1}{1}$ $\quad c_2 = \dfrac{1}{1}$ $\quad c_3 = \dfrac{1}{3}$, $c_4 = 0$

$\langle c \rangle = \dfrac{7}{12}$

$C_{glob} = \dfrac{3}{5}$

5.b

Let A be the adj. matrix of a network with λ its largest eigenvalue and v, u the corresponding left and right eigenvectors. What happens to λ if we remove an edge? What happens if we remove a node?

Show that

$$I_{ij} := -\frac{\Delta \lambda_{ij}}{\lambda} = \frac{A_{ij} v_i u_j}{\lambda v^T u}$$

$$I_K = -\frac{\Delta \lambda_K}{\lambda} = \frac{v_K u_K}{v^T u}$$

We can ~~imagine that removing a node/edg~~ write A, v, and u after the removal as

$A \to A + \Delta A$

$\lambda \to \lambda + \Delta \lambda$

$u \to u + \Delta u$

$v \to v + \Delta v$ ...

i.e., as the previous values + a perturbation.

1) edge removal

Since $u + \Delta u$ is the e.v. of $A + \Delta A$, we can write

$$(A + \Delta A)(u + \Delta u) = (\lambda + \Delta\lambda)(u + \Delta u)$$

$$\Rightarrow \cancel{Au} + \Delta A u + A \Delta u + \Delta A \Delta u = \cancel{\lambda u} + \lambda \Delta u + \Delta\lambda u + \Delta\lambda \Delta u$$

Ignoring 2^{nd} order terms,

$$\Delta A u + A \Delta u = \lambda \Delta u + \Delta\lambda u$$

Left-multiplying by v

$$v^T \Delta A u + v^T \cancel{A \Delta u} = \cancel{\lambda v^T \Delta u} + \Delta\lambda v^T u$$

$$\Rightarrow \Delta\lambda = \frac{v^T \Delta A u}{v^T u}$$

Removing (i,j) is setting A_{ij} to zero $\Rightarrow (\Delta A)_{lm} = -A_{ij} \delta_{li} \delta_{mj}$

$$\Rightarrow \Delta\lambda = -\frac{\sum_{lm} A_{ij} \delta_{il} \delta_{jm} v_l u_m}{v^T u} = -\frac{A_{ij} v_i u_j}{v^T u}$$

2) Node removal

If we remove node K, we set both the K-th row and the Kth column to zero

$$\Rightarrow (\Delta A)_{lm} = -A_{lm} (\delta_{lK} + \delta_{mK}).$$

Since the K-th row and K-th columns are $=0$, the K-th entry of u must be set to zero too. Consequently, we write $\Delta u = \underbrace{\delta u}_{\text{"small" perturbation to all the entries}} - \underbrace{u_K \hat{e}_K}_{\to K\text{-th entry to zero}}$

After some algebra we are left with:

$$\Delta\lambda_K = \frac{v^T \Delta A u - v^T \Delta A u_K \hat{e}_K}{v^T u - v_K u_K}$$

Now

$$v^T \Delta A u = \sum_{l,m} (\Delta A)_{lm} v_l u_m = -\sum_{l,m} \left[(v_l A_{lm} \delta_{lK}) u_m + (A_{lm} u_m \delta_{mK}) v_l \right]$$

$$= -\sum_l (v_l A_{lK}) u_K - \sum_m v_K (A_{Km} u_m) = -v_K \lambda u_K - v_K \lambda u_K = -2\lambda v_K u_K$$

And similarly

$$v^T \Delta A \hat{e}_K = \dots = -\lambda v_K u_K$$

Finally, assuming a large network, $v^T u \gg v_K u_K$

$$\Rightarrow \Delta\lambda_K = -\frac{\lambda v_K u_K}{v^T u} \Rightarrow I_K = -\frac{\cancel{\lambda} v_K u_K}{v^T u}$$

After some algebra we are left with:

$$\Delta\lambda_K = \frac{V^T \Delta A\, U - V^T \Delta A\, U_K \hat{e}_K}{V^T U - V_K U_K}$$

Now

$$V^T \Delta A\, U = \sum_{l,m} (\Delta A)_{lm} V_l U_m = -\sum_{l,m} \Big[(V_l A_{lm} \delta_{lK}) U_m + (A_{lm} U_m \delta_{mK}) V_l \Big]$$

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And similarly

$$V^T \Delta A\, \hat{e}_K = \dots = -\lambda V_K U_K$$

Finally, assuming a large network, $V^T U \gg V_K U_K$

$$\Rightarrow \Delta\lambda_K = -\frac{\lambda V_K U_K}{V^T U} \Rightarrow I_K = -\frac{\lambda V_K U_K}{V^T U}$$