Math C5.4, Networks, University of Oxford, MT 2022
Problem Sheet 2

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1. *Erdos-Renyi.* The simplest and most famous type of random graph is an Erdős–Rényi (ER) graph (which was first studied by Solomonoff and Rapoport). In particular, consider the random-graph ensemble $\mathcal{G}(N,p)$, which is defined as follows: Suppose that there are N nodes. Between each pair of distinct nodes, a single edge exists with uniform and independent probability p. There are no self-edges. A single graph $G \in \mathcal{G}(N,p)$ is generated using this process, and it is interesting to study the properties of collections ("ensembles") of graphs that are generated in this way. The probability in which each simple graph G with m edges appears in a graph $G \in \mathcal{G}(N,p)$ is

$$P(G) = p^m (1-p)^{{}^nC_2 - m} \,. \tag{1}$$

   (a) Write down the total probability of drawing a graph G with exactly m edges from the ensemble $\mathcal{G}(N,p)$, and use it to find the mean value of edges $\langle m \rangle$.

   (b) Calculate the expected mean degree of an ER graph.

   (c) Show, under an appropriate assumption (which you should state), that the degree distribution for an ER graph (in expectation over the ensemble) satisfies

$$p_k \sim e^{-c}\frac{c^k}{k!}\,, \quad N \to \infty\,, \tag{2}$$

   where $c = (N-1)p$.

   (d) Reproduce numerically the results of Figure (14) in the lecture notes.

   (e) Calculate the global clustering coefficient C of an ER graph (in expectation over the ensemble), and verify your prediction numerically.

   (f) Show that the diameter of an ER graph is $B + \frac{\ln N}{\ln c}$ as $N \to \infty$, where B is a constant and $c = (N-1)p$.

2. *Configuration model.*

   (a) One way to create an undirected network having a desired vertex degree distribution, is to create new undirected networks from an existing one (with the required distribution). Suppose that we have such a undirected network: select two separate existing edges at random, independently, say (v_a, v_b) and (v_c, v_d). We will delete the edge (v_a, v_b) and (v_c, v_d) and replace them with new edges (v_a, v_d) and (v_c, v_b) providing that neither of the latter already exists. This is easy to check if we hold the present network by keeping a list of edges. Consider a sparse graph on n vertices with a number $m << n(n-1)/2$ of edges in total, and doing this "swap" independently a total of K consecutive times. What is the probability, q, that any particular edge has NOT been replaced after after exactly K edge-pair swaps? Show that if we wish to have $q < 1/10m$ (i.e. that we expect to have all edges replaced) we should have $K > m\ln(10m)/2$.

   (b) Check this with a numerical implementation.

3. *Range Dependent networks.*

   (a) Consider an undirected network on an infinite number of vertices indexed by both the negative and positive integers $i = \ldots -2, -1, 0, 1, 2, \ldots$ Consider the random network where an edge exist between vertex v_i and vertex v_j $(i \neq j)$ with independent probability equal to $f(|i-j|) \equiv \alpha\lambda^{|i-j|-1}$. Here $0 < \alpha, \lambda \leq 1$, are fixed parameters. For any pair of vertices, v_i and vertex v_j, $|i-j|$ is called the range of the possible connecting edge. How many edges is a vertex expected to have to the right? What is the expected degree of each vertex?

   (b) Generate instances of such a graph on a large number n of vertices, with $\alpha = 1$, so that neighbours are always connected. Calculate the Watts-Strogatz clustering coefficient. How does that vary with $\lambda \in (0,1]$? Can you produce a figure like Figure 1, below?

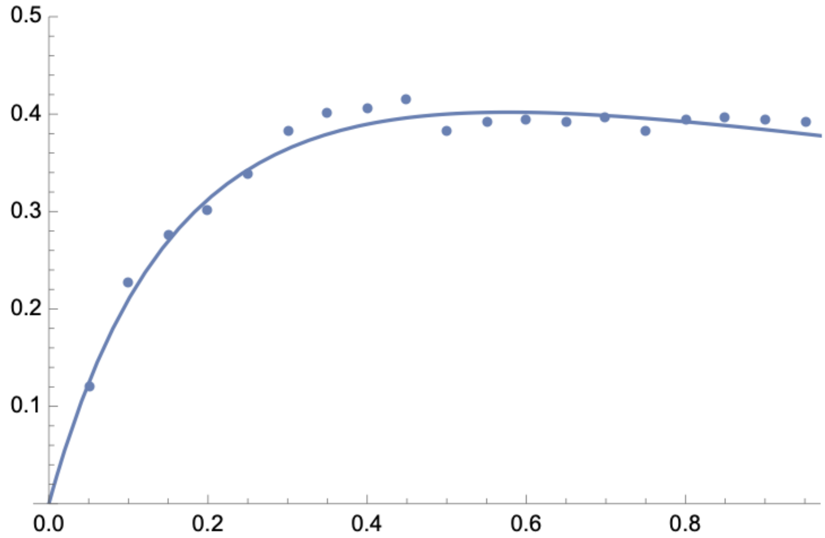

FIG. 1: Clustering coefficient versus $\lambda = 0.05, 0.01, 0.15, \ldots$ (for $\alpha = 1$). Experiments are all for $n = 500$ plotted as points; solid curve is $3\lambda/(1+\lambda)(1+3\lambda)$ from `https://tinyurl.com/yc5rxnjc`.