Navier-Stokes Equations – 2d case

SOE3211/2 Fluid Mechanics lecture 3

NSE (A)

- conservation of mass, momentum.
- often written as set of pde's
- differential form – fluid flow at a point
- 2d case, incompressible flow :

Continuity equation :

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0$$

- conservation of mass
- seen before – potential flow

Momentum equations :

$$\begin{aligned}
\frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \\
&+ \nu \left[\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2}\right] + f_x \\
\frac{\partial u_y}{\partial t} + u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} \\
&+ \nu \left[\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2}\right] + f_y
\end{aligned}$$

- (x and y cmpts)
- 3 variables, u_x, u_y, p
- linked equations
- need to simplfy by considering details of problem

The NSE are

- Non-linear – terms involving $u_x \frac{\partial u_x}{\partial x}$
- Partial differential equations – u_x, p functions of x, y, t
- 2nd order – highest order derivatives $\frac{\partial^2 u_x}{\partial x^2}$
- Coupled – momentum equation involves p, u_x, u_y

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Two ways to solve these equations

1. Apply to *simple* cases – simple geometry, simple conditions – and reduce equations until we can solve them
2. Use computational methods – CFD (SOE3212/3)

Equation analysis (A)

Consider the various terms :

$$\frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \left[\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2}\right] + f_x$$

$$\frac{\partial u_x}{\partial t}$$

- change of u_x at a point

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$$u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y}$$

- *transport/advection* term
- how does flow (u_x, u_y) move u_x?
- non-linear

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$$-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

- pressure gradient – usually drives flow

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$$\nu \left[\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} \right]$$

- viscous term – effect of viscosity ν on flow
- has a diffusive effect

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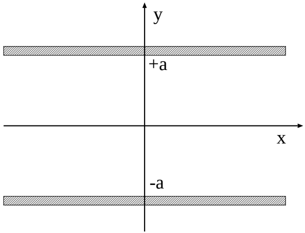
f_x

- external *body forces* – eg. gravity

Laminar flow between plates (A)

Fully developed laminar flow between infinite plates at $y = \pm a$



What do we expect from the flow?

- $\underline{u} = 0$ at walls
- Flow symmetric around $y = 0$
- Flow parallel to walls

$$
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Flow parallel to walls – we expect

$$u_y = 0, \quad \frac{dp}{dy} = 0 \qquad \text{and} \qquad u_x = u_x(y)$$

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Flow fully developed – no change in profile in streamwise direction

$$\text{i.e.} \quad \frac{\partial}{\partial t} = 0, \qquad \frac{\partial}{\partial x} = 0$$

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\begin{aligned}
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&+ \nu\left[\frac{\partial^2 u_x}{\partial y^2}\right]
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$$\frac{1}{2}y^2\frac{dp}{dx} = \rho\nu u_x + C_2$$

But at $y = \pm a$, $u_x = 0$, so

$$C_2 = \frac{1}{2}a^2\frac{dp}{dx}$$

Final solution

$$u_x(y) = \frac{1}{2\rho\nu}\left(y^2 - a^2\right)\frac{dp}{dx}$$

– equation of a parabola

Also, remember that

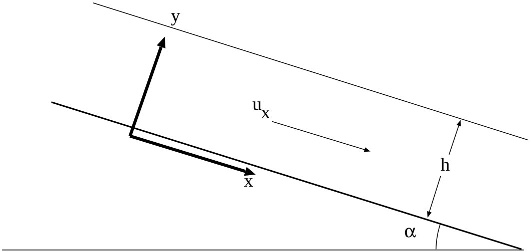
$$\tau = \mu\frac{\partial u_x}{\partial y}$$

So from this we see that in this case

$$\tau = y\frac{dp}{dx}$$

Flow down inclined plane (A)

– Flow of liquid down inclined plane



Take x-component momentum equation

$$\frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \left[\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2}\right] + f_x$$

Note :

1. Steady flow
2. $u_x(y)$ only
3. *No pressure gradient*
4. $f_x = g \sin \alpha$

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Equation becomes

$$\frac{d^2 u_x}{dy^2} = -\frac{g}{\nu} \sin \alpha$$

which we can integrate easilly.

Boundary conditions :

- lower surface – $u_x(0) = 0$
- upper surface – $\frac{du_x}{dy} = 0$

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Solution

$$u_x = \frac{g}{\nu} \sin \alpha \left(hy - \frac{y^2}{2} \right)$$

Tips (A)

Most NSE problems will be time-independent. They will probably only involve one direction of flow, and one coordinate direction. They will probably be either *pressure driven* (so no viscous term) or *shear driven* (ie. viscous related, so no pressure term).

Thus, most NSE problems will lead to a 2nd order ODE for a velocity component (u_x or u_y) as a function of one coordinate (x or y).

Thus we would expect to integrate twice, and to impose two boundary conditions.

- A *wall* boundary condition produces a fixed value : eg. $u_x = 0$.
- A *free surface* produces a *zero gradient* condition, eg. $\frac{du_x}{dy} = 0$.