

1 Comparison to Discrete Random Variables

1.1 Example

For a uniformly distributed interval $(0, 1)$, what is the probability of picking $\frac{1}{2}$? But of picking something larger than $\frac{1}{2}$?

$$P(X = \frac{1}{2}) = \frac{1}{\infty} = 0$$

$$P(X > \frac{1}{2}) = P(X \in (\frac{1}{2}, 1)) = \frac{1}{2}$$

1.2 Probability Density Function

This is how the distribution of a discrete random variable usually looks (segmented/sampled):

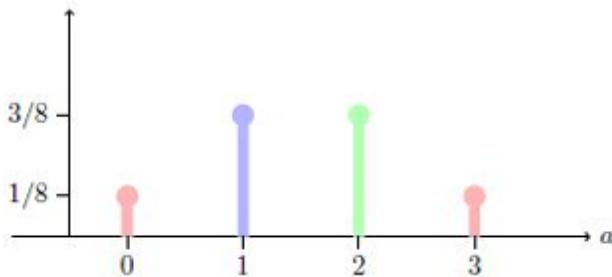


Figure 1: Discrete distribution pmf

Here, x is the value and y is the probability of x . We can directly compute $P(X = k)$. This is how a continuous distribution usually looks (all points connected):

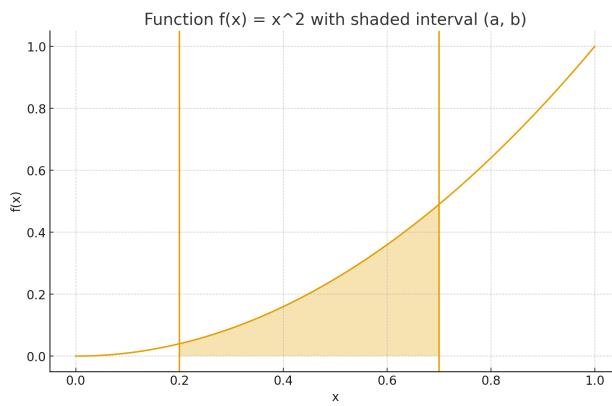


Figure 2: Continuous distribution pdf

In this scenario, we cannot pick a point and assign probability, but we can pick intervals. The analogue of the pmf is the **probability density function** $f(x)$.

To calculate the probability for the interval (a, b) :

$$\int_a^b f(x) dx$$

1.2.1 Properties

For a pdf $f(x)$:

1. $f(x) \geq 0$
- 2.

$$P(\Omega) = \int_{-\infty}^{\infty} f(x) dx = 1$$

Note: If $f(x)$ is defined on (a, b) , then $f(x) = 0$ for $x \notin (a, b)$.

The main differences from discrete distributions: we use integrals instead of sums, and $f(x)$ does not need to be ≤ 1 .

1.3 Cumulative Distribution Function

This is how the cdf looks for the discrete variable:

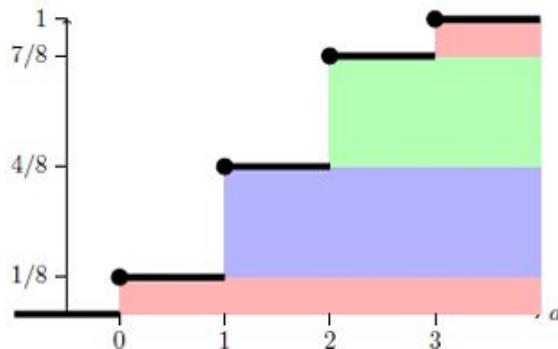


Figure 3: Discrete distribution cdf

And this is the cdf of the continuous distribution:

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(t) dt$$

The cdf is also called the repartition function.

We also have the tail probability which is the complementary of the cdf
 $P(X \geq x) = \int_x^{\infty} f(t) dt = 1 - F(x)$

1.3.1 Properties

Same as discrete, with additions:

1. $P(a \leq X \leq b) = F(b) - F(a)$
2. $F'(x) = f(x)$

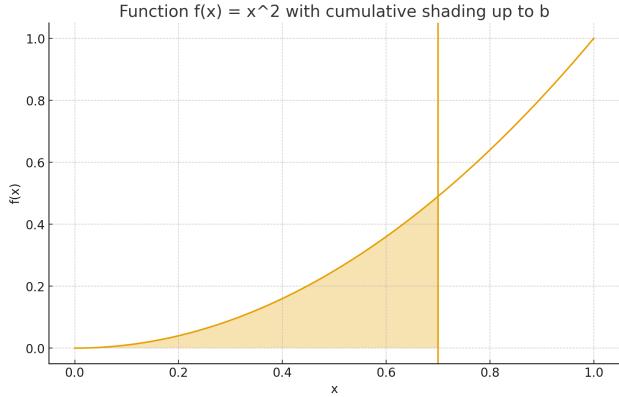


Figure 4: Continuous distribution cdf

1.4 Exercises

Exercise 1

For a continuous random variable X taking values in $(0, 1)$, with pdf $f(x) = Cx^2$, calculate:

- a) $C = ?$
- b) $F(x) = ?$
- c) $P(0.3 \leq x \leq 0.7) = ?$

Solution

a) Using the pdf property:

$$\int_0^1 Cx^2 dx = 1$$

$$\frac{C}{3}x^3 \Big|_0^1 = \frac{C}{3} = 1 \quad \Rightarrow \quad C = 3$$

Thus $f(x) = 3x^2$.

b) The cdf:

$$F(a) = \int_0^a 3x^2 dx = a^3$$

Cases:

1. $a < 0 \Rightarrow F(a) = 0$
2. $a \in (0, 1) \Rightarrow F(a) = a^3$
3. $a > 1 \Rightarrow F(a) = 1$
- c)

$$P(0.3 \leq x \leq 0.7) = F(0.7) - F(0.3)$$

$$= 0.7^3 - 0.3^3 = 0.316$$

The graphs for $f(x)$ and $F(x)$:

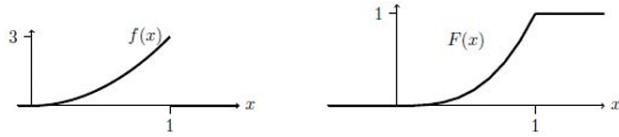


Figure 5: Exercise 1: $f(x)$ and $F(x)$

2 Classic distributions

2.1 Uniform distribution

This is the distribution you mostly used throughout your labs to generate numbers between $(0, 1)$ to simulate experiments.

Notation: $U(a, b)$, where (a, b) is the interval from which samples will be selected with the same probability.

pdf: $f(x) = \frac{1}{b-a}$

cdf: $F(x) = \frac{x-a}{b-a}$

Visualisation:

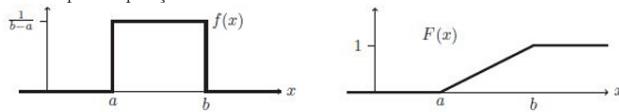


Figure 6: Example of Uniform random variable pdf and pmf

TODO: Teorema de universalitate a repartitiei uniforme (Paul Levy)

2.2 Exponential Distribution

This distribution models waiting times until a certain event will happen (like waiting for the ambulance to come).

Notation: $\exp(\lambda)$, defined on $(0, \infty)$, where λ represented the rate of occurrence.

pdf: $f(x) = \lambda e^{-\lambda x}$

cdf: $F(x) = 1 - e^{-x\lambda}$

Visualisation:

Example: What is the probability of having to wait less than 7 minutes for a bus, taking into account there's two buses I can take (B1, B2, they are independent), on average having 10 buses B1 each hour and 6 buses B2 each hour? ?

First bus is modeled by $X_1 = \exp(\lambda_1)$ and the second by $X_2 = \exp(\lambda_2)$

The probability of no bus coming in 7 minutes is:

$$P(X_1 > 7, X_2 > 7) = P(X_1 > 7)P(X_2 > 7) = e^{-\lambda_1 7} e^{-\lambda_2 7} = e^{-7(\lambda_1 + \lambda_2)}$$

So the probability of any coming in 7 minutes is:

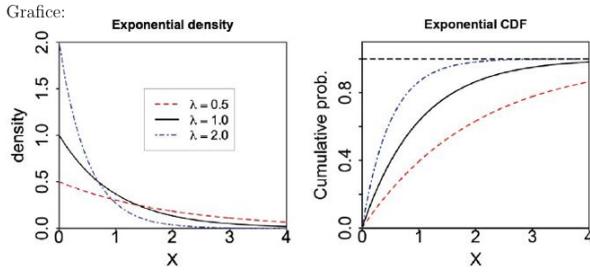


Figure 7: Example of Exponential random variable pdf and cdf

$$1 - e^{-7(\lambda_1 + \lambda_2)}$$

Let's calculate the λ now

$$\lambda_1 = \frac{60}{10} = 6$$

$$\lambda_2 = \frac{60}{6} = 10$$

Pop it into the formula and you get the result

2.2.1 Memorylessness

Let's say I'm only waiting for the first bus (B1) now. Knowing that I waited 5 minutes already without the bus coming, what is the probability that I will have to wait at least 3 more minutes?

$$P(X \geq 3 + 5 \mid X \geq 5) = \frac{P(X \geq 8 \cap X \geq 5)}{P(X \geq 5)} = \frac{P(X \geq 8)}{P(X \geq 5)} = \frac{e^{-8\lambda}}{e^{-5\lambda}} = e^{-3\lambda} = P(X \geq 3)$$

Note: As you might have guessed by now, the exponential is the continuous representation of the geometric distribution

2.3 Normal distribution

The most important continuous distribution. Will appear again as we go towards statistics.

Notation: $N(\mu, \sigma^2)$ is a continuous random variable that is defined by its expected value (μ) and standard deviation (σ)

$$\text{pdf: } f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

cdf: e^{x^2} does not have a primitive so we don't know who $F(x)$ is. For that we use tables

Visualization:

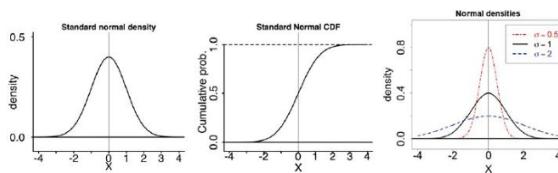


Figure 8: Different normal distributions

$Z = N(0, 1)$ is called the standard normal distribution, having $\phi(x)$ as the pdf and $\Phi(x)$ as the cdf.

Visualization:

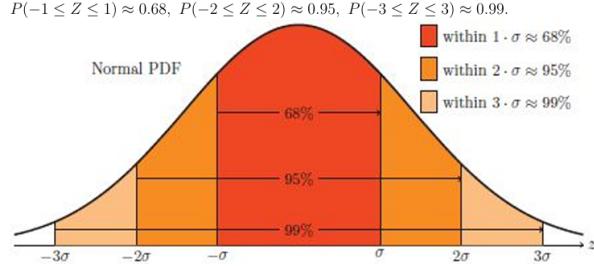


Figure 9: Normal distribution with big thumb rule

Those above probabilities also present **the big thumb rule**.

You might also notice that the probabilities are symmetric with respect to 0.

Exercise:

What is the value of $\Phi(1)$?

Solution

$$\Phi(1) = P(Z \leq 1) = P(Z \leq -1) + P(-1 \leq Z \leq 1)$$

We know the ladder from the big thumb rule, but how do we find the first?

As I said, the variable is symmetric with respect to 0, which means $P(Z \leq -1) = P(Z \geq 1)$

$$P(\Omega) = P(Z \leq -1) + P(-1 \leq Z \leq 1) + P(Z \geq 1) = 2P(Z \leq -1) + P(-1 \leq Z \leq 1) = 1$$

We already know the value of $P(-1 \leq Z \leq 1)$ so we can get $2P(Z \leq -1)$, and so we have everything we need to make the calculation

2.4 Pareto distribution

2.5 Variable substitution

2.5.1 Uniform distribution example

Let $X \sim U(0, 2)$, so the density is $f_X(x) = 1/2$ and the cdf is $F_X(x) = x/2$ on the interval $[0, 2]$.

What are the domain of values, pdf, and cdf of $Y = X^2$?

Answer: The domain of values of Y is $[0, 4]$.

To find the cdf, we use the definition:

$$F_Y(y) = P(Y \leq y) = P(X^2 \leq y) = P(X \leq \sqrt{y}) = F_X(\sqrt{y}) = \frac{\sqrt{y}}{2}.$$

To find the pdf, we differentiate the cdf:

$$f_Y(y) = F'_Y(y) = \left(\frac{\sqrt{y}}{2} \right)' = \frac{1}{4\sqrt{y}}.$$

2.5.2 Exponential distribution example

Let $X \sim \exp(\lambda)$, so $f_X(x) = \lambda e^{-\lambda x}$ on $[0, \infty)$. What is the density of $Y = X^2$?

Answer: We use the change of variables:

$$y = x^2 \Rightarrow dy = 2x dx \Rightarrow dx = \frac{dy}{2\sqrt{y}}.$$

Thus,

$$f_X(x)dx = \lambda e^{-\lambda x}dx = \lambda e^{-\lambda\sqrt{y}} \frac{dy}{2\sqrt{y}} = f_Y(y)dy.$$

Therefore,

$$f_Y(y) = \frac{\lambda}{2\sqrt{y}} e^{-\lambda\sqrt{y}}.$$

2.5.3 Normal distribution example

Assume $X \sim N(5, 3^2)$. Show that $Z = \frac{X-5}{3}$ is a standard normal random variable, i.e., $Z \sim N(0, 1)$.

Answer: By using the change of variables and the formula for $f_X(x)$, we have:

$$z = \frac{x-5}{3} \Rightarrow x = 3z + 5 \Rightarrow dx = 3dz.$$

Then

$$f_X(x)dx = \frac{1}{3\sqrt{2\pi}} e^{-(x-5)^2/(2\cdot 3^2)} dx = \frac{1}{3\sqrt{2\pi}} e^{-z^2/2} 3dz = f_Z(z)dz.$$

Thus Z has the standard normal density.