Lecture 3 Special Relativity

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Lecture 3: Relativistic Laws of Motion and $E = mc^2$

Binomial Development

- When v is much less than c, we can use the binomial expansion:

$$\frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = 1 + \frac{1}{2}\left(\frac{v}{c}\right)^2 + \frac{3}{8}\left(\frac{v}{c}\right)^4 + \ldots$$

- This expansion is useful for understanding relativistic effects when velocities are not extreme.

Time-like Trajectory

- For a particle moving along a time-like trajectory, the proper time τ is defined as the time experienced by the particle itself.
- The relation between the proper time τ and the coordinate time t is given by:

$$\tau = \int \sqrt{1 - \frac{v^2}{c^2}}\, dt$$

- The proper time is always positive and is the longest possible time interval between two events.

Worldline of a Particle

- The worldline of a particle is the trajectory of the particle in spacetime, described by the coordinates (x, y, z, ct).
- The interval $d^{\,2}$ along the worldline is given by:

$$d^{\,2} = c^2\, dt^2 - dx^2 - dy^2 - dz^2$$

- The spacetime interval is invariant under Lorentz transformations and is independent of the observer's reference frame.

Relation of 4-Velocity to Regular Velocity

- The 4-velocity U^μ of a particle is defined as:

$$U^\mu = \frac{dx^\mu}{d\tau} = \gamma\left(c, \frac{d\mathbf{r}}{dt}\right)$$

- Here, $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ is the Lorentz factor.
- The components of the 4-velocity relate to the regular velocity $\boldsymbol{v}$ as:

$$U^0 = \gamma c, \quad U^i = \gamma v^i$$

Least Action

- The principle of least action states that the path taken by a particle between two events minimizes the action, which is the integral of the Lagrangian over the proper time:

$$S = \int L \, d\tau$$

- The action is stationary for the actual path taken by the particle.

Lagrangian

- The Lagrangian L is defined as the difference between the kinetic energy T and the potential energy V of a system: $L = T - V$.
- In special relativity, the Lagrangian is given by:

$$L = -m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}}$$

- Here, m_0 is the rest mass of the particle.

Momentum Conservation

- Momentum is defined as the derivative of the Lagrangian with respect to velocity: $\boldsymbol{p} = \frac{\partial L}{\partial \boldsymbol{v}}$.
- The conservation of momentum arises from the invariance of the Lagrangian under translations in space.

Hamiltonian and Energy

- The Hamiltonian H is defined as the Legendre transform of the Lagrangian: $H = \boldsymbol{p} \cdot \boldsymbol{v} - L$.
- The Hamiltonian is related to the energy of the system: $H = E - \boldsymbol{p} \cdot \boldsymbol{v}$.

Zero Mass Particles and Positronium

- Zero mass particles, such as photons, follow null trajectories in spacetime.
- The relativistic energy-momentum relation for zero mass particles is given by $E = pc$.
- Positronium is a bound state of an electron and a positron. Its total energy can be expressed as the sum of the electron and positron energies.