Lecture 3 Special Relativity (2)

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Invariance of the Laws of Nature

Laws take the same form in every reference frame. Definition of a straight line for example is the « shortest distance between two points » does not reference a coordinate system. Apply the Pythagorean Theorem at the infinitesimal level.

$$ds = \sqrt{dx^2 + dy^2}$$

The distance between two points is

$$s = \int ds = \int dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

find the curve that minimizes the distance s. Find quantities which are invariant under coordinate transformations The vector (dx, dy) for example is not invariant under a rotation, but ds^2 is invariant. For example, a vector (tensor) rotation (transformation):

$$\begin{aligned} A_s &= A_x \cos\theta - A_y \sin\theta \\ A_y &= A_x \sin\theta + A_y \cos\theta \end{aligned}$$

Compare to a Lorentz transformation:

$$\begin{aligned} &x' = x\cosh\omega - T\sinh\omega \\ &T' = -x\sinh\omega + T\cosh\omega \\ &y' = y \\ &z' = z \end{aligned}$$

The invariant combination is

$$d\tau^2 = T^2 - x^2 - y^2 - z^2$$

Time T does not change under a spatial rotation. All 4-vectors transform as indicated under a Lorentz transformation. How to construct quantities which are scalar invariants with respect to a Lorentz transformation? Consider the contravariant 4-vector

$$dx^\mu = (dT, dx, dy, dz)$$

Construct the covariant vector by reversing the sign of the time coordinate.

$$dx_\mu = (-dT, dx, dy, dz)$$

Note that the product between these two vectors is invariant. Use the summation convention.

$$d\tau^2 = dx^4 dx_\mu = -dT^2 + dx^2 + dy^2 + dz^2$$

In general, given the contravariant vector A^μ and the covariant vector B_μ, then $A^\mu B_\mu$ is invariant by construction under the Lorentz Transformation.

$$A^\mu B_\mu = -A_T B_T + A_x B_x + A_y B_y + A_z B_z$$

The inertial reference frame implies Cartesian coordinates and that Newton's Laws are satisfied.

Transformation rule

If A^μ is a 4-vector and B_μ has unknown transformation properties, then if $A^\mu B_\mu$ is a scalar then B_μ is a 4-vector.

Field Theory

Fields in nature: * Higgs field is a scalar. * 4-vector electromagnetic field is a 4-vector. * Energy and momentum are parts of a 4-vector. * Temperature field in a room is a scalar field. Lagrangians must be scalars. The partial derivative of a scalar is a covariant 4 -vector.

$$\frac{\partial\phi}{\partial x^{2x}} = \phi$$

The quantity dx^μ is a contravariant vector, so then

$$\frac{\partial\phi}{\partial x^\alpha} dx^\alpha$$

should be a scalar. Identify this expression as the total derivative.

$$\frac{\partial\phi}{\partial x^\mu} dx^\mu = d\phi$$

The right hand side is scalar, therefore the left hand side is a scalar, therefore

$$\frac{\partial\phi}{\partial x^x}$$

is a covariant 4 -vector. Invent ϕ^μ by changing the sign of the time component.

$$\phi_{\mu\mu} = \left(\frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$
$$\phi_{\mu\mu} = \left(-\frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

then $\phi_{\mu\mu}\phi^\mu$ is a scalar.

$$\phi_{\mu\mu}\phi^\mu\left(-\left(\frac{\partial\phi}{\partial t}\right)^2, \left(\frac{\partial\phi}{\partial x}\right)^2, \left(\frac{\partial\phi}{\partial y}\right)^2, \left(\frac{\partial\phi}{\partial z}\right)^2\right)$$

The four-dimensional Lagrangian

Generalize the action from

$$\text{Action} = \int dt\mathcal{L}$$

to a four-dimensional integration.

$$\text{Action} = \int dxdtL \quad \text{Integrate over spacetime.}$$

Return to the vibrating string problem.

$$L = \frac{\dot{\phi}^2}{2} - \frac{c^2(\partial_x\phi)^2}{2}$$

The Lagrangian L is a scalar which implies Lorentz invariant equations of motion.

$$\text{Action} = \int dx^4[\phi_i^2 - \phi_x^2 - \phi_y^2 - \phi_z^2]$$

or

$$\text{Action} = \int dx^4[\phi^{\mu}\phi_{j\mu}]$$

Minimize the action. Start with minimizing a little region... to be continued.