

# Symmetry Groups and Degeneracy

30 Octobre, 2023

**Lucas Duchet-Annez**

## Symmetry Groups and Degeneracy in Advanced Quantum Mechanics

### 1. Rotational Symmetry:

- Rotational symmetry is a fundamental concept in quantum mechanics that arises from the isotropy of space, i.e., the laws of physics remain unchanged under rotations.
- Rotational symmetry is closely related to angular momentum, which quantifies the rotational motion of a system.
- Angular momentum operators, denoted by  $J$ , are defined by the commutation relations  $[J_i, J_j] = i\hbar\epsilon_{ijk}J_k$ , where  $i, j$ , and  $k$  are the Cartesian components and  $\epsilon_{ijk}$  is the Levi-Civita symbol.

### 2. Angular Momentum:

- Angular momentum is a conserved quantity in quantum mechanics and plays a crucial role in the description of particles and their interactions.
- In quantum mechanics, angular momentum is quantized, meaning it can only take certain discrete values.
- The total angular momentum operator, denoted by  $J$ , is the sum of the orbital angular momentum operator ( $L$ ) and the spin angular momentum operator ( $S$ ).

### 3. Commutator:

- The commutator between two operators  $A$  and  $B$ , denoted by  $[A, B]$ , is defined as  $[A, B] = AB - BA$ .
- The commutator quantifies the non-commutativity of operators and determines the order in which they act.
- In quantum mechanics, the commutator between two observables represents the uncertainty relation between them.

### 4. Degeneracy:

- Degeneracy refers to the phenomenon where multiple states of a quantum system have the same energy.
- Degeneracy arises due to the existence of symmetry in the system, leading to multiple states with indistinguishable energies.
- Degenerate states form a subspace within the larger Hilbert space associated with the system.
- Degeneracy can have profound implications for the behavior and properties of quantum systems.

### 5. Symmetry Generators:

- Symmetry generators are operators that generate symmetry transformations on a quantum system.
- For rotational symmetry, the symmetry generators are the components of angular momentum operators ( $J_x, J_y, J_z$ ).
- Symmetry generators act on quantum states to produce transformed states that belong to the same symmetry class.

### 6. Symmetry Groups:

- Symmetry groups are mathematical structures that describe the collection of all symmetry transformations that leave a physical system invariant.
- In quantum mechanics, symmetry groups play a fundamental role in determining the properties and behaviors of quantum systems.
- The symmetry group associated with rotational symmetry is the special unitary group in three dimensions, denoted by  $SU(2)$ .
- Symmetry groups provide a powerful framework for understanding the degeneracy and symmetry-related properties of quantum systems.

#### 7. Lie Algebra:

- The Lie algebra of a symmetry group is a vector space that captures the algebraic properties of the group's generators.
- In the case of rotational symmetry, the Lie algebra associated with the special unitary group in three dimensions ( $SU(2)$ ) is the algebra of Pauli spin matrices.
- The Lie algebra provides a mathematical foundation for studying the symmetry transformations and their algebraic relations.

#### 8. Raising and Lowering Operators:

- Raising and lowering operators are operators that allow the generation of new states with different angular momentum quantum numbers from a given state.
- In the context of angular momentum, raising operators ( $J_+$ ) increase the angular momentum quantum number, while lowering operators ( $J_-$ ) decrease it.
- The action of raising and lowering operators on angular momentum eigenstates leads to the construction

of multiplets and facilitates the understanding of degeneracy.

#### Maths