Category Theory

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Outline

Introduction

- Definition of a category, objects, morphisms, composition, identity morphisms
- Examples of categories (sets, groups, topological spaces, etc.)

Properties of Categories

- Initial and terminal objects
- Products and coproducts
- Equalizers and coequalizers
- Limits and colimits
- Functors and natural transformations

Adjoint Functors

- Definition and properties
- Examples

Universal Properties - Definition and motivation - Examples (free groups, tensor products, etc)

Yoneda Lemma

- Statement and proof
- Consequences

Monads

- Definition
- Examples (maybe monad, list monad)
- Kleisli categories

Enriched Categories

- Motivation and definition
- Examples of enriched categories

Higher Category Theory

- Motivation
- Definition of 2-categories and n-categories
- Examples

Applications

- Categories in computer science (type theory, semantics)
- Categories in physics (topological quantum field theories)

Introduction to Category Theory

What is a Category?

A category C consists of:

- A collection of objects $\mathrm{ob}\,(C)$
- A collection of morphisms (also called arrows) $\hom\,(A, B)$ between objects A and B
- A composition operation: if $f : A \to B$ and $g : B \to C$ are morphisms, then there is a composite morphism $g \circ f : A \to C$
- An identity morphism $\mathrm{id}_A : A \to A$ for each object

Composition must be:

- Associative: $h \circ (g \circ f) = (h \circ g) \circ f$
- Unital: $f \circ \mathrm{id}_A = f$ and $\mathrm{id}_B \circ f = f$

$$
\begin{tikzcd}
\bullet A \arrow[r, "f"] \arrow[dr, "g \circ f"'] & \bullet B \arrow[d, "g"] \\
& \bullet C
\end{tikzcd}
$$

Figure 1: Simple Category

Examples of Categories

- Set
  - Objects are sets that satisfy the axioms of set theory:
    - Axiom of extensionality: Two sets are equal if they contain the same elements
    - Axiom of pairing: For any a and b, there exists a set {a,b} containing a and b
    - Axiom of union: For any sets A and B, there is a set C = A ∪ B containing all elements of A and B
  - Morphisms are functions between sets that map elements of one set to another
- Grp
  - Objects are groups that satisfy the group axioms:
    - Closure under an associative binary operation

    - Existence of identity element
    - Existence of inverse for each element
  - Morphisms are group homomorphisms that preserve the group structure
- Top
  - Objects are topological spaces that satisfy the axioms of topology:
    - The union of any number of open sets is open
    - The intersection of a finite number of open sets is open
    - The empty set and the whole space are open
  - Morphisms are continuous maps between topological spaces
- Vect
  - Objects are vector spaces over a field F that satisfy:
    - Closure under vector addition and scalar multiplication
    - Vector addition and scalar multiplication obey field axioms
  - Morphisms are linear maps that preserve vector space structure

Intuition

Categories formalize mathematical structure and transformations that preserve that structure. The objects represent mathematical concepts, while the morphisms represent relationships between objects.

Properties of Categories

Categories can have additional structure and properties that reveal relationships between objects. We will explain some important properties and constructions.

Initial and Terminal Objects

- An initial object is a special object that has exactly one morphism $init_A : I \to A$ going to every other object A in the category. It is like a « source » object that every object can be mapped from in a unique way.
- A terminal object has exactly one morphism $term_A : A \to T$ coming from every other object A. It is like a « sink » or « ending » object that everything maps to uniquely.
- For example, in the category of sets, the empty set $\varnothing$ is initial - there is only one function $\varnothing \to A$ from the empty set to any set A. A one element set $\{*\}$ is terminal - there is only one function $A \to \{*\}$ from any set A to the one element set.

Products

- The product of two objects A and B captures the idea of combining or « multiplying » A and B together. It is an object P and two morphisms $p_1 : P \to A$ and $p_2 : P \to B$.
- P has to satisfy a universal property: for any other object C with morphisms $f : C \to A$ and $g : C \to B$, there must be a unique morphism $h : C \to P$ that makes the whole diagram commute. This uniquely characterizes the product.
- For example, in sets the product of A and B is their cartesian product $A \times B$. In groups it is the direct product of groups.

$$\begin{tikzcd}
& P \arrow[dl, "p_1"] \arrow[dr, "p_2"] & \\
A & & B \\
& X \arrow[uu, "h"] \arrow[ul, "f"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Figure 2: Product

Equalizers

- An equalizer of $f, g : A \to B$ embodies the idea of f and g « being equal ». It is an object E and morphism $e : E \to A$ such that $f \circ e = g \circ e$. So E « equalizes » f and g.
- E has a universal property like products. Intuitively, E contains elements of A that f and g map identically.

$$\begin{tikzcd}
& E \arrow[d, "e"] & \\
A \arrow[r, "f"] & B & A \arrow[l, "g"']
\end{tikzcd}$$

Figure 3: Equalizer

Here are draft sections on adjoint functors and universal properties:

Adjoint Functors

Adjoint functors are a powerful concept in category theory that formalize a relationship between two functors.

- Two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ between categories $\mathcal{C}$ and $\mathcal{D}$ are adjoints if there is a natural bijection:

$$\hom_{\mathcal{D}}(F(c), d) \cong \hom_{\mathcal{C}}(c, G(d))$$

for all objects $c \in \mathcal{C}, d \in \mathcal{D}$.

- F is called the left adjoint and G the right adjoint. Intuitively, F preserves sources and G preserves sinks.
- Examples of adjoint functor pairs:
  - Free/forgetful functors between Sets and Grps
  - Hom/tensor product between vector spaces
  - Direct/inverse image functors in topology

Adjoints formalize the idea of two functors being « inverses » in a constructive way that is weaker than isomorphism. They show up often in mathematics and imply many deeper properties.

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

Figure 4: Adjoint Functors

Universal Properties

Many constructions in categories are defined by universal properties, which capture the essence of an object or morphism uniquely up to isomorphism.

- Products, equalizers, limits, and other concepts are defined by universal properties. These specify a mapping property that an object or morphism must satisfy.
- For example, a product P of objects A and B has projections $p_1 : P \to A$ and $p_2 : P \to B$ such that for any other object X with maps $f : X \to A$ and $g : X \to B$, there exists a unique map $h : X \to P$ making the diagram commute.
- Universal properties allow defining concepts intrinsically without referring to concrete representations. This is powerful for proving theorems.
- Many basic algebraic constructions are characterized by universal properties:
  - Free groups, rings, modules
  - Tensor products
  - Kernels and images of morphisms

Universal properties abstract the key aspects of mathematical notions and their relationships. Understanding objects and morphisms via universal properties is fundamental to categorical thinking.

Yoneda Lemma

The Yoneda lemma relates an object in a category to the functor it generates.

- For an object A in a category $\mathcal{C}$, there is a **representable functor** $y_A : \mathcal{C} \to Set$ defined by $y_A(B) = \hom(A, B)$.
- That is, y_A maps an object B to the set of all morphisms from A to B. We can visualize this mapping as:
- The Yoneda lemma states that the natural transformations from y_A to any other functor $F : \mathcal{C} \to Set$ are in bijection with the elements of $F(A)$.
- So the object A is uniquely determined up to isomorphism by its associated representable functor y_A. Objects are characterized by their mapping properties.

$$
\begin{tikzcd}
 & A & \\
B \arrow[ur, "f"] & C \arrow[u, "g"'] & D \arrow[ul, "h"']
\end{tikzcd}
$$

Figure 5: Yoneda Lemma

Monads

A monad on a category $\mathcal{C}$ is a triple (T, η, μ):

- $T : \mathcal{C} \to \mathcal{C}$ is an **endofunctor**, mapping objects and morphisms to themselves.
- $\eta : 1_{\mathcal{C}} \to T$ is a **unit** natural transformation from the identity functor to T.
- $\mu : T^2 \to T$ is a **multiplication** natural transformation, mapping from T applied twice to once.

These satisfy monad axioms. Intuitively:

- T « enhances » objects in $\mathcal{C}$
- η embeds an object into its T-enhanced version
- μ « flattens » double enhancement T^2 to single T

Examples formalize data augmentation, effects, semantics.

$$
\begin{tikzcd}
 & T \arrow[loop right, "\mu"] & \\
C \arrow[ur, "\eta"] \arrow[r, "T"] & T \arrow[u, "\mu"] & C \arrow[ul, "\eta"'] \arrow[l, "T"']
\end{tikzcd}
$$

Figure 6: Monad

Applications

- **Programming languages**: Category theory used in type theory, semantics. Monads in functional programming.
- **Physics**: Topological quantum field theories are functorial theories based on higher categories.
- **Mathematics**: Category theory clarifies foundations and connections between diverse fields.

Maths