

# Category Theory

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## Outline

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- Definition of a category, objects, morphisms, composition, identity morphisms
- Examples of categories (sets, groups, topological spaces, etc.)

### Properties of Categories

- Initial and terminal objects
- Products and coproducts
- Equalizers and coequalizers
- Limits and colimits
- Functors and natural transformations

### Adjoint Functors

- Definition and properties
- Examples

Universal Properties - Definition and motivation - Examples (free groups, tensor products, etc)

### Yoneda Lemma

- Statement and proof
- Consequences

### Monads

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- Examples (maybe monad, list monad)
- Kleisli categories

### Enriched Categories

- Motivation and definition
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### Applications

- Categories in computer science (type theory, semantics)
- Categories in physics (topological quantum field theories)

# Introduction to Category Theory

## What is a Category?

A category  $\mathcal{C}$  consists of:

- A collection of objects  $\text{ob}(\mathcal{C})$
- A collection of morphisms (also called arrows)  $\text{hom}(A, B)$  between objects  $A$  and  $B$
- A composition operation: if  $f : A \rightarrow B$  and  $g : B \rightarrow C$  are morphisms, then there is a composite morphism  $g \circ f : A \rightarrow C$
- An identity morphism  $\text{id}_A : A \rightarrow A$  for each object

Composition must be:

- Associative:  $h \circ (g \circ f) = (h \circ g) \circ f$
- Unital:  $f \circ \text{id}_A = f$  and  $\text{id}_B \circ f = f$

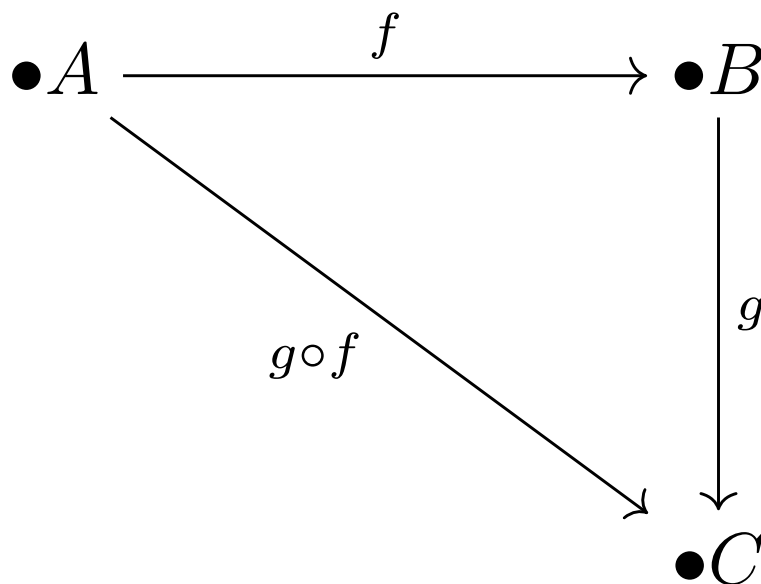


Figure 1: Simple Category

## Examples of Categories

- Set
  - Objects are sets that satisfy the axioms of set theory:
    - Axiom of extensionality: Two sets are equal if they contain the same elements
    - Axiom of pairing: For any  $a$  and  $b$ , there exists a set  $\{a, b\}$  containing  $a$  and  $b$
    - Axiom of union: For any sets  $A$  and  $B$ , there is a set  $C = A \cup B$  containing all elements of  $A$  and  $B$
  - Morphisms are functions between sets that map elements of one set to another
- Grp
  - Objects are groups that satisfy the group axioms:

- Closure under an associative binary operation
- Existence of identity element
- Existence of inverse for each element
- Morphisms are group homomorphisms that preserve the group structure
- Top
  - Objects are topological spaces that satisfy the axioms of topology:
    - The union of any number of open sets is open
    - The intersection of a finite number of open sets is open
    - The empty set and the whole space are open
  - Morphisms are continuous maps between topological spaces
- Vect
  - Objects are vector spaces over a field  $F$  that satisfy:
    - Closure under vector addition and scalar multiplication
    - Vector addition and scalar multiplication obey field axioms
  - Morphisms are linear maps that preserve vector space structure

## Intuition

Categories formalize mathematical structure and transformations that preserve that structure. The objects represent mathematical concepts, while the morphisms represent relationships between objects.

## Properties of Categories

Categories can have additional structure and properties that reveal relationships between objects. We will explain some important properties and constructions.

### Initial and Terminal Objects

- An initial object is a special object that has exactly one morphism  $init_A : I \rightarrow A$  going to every other object  $A$  in the category. It is like a « source » object that every object can be mapped from in a unique way.
- A terminal object has exactly one morphism  $term_A : A \rightarrow T$  coming from every other object  $A$ . It is like a « sink » or « ending » object that everything maps to uniquely.
- For example, in the category of sets, the empty set  $\emptyset$  is initial - there is only one function  $\emptyset \rightarrow A$  from the empty set to any set  $A$ . A one element set  $\{*\}$  is terminal - there is only one function  $A \rightarrow \{*\}$  from any set  $A$  to the one element set.

### Products

- The product of two objects  $A$  and  $B$  captures the idea of combining or « multiplying »  $A$  and  $B$  together. It is an object  $P$  and two morphisms  $p_1 : P \rightarrow A$  and  $p_2 : P \rightarrow B$ .
- $P$  has to satisfy a universal property: for any other object  $C$  with morphisms  $f : C \rightarrow A$  and  $g : C \rightarrow B$ , there must be a unique morphism  $h : C \rightarrow P$  that makes the whole diagram commute. This uniquely characterizes the product.

- For example, in sets the product of A and B is their cartesian product  $A \times B$ . In groups it is the direct product of groups.

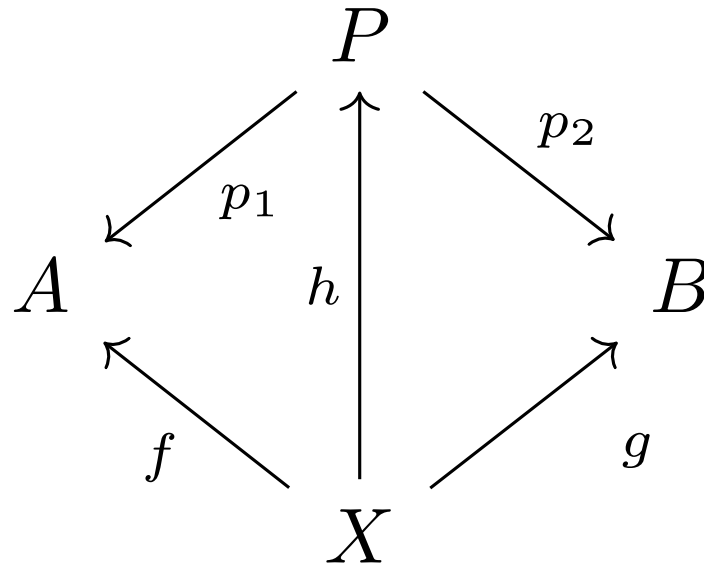


Figure 2: Product

### Equalizers

- An equalizer of  $f, g : A \rightarrow B$  embodies the idea of  $f$  and  $g$  « being equal ». It is an object E and morphism  $e : E \rightarrow A$  such that  $f \circ e = g \circ e$ . So E « equalizes » f and g.
- E has a universal property like products. Intuitively, E contains elements of A that f and g map identically.

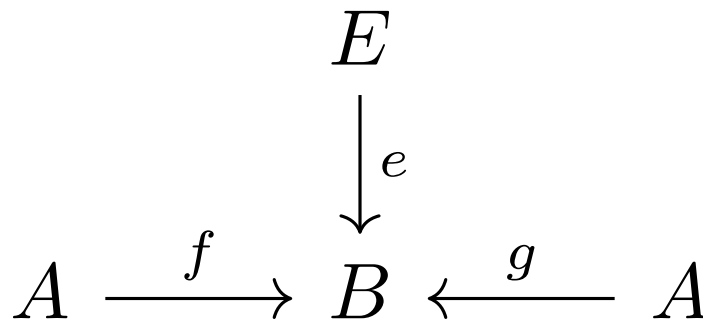


Figure 3: Equalizer

Here are draft sections on adjoint functors and universal properties:

## Adjoint Functors

Adjoint functors are a powerful concept in category theory that formalize a relationship between two functors.

- Two functors  $F : \mathcal{C} \rightarrow \mathcal{D}$  and  $G : \mathcal{D} \rightarrow \mathcal{C}$  between categories  $\mathcal{C}$  and  $\mathcal{D}$  are adjoints if there is a natural bijection:

$$\text{hom}_{\mathcal{D}}(F(c), d) \cong \text{hom}_{\mathcal{C}}(c, G(d))$$

for all objects  $c \in \mathcal{C}, d \in \mathcal{D}$ .

- $F$  is called the left adjoint and  $G$  the right adjoint. Intuitively,  $F$  preserves sources and  $G$  preserves sinks.
- Examples of adjoint functor pairs:
  - Free/forgetful functors between Sets and Grps
  - Hom/tensor product between vector spaces
  - Direct/inverse image functors in topology

Adjoint functors formalize the idea of two functors being « inverses » in a constructive way that is weaker than isomorphism. They show up often in mathematics and imply many deeper properties.

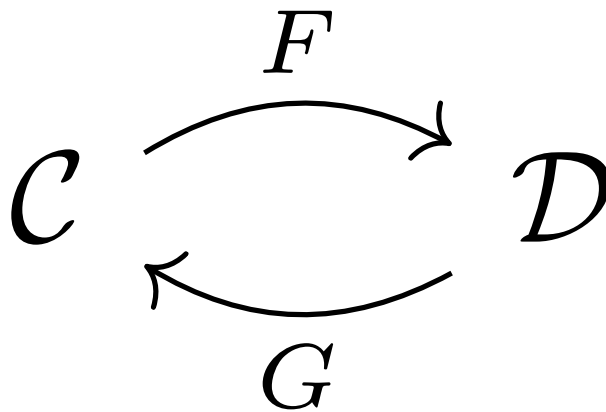


Figure 4: Adjoint Functors

## Universal Properties

Many constructions in categories are defined by universal properties, which capture the essence of an object or morphism uniquely up to isomorphism.

- Products, equalizers, limits, and other concepts are defined by universal properties. These specify a mapping property that an object or morphism must satisfy.
- For example, a product  $P$  of objects  $A$  and  $B$  has projections  $p_1 : P \rightarrow A$  and  $p_2 : P \rightarrow B$  such that for any other object  $X$  with maps  $f : X \rightarrow A$  and  $g : X \rightarrow B$ , there exists a unique map  $h : X \rightarrow P$  making the diagram commute.
- Universal properties allow defining concepts intrinsically without referring to concrete representations. This is powerful for proving theorems.
- Many basic algebraic constructions are characterized by universal properties:
  - Free groups, rings, modules
  - Tensor products
  - Kernels and images of morphisms

Universal properties abstract the key aspects of mathematical notions and their relationships. Understanding objects and morphisms via universal properties is fundamental to categorical thinking.

## Yoneda Lemma

The Yoneda lemma relates an object in a category to the functor it generates.

- For an object  $A$  in a category  $\mathcal{C}$ , there is a **representable functor**  $y_A : \mathcal{C} \rightarrow \mathbf{Set}$  defined by  $y_A(B) = \text{hom}(A, B)$ .
- That is,  $y_A$  maps an object  $B$  to the set of all morphisms from  $A$  to  $B$ . We can visualize this mapping as:
- The Yoneda lemma states that the natural transformations from  $y_A$  to any other functor  $F : \mathcal{C} \rightarrow \mathbf{Set}$  are in bijection with the elements of  $F(A)$ .
- So the object  $A$  is uniquely determined up to isomorphism by its associated representable functor  $y_A$ . Objects are characterized by their mapping properties.

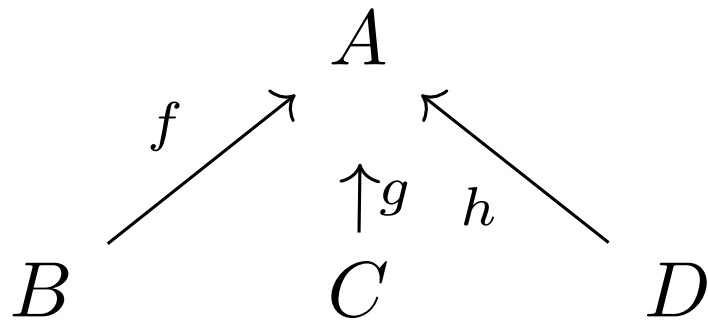


Figure 5: Yoneda Lemma

## Monads

A monad on a category  $\mathcal{C}$  is a triple  $(T, \eta, \mu)$ :

- $T : \mathcal{C} \rightarrow \mathcal{C}$  is an **endofunctor**, mapping objects and morphisms to themselves.
- $\eta : 1_{\mathcal{C}} \rightarrow T$  is a **unit** natural transformation from the identity functor to  $T$ .
- $\mu : T^2 \rightarrow T$  is a **multiplication** natural transformation, mapping from  $T$  applied twice to once.

These satisfy monad axioms. Intuitively:

- $T$  « enhances » objects in  $\mathcal{C}$
- $\eta$  embeds an object into its  $T$ -enhanced version
- $\mu$  « flattens » double enhancement  $T^2$  to single  $T$

Examples formalize data augmentation, effects, semantics.

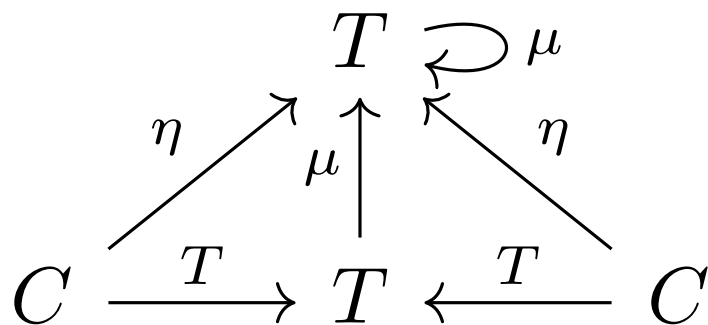


Figure 6: Monad

## Applications

- **Programming languages:** Category theory used in type theory, semantics. Monads in functional programming.
- **Physics:** Topological quantum field theories are functorial theories based on higher categories.
- **Mathematics:** Category theory clarifies foundations and connections between diverse fields.

Maths