Gradient Vector

22 Juin, 2023

Lucas

1. Gradient Vector

1.1. Definition

In multivariable calculus, the gradient vector of a scalar-valued function is a vector-valued function that points in the direction of the greatest rate of increase of the function at a given point. It is a generalization of the concept of the derivative of a function of one variable, which points in the direction of the greatest rate of increase of the function at a given point.

The gradient vector of a function $f(x, y)$ at a point $\left(x_0, y_0\right)$ is given by:

$$\nabla f\left(x_0, y_0\right) = \left(\frac{\partial f}{\partial x}\left(x_0, y_0\right) \ \frac{\partial f}{\partial y}\left(x_0, y_0\right)\right)$$

Where $\frac{\partial f}{\partial x}\left(x_0, y_0\right)$ and $\frac{\partial f}{\partial y}\left(x_0, y_0\right)$ are the partial derivatives of the function with respect to x and y respectively.

The gradient vector is a useful concept in multivariable calculus, as it allows us to find the direction of maximum increase of a function at a given point, and to calculate the directional derivative of a function in a particular direction.

1.2. Example

To find the gradient vector of the function $f(x, y) = y \ln x + xy^2$ at a point $\left(x_0, y_0\right)$, we need to take the partial derivatives of the function with respect to x and y and evaluate them at the point $\left(x_0, y_0\right)$.

The partial derivatives of f with respect to x and y are:

$$\frac{\partial f}{\partial x} = y^2 + y\frac{1}{x}$$

$$\frac{\partial f}{\partial y} = \ln x + 2xy$$

The gradient vector of the function at the point $\left(x_0, y_0\right)$ is then given by:

$$\nabla f\left(x_0, y_0\right) = \left(\frac{\partial f}{\partial x}\left(x_0, y_0\right) \ \frac{\partial f}{\partial y}\left(x_0, y_0\right)\right) = \left(y_0^2 + \frac{y_0}{x_0} \ln x_0 + 2x_0y_0\right)$$

For example, if $\left(x_0, y_0\right) = (1, 1)$, then the gradient vector is:

$$\nabla f(1, 1) = \left(1^2 + \tfrac{1}{1} \ln 1 + 2 \cdot 1 \cdot 1\right) = (2 \ 2)$$

Evaluating these partial derivatives at the point $(1, 2)$ gives us:

$$\frac{\partial f}{\partial x}(1, 2) = 2^2 + 2 \cdot \frac{1}{1} = 6$$

$$\frac{\partial f}{\partial y}(1, 2) = \ln 1 + 2 \cdot 1 \cdot 2 = 4$$

The gradient vector at the point $(1, 2)$ is then given by:

$$\nabla f(1,2) = \left(\frac{\partial f}{\partial x}(1,2)\ \frac{\partial f}{\partial y}(1,2)\right) = (6\ 4)$$

Therefore, the gradient vector at the point $(1,2)$ is $(6\ 4)$.

1.3. Link

- Partial Derivative