Symmetry Groups and Degeneracy

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Symmetry Groups and Degeneracy in Advanced Quantum Mechanics

1. Rotational Symmetry:

- Rotational symmetry is a fundamental concept in quantum mechanics that arises from the isotropy of space, i.e., the laws of physics remain unchanged under rotations.
- Rotational symmetry is closely related to angular momentum, which quantifies the rotational motion of a system.
- Angular momentum operators, denoted by J, are defined by the commutation relations $[J_i, J_j] = i\hbar\epsilon_{ijk}J_k$, where i, j, and k are the Cartesian components and ϵ_{ijk} is the Levi-Civita symbol.

2. Angular Momentum:

- Angular momentum is a conserved quantity in quantum mechanics and plays a crucial role in the description of particles and their interactions.
- In quantum mechanics, angular momentum is quantized, meaning it can only take certain discrete values.
- The total angular momentum operator, denoted by J, is the sum of the orbital angular momentum operator (L) and the spin angular momentum operator (S).

3. Commutator:

- The commutator between two operators A and B, denoted by $[A, B]$, is defined as $[A, B] = AB - BA$.
- The commutator quantifies the non-commutativity of operators and determines the order in which they act.
- In quantum mechanics, the commutator between two observables represents the uncertainty relation between them.

4. Degeneracy:

- Degeneracy refers to the phenomenon where multiple states of a quantum system have the same energy.
- Degeneracy arises due to the existence of symmetry in the system, leading to multiple states with indistinguishable energies.
- Degenerate states form a subspace within the larger Hilbert space associated with the system.
- Degeneracy can have profound implications for the behavior and properties of quantum systems.

5. Symmetry Generators:

- Symmetry generators are operators that generate symmetry transformations on a quantum system.
- For rotational symmetry, the symmetry generators are the components of angular momentum operators (J_x, J_y, J_z).

- Symmetry generators act on quantum states to produce transformed states that belong to the same symmetry class.

6. Symmetry Groups:

- Symmetry groups are mathematical structures that describe the collection of all symmetry transformations that leave a physical system invariant.
- In quantum mechanics, symmetry groups play a fundamental role in determining the properties and behaviors of quantum systems.
- The symmetry group associated with rotational symmetry is the special unitary group in three dimensions, denoted by $SU(2)$.
- Symmetry groups provide a powerful framework for understanding the degeneracy and symmetry-related properties of quantum systems.

7. Lie Algebra:

- The Lie algebra of a symmetry group is a vector space that captures the algebraic properties of the group's generators.
- In the case of rotational symmetry, the Lie algebra associated with the special unitary group in three dimensions ($SU(2)$) is the algebra of Pauli spin matrices.
- The Lie algebra provides a mathematical foundation for studying the symmetry transformations and their algebraic relations.

8. Raising and Lowering Operators:

- Raising and lowering operators are operators that allow the generation of new states with different angular momentum quantum numbers from a given state.
- In the context of angular momentum, raising operators (J_+) increase the angular momentum quantum number, while lowering operators (J_-) decrease it.
- The action of raising and lowering operators on angular momentum eigenstates leads to the construction

of multiplets and facilitates the understanding of degeneracy.

<u>Maths</u>