

# Lecture 3 Special Relativity

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## Lecture 3: Relativistic Laws of Motion and $E = mc^2$

### Binomial Development

- When  $v$  is much less than  $c$ , we can use the binomial expansion:

$$\frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = 1 + \frac{1}{2}\left(\frac{v}{c}\right)^2 + \frac{3}{8}\left(\frac{v}{c}\right)^4 + \dots$$

- This expansion is useful for understanding relativistic effects when velocities are not extreme.

### Time-like Trajectory

- For a particle moving along a time-like trajectory, the proper time  $\tau$  is defined as the time experienced by the particle itself.
- The relation between the proper time  $\tau$  and the coordinate time  $t$  is given by:

$$\tau = \int \sqrt{1 - \frac{v^2}{c^2}} dt$$

- The proper time is always positive and is the longest possible time interval between two events.

### Worldline of a Particle

- The worldline of a particle is the trajectory of the particle in spacetime, described by the coordinates  $(x, y, z, ct)$ .
- The interval  $ds^2$  along the worldline is given by:

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

- The spacetime interval is invariant under Lorentz transformations and is independent of the observer's reference frame.

### Relation of 4-Velocity to Regular Velocity

- The 4-velocity  $U^\mu$  of a particle is defined as:

$$U^\mu = \frac{dx^\mu}{d\tau} = \gamma \left( c, \frac{d\mathbf{r}}{dt} \right)$$

- Here,  $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$  is the Lorentz factor.
- The components of the 4-velocity relate to the regular velocity  $\mathbf{v}$  as:

$$U^0 = \gamma c, \quad U^i = \gamma v^i$$

### Least Action

- The principle of least action states that the path taken by a particle between two events minimizes the action, which is the integral of the Lagrangian over the proper time:

$$S = \int L d\tau$$

- The action is stationary for the actual path taken by the particle.

## Lagrangian

- The Lagrangian  $L$  is defined as the difference between the kinetic energy  $T$  and the potential energy  $V$  of a system:  $L = T - V$ .
- In special relativity, the Lagrangian is given by:

$$L = -m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}}$$

- Here,  $m_0$  is the rest mass of the particle.

## Momentum Conservation

- Momentum is defined as the derivative of the Lagrangian with respect to velocity:  $\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}}$ .
- The conservation of momentum arises from the invariance of the Lagrangian under translations in space.

## Hamiltonian and Energy

- The Hamiltonian  $H$  is defined as the Legendre transform of the Lagrangian:  $H = \mathbf{p} \cdot \mathbf{v} - L$ .
- The Hamiltonian is related to the energy of the system:  $H = E - \mathbf{p} \cdot \mathbf{v}$ .

## Zero Mass Particles and Positronium

- Zero mass particles, such as photons, follow null trajectories in spacetime.
- The relativistic energy-momentum relation for zero mass particles is given by  $E = pc$ .
- Positronium is a bound state of an electron and a positron. Its total energy can be expressed as the sum of the electron and positron energies.