Lecture 3 Special Relativity (2)

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Invariance of the Laws of Nature

Laws take the same form in every reference frame. Definition of a straight line for example is the « shortest distance between two points » does not reference a coordinate system. Apply the Pythagorean Theorem at the infinitesimal level.

$$d \ = \sqrt{dx^2 + dy^2}$$

The distance between two points is

$$= \int d \ = \int dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

find the curve that minimizes the distance . Find quantities which are invariant under coordinate transformations The vector (dx, dy) for example is not invariant under a rotation, but $d^{\ 2}$ is invariant. For example, a vector (tensor) rotation (transformation):

$$\begin{aligned} A \ &= A_x \cos\theta - A_y \sin\theta \\ A_y &= A_x \sin\theta + A_y \cos\theta \end{aligned}$$

Compare to a Lorentz transformation:

$$\begin{aligned} x^{'} &= x \cosh\omega - T \sinh\omega \\ T^{'} &= -x \sinh\omega + T \cosh\omega \\ y^{'} &= y \\ z^{'} &= z \end{aligned}$$

The invariant combination is

$$d\tau^2 = T^2 - x^2 - y^2 - z^2$$

Time T does not change under a spatial rotation. All 4-vectors transform as indicated under a Lorentz transformation. How to construct quantities which are scalar invariants with respect to a Lorentz transformation? Consider the contravariant 4-vector

$$dx^\mu = (dT, dx, dy, dz)$$

Construct the covariant vector by reversing the sign of the time coordinate.

$$dx_\mu = (-dT, dx, dy, dz)$$

Note that the product between these two vectors is invariant. Use the summation convention.

$$d\tau^2 = dx^4 dx_\mu = -dT^2 + dx^2 + dy^2 + dz^2$$

In general, given the contravariant vector A^μ and the covariant vector $()^{B}_{\mu'}$, then $()^{A^\mu B_\mu}$ is invariant by construction under the Lorentz Transformation.

$$A^{\mu}B_{\mu} = -A_T B_T + A_x B_x + A_y B_y + A_z B_z$$

The inertial reference frame implies Cartesian coordinates and that Newton's Laws are satisfied.

Transformation rule

If A^{μ} is a 4-vector and $()_{\mu}^{B}$ has unknown transformation properties, then if $()^{A^{\mu}B_{\mu}}$ is a scalar then $()_{\mu}^{B}$ is a 4-vector.

Field Theory

Fields in nature: * Higgs field is a scalar. * 4-vector electromagnetic field is a 4-vector. * Energy and momentum are parts of a 4-vector. * Temperature field in a room is a scalar field. Lagrangians must be scalars. The partial derivative of a scalar is a covariant 4 -vector.

$$\frac{\partial \phi}{\partial x^{2x}} = \phi$$

The quantity dx^{μ} is a contravariant vector, so then

$$\frac{\partial \phi}{\partial x^{\alpha}} dx^{\alpha}$$

should be a scalar. Identify this expression as the total derivative.

$$\frac{\partial \phi}{\partial x^{\mu}} dx^{\mu} = d\phi$$

The right hand side is scalar, therefore the left hand side is a scalar, therefore

$$\frac{\partial \phi}{\partial x^{x}}$$

is a covariant 4 -vector. Invent ϕ^{μ} by changing the sign of the time component.

$$\phi_{\mu\mu} = \left(\frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right)$$
$$\phi_{\mu\mu} = \left(-\frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right)$$

then $\phi_{\mu\mu}\phi^{\mu}$ is a scalar.

$$\phi_{\mu\mu}\phi^{\mu}\left(-\left(\frac{\partial \phi}{\partial t}\right)^2, \left(\frac{\partial \phi}{\partial x}\right)^2, \left(\frac{\partial \phi}{\partial y}\right)^2, \left(\frac{\partial \phi}{\partial z}\right)^2\right)$$

The four-dimensional Lagrangian

Generalize the action from

$$\text{Action} = \int dt \mathcal{L}$$

to a four-dimensional integration.

$$\text{Action} = \int dx dt L \quad \text{Integrate over spacetime.}$$

Return to the vibrating string problem.

$$L = \frac{\dot{\phi}^2}{2} - \frac{c^2(\partial_x \phi)^2}{2}$$

The Lagrangian L is a scalar which implies Lorentz invariant equations of motion.

$$\text{Action} = \int dx^4 \left[\phi_i^2 - \phi_x^2 - \phi_y^2 - \phi_z^2\right]$$

or

$$\text{Action} = \int dx^4 \left[\phi^{\mu} \phi_{j\mu}\right]$$

Minimize the action. Start with minimizing a little region... to be continued.