Directional Derivative

22 Juin, 2023

Lucas

1. Directional Derivative

1.1. Definition

The directional derivative of a multivariable function at a given point is a measure of the rate of change of the function in a particular direction at that point. It is a generalization of the concept of the derivative of a function of one variable, which measures the rate of change of the function in the direction of the x-axis.

The directional derivative of a function $f(x, y)$ at a point $\left(x_0, y_0\right)$ in the direction of a unit vector $\boldsymbol{u} = (a\ b)$ is given by:

$$\frac{\partial f}{\partial \boldsymbol{u}}\left(x_0, y_0\right) = \nabla f\left(x_0, y_0\right) \cdot \boldsymbol{u} = \left(\frac{\partial f}{\partial x}\left(x_0, y_0\right)\ \frac{\partial f}{\partial y}\left(x_0, y_0\right)\right) \cdot (a\ b)$$

Where $\nabla f\left(x_0, y_0\right)$ is the gradient vector of the function at the point $\left(x_0, y_0\right)$, and $\frac{\partial f}{\partial x}\left(x_0, y_0\right)$ and $\frac{\partial f}{\partial y}\left(x_0, y_0\right)$ are the partial derivatives of the function with respect to x and y respectively.

The directional derivative can be thought of as the projection of the gradient vector onto the direction specified by the unit vector $\boldsymbol{u}$. It tells us the rate of change of the function in the direction of $\boldsymbol{u}$ at the point $\left(x_0, y_0\right)$.

1.2. Example

To find the directional derivative of the function $f(x, y) = x^2 \sin 2y$ in the direction of the vector $\boldsymbol{u} = (\cos \sin)$, we need to first find the gradient vector of the function at the given point.

The partial derivatives of f with respect to x and y are:

$$\frac{\partial f}{\partial x} = 2x \sin 2y$$

$$\frac{\partial f}{\partial y} = 2x^2 \cos 2y$$

The gradient vector of the function at a point $\left(x_0, y_0\right)$ is given by:

$$\nabla f\left(x_0, y_0\right) = \left(\frac{\partial f}{\partial x}\left(x_0, y_0\right)\ \frac{\partial f}{\partial y}\left(x_0, y_0\right)\right) = \left(2x_0 \sin 2y_0\ 2x_0^2 \cos 2y_0\right)$$

The directional derivative of the function in the direction of $\boldsymbol{u}$ at the point $\left(x_0, y_0\right)$ is then given by:

$$\frac{\partial f}{\partial \boldsymbol{u}}\left(x_0, y_0\right) = \nabla f\left(x_0, y_0\right) \cdot \boldsymbol{u} = \left(2x_0 \sin 2y_0\ 2x_0^2 \cos 2y_0\right) \cdot (\cos \sin)$$

We can evaluate this expression to find the directional derivative. For example, if $\left(x_0, y_0\right) = (1, 1)$, then the directional derivative is:

$$\frac{\partial f}{\partial \boldsymbol{u}}(1, 1) = (2 \cdot 1 \cdot \sin 2 \cdot 1\ 2 \cdot 1^2 \cos 2 \cdot 1) \cdot (\cos \sin) = (\sin 2\ \cos 2) \cdot (\cos \sin) = \sin^2 2 + \cos^2 2 = 1$$

1.3. Link

- Gradient Vector
- Partial Derivative