Analyse 1

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Génie Mécanique

1 Exercice 1

1.1

1. $A = \{n \text{ impair}, n \leq 19\}$
2. $V = \{n \text{ pair}, n \in \mathbb{N}\} \cup \{n, n \geq 41\}$
3. $C = (\{n \text{ pair}\} \cap \{n \leq 10\}) \cup (\{n \text{ impair}\} \cap \{n \leq 53\})$
4. $N = 61$

1.2

1. $A' = \{1, 2, 8\}$
2. $B' = \{5, 8, 12, 14, 16, 17, 20, 21\}$
3.
4. $C' = \{6, 9, 13, 15, 19\} \cup \{n \geq 22\}$
5. $N = 25$

2 Exercice 2

$$x_{n+1} - x_n = \frac{an + a + b}{cn + c + d} - \frac{an + b}{cn + d}$$

$$= \frac{acn^2 + acn + and + ad + bcn + bd - acn^2 - acn - and - bcn - bc - bd}{(cn + d)(cn + c + d)}$$

$$= \frac{ad - bc}{(cn + d)(c(n + 1) + d)}$$

Ansi $x_{n+1} - x_n \geq 0 \iff ad - bc \geq 0$

3 Exercice 3

3.1

On sait que $a_{n+1} - a_n \geq 0$ car $ad - bc = 1 - 0 = 1$

$$\left|\frac{n}{n} + 1 - 1\right| = \frac{1}{n + 1}$$

$$\frac{1}{n + 1} \leq \varepsilon$$

$$\frac{1}{n + 1} \leq \frac{1}{10}$$

$$9 \leq n$$

Si on pose $N = 9$ $\forall n \geq N$ $|a_n - l| \leq \varepsilon$

3.2

$$\left|\frac{n}{n}+1-1\right| \leq \varepsilon$$

$$\frac{1}{n+1} \leq \frac{1}{100}$$

$$99 \leq n$$

Si on pose $N = 99$ $\forall n \geq N$ $|a_n - l| \leq \varepsilon$

3.3

$$\left|\frac{(-1)^n}{n} - 0\right| \leq \varepsilon$$

$$\frac{1}{n} \leq \varepsilon$$

$$\frac{1}{n} \leq \frac{1}{100}$$

$$100 \leq n$$

Si on pose $N = 100$ $\forall n \geq N$ $|a_n - l| \leq \varepsilon$

3.4

$$\left|\frac{n}{n^2+1} - 0\right| \leq \varepsilon$$

$$\frac{n}{n^2+1} \leq \frac{1}{4}$$

$$4n \leq n^2 + 1$$

$$0 \leq n^2 + 1 - 4n$$

Dans $\mathbb{R}$

$$\Delta = 16 - 4 = 12 = \left(2\sqrt{3}\right)^2$$

$x_1 = \frac{4+2\sqrt{3}}{2} = 2 + \sqrt{3} \approx 3.73$

$$x_2 = 4 - 2\frac{\sqrt{3}}{2} = 2 - \sqrt{3}$$

Soit $N = 4$ $\forall n \geq N$ $|a_n - l| \leq \varepsilon$

3.5

$$\left|2\frac{n}{n+1} - 1\right| \leq \varepsilon$$

Pour $n \geq 1$

$$\frac{n-1}{n+1} \leq \varepsilon$$

$$n-1 \leq n+1\left(\frac{3}{4}\right)$$

$$0 \leq \frac{3}{4}(n) + \frac{3}{4} + 1 - n$$

$$0 \leq -\frac{1}{4}(n) + \frac{7}{4}$$

$$-\frac{7}{4} \leq -\frac{1}{4}(n)$$

$$7 \geq n$$

$$\{n \in \mathbb{N}, n \leq 7\}$$

4 Exercice 4

$\lim(a_2 k) = L \Longleftrightarrow \exists N_1, \forall \varepsilon > 0 |a_2 k - L| \leq \varepsilon$ $\quad \lim(a_{2k+1}) = L \Longleftrightarrow \exists N_2, \forall \varepsilon > 0 \left|a_{2k+1} - L\right| \leq \varepsilon$

Soit $N = \max(N_1, N_2)$ Ainsi $\forall n \geq N$ Si n pair on a $n = 2k, k \in \mathbb{N}$ et $|a_n - L| = |a_{2k} - L| \leq \varepsilon$ Si n impair on a $n = 2k+1, k \in \mathbb{N}$ et $|a_n - L| = \left|a_{2k+1} - L\right| \leq \varepsilon$

Ainsi $|a_n - L| \leq \varepsilon, \forall n \geq N$

5 Exercice 5

$$|x_n - l| = |x_n - 1| = 10^{-n}$$

On pose

$$\varepsilon \geq 10^{-n}$$

$n \geq -\log(\varepsilon)$

Alors Si $N = \lfloor -\log(\varepsilon) \rceil + 1$

$$\forall n \geq N, |x_n - 1| \leq \varepsilon$$

6 Exercice 6

6.1

$$\left|\frac{2n-3}{3n+7} - \frac{2}{3}\right| = \left|-\frac{23}{3}(3n+7)\right| = \frac{23}{3}(3n+7)$$

$$\frac{23}{3}(3n+7) \leq \varepsilon$$

$$\frac{23}{3\varepsilon} \leq 3n + 7$$

$$\frac{23 - 21\varepsilon}{9\varepsilon} \leq n$$

Soit $N = \left\lfloor \frac{23-21\varepsilon}{9\varepsilon} \right\rceil + 1 \ \forall n \geq N, \left|x_n - \frac{2}{3}\right| \leq \varepsilon$

6.2

$$\left|\frac{an+b}{cn+d} - \left(\frac{a}{c}\right)\right| = \left|\frac{acn + bc - acn - da}{c(cn+d)}\right| = \frac{|bc - da|}{c(cn+d)}) = \frac{|da - bc|}{(c(cn+d)}$$

$$\frac{|ad - bc|}{c(cn+d)} \leq \varepsilon$$

$\frac{1}{c\left(\frac{|ad-bc|}{c\varepsilon} - d\right)} \leq n$

Soit $N = \left\lfloor \frac{1}{c\left(\frac{|ad-bc|}{c\varepsilon} - d\right)} \right\rceil$

$$\forall n \geq N, \left|x_n - \frac{a}{c}\right| \leq \varepsilon$$

7 Exercice 7

7.1

Equivalent

7.2

Equivalent

7.3

Equivalent

7.4

Equivalent

7.5

Equivalent

8 Exercice 8

8.1

Faux

8.2

Faux

8.3
Faux

8.4
Vrai

8.5
Vrai

8.6
Faux

8.7
Vrai

8.8
Faux

8.9
Faux

8.10
Faux

9 Exercice 9

9.1

$$0 \leq \left|\frac{\cos(\sqrt{n})}{n}\right| \leq \frac{1}{n}$$

$$\left|\frac{1}{n}\right| \leq \varepsilon$$

$$\frac{1}{\varepsilon} \leq n$$

Soit $N = \left\lfloor \frac{1}{\varepsilon} \right\rceil$

$$\forall n \geq N, \left|\frac{1}{n} - 0\right| \leq \varepsilon$$

Ainsi $\lim_{n \to +\infty} 0 = 0$ et $\lim_{n \to +\infty} \frac{1}{n} = 0$ donc $\lim_{n \to +\infty} \left|\frac{\cos(\sqrt{n})}{n}\right| = 0 = \lim_{n \to +\infty} \frac{\cos(\sqrt{n})}{n}$

9.2

$$0 \leq \frac{n!}{n^n}$$

$$1(2)(3)(4)(5)(...)\frac{n}{n}(n)(n)(n)(...) = \left(\frac{1}{n}\right)\left(\frac{2}{n}\right)\left(\frac{3}{n}\right)(...) \leq \frac{1}{n}$$

Ainsi $\lim_{n\to+\infty} 0 = 0$ et $\lim_{n\to+\infty} \frac{1}{n} = 0$

$$\lim_{n\to+\infty} \frac{n!}{n^n} = 0$$

9.3

$$\sqrt[n]{1+2^n} \geq \sqrt[n]{2^n} = 2$$

$$\sqrt[n]{1+2^n} \leq \sqrt[n]{2^n+1} = 2^{\frac{n+1}{n}}$$

On sait que $\lim_{n\to+\infty} \frac{n+1}{n} = 1$ Ainsi $\lim_{n\to+\infty} 2^{\frac{n+1}{n}} = 2$ D'après le théorème des gendarmes

$$\lim_{n\to+\infty} \sqrt[n]{1+2^n} = 2$$

10 Exercice 10

10.1

$$x \geq 0$$

$$1 + x \geq 1$$

$$\sqrt{1+x} \geq 1$$

$$\sqrt{1+x} = \sqrt{1+2\frac{x}{2}} \leq \sqrt{1+2\frac{x}{2}+\left(\frac{x}{2}\right)^2} = \sqrt{\left(1+\frac{x}{2}\right)^2} = 1+\frac{x}{2}$$

10.2

$\frac{\sqrt{n^2+2}}{2}n = \frac{\sqrt{n^2\left(1+\frac{2}{n^2}\right)}}{2}n = \frac{\sqrt{1+\frac{2}{n^2}}}{2}$

Soit $x = \frac{2}{n^2}$ $\lim_{n\to+\infty} \sqrt{1+\frac{2}{n^2}} = 1$ $\lim_{n\to+\infty} \frac{\sqrt{n^2+2}}{2}n = \frac{1}{2}$

11 Exercice 11

$$\lim_{n\to+\infty} a_n = \frac{3}{1} = 3$$

11.1

$$\lim_{n\to+\infty} \frac{1}{a_n} = \frac{1}{3}$$

11.2

$$\lim_{n\to+\infty}\left(\frac{a_n}{3}+\frac{3}{a_n}\right)=1+1=2$$

11.3

$$\lim_{n\to+\infty}\frac{9n^2}{n^2+4n+4}=\lim_{n\to+\infty}(a_n)(a_n)=3^2=9$$

12 Exercice 12

On sait que $\forall x \in A x \leq s, \forall \varepsilon > 0 \exists x \in A, s-\varepsilon < x \leq s$ Soit $\varepsilon_n = \frac{1}{n}, N = \frac{1}{\varepsilon_n}$

$$\forall n \geq N, s-\frac{1}{n} < x_n \leq s$$

$$-\frac{1}{n} < x_n - s \leq 0$$

On sait que $\lim_{n\to+\infty} -\frac{1}{n} = 0$ Donc d'après le théorème des gendarmes

$$\lim_{n\to+\infty} x_n - s = 0 \Longleftrightarrow \forall \varepsilon > 0\ \exists N, \forall n \geq N, |x_n - s| \leq \varepsilon$$

Ainsi

$$\lim_{n\to+\infty} x_n = s$$