HOMOMORPHISM TESTING LEMMAS AND COSLICES

LUCA REGGIO

The aim of this note is to prove a 'homomorphism testing lemma with constants', see Theorem 2.3 below. The term *constants* refers to the fact that we work in coslice categories. We first discuss homomorphism testing lemmas in Section 1, and then lift them to coslice categories in Section 2.

Notation: Given objects e_1, e_2 in a category $\mathcal{E}$, the expression $e_1 \to e_2$ denotes the existence of an arrow from e_1 to e_2. That is, $e_1 \to e_2$ precisely when $\mathcal{E}(e_1, e_2) \neq \emptyset$.

Given an adjunction $L \dashv R$, the unit and counit of the adjunction are denoted, respectively, by η and ε.

1. Homomorphism Testing Lemmas

Suppose that G is a comonad on a category $\mathcal{E}$. A *homomorphism testing lemma* states that, given arbitrary objects $e_1, e_2 \in \mathcal{E}$, $Ge_1 \to e_2$ if and only if, for all objects d in some class $\mathcal{D} \subseteq \mathcal{E}$, $d \to e_1$ entails $d \to e_2$.

The significance of such a result can be explained as follows. The object Ge_1 is typically a colimit of objects $\{d_i \mid i \in I\}$ from $\mathcal{D}$. The existence of an arrow $Ge_1 \to e_2$ then amounts to the existence of a *compatible* cocone

$$\{d_i \to e_2 \mid i \in I\}.$$

A homomorphism testing lemma essentially says that we can dispense with the compatibility requirement, making it easier to show that $Ge_1 \to e_2$. The smaller the class $\mathcal{D}$ of 'test structures', the more useful the result is.

We start by recalling a homomorphism testing lemma in which the class $\mathcal{D}$ is in a sense maximal, and in particular contains the object Ge_1.

Lemma 1.1 ([1, Proposition 7.1])**.** *Let (G, δ, ε) be a comonad on a category $\mathcal{E}$. The following statements are equivalent for all objects e_1, e_2 of $\mathcal{E}$:*

(1) $Ge_1 \to e_2$.
(2) For all $e \in \mathcal{E}$ admitting a G-coalgebra structure, $e \to e_1 \Rightarrow e \to e_2$.

Proof. For (1) $\Rightarrow$ (2), given a G-coalgebra $(e, \alpha\colon e \to Ge)$ and morphisms $f\colon Ge_1 \to e_2$ and $g\colon e \to e_1$, consider the morphism

$$e \xrightarrow{\alpha} Ge \xrightarrow{Gg} Ge_1 \xrightarrow{f} e_2.$$

For the converse implication, just note that (Ge_1, δ_{e_1}) is a G-coalgebra and $Ge_1 \to e_1$ (consider, e.g., ε_{e_1}). So, $Ge_1 \to e_2$. $\square$

In the next proposition, we prove a homomorphism testing lemma in which the class $\mathcal{D}$ of test structures need not contain Ge_1.

Recall that an adjunction $L \dashv R$ is *finitary* if the right adjoint R preserves filtered colimits. Further, given a category $\mathcal{E}$, we denote by $\mathcal{E}_{\mathrm{fp}}$ its full subcategory defined by the finitely presentable objects. An object $e \in \mathcal{E}$ is of *finite* $\mathcal{E}_{\mathrm{fp}}$*-type* if, for all $d \in \mathcal{E}_{\mathrm{fp}}$, the hom-set $\mathcal{E}(d, e)$ is finite.

Proposition 1.2. *Consider a finitary adjunction $L \dashv R\colon \mathcal{E} \to \mathcal{C}$ with $\mathcal{C}$ locally finitely presentable. Let e_1, e_2 be objects of $\mathcal{E}$ and suppose that e_2 is of finite $\mathcal{E}_{\mathrm{fp}}$-type. The following statements are equivalent:*

(1) $LRe_1 \to e_2$.
(2) For all $d \in \mathcal{C}_{\mathrm{fp}}$, $Ld \to e_1 \;\Rightarrow\; Ld \to e_2$.

Proof. As $\mathcal{C}$ is locally finitely presentable, Re_1 is the colimit of the canonical (filtered) diagram given by the forgetful functor $\mathcal{C}_{\mathrm{fp}}/Re_1 \to \mathcal{C}$. We write

$$Re_1 \cong \operatorname*{colim}_{\mathcal{C}_{\mathrm{fp}}/Re_1} d, \tag{1}$$

where d ranges over the objects in the image of $\mathcal{C}_{\mathrm{fp}}/Re_1 \to \mathcal{C}$.

Now, fix arbitrary objects $e_1, e_2 \in \mathcal{E}$. We compute:

$$\begin{aligned}
\mathcal{E}(LRe_1, e_2) &\cong \mathcal{C}(Re_1, Re_2) && (L \dashv R)\\
&\cong \mathcal{C}(\operatorname*{colim}_{\mathcal{C}_{\mathrm{fp}}/Re_1} d, Re_2) && (\text{Eq. } (1))\\
&\cong \lim_{\mathcal{C}_{\mathrm{fp}}/Re_1} \mathcal{C}(d, Re_2)\\
&\cong \lim_{\mathcal{C}_{\mathrm{fp}}/Re_1} \mathcal{E}(Ld, e_2). && (L \dashv R)
\end{aligned}$$

Because the adjunction $L \dashv R$ is finitary, L preserves finitely presentable objects. Thus, if e_2 is of finite $\mathcal{E}_{\mathrm{fp}}$-type, the set $\mathcal{E}(Ld, e_2)$ is finite whenever $d \in \mathcal{C}_{\mathrm{fp}}$. It follows from the computation above that $\mathcal{E}(LRe_1, e_2)$ is an inverse limit of finite sets. Recall that an inverse limit of finite sets X_i is non-empty precisely when each X_i is non-empty. So, $LRe_1 \to e_2$ if and only if, for all objects d in the image of $\mathcal{C}_{\mathrm{fp}}/Re_1 \to \mathcal{C}$, we have $Ld \to e_2$.

Finally, in view of the adjunction $L \dashv R$, for all $d \in \mathcal{C}$ we have $d \to Re_1$ if and only if $Ld \to e_1$. Together, these observations yield the statement. □

We specialise the previous proposition to the case of a comonad of finite rank on a locally finitely presentable category $\mathcal{E}$.

Corollary 1.3. *Let G be a comonad of finite rank on a locally finitely presentable category $\mathcal{E}$. Let e_1, e_2 be objects of $\mathcal{E}$ and suppose that e_2 is of finite $\mathcal{E}_{\mathrm{fp}}$-type. The following statements are equivalent:*

(1) $Ge_1 \to e_2$.
(2) For all $e \in \mathcal{E}_{\mathrm{fp}}$ admitting a G-coalgebra structure, $e \to e_1 \;\Rightarrow\; e \to e_2$.

Proof. If G is of finite rank, then its category EM of Eilenberg-Moore coalgebras is locally finitely presentable, the associated adjunction

$$L \dashv R\colon \mathcal{E} \to \mathsf{EM}$$

is finitary, and L preserves and reflects finitely presentable objects (see e.g., [3] or [5, Appendix B]). Hence, an object $e \in \mathcal{E}_{\mathrm{fp}}$ admits a coalgebra structure precisely when it is of the form Lc for some $c \in \mathsf{EM}_{\mathrm{fp}}$. Therefore, the statement follows at once from Proposition 1.2 by setting $\mathcal{C} := \mathsf{EM}$. □

2. Coslices

In this section, we prove a version of Proposition 1.2 with ‘constants’, in the sense that the category $\mathcal{E}$ is replaced with one of its *coslice* (or *under*) categories. This is Theorem 2.3 below.

To start with, given an adjunction $L \dashv R\colon\ \mathcal{E} \to \mathcal{C}$ and objects $c \in \mathcal{C}$ and $e \in \mathcal{E}$, recall that there is a natural hom-set bijection

$$\mathcal{E}(Lc, e) \to \mathcal{C}(c, Re), \quad g \mapsto g^\flat$$

with inverse

$$\mathcal{C}(c, Re) \to \mathcal{E}(Lc, e), \quad k \mapsto k^\#.$$

Naturality in e amounts to the following property: for all arrows $g_1\colon e \to e'$ and $g_2\colon Lc \to e$ in $\mathcal{E}$,

$$(g_1 \circ g_2)^\flat = Rg_1 \circ g_2^\flat.$$

Similarly, naturality in c means that for all morphisms $k_1\colon c \to Re$ and $k_2\colon c' \to c$ in $\mathcal{C}$,

$$(k_1 \circ k_2)^\# = k_1^\# \circ Lk_2.$$

Next, we recall how to lift the adjunction $L \dashv R$ to the coslice categories.

Lemma 2.1. *Consider an arbitrary adjunction*

$$L \dashv R\colon\ \mathcal{E} \to \mathcal{C}\,.$$

For all objects $c \in \mathcal{C}$, there is an adjunction

$$L_c \dashv R_c\colon Lc/\mathcal{E} \to c/\mathcal{C}$$

where L_c and R_c are the functors defined (on objects), respectively, by

$$L_c(c \xrightarrow{h} c') = Lc \xrightarrow{Lh} Lc'$$

and

$$R_c(Lc \xrightarrow{f} e) = c \xrightarrow{\eta_c} RLc \xrightarrow{Rf} Re = c \xrightarrow{f^\flat} RLc.$$

Proof. The actions of L_c and R_c on morphisms are the obvious ones induced by L and R, respectively. Now, fix arbitrary objects $h\colon c \to c'$ and $f\colon Lc \to e$ in $c/\mathcal{C}$ and $Lc/\mathcal{E}$, respectively. We define a function

$$Lc/\mathcal{E}(L_c h, f) \to c/\mathcal{C}(h, R_c f) \tag{2}$$

by sending a morphism $g \in Lc/\mathcal{E}(L_c h, f)$, as displayed in the left-hand diagram below, to the morphism $g^\flat \in c/\mathcal{C}(h, R_c f)$.

$$\begin{tikzcd} & Lc \arrow[dl, "Lh"'] \arrow[dr, "f"] & \\ Lc' \arrow[rr, "g"] & & e \end{tikzcd} \qquad \begin{tikzcd} & c \arrow[dl, "h"'] \arrow[dr, "Rf \circ \eta_c"] & \\ c' \arrow[rr, "g^\flat"] & & Re \end{tikzcd}$$

Note that this function is well-defined because

$$Rf \circ \eta_c = Rf \circ \mathrm{id}_{Lc}^{\flat} = (f \circ \mathrm{id}_{Lc})^{\flat} = (g \circ Lh)^{\flat} = ((g^{\flat} \circ h)^{\#})^{\flat} = g^{\flat} \circ h,$$

i.e. the right-hand diagram above commutes.

Reasoning in a similar manner, it is not difficult to see that the map in (2) is a bijection, whose inverse sends $k \in c/\mathcal{C}(h, R_c f)$ to $k^{\#} \in Lc/\mathcal{E}(L_c h, f)$. Naturality of these bijections follows by the corresponding property for the adjunction $L \dashv R$. □

Remark 2.2. Consider an adjunction $L \dashv R\colon \mathcal{E} \to \mathcal{C}$ and an object $c \in \mathcal{C}$. In view of Lemma 2.1, there is a comonad $G_c := L_c R_c$ on $Lc/\mathcal{E}$. Explicitly, G_c sends an object $f\colon Lc \to e$ of $Lc/\mathcal{E}$ to the composite

$$Lc \xrightarrow{L\eta_c} GLc \xrightarrow{Gf} Ge,$$

regarded as an object of $Lc/\mathcal{E}$.

As locally finitely presentable categories are stable under taking coslices, one may consider applying Proposition 1.2 or Corollary 1.3 to this situation. However, assuming we have done so, we would be able to characterise only the existence of morphisms of the form $G_c f \to t$. Instead, in Theorem 2.3 we take a different route which allows us to consider, more generally, morphisms of the form $L_c s \to t$ for some $s\colon c \to Re_1$.

Theorem 2.3. *Consider a finitary adjunction $L \dashv R\colon \mathcal{E} \to \mathcal{C}$ with $\mathcal{C}$ locally finitely presentable. Let e_1, e_2 be objects of $\mathcal{E}$ and suppose that e_2 is of finite $\mathcal{E}_{\mathrm{fp}}$-type. The following statements are equivalent for all objects $c \in \mathcal{C}_{\mathrm{fp}}$ and morphisms $s\colon c \to Re_1$ and $t\colon Lc \to e_2$:*

1. *$Ls \to t$ in $Lc/\mathcal{E}$.*
2. *For all morphisms $u\colon c \to d$ with $d \in \mathcal{C}_{\mathrm{fp}}$, $Lu \to s^{\#} \Rightarrow Lu \to t$.*

Proof. The following argument is a straightforward generalisation of the proof of Proposition 1.2. By Lemma 2.1, there is an adjunction

$$L_c \dashv R_c\colon Lc/\mathcal{E} \to c/\mathcal{C}.$$

Recall that coslices of locally finitely presentable categories are locally finitely presentable, see e.g. [2, Proposition 1.57]. In particular, $c/\mathcal{C}$ is locally finitely presentable. Furthermore, because c is a finitely presentable object, there is an equivalence of categories

$$(c/\mathcal{C})_{\mathrm{fp}} \simeq c/\mathcal{C}_{\mathrm{fp}}.$$

See e.g. [4, Corollary 2.14].

Thus, we have:

$$\begin{aligned} Lc/\mathcal{E}(Ls,t) &= Lc/\mathcal{E}(L_c s,t) \\ &\cong c/\mathcal{C}(s,R_c t) && (L_c \dashv R_c) \\ &\cong c/\mathcal{C}(\operatorname*{colim}_{(c/\mathcal{C})_{\mathrm{fp}}/s} u, R_c t) \\ &\cong c/\mathcal{C}(\operatorname*{colim}_{(c/\mathcal{C}_{\mathrm{fp}})/s} u, R_c t) && ((c/\mathcal{C})_{\mathrm{fp}} \simeq c/\mathcal{C}_{\mathrm{fp}}) \\ &\cong \lim_{(c/\mathcal{C}_{\mathrm{fp}})/s} c/\mathcal{C}(u, R_c t) \\ &\cong \lim_{(c/\mathcal{C}_{\mathrm{fp}})/s} Lc/\mathcal{E}(Lu,t). && (L_c \dashv R_c) \end{aligned}$$

Note that the hom-sets $Lc/\mathcal{E}(Lu,t)$ are finite. Just observe that (i) the codomain of Lu is finitely presentable because the adjunction $L \dashv R$ is finitary and so L preserves finitely presentable objects, and (ii) e_2, the codomain of t, is of finite $\mathcal{E}_{\mathrm{fp}}$-type.

Hence, $Ls \to t$ if and only if, for all morphisms $u\colon c \to d$ with $d \in \mathcal{C}_{\mathrm{fp}}$, if $u \to s$ then $Lu \to t$. The statement follows by observing that $u \to s$ precisely when $Lu \to s^\#$. Just note that, if $s = f \circ u$ for some f, then

$$s^\# = (f \circ u)^\# = f^\# \circ Lu.$$

Conversely, if $s^\# = g \circ Lu$ for some g, then

$$(g^\flat \circ u)^\# = g \circ Lu = s^\#$$

and so $g^\flat \circ u = s$. □

Remark 2.4. Note that Proposition 1.2 is a special case of Theorem 2.3, as it can be seen by letting c be the initial object of $\mathcal{C}$ in the latter result.

Lemma 2.5. *Consider an adjunction $L \dashv R\colon \mathcal{E} \to \mathcal{C}$ and an object c of $\mathcal{C}$. Let $s\colon c \to Re_1$ and $t\colon c \to Re_2$ satisfy $s \leftrightarrows t$ in $c/\mathcal{C}$. Then*

$$R_c L_c t \leftrightarrows R_c(s^\# + L_c t) \quad \text{in } c/\mathcal{C}$$

whenever the coproduct $s^\# + L_c t$ of $s^\#$ and $L_c t$ in $Lc/\mathcal{E}$ exists.

Proof. Observe that $Lt \to s^\# + Lt$ in $Lc/\mathcal{E}$ (just take the coproduct injection). Applying the functor R_c, we get $R_c L_c t \to R_c(s^\# + L_c t)$.

It remains to prove that $R_c(s^\# + L_c t) \to R_c L_c t$. Recall that $t \to s$ precisely when $Lt \to s^\#$, cf. the proof of Theorem 2.3. Since $t \to s$ by hypothesis, the copairing of any arrow $Lt \to s^\#$ with the identity $s^\# \to s^\#$ yields an arrow $s^\# + Lt \to s^\#$. Applying the functor R_c, we get

$$R_c(s^\# + Lt) \to R_c s^\# = s.$$

To conclude the proof, it is enough to show that $s \to R_c L_c t$. In turn, in view of the adjunction $L_c \dashv R_c$, this is equivalent to $L_c s \to L_c t$. But this is immediate from $s \to t$. □

3. Moving Across Coslices

Let $i\colon \tilde{c} \to c$ be any morphism in $\mathcal{C}$. If the category $\mathcal{C}$ has all pushouts, then the functor
$$i^{!}\colon c/\mathcal{C} \to \tilde{c}/\mathcal{C}, \ \ h \mapsto h \circ i$$
has a left adjoint
$$i_{*}\colon \tilde{c}/\mathcal{C} \to c/\mathcal{C}$$
that sends a morphism $k\colon \tilde{c} \to c'$ to its pushout along i:

$$\begin{tikzcd} \tilde{c} \arrow[r, "i"] \arrow[d, "k"'] & c \arrow[d, "i_* k"] \\ c' \arrow[r] & \cdot \end{tikzcd}$$

In this case we have a diagram as follows, where both the square of right adjoints (consisting of the solid arrows) and the square of left adjoints (consisting of the dotted arrows) commute.

$$\begin{tikzcd} c/\mathcal{C} \arrow[r, shift left, "i^!"] \arrow[d, dotted, shift right, "L_c"'] & \tilde{c}/\mathcal{C} \arrow[l, dotted, shift left, "i_*"] \arrow[d, dotted, shift right, "L_{\tilde{c}}"'] \\ Lc/\mathcal{E} \arrow[r, shift left, "(Li)^!"] \arrow[u, shift right, "R_c"'] & L\tilde{c}/\mathcal{E} \arrow[l, dotted, shift left, "(Li)_*"] \arrow[u, shift right, "R_{\tilde{c}}"'] \end{tikzcd}$$

References

1. S. Abramsky and N. Shah, *Relating structure and power: Comonadic semantics for computational resources*, Journal of Logic and Computation **31** (2021), no. 6, 1390–1428.
2. J. Adámek and J. Rosický, *Locally presentable and accessible categories*, London Mathematical Society Lecture Note Series, vol. 189, Cambridge University Press, Cambridge, 1994.
3. Y. Diers, *Categories of Boolean sheaves of simple algebras*, Lecture Notes in Mathematics, vol. 1187, Springer-Verlag, Berlin, 1986.
4. A. Osmond, *On coslices and commas of locally finitely presentable categories*, preprint available at `https://arxiv.org/abs/2104.06537`, 2021.
5. L. Reggio, *Polyadic sets and homomorphism counting*, preprint available at `https://arxiv.org/abs/2110.11061`, 2021.

Department of Computer Science, University of Oxford, United Kingdom
Email address: `luca.reggio@cs.ox.ac.uk`