Everything Everywhere All in One

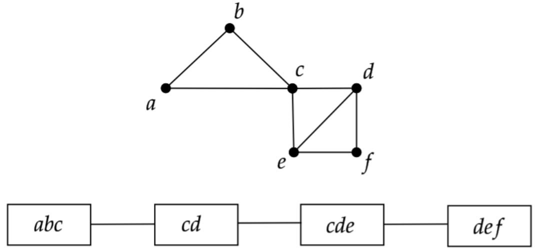
Yoàv Montacute

University of Cambridge

Joint work with Nihil Shah
Resources in Computation 2022

Pathwidth

Path decomposition (Robertson and Seymour)



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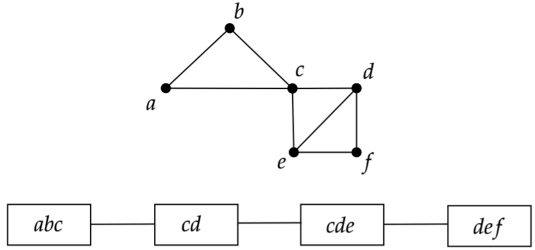

We define a **coalgebra number** $\kappa^{\mathbb{PR}}(\mathcal{A})$ of a finite structure $\mathcal{A}$ to be the least k such that there exists a coalgebra $\alpha : \mathcal{A} \to \mathbb{PR}_k\mathcal{A}$.

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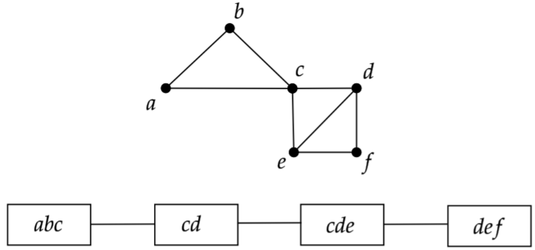

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Theorem (coalgebraic characterisation of pathwidth)

For all σ-structures $\mathcal{A}$, $\mathsf{pw}(\mathcal{A}) = \kappa^{\mathbb{PR}}(\mathcal{A}) - 1$.

The comonad $\mathbb{PR}_k$

Given a relational structure $\mathcal{A} = (A, R_1, \ldots, R_m)$, we define $\mathbb{PR}_k\mathcal{A}$ as follows:

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- The universe $\mathbb{PR}_k A$ consists of all pairs of non-empty sequences and indices
$$(s, i) = ([(p_1, a_1), \ldots, (p_n, a_n)], i),$$
where $p_i \in [k]$, $a_i \in A$ and $i \in [n]$.

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where $p_i \in [k]$, $a_i \in A$ and $i \in [n]$.
- For each relation R (for simplicity let R be binary), we define $((s, i), (t, j)) \in R^{\mathbb{PR}_k\mathcal{A}}$ if
  1. $s = t$;
  2. If $j > i$ (resp. $i > j$), then the i-th pebble of s does not appear in $s(i, j]$;
  3. $R^{\mathcal{A}}(\varepsilon_{\mathcal{A}}(s, i), \varepsilon_{\mathcal{A}}(t, j))$, where $\varepsilon_{\mathcal{A}}([(p_1, a_1), \ldots, (p_n, a_n)], i) = a_i$.

The coKleisli category

Consider the category $\mathcal{K}(\mathbb{PR}_k)$ which is the coKleisli category over the comonad $\mathbb{PR}_k$. Its objects are the same as $\mathcal{R}(\sigma)$ and morphisms from $\mathcal{A}$ to $\mathcal{B}$ in the category are homomorphisms $f : \mathbb{PR}_k\mathcal{A} \to \mathcal{B}$.

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Composition of morphisms $g \circ_{\mathcal{K}(\mathbb{PR}_k)} f = g \circ f^*$, where

$$f^* : ([(p_1, a_1), \dots, (p_n, a_n)], i) \mapsto ([(p_1, f(s_1)), \dots, (p_n, f(s_n))], i)$$

and $s_j = ([(p_1, a_1), \dots, (p_n, a_n)], j)$, for all $j \in [n]$.

All-in-one k-pebble game

Let $\exists^+ \curlywedge \mathcal{L}^k$ denote the fragment of $\exists^+ \mathcal{L}^k$ with restricted conjunction. i.e. every conjunction has at most one quantified formula with free variables.

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1. Spoiler provides a sequence of pebble placements $[(p_1, a_1), \ldots, (p_n, a_n)]$.
2. Duplicator answers with a sequence $[(p_1, b_1), \ldots, (p_n, b_n)]$.

If every prefix induces a partial homomorphism from $\mathcal{A}$ to $\mathcal{B}$, then Duplicator wins the game.

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Theorem (morphism power theorem)

Given two σ-structures $\mathcal{A}$ and $\mathcal{B}$, the following are equivalent:

- *Duplicator has a winning strategy in $\exists \mathbf{PPeb}_k(\mathcal{A}, \mathcal{B})$.*
- $\mathcal{A} \Rrightarrow^{\exists^+ \curlywedge \mathcal{L}^k} \mathcal{B}$.
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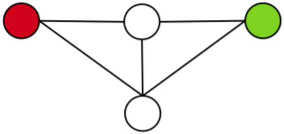
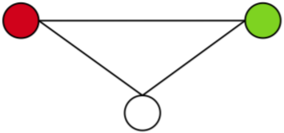
2-pebble game (standard vs. all-in-one)



$\mathcal{A}$



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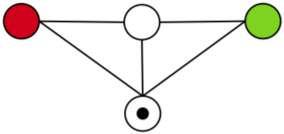

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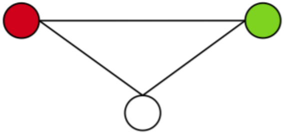

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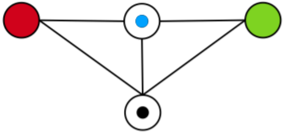

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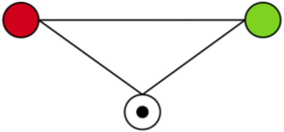

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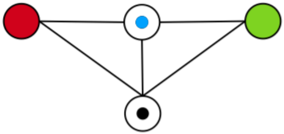

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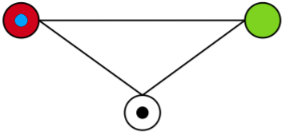

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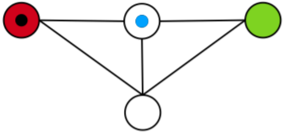

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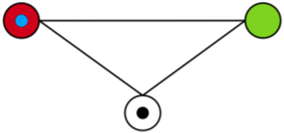

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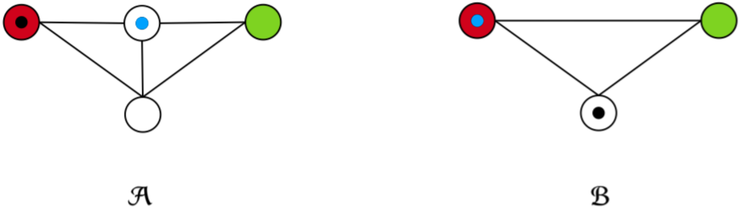

- In this example Duplicator loses the 2-pebble game but wins the all-in-one 2-pebble game.

All-in-one bijective k-pebble game

Consider the all-in-one bijective k-pebble game $\#\mathbf{PPeb}_k(\mathcal{A}, \mathcal{B})$. The game is played in one round during which:

1. Spoiler provides a sequence of pebble placements with one pebble placement hidden $[(p_1, a_1), \ldots, (p_j, \_), \ldots, (p_n, a_n)]$.
2. Duplicator answers with a sequence $[(p_1, \psi_1), \ldots, (p_n, \psi_n)]$ of pebble placements and bijections $\psi_i : A \to B$.

If every prefix induces a partial isomorphism from $\mathcal{A}$ to $\mathcal{B}$, then Duplicator wins the game.

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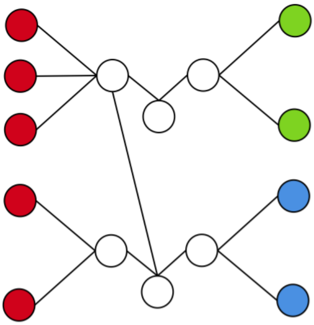
- *There exists a coKleisli isomorphism $f : \mathbb{PR}_k\mathcal{A} \to \mathcal{B}$.*

Bijective 2-pebble game (standard vs. all-in-one)



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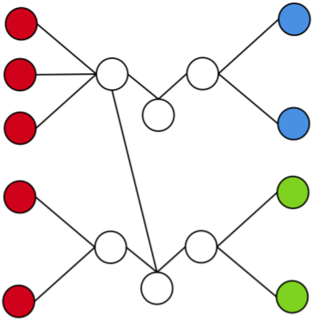

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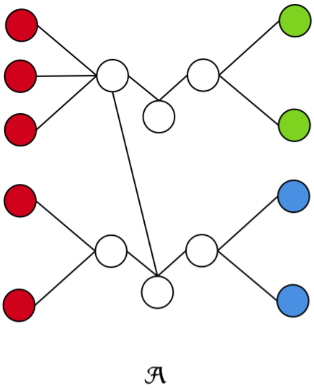

- The $\#\mathcal{L}^k$-formula

$$\exists x\Big(\exists y\big(Exy \wedge \exists_{\leq 2} x(Eyx \wedge Rx)\big) \wedge \exists y\big(Exy \wedge \exists_{\geq 2} x(Eyx \wedge Bx)\big)\Big)$$

is true in $\mathcal{A}$ but not in $\mathcal{B}$.

Lovász-type theorem for pathwidth

Theorem (Dawar, Jakl and Reggio)

Given a locally finite category $\mathscr{C}$ *with pushout and proper factorisation system, for all* $\mathcal{A}, \mathcal{B} \in \mathscr{C}$,

$$\mathcal{A} \cong \mathcal{B} \iff |\mathbf{hom}_{\mathscr{C}}(\mathcal{C}, \mathcal{A})| = |\mathbf{hom}_{\mathscr{C}}(\mathcal{C}, \mathcal{B})|, \forall \mathcal{C} \in \mathscr{C}.$$

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For every finite σ-structures $\mathcal{A}$ and $\mathcal{B}$:

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for every finite σ-structure $\mathcal{C}$ with pathwidth at most k.

Computational complexity

Ongoing work with Anuj Dawar and Nihil Shah.

Figure: Bisimulation vs. trace-equivalence

Computational complexity

Theorem (Balcázar, Gabarró and Sántha)

Deciding bisimulation is P-complete.

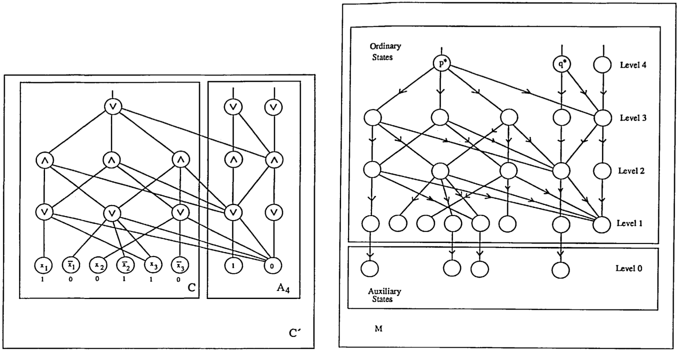

Figure: Balcázar, Gabarró and Sántha (1992)

Computational complexity

Theorem (Kolaitis and Panttaja)

Determining the winner of the k-pebble game for a fixed k is P-complete

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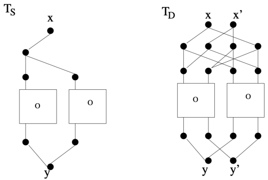

Figure: Kolaitis and Panttaja (2003)

Computational Complexity

Theorem (Chandra and Stockmeyer)

Deciding trace-equivalence is $PSPACE$*-complete*

Computational Complexity

Theorem (Chandra and Stockmeyer)

Deciding trace-equivalence is PSPACE-complete

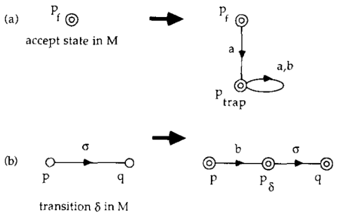

Figure: Kanellakis and Smolka (1990)

Computational complexity

Conjecture

Determining the winner of the all-in-one k-pebble game for a fixed k is $PSPACE$-complete.

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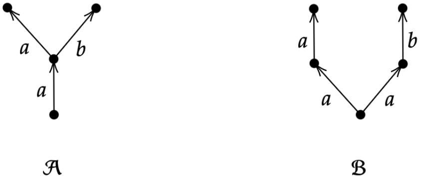

Thank you