Notes on three-body bound states

August 10, 2021

Abstract

We use the effective field theory (EFT) approach to investigate the three-body problem. Our particular interest is to compute bound states.

1 Introduction

The Fig. 1, taken from Ref. [1], ilustrate diagrammatically the Faddeev equations for the case of ^{6}He bound-state. The single dashed (or solid) line denotes the c (or n) one-body propagator. The thick black and shaded lines represents the nc and nn two-body propagators. The F_c and F_n are the corresponding Faddeev components. In summary, these components are the partial-wave-decomposed states that are obtained by solving the two coupled-channel integral equations. For the general case of a two-neutron halo ground-state (core-neutron-neutron), they are given as

$$F_c(q) = 8\pi \int_0^\Lambda q'^2 dq' X_{cn}\left(q,q';-B_3\right) \tau_n\left(q';-B_3\right) F_n\left(q'\right), \tag{1}$$

$$F_n(q) = 4\pi \int_0^\Lambda q'^2 dq' X_{nc}\left(q,q';-B_3\right) \tau_c\left(q';-B_3\right) F_c\left(q'\right) + 4\pi \int_0^\Lambda q'^2 dq' X_{nn}\left(q,q';-B_3\right) \tau_n\left(q';-B_3\right) F_n\left(q'\right), \tag{2}$$

where Λ is the ultraviolet cutoff. The kernel function X_{ij} is defined as

$$X_{ij}\left(q,q';E\right) = \iint p^2 dp p'^2 dp' g_{l_i}(p) G_0^{(i)}(p,q;E) g_{l_j}\left(p'\right) {}_i\left\langle p,q;\Omega_i \mid p',q';\Omega_j\right\rangle_j,$$

which includes the three-body Green's function

$$G_0^{(i)}(p,q;E) = \left(E - \frac{p^2}{2\mu_{jk}} - \frac{q^2}{2\mu_{i(jk)}}\right)^{-1},$$

and the two-body form factors g_{l_i} and g_{l_j} (**falar sobre Yamagushi**). The factor ${}_i\left\langle p,q;\Omega_i \mid p',q';\Omega_j\right\rangle_j$ is the projection of the eigenstate of the free Hamiltonian in the partition of spectator i onto the free eigenstate in the partition of spectator j:

$${}_i\left\langle p,q;\Omega_i \left|t_i\right| p',q';\Omega_i'\right\rangle_i = 4\pi g_{l_i}(p) \tau_i(q;E) g_{l_i}\left(p'\right) \delta_{\Omega_i\Omega_i'} \frac{1}{q^2} \delta\left(q-q'\right).$$

$$\tau_{nn}(E) = \frac{1}{4\pi^2 \mu_{nn}\left(\gamma_0 + ik\right)},$$

$$\tau_{nc}(E) = \frac{1}{4\pi^2 \mu_{nc}(k^2 - k_R^2)}$$

where $k = \sqrt{2\mu E}$, $\gamma_0 = 1/a_0$ with nn scattering length $a_0 = -18.7$ fm, since nn interaction is dominated by an s-wave virtual state. In addition, $k_R = \sqrt{2/(a_1 r_1)}$ and $\gamma_1 = -r_1/2$, in which

$$\begin{aligned} \frac{1}{a_1} &= \frac{1}{4\pi^2 \mu_{nc} \lambda_1} + \frac{2\Lambda^3}{3\pi} \\ \frac{r_1}{2} &= -\frac{2\Lambda}{\pi} \end{aligned}$$

We look the solution for an energy of $E = -B_3$ where the eigenvalue of the kernel is one.

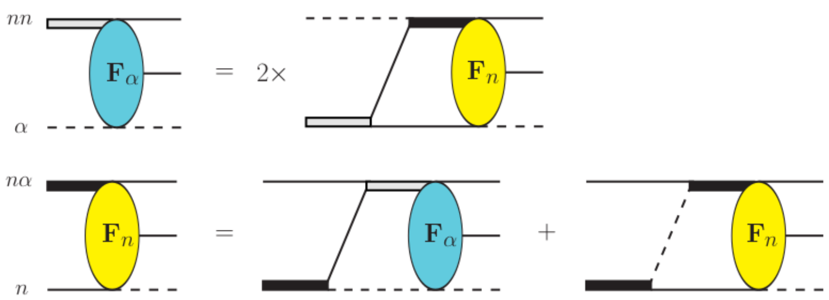

Figure 1: The Faddeev equations for the ^{6}He bound-state where F_c and F_n are the corresponding Faddeev components. Figure taken from Ref. [1].

.... and finally

$$\tau_c(q; -B_3) = \frac{1}{2\pi^2 m_n} \frac{1}{\gamma_0 - \mathscr{K}_c(q; -B_3)}$$

where the two-body binding momentum $\mathscr{K}_c$ is related to q and B_3 by

$$\mathscr{K}_c(q; -B_3) = \sqrt{m_n B_3 + \frac{A+2}{4A} q^2}$$

Here A indicates the mass ratio between the c -core and a neutron $A = m_c/m_n$. Similarly, we can write τ_n as a function of B_3 and the Jacobi momentum q :

$$\tau_n(q; -B_3) = -\frac{1}{4\pi^2 m_n \gamma_1} \left(\frac{A+1}{A} \right) \frac{1}{\mathscr{K}_n^2(q; -B_3) + k_R^2}$$

with

$$\mathscr{K}_n(q; -B_3) = \sqrt{\frac{2A}{A+1} \left(m_n B_3 + \frac{A+2}{2(A+1)} q^2 \right)}$$

$$\begin{aligned} X_{nc}(q, q'; -B_3) &= -\sqrt{2} m_n \left[\tfrac{A}{A+1} \tfrac{1}{q'} Q_0(z_{nc}) + \tfrac{1}{q} Q_1(z_{nc}) \right] \\ X_{cn}(q, q'; -B_3) &= -\sqrt{2} m_n \left[\tfrac{A}{A+1} \tfrac{1}{q} Q_0(z_{cn}) + \tfrac{1}{q'} Q_1(z_{cn}) \right] \\ X_{nn}(q, q'; -B_3) &= A m_n \left[\tfrac{A^2+2A+3}{(A+1)^2} Q_0(z_{nn}) + \tfrac{2}{A+1} \tfrac{q^2+q'^2}{qq'} Q_1(z_{nn}) + Q_2(z_{nn}) \right] \end{aligned}$$

with $A = m_c/m_n$ and Q_l are the Legendre functions of the second kind, which are related the ordinary Legendre polynomials P_l by

$$Q_l(z) = \frac{1}{2} \int_{-1}^{1} dx \frac{P_l(x)}{z - x}$$

for $|z| > 1$. The arguments z_{nc}, z_{cn} and z_{nn} in are defined as

$$\begin{aligned} z_{nc} &= -\tfrac{1}{qq'} \left(m_n B_3 + q^2 + \tfrac{A+1}{2A} q'^2 \right) \\ z_{cn} &= -\tfrac{1}{qq'} \left(m_n B_3 + q'^2 + \tfrac{A+1}{2A} q^2 \right) \\ z_{nn} &= -\tfrac{A}{qq'} \left(m_n B_3 + \tfrac{A+1}{2A} \left(q^2 + q'^2 \right) \right) \end{aligned}$$

About the parameters (to do a table): "A more recent analysis of $n\alpha$ data gives $a_1 = -65.7$ fm^3, $r_1 = -0.84$ fm^{-1}".

2 Numerical Solution

2.1 Method 1

For simplicity, we write the Eqs. (1) and (2) as

$$
\begin{aligned}
F_c(q) &= 2\int_0^{\Lambda} dq' K_{cn}\left(q,q';-B_3\right) F_n\left(q'\right), && (3)\\
F_n(q) &= \int_0^{\Lambda} dq' K_{nc}\left(q,q';-B_3\right) F_c\left(q'\right) + \int_0^{\Lambda} dq' K_{nn}\left(q,q';-B_3\right) F_n\left(q'\right). && (4)
\end{aligned}
$$

where

$$
\begin{aligned}
K_{cn}\left(q,q';-B_3\right) &= 4\pi\, q'^2\, X_{cn}\left(q,q';-B_3\right)\tau_n\left(q';-B_3\right)\\
K_{nc}\left(q,q';-B_3\right) &= 4\pi\, q'^2\, X_{nc}\left(q,q';-B_3\right)\tau_c\left(q';-B_3\right)\\
K_{nn}\left(q,q';-B_3\right) &= 4\pi\, q'^2\, X_{nn}\left(q,q';-B_3\right)\tau_n\left(q';-B_3\right)
\end{aligned}
$$

Discretizing expressions (16) and (17), it is possible to write

$$
\begin{aligned}
F_c(q_i) &= 2\sum_{j=1}^{N} K_{cn}\left(q_i,q_j;-B_3\right) F_n\left(q_j\right) w_j && (5)\\
F_n(q_i) &= \sum_{j=1}^{N} K_{nc}\left(q_i,q_j;-B_3\right) F_c\left(q_j\right) w_j + \sum_{j=1}^{N} K_{nn}\left(q_i,q_j;-B_3\right) F_n\left(q_j\right) w_j, && (6)
\end{aligned}
$$

with

$$
\begin{aligned}
K_{cn}\left(q_i,q_j;-B_3\right) &= 4\pi\, q_j^2\, X_{cn}\left(q_i,q_j;-B_3\right)\tau_n\left(q_j;-B_3\right)\\
K_{nc}\left(q_i,q_j;-B_3\right) &= 4\pi\, q_j^2\, X_{nc}\left(q_i,q_j;-B_3\right)\tau_c\left(q_j;-B_3\right)\\
K_{nn}\left(q_i,q_j;-B_3\right) &= 4\pi\, q_j^2\, X_{nn}\left(q_i,q_j;-B_3\right)\tau_n\left(q_j;-B_3\right).
\end{aligned}
$$

We can write the expressions

$$
\begin{aligned}
F_c(q_i) - 2\sum_{j=1}^{N} K_{cn}\left(q_i,q_j;-B_3\right) F_n\left(q_j\right) w_j &= 0, && (7)\\
F_n(q_i) - \sum_{j=1}^{N} K_{nc}\left(q_i,q_j;-B_3\right) F_c\left(q_j\right) w_j - \sum_{j=1}^{N} K_{nn}\left(q_i,q_j;-B_3\right) F_n\left(q_j\right) w_j &= 0, && (8)
\end{aligned}
$$

that can be given in the matrix form, such as

$$
\begin{aligned}
\left(\delta_{ij} - 2\sum_{j=1}^{N} K_{cn}\left(q_i,q_j;-B_3\right) w_j\right)\begin{pmatrix} F_c\left(q_i\right)\\ F_n\left(q_i\right)\end{pmatrix} &= 0, && (9)\\
\left(\delta_{ij} - \sum_{j=1}^{N} K_{nc}\left(q_i,q_j;-B_3\right) w_j - \sum_{j=1}^{N} K_{nn}\left(q_i,q_j;-B_3\right) w_j\right)\begin{pmatrix} F_n\left(q_i\right)\\ F_c\left(q_i\right)\\ F_n\left(q_i\right)\end{pmatrix} &= 0, && (10)
\end{aligned}
$$

in which the Kronecker's delta was introduced. We can write

$$
\begin{bmatrix} -\delta_{ij} & 2\sum_{j=1}^{N} K_{cn}\left(q_i,q_j;-B_3\right) w_j \\ \sum_{j=1}^{N} K_{nc}\left(q_i,q_j;-B_3\right) w_j & \sum_{j=1}^{N} K_{nn}\left(q_i,q_j;-B_3\right) w_j - \delta_{ij}\end{bmatrix}\begin{pmatrix} F_c\left(q_i\right)\\ F_n\left(q_i\right)\end{pmatrix} = 0,
$$

$$\begin{bmatrix} -\hat{1} & 2\sum_{j=1}^{N} K_{cn}\left(q_i,q_j;-B_3\right)w_j \\ \sum_{j=1}^{N} K_{nc}\left(q_i,q_j;-B_3\right)w_j & \sum_{j=1}^{N} K_{nn}\left(q_i,q_j;-B_3\right)w_j-\hat{1} \end{bmatrix} \begin{pmatrix} F_c\left(q_i\right) \\ F_n\left(q_i\right) \end{pmatrix}=0. \tag{11}$$

We define

$$\mathscr{M}_{(2N\times 2N)}=\begin{bmatrix} -\hat{1}_{(N\times N)} & 2M_{cn} \\ M_{nc} & M_{nn}-\hat{1}_{(N\times N)} \end{bmatrix} \tag{12}$$

to rewrite the expression 11 as

$$\mathscr{M}_{(2N\times 2N)}\times\begin{pmatrix} \mathscr{F}_c \\ \mathscr{F}_n \end{pmatrix}=0, \tag{13}$$

where

$$M_{xy}=\begin{bmatrix} -K_{xy}\left(q_1,q_1;-B_3\right)w_1 & -K_{xy}\left(q_1,q_2;-B_3\right)w_2 & \cdots & -K_{xy}\left(q_1,q_N;-B_3\right)w_N \\ -K_{xy}\left(q_2,q_1;-B_3\right)w_1 & -K_{xy}\left(q_2,q_2;-B_3\right)w_2 & \cdots & -K_{xy}\left(q_2,q_N;-B_3\right)w_N \\ \vdots & \vdots & \ddots & \vdots \\ -K_{xy}\left(q_N,q_1;-B_3\right)w_1 & -K_{xy}\left(q_N,q_2;-B_3\right)w_2 & \cdots & -K_{xy}\left(q_N,q_N;-B_3\right)w_N \end{bmatrix}, \tag{14}$$

$$\hat{1}_{(N\times N)}=\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{(N\times N)} \tag{15}$$

with

$$\mathscr{F}_n=\begin{bmatrix} F_n(q_1) \\ F_n(q_2) \\ \vdots \\ F_n(q_N) \end{bmatrix} \quad \text{and also} \quad \mathscr{F}_c=\begin{bmatrix} F_c(q_1) \\ F_c(q_2) \\ \vdots \\ F_c(q_N) \end{bmatrix}.$$

2.2 Method 2

$$F_c(q) = 2\int_0^{\Lambda} dq' K_{cn}\left(q,q';-B_3\right)F_n\left(q'\right), \tag{16}$$

$$F_n(q) = \int_0^{\Lambda} dq' K_{nc}\left(q,q';-B_3\right)F_c\left(q'\right)+\int_0^{\Lambda} dq' K_{nn}\left(q,q';-B_3\right)F_n\left(q'\right). \tag{17}$$

where

$$K_{cn}\left(q,q';-B_3\right)=4\pi\, q'^2\, X_{cn}\left(q,q';-B_3\right)\tau_n\left(q';-B_3\right)$$
$$K_{nc}\left(q,q';-B_3\right)=4\pi\, q'^2\, X_{nc}\left(q,q';-B_3\right)\tau_c\left(q';-B_3\right)$$
$$K_{nn}\left(q,q';-B_3\right)=4\pi\, q'^2\, X_{nn}\left(q,q';-B_3\right)\tau_n\left(q';-B_3\right)$$

Taking Eq. (16) into Eq. (17):

$$\begin{aligned} F_n(q) &= 2\int_0^{\Lambda} dq' \tau_n\left(q';-B_3\right)\left[\int_0^{\Lambda} dq'' K_{nc}\left(q,q'';-B_3\right)X_{cn}\left(q'',q';-B_3\right)\right]F_n\left(q'\right) \\ &\quad +\int_0^{\Lambda} dq' K_{nn}\left(q,q';-B_3\right)F_n\left(q'\right). \end{aligned} \tag{18}$$

References

[1] C. Ji, C. Elster, and D. R. Phillips. ^{6}He nucleus in halo effective field theory. *Phys. Rev. C*, 90:044004, Oct 2014. doi:10.1103/PhysRevC.90.044004.