Distribution	Probability Density Function	$\mathbb{E}(X)$	Var(X)
$X \sim \text{Bi}(n,p)$	$p_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$	np	$np(1-p)$
$X \sim \text{U}(m,n)$ - Discrete Uniform	$p_X(x) = \frac{1}{n-m+1}$	$\frac{m+n}{2}$	$\frac{1}{12}((n-m+1)^2-1)$
$X \sim \text{R}(a,b)$ - Continuous Uniform	$f_X(x) = \frac{1}{b-a}$	$\frac{a+b}{2}$	$\frac{1}{12}(b-a)^2$
$T \sim \exp(\alpha)$ - Rate Parameter	$f_T(t) = \alpha e^{-\alpha t}$, for $t \geq 0$	$\frac{1}{\alpha}$	$\frac{1}{\alpha^2}$
$T \sim \exp(\theta)$ - Scale Parameter	$f_T(t) = \frac{1}{\theta} e^{-\frac{1}{\theta} t}$, for $t \geq 0$	θ	θ^2
$T \sim \gamma(r,\alpha)$	$f_T(t) = \frac{\alpha^r t^{r-1} e^{-\alpha t}}{\Gamma(r)}$, for $t \geq 0$	$\frac{r}{\alpha}$	$\frac{r}{\alpha^2}$
$X \sim \text{N}(\mu,\sigma^2)$	$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$, for $x \in \mathbb{R}$	μ	σ^2
$X \sim \text{Beta}(\alpha,\beta)$	$f_X(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$, for $x \in [0,1]$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

Confidence Intervals

Estimating Means

For large r, $t_r \longrightarrow \text{N}(0,1)$ is a good approximation.

Normal, Single Mean, Known σ

$\left(\bar{x} \pm c\frac{\sigma}{\sqrt{n}}\right)$; with $c = \text{F}^{-1}(1-\frac{\alpha}{2})$ from pivot $\text{N}(0,1)$.

Normal, Single Mean, Unknown σ

$\left(\bar{x} \pm c\frac{s}{\sqrt{n}}\right)$; with $c = \text{F}^{-1}(1-\frac{\alpha}{2})$ from pivot t_{n-1}.

Normal, Two Means, Two Known σ's

$\left((\bar{x}-\bar{y}) \pm c\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}\right)$; with c from pivot $\text{N}(0,1)$.

Normal, Two Means, Unknown σ's, Many Samples

$\left((\bar{x}-\bar{y}) \pm c\sqrt{\frac{s_X^2}{n} + \frac{s_Y^2}{m}}\right)$; with c from pivot $\text{N}(0,1)$.

Normal, Two Means, Unknown σ's, Common Variance

$\left((\bar{x}-\bar{y}) \pm c \cdot s_P\sqrt{\frac{1}{n} + \frac{1}{m}}\right)$; with c from pivot t_{n+m-2}.

$$s_P = \sqrt{\frac{(n-1)s_X^2 + (m-1)s_Y^2}{n+m-2}}$$

Normal, Two Means, Unknown σ's, $\neq$ Variances

$\left((\bar{x}-\bar{y}) \pm c\sqrt{\frac{s_X^2}{n} + \frac{s_Y^2}{m}}\right)$; with c from pivot t_r.

$$r = \left(\frac{s_X^2}{n} + \frac{s_Y^2}{m}\right)^2 \Big/ \left(\frac{s_X^4}{n^2(n-1)} + \frac{s_Y^4}{m^2(m-1)}\right)$$

Normal, Paired Samples

For pairs (X_i, Y_i), let $D_i = X_i - Y_i$. $D_i \sim \text{N}(\mu_D, \sigma_D^2)$.

$\left(\bar{d} \pm c\frac{s_d}{\sqrt{n}}\right)$; with c from pivot t_{n-1}.

Estimating Variance

Normal, Single Variance - estimate of σ^2 not σ!

$\left(\frac{(n-1)s^2}{b}, \frac{(n-1)s^2}{a}\right)$; with a, b from pivot χ^2_{n-1}.

Normal, Two Variances

A confidence interval for the ratio of the variances σ_X^2/σ_Y^2 is $\left(a \cdot \frac{s_x^2}{s_y^2}, b \cdot \frac{s_x^2}{s_y^2}\right)$; with a, b from pivot $\text{F}_{m-1,n-1}$.

Estimating Proportions (Large n - Normal Approx.)

Single Proportion

$\approx \left(\hat{p} \pm c\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$; with c from pivot $\text{N}(0,1)$.

Two Proportions

$\approx \left(\hat{p_1} - \hat{p_2} \pm c\sqrt{\frac{\hat{p_1}(1-\hat{p_1})}{n} + \frac{\hat{p_2}(1-\hat{p_2})}{n}}\right)$; c from pivot $\text{N}(0,1)$.

Can also derive a confidence interval from the exact distribution of the binomial RV $n \cdot \hat{p} \sim \text{Bi}(n,p)$.

Prediction Intervals

Let X^* be a future realisation of X.

If $X \sim \text{N}(\mu,\sigma^2), \bar{X} \sim \text{N}(\mu, \frac{\sigma^2}{n})$, and $(\bar{X} - X^*) \sim \text{N}(\mu, \sigma^2 + \frac{\sigma^2}{n})$.

$\left(\bar{x} \pm c\sqrt{s^2 + \frac{s^2}{n}}\right)$ is a prediction interval with c from t_{n-1}.

If σ is known, use pivot $\text{N}(0,1)$. If μ is known use χ^2_{n-1}.

Pivots

Sampling from a normal distribution:

- $Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim \text{N}(0,1)$
- $T = \frac{\bar{X}-\mu}{s/\sqrt{n}} \sim t_{n-1}$
- $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$

Regression

Ordinary Least Square (OLS) Estimators

All parameters are unbiased and except $\hat{\sigma}^2$ are normally distributed (Variances as below).

Let $K = \sum_{i=1}^n (x_i - \bar{x})^2$ and $D^2 = \sum_{i=1}^n (Y_i - \hat{\alpha}_0 - \hat{\beta}(x_i - \bar{x}))^2$:

$$\hat{\alpha}_0 = \bar{Y}; \quad \hat{\beta} = \frac{\sum_{i=1}^n (x_i - \bar{x})Y_i}{\sum_{i=1}^n (x_i - \bar{x})^2}; \quad \hat{\alpha} = \hat{\alpha}_0 - \hat{\beta}\bar{x}; \quad \hat{\sigma}^2 = \frac{D^2}{n-2}$$

$$\text{Var}(\hat{\alpha}) = \left(\frac{1}{n} + \frac{\bar{x}^2}{K}\right)\sigma^2; \quad \text{Var}(\hat{\beta}) = \frac{\sigma^2}{K}; \quad \text{Var}(\hat{\alpha}_0) = \frac{\sigma^2}{n}$$

$$\text{Cov}(\hat{\alpha}_0, \hat{\beta}) = 0; \quad \text{Var}(\hat{\mu}(x)) = \left(\frac{1}{n} + \frac{(x-\bar{x})^2}{K}\right)\sigma^2$$

Regression Pivots

$$\frac{(n-2)\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{n-2}; \quad \frac{\hat{\beta} - \beta}{\hat{\sigma}/\sqrt{K}} \sim t_{n-2}; \quad \frac{\hat{\mu}(x) - \mu(x)}{\hat{\sigma}\sqrt{\frac{1}{n} + \frac{(x-\bar{x})^2}{K}}} \sim t_{n-2}$$

Bayesian Inference

Law of Total Probability

$f(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{-\infty}^{\infty} f(x \mid y)f(y)dy$

Normal Distribution, Known σ, Inference for μ

$X_1, \ldots, X_n \sim \text{N}(\mu, \sigma^2)$. Let $Y = \bar{X} \sim \text{N}(\mu, \sigma^2/n)$.

Prior: $\mu \sim \text{N}(\mu_0, \sigma_0^2)$

Posterior: $f(\mu \mid y) \propto f(y \mid \mu)f(\mu) \propto \exp\left[-\frac{(\mu-\mu_1)^2}{2\sigma_1^2}\right]$

Where:

$$\mu_1 = \frac{\frac{\mu_0}{\sigma_0^2} + \frac{y}{\sigma^2/n}}{\frac{1}{\sigma_0^2} + \frac{1}{\sigma^2/n}}; \quad \frac{1}{\sigma_1^2} = \frac{1}{\sigma_0^2} + \frac{1}{\sigma^2/n}$$

So $\mu \mid y \sim \text{N}(\mu_1, \sigma_1^2)$

Binomial Distribution, Beta / Uniform Prior

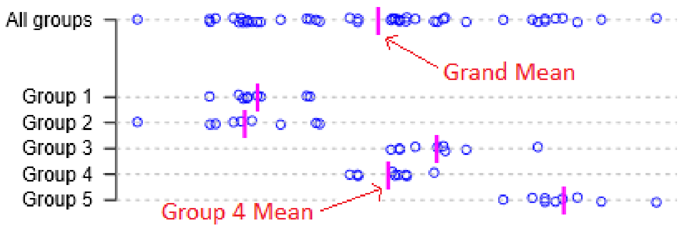
$X \sim \text{Bi}(n, \theta)$. Prior: $\theta \sim \text{Beta}(\alpha, \beta)$

Posterior: $\theta \mid x \sim \text{Beta}(\alpha + x, \beta + n - x)$

ANOVA

All populations assumed to have equal variance.
Total Sample Size: $n = n_1 + \cdots + n_k$
Group Means: $\bar{X}_{i\cdot} = \frac{1}{n_i}\sum_{j=1}^{n_i} X_{ij}$
Grand Mean: $\bar{X}_{\cdot\cdot} = \frac{1}{n}\sum_{i=1}^{k}\sum_{j=1}^{n_i} X_{ij} = \frac{1}{n}\sum_{i=1}^{k} n_i \bar{X}_{i\cdot}$

Single Factor (One Way) ANOVA



Hypotheses: $H_0 : \mu_1 = \mu_2 = \cdots = \mu_k$ vs. $H_1 : \bar{H}_0$
Sum of Squares:
Treatment SS:
$SS(T) = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(\bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot})^2 = \sum_{i=1}^{k} n_i(\bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot})^2$
Error SS:
$SS(E) = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(X_{ij} - \bar{X}_{i\cdot})^2 = \sum_{i=1}^{k}(n_i - 1)S_i^2$
Total SS:
$SS(TO) = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(X_{ij} - \bar{X}_{\cdot\cdot})^2 = SS(T) + SS(E)$
Null Distributions of Sum of Squares Terms:

$$\frac{SS(T)}{\sigma^2} \sim \chi^2_{k-1}; \quad \frac{SS(E)}{\sigma^2} \sim \chi^2_{n-k}; \quad \frac{SS(TO)}{\sigma^2} \sim \chi^2_{n-1}$$

Test Statistic (F-test):

$$F = \frac{SS(T)/(k-1)}{SS(E)/(n-k)}$$

Under H_0, $F \sim \text{F}_{k-1,n-1}$, so since the numerator will be larger when H_0 is false reject H_0 for $F > c$.

Two Factor (Two Way) ANOVA

Assume $X_{ij} \sim N(\mu_{ij}, \sigma^2)$. $\mu_{ij} = \mu + \alpha_i + \beta_j$. $\sum_{i=1}^{a}\alpha_i = 0$ and $\sum_{j=1}^{b}\beta_j = 0$. Take one observation from each combination of a and b ($n = ab$).
Hypotheses: $H_{0A} : \alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$ vs. $H_1 : \bar{H}_0$
or $H_{0B} : \beta_1 = \beta_2 = \cdots = \beta_k = 0$ vs. $H_1 : \bar{H}_0$
Means:

$$\bar{X}_{\cdot\cdot} = \frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b} X_{ij}; \quad \bar{X}_{i\cdot} = \frac{1}{b}\sum_{j=1}^{b} X_{ij}; \quad \bar{X}_{\cdot j} = \frac{1}{a}\sum_{i=1}^{a} X_{ij}$$

Sum of Squares:
$SS(A) = b\sum_{i=1}^{a}(\bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot})^2$
$SS(B) = a\sum_{j=1}^{b}(\bar{X}_{\cdot j} - \bar{X}_{\cdot\cdot})^2$
Error SS:
$SS(E) = \sum_{i=1}^{a}\sum_{j=1}^{b}(X_{ij} - \bar{X}_{i\cdot} - \bar{X}_{\cdot j} + \bar{X}_{\cdot\cdot})^2$
Total SS:
$SS(TO) = SS(A) + SS(B) + SS(E)$
Test Statistics (F-test):
The test statistics for H_{0A} and H_{0B} are:

$$F = \frac{SS(A)/(a-1)}{SS(E)/(a-1)(b-1)} \quad \text{or} \quad \frac{SS(B)/(b-1)}{SS(E)/(a-1)(b-1)} > c$$

With c from $\text{F}_{a-1,(a-1)(b-1)}$ or $\text{F}_{b-1,(a-1)(b-1)}$ respectively.

Two Factor ANOVA with Interaction Terms

Take samples over two factors, but c samples for each factor pair. $\mu_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij}$.
Hypotheses: $H_{0A} : \alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$ vs. $H_1 : \bar{H}_0$
or $H_{0B} : \beta_1 = \beta_2 = \cdots = \beta_k = 0$ vs. $H_1 : \bar{H}_0$
or $H_{0AB} : \gamma_{ij} = 0 \; \forall \, i, j$ vs. $H_1 : \bar{H}_0$
Means:

$$\bar{X}_{ij\cdot} = \frac{1}{c}\sum_{k=1}^{c} X_{ijk}; \quad \bar{X}_{i\cdot\cdot} = \frac{1}{bc}\sum_{j=1}^{b}\sum_{k=1}^{c} X_{ijk}$$

$$\bar{X}_{\cdot j\cdot} = \frac{1}{ac}\sum_{i=1}^{a}\sum_{k=1}^{c} X_{ijk}; \quad \bar{X}_{\cdot\cdot\cdot} = \frac{1}{abc}\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c} X_{ijk}$$

Sum of Squares:
$SS(A) = bc\sum_{i=1}^{a}(\bar{X}_{i\cdot\cdot} - \bar{X}_{\cdot\cdot\cdot})^2$
$SS(B) = ac\sum_{j=1}^{b}(\bar{X}_{\cdot j\cdot} - \bar{X}_{\cdot\cdot\cdot})^2$
$SS(AB) = c\sum_{i=1}^{a}\sum_{j=1}^{b}(\bar{X}_{ij\cdot} - \bar{X}_{i\cdot\cdot} - \bar{X}_{\cdot j\cdot} + \bar{X}_{\cdot\cdot\cdot})^2$
Error SS:
$SS(E) = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}(X_{ijk} - \bar{X}_{ij\cdot})^2$
Total SS:
$SS(TO) = SS(A) + SS(B) + SS(AB) + SS(E)$
Test Statistics (F-test):
The test statistics for H_{0A} and H_{0B} are:

$$F = \frac{SS(A)/(a-1)}{SS(E)/ab(c-1)} \quad \text{or} \quad \frac{SS(B)/(b-1)}{SS(E)/ab(c-1)} > c$$

With c from $\text{F}_{a-1,ab(c-1)}$ or $\text{F}_{b-1,ab(c-1)}$ respectively.
And for H_{0AB} with c from $\text{F}_{(a-1)(b-1),ab(c-1)}$:

$$F = \frac{SS(AB)/(a-1)(b-1)}{SS(E)/ab(c-1)} > c$$

Mean Squares / Mean Square Error

MSE is found by dividing the $SS(E)$ by its degrees of freedom.
$\hat{\sigma}^2 = MS(E)$ is an unbiased estimator for the variance.

Goodness-of-fit Test (χ^2)

O_i is the Observed count of data in class i, E_i is the Expected count under H_0. Be careful to use the count and not the proportion. For k classes:

$$Q_{k-1} = \sum_{i=1}^{k}\frac{(O_i - E_i)^2}{E_i} \approx \chi^2_{k-1}$$

Since the numerator $(O_i - E_i)^2$ will be larger when H_0 is false, we reject H_0 for $Q_{k-1} > c$, with c from χ^2_{k-1}.
Two Classes ($k = 2$)
Testing $H_0 : p = p_1$ vs. $H_1 : p \neq p_1$
More than Two Classes ($k > 2$)
Let $p_1, \ldots, p_k$ define the proportions of a categorical distribution. H_0 : "$p_1, \ldots, p_k$ do define the distribution" vs. $H_1 : \bar{H}_0$

Other Formulae

Sample Variance
$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{n-1}\left(\left(\sum_{i=1}^{n} x_i^2\right) - n\bar{x}^2\right)$
Cramér-Rao Lower Bound
The CR Lower Bound is the minimum possible variance of an estimator $\hat{\theta}$ of θ. It is the asymptotic variance of the MLE.

$$\ell(\theta) = \ln L(\theta); \quad U(\theta) = \frac{\partial \ell}{\partial \theta} \quad \text{(Score Function)}$$

$$V(\theta) = -\frac{\partial U}{\partial \theta}; \quad I(\theta) = \mathbb{E}(V(\theta)) \quad \text{(Fisher Information)}$$

$$\text{Var}(\hat{\theta}) \geq \frac{1}{I(\theta)} \quad \text{(CR Lower Bound)}$$

Gamma Function in Beta + γ Distributions
$\Gamma(n) = (n-1)!$