Putnam A3

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February 2026

A3. Alice and Bob play a game with a string of n digits, each of which is restricted to be 0, 1, or 2. Initially all the digits are 0. A legal move is to add or subtract 1 from one digit to create a new string that has not appeared before. A player with no legal move loses, and the other player wins. Alice goes first, and the players alternate moves. For each $n \geq 1$, determine which player has a strategy that guarantees winning.

Answer. For every $n \geq 1$, **Player 2 (Bob)** has a winning strategy.

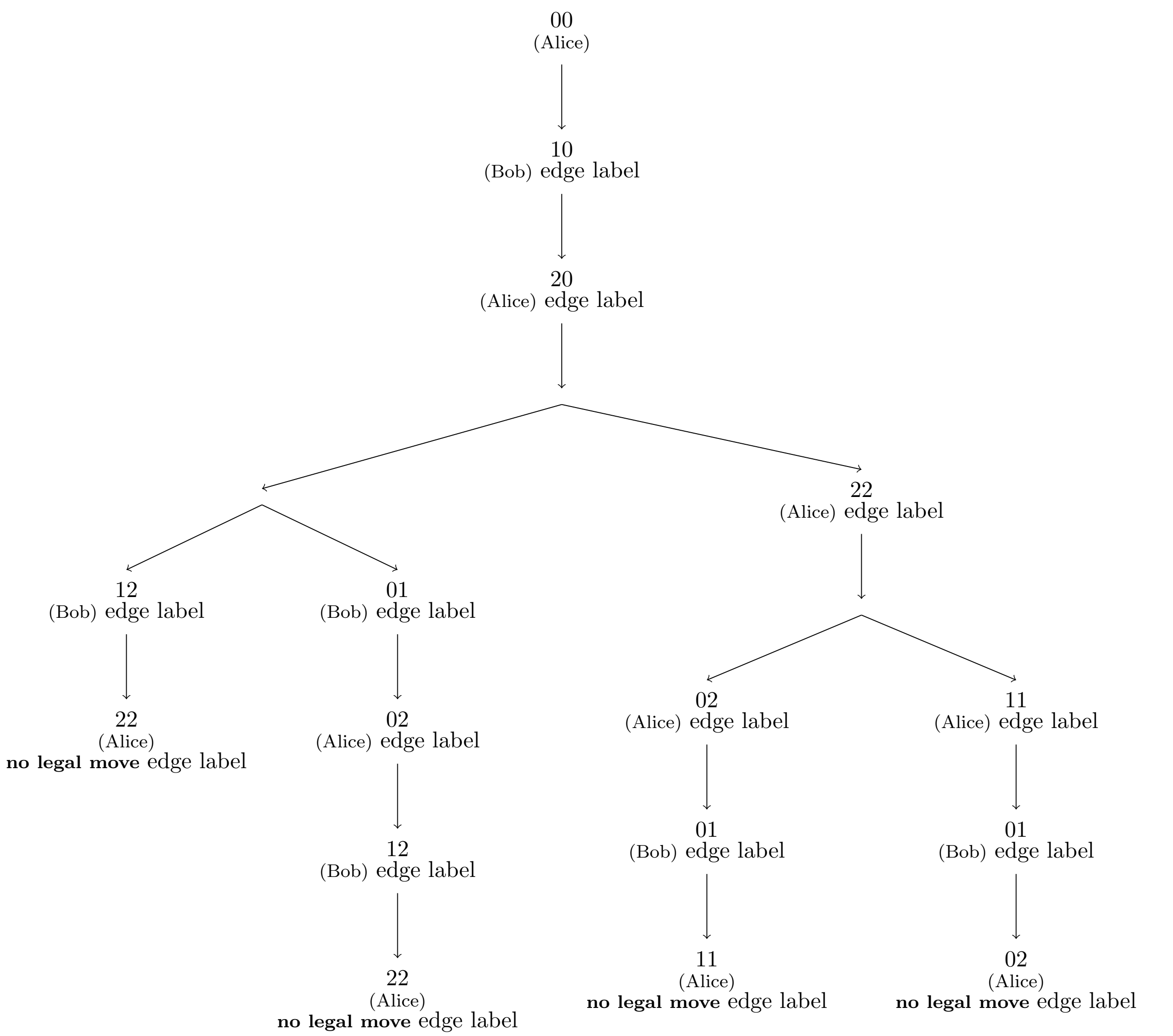
Base case $n = 1$.

$$\text{P1: } 0 \to 1, \qquad \text{P2: } 1 \to 2,$$

and then P1 has no legal move (the only neighbor is 1, already used), so **P2 wins**.

Decision tree for $n = 2$ (showing half, by symmetry).

We start at (00). By symmetry, it suffices to show the case where Alice's first move is $(00) \to (10)$ (the case $(00) \to (01)$ is identical under swapping the two coordinates).



Every leaf shown is a position where *Alice* has no legal move, so Bob wins in this entire (representative) half.

Graph theory proof for general n.

Let G_n be the graph whose vertex set is

$$V = \{0, 1, 2\}^n$$

and where two strings are adjacent iff they differ by adding or subtracting 1 in exactly one coordinate. The game starts at

$$s = (0, 0, \ldots, 0),$$

and each move is a step along an edge to a *new* (previously unvisited) vertex.

Consider the set

$$V \setminus \{s\} = \{0, 1, 2\}^n \setminus \{(0, \ldots, 0)\}.$$

For each $v = (v_1, \ldots, v_n) \in V \setminus \{s\}$, let

$$i(v) = \min\{k : \ v_k \neq 0\},$$

and define $\mu(v)$ by flipping that first nonzero coordinate between 1 and 2:

$$\mu(v)_k = \begin{cases} v_k, & k \neq i(v), \\ 2, & k = i(v) \text{ and } v_{i(v)} = 1, \\ 1, & k = i(v) \text{ and } v_{i(v)} = 2. \end{cases}$$

Then $\mu(\mu(v)) = v$ and $v \sim \mu(v)$ (they differ by ± 1 in exactly one position), so the edges $\{v, \mu(v)\}$ form a **perfect matching** of $V \setminus \{s\}$.

Bob's strategy (pairing/matching strategy). Whenever Alice moves to a new vertex $v \neq s$, Bob replies by moving to $\mu(v)$.

This reply is always legal:

- $\mu(v)$ is adjacent to v, so the move exists in G_n;
- $\mu(v)$ cannot have been visited earlier without v also having been visited earlier (they are paired uniquely), so since v is new, $\mu(v)$ is new as well.

Hence Bob always has a move after every Alice move. Because V is finite, eventually someone gets stuck; it cannot be Bob, so it must be Alice. Therefore **Bob wins for all** $n \geq 1$.

Diagrams of G_n for $n = 1, 2, 3$.

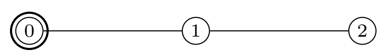


Figure 1: G_1: vertices $\{0, 1, 2\}$ with edges between numbers differing by ± 1. Start vertex is 0.

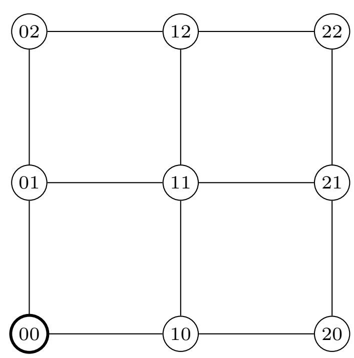


Figure 2: $G_2 \cong P_3 \square P_3$: the 3×3 grid on vertices $\{0, 1, 2\}^2$. Start vertex is 00.

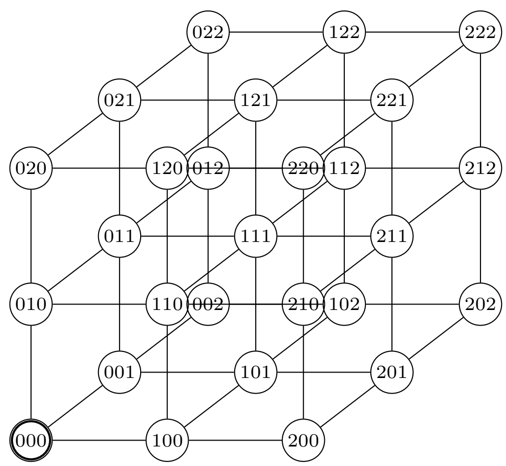


Figure 3: $G_3 \cong P_3 \Box P_3 \Box P_3$: a $3 \times 3 \times 3$ grid drawn as three offset layers. Start vertex is 000.