Favorite Putnam Problems

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2025 - A3 Alice and Bob play a game with a string of n digits, each of which is restricted to be 0, 1, or 2. Initially all the digits are 0. A legal move is to add or subtract 1 from one digit to create a new string that has not appeared before. A player with no legal move loses, and the other player wins. Alice goes first, and the players alternate moves. For each $n \geq 1$, determine which player has a strategy that guarantees winning.

Answer. For every $n \geq 1$, **Player 2 (Bob)** has a winning strategy.

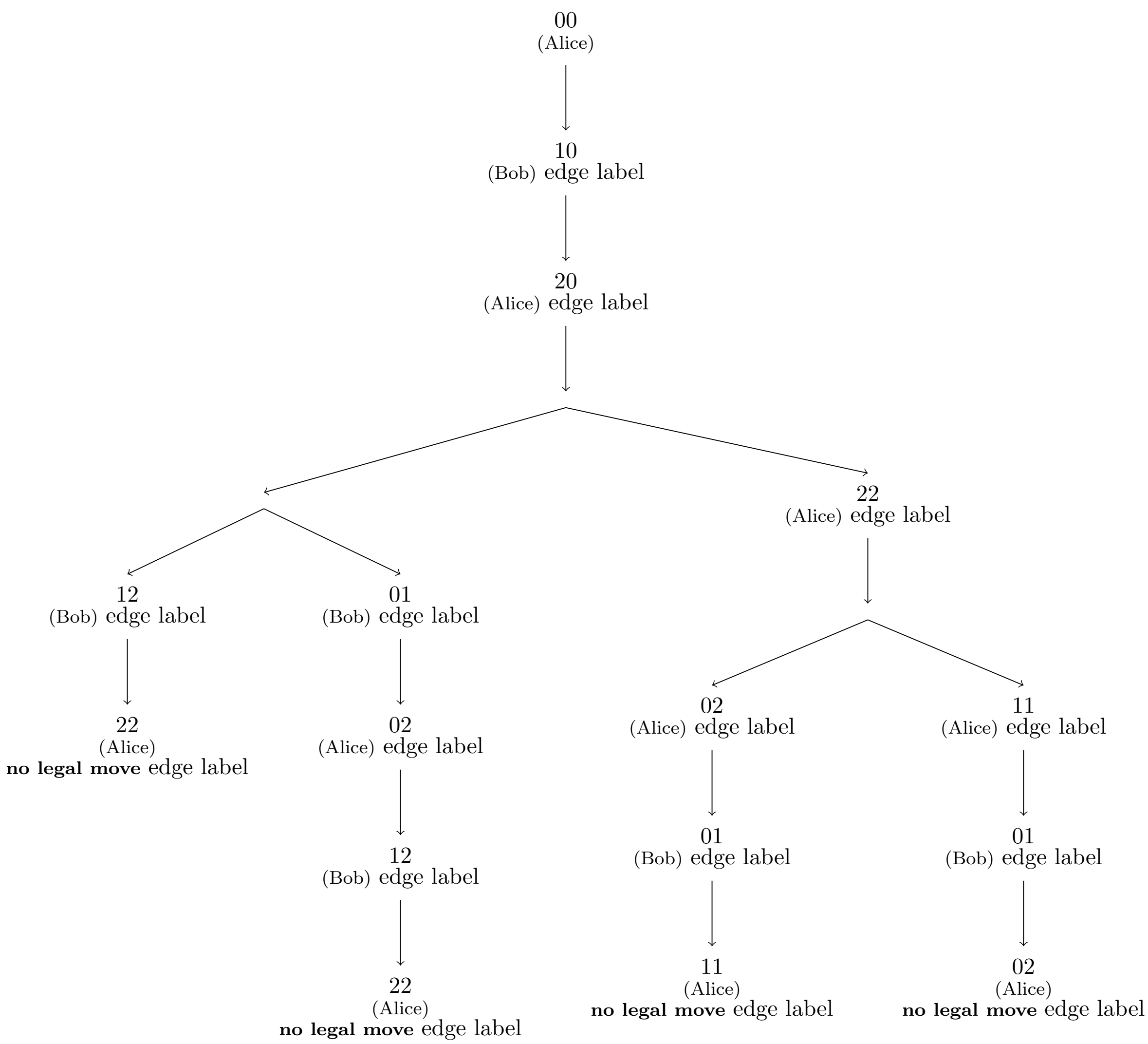
Base case $n = 1$.

$$\text{P1: } 0 \to 1, \qquad \text{P2: } 1 \to 2,$$

and then P1 has no legal move (the only neighbor is 1, already used), so **P2 wins**.

Decision tree for $n = 2$ (showing half, by symmetry).

We start at (00). By symmetry, it suffices to show the case where Alice's first move is $(00) \to (10)$ (the case $(00) \to (01)$ is identical under swapping the two coordinates).



Every leaf shown is a position where *Alice* has no legal move, so Bob wins in this entire (representative) half.

Graph theory proof for general n.

Let G_n be the graph whose vertex set is

$$V = \{0, 1, 2\}^n$$

and where two strings are adjacent iff they differ by adding or subtracting 1 in exactly one coordinate. The game starts at

$$s = (0, 0, \dots, 0),$$

and each move is a step along an edge to a *new* (previously unvisited) vertex.

Consider the set

$$V \setminus \{s\} = \{0, 1, 2\}^n \setminus \{(0, \dots, 0)\}.$$

For each $v = (v_1, \dots, v_n) \in V \setminus \{s\}$, let

$$i(v) = \min\{k : \ v_k \neq 0\},$$

and define $\mu(v)$ by flipping that first nonzero coordinate between 1 and 2:

$$\mu(v)_k = \begin{cases} v_k, & k \neq i(v), \\ 2, & k = i(v) \text{ and } v_{i(v)} = 1, \\ 1, & k = i(v) \text{ and } v_{i(v)} = 2. \end{cases}$$

Then $\mu(\mu(v)) = v$ and $v \sim \mu(v)$ (they differ by ± 1 in exactly one position), so the edges $\{v, \mu(v)\}$ form a **perfect matching** of $V \setminus \{s\}$.

Bob's strategy (pairing/matching strategy). Whenever Alice moves to a new vertex $v \neq s$, Bob replies by moving to $\mu(v)$.

This reply is always legal:

- $\mu(v)$ is adjacent to v, so the move exists in G_n;
- $\mu(v)$ cannot have been visited earlier without v also having been visited earlier (they are paired uniquely), so since v is new, $\mu(v)$ is new as well.

Hence Bob always has a move after every Alice move. Because V is finite, eventually someone gets stuck; it cannot be Bob, so it must be Alice. Therefore **Bob wins for all** $n \geq 1$.

Diagrams of G_n for $n = 1, 2, 3$.

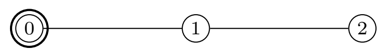


Figure 1: G_1: vertices $\{0, 1, 2\}$ with edges between numbers differing by ± 1. Start vertex is 0.

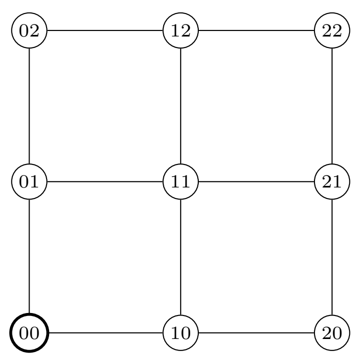


Figure 2: $G_2 \cong P_3 \square P_3$: the 3×3 grid on vertices $\{0, 1, 2\}^2$. Start vertex is 00.

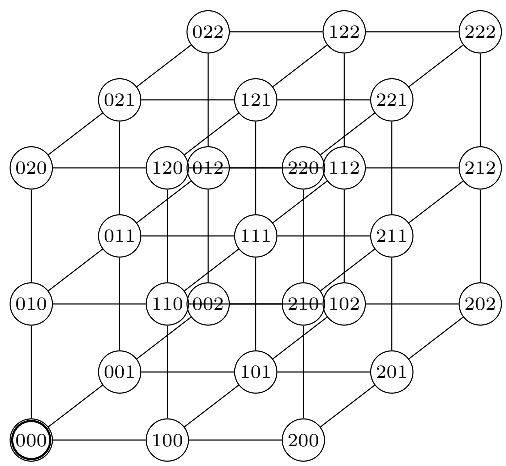


Figure 3: $G_3 \cong P_3 \square P_3 \square P_3$: a $3 \times 3 \times 3$ grid drawn as three offset layers. Start vertex is 000.

1968 - A1 Prove that

$$\frac{22}{7} - \pi = \int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx.$$

Solution. Expand the numerator:

$$x^4(1-x)^4 = x^4 - 4x^5 + 6x^6 - 4x^7 + x^8.$$

A quick polynomial division by $1+x^2$ gives the identity

$$x^4(1-x)^4 = \big(x^6 - 4x^5 + 5x^4 - 4x^2 + 4\big)(1+x^2) - 4.$$

Hence

$$\frac{x^4(1-x)^4}{1+x^2} = \big(x^6 - 4x^5 + 5x^4 - 4x^2 + 4\big) - \frac{4}{1+x^2}.$$

Integrating term-by-term on $[0,1]$,

$$\int_0^1 \big(x^6 - 4x^5 + 5x^4 - 4x^2 + 4\big)\,dx = \left[\frac{x^7}{7} - \frac{4x^6}{6} + \frac{5x^5}{5} - \frac{4x^3}{3} + 4x\right]_0^1 = \frac{22}{7},$$

and

$$\int_0^1 \frac{4}{1+x^2}\,dx = 4\big[\arctan x\big]_0^1 = 4\cdot\frac{\pi}{4} = \pi.$$

Therefore,

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx = \frac{22}{7} - \pi,$$

as desired.

1992 - B1 Let S be a set of n distinct real numbers. Let A_S be the set of numbers that occur as averages of two distinct elements of S. For a given $n \geq 2$, what is the smallest possible number of elements in A_S?

Answer. The minimum possible value of $|A_S|$ is $2n-3$.

Solution (extremes / "one end of the average set"). Write the elements in increasing order:

$$x_1 < x_2 < \cdots < x_n.$$

Look at the "left end" of the average set. The smallest possible averages must come from pairing the smallest element x_1 with the others:

$$\frac{x_1+x_2}{2},\ \frac{x_1+x_3}{2},\ \ldots,\ \frac{x_1+x_n}{2}.$$

These are strictly increasing (since the x_i are distinct), so they contribute $n-1$ **distinct** elements of A_S.

Now look at the "right end". Pair the largest element x_n with the others:

$$\frac{x_1+x_n}{2},\ \frac{x_2+x_n}{2},\ \ldots,\ \frac{x_{n-1}+x_n}{2}.$$

These are also strictly increasing, hence give $n-1$ **distinct** averages.

Crucially, these two blocks overlap in at most one place:

- Every average $\frac{x_1+x_k}{2}$ with $k<n$ is *strictly less* than $\frac{x_1+x_n}{2}$.
- Every average $\frac{x_k+x_n}{2}$ with $k>1$ is *strictly greater* than $\frac{x_1+x_n}{2}$.

So the only possible common element is the midpoint of the extremes $\frac{x_1+x_n}{2}$. Therefore

$$|A_S| \;\geq\; (n-1)+(n-1)-1 = 2n-3.$$

Sharpness. Take S to be an arithmetic progression, e.g.

$$S = \{0, 1, 2, \ldots, n-1\}.$$

Then every average has the form $\frac{i+j}{2}$ with $0 \leq i < j \leq n-1$, so $i+j$ ranges over all integers from 1 to $2n-3$. Hence A_S is exactly the set

$$\left\{\frac{1}{2}, 1, \frac{3}{2}, \ldots, n-\frac{3}{2}\right\},$$

which has $2n-3$ elements. So the minimum is indeed $2n-3$.

2016 - B1 Let $x_0, x_1, x_2, \ldots$ be the sequence such that $x_0 = 1$ and for $n \geq 0$,

$$x_{n+1} = \ln\bigl(e^{x_n} - x_n\bigr)$$

(as usual, ln is the natural logarithm). Show that the infinite series $x_0 + x_1 + x_2 + \cdots$ converges and find its sum.

Answer. The series converges and

$$\sum_{n=0}^{\infty} x_n = e - 1.$$

Solution (telescoping after exponentiating). Define

$$y_n := e^{x_n} \qquad (n \geq 0).$$

Exponentiating the recursion gives

$$y_{n+1} = e^{x_{n+1}} = e^{\ln(e^{x_n} - x_n)} = e^{x_n} - x_n = y_n - x_n,$$

so

$$x_n = y_n - y_{y_{n+1}}.$$

Therefore the partial sums telescope:

$$\sum_{k=0}^{N} x_k = \sum_{k=0}^{N}(y_k - y_{k+1}) = y_0 - y_{N+1} = e^{x_0} - e^{x_{N+1}} = e - e^{x_{N+1}}.$$

It remains to show $x_{N+1} \to 0$. First, $x_0 = 1 > 0$, and if $x_n > 0$ then $e^{x_n} - x_n > 1$ (since $e^t > 1 + t$), so $x_{n+1} = \ln(e^{x_n} - x_n) > 0$. Also, for $x_n > 0$ we have $e^{x_n} - x_n < e^{x_n}$, hence

$$x_{n+1} = \ln(e^{x_n} - x_n) < \ln(e^{x_n}) = x_n.$$

So (x_n) is positive and strictly decreasing, hence converges to some $L \geq 0$. Taking limits in

$$x_{n+1} = \ln(e^{x_n} - x_n)$$

gives

$$L = \ln(e^L - L) \quad\Longrightarrow\quad e^L = e^L - L \quad\Longrightarrow\quad L = 0.$$

Thus $x_{N+1} \to 0$, so $e^{x_{N+1}} \to 1$, and the telescoping identity yields

$$\sum_{n=0}^{\infty} x_n = \lim_{N\to\infty}\bigl(e - e^{x_{N+1}}\bigr) = e - 1.$$