Mathematical Explanation

The equation for the probability to roll a particular set of six numbers is

$$\boldsymbol{P}_{set} = \frac{6!}{(r_1)!\,(r_2)!\,(r_3)!} \prod_{i=1}^{6} P\left(n_i\right)$$

where $\{n_1, n_2, n_3, n_4, n_5, n_6\}$ denote the six numbers rolled, $P(n_i)$ denotes the *independent* probably to obtain number n_i, and $\{r_1, r_2, r_3\}$ are factors that I'll call "repeating factors".

Statistical Mechanics

In general, to obtain the total probability of something being in a certain "state", you need to count the total number of "microstates" that result in that state (or rather that "macrostate"). The probability for your system to be in a particular macrostate is then just the number of microstates resulting in that macrostate divided by the total number of all possible microstates your system can be in. This fact arises from the fundamental axiom of statistical mechanics: all accessbile microstates are equally likely (which, in a way, is how you can define what a "microstate" is).

For example, suppose I am flipping a coin twice and tracking the total number of times I get tails.
I'll choose to define my macrostates based on how many tails I get.
In this example there are 3 macrostates: I get 2 tails, 1 tails, or no tails.
However, there are 4 total microstates for this system:
tails first then another tails $[T, T]$, tails first then heads $[T, H]$, heads first then tails $[H, T]$, heads first then another heads $[H, H]$.

You may have noticed that some of these outcomes will amount to the same number of tails (ex: $[T, H]$ and $[H, T]$). However, even though they are the same in terms of what I said I was tracking (ie in terms of the "macrostate" as I have chosen to define it), each of these outcomes really represents two distinct realities. In one, I flipped and got heads first. In the other, I got tails first then heads. They are different scenarios. This is exactly the concept this example is meant to show. Different "microstates" can correspond to the same "macrostate".

To summarize, each microstate corresponds to a macrostate like so:
2 tails: $[T, T]$
1 tail: $[T, H]$, $[H, T]$
0 tails: $[H, H]$

Because there is 1 microstate (out of 4) that corresponds to getting "2 tails", the odds of getting 2 tails is $\frac{1}{4}$.
Because there is 2 microstate (out of 4) that corresponds to getting "1 tails", the odds of getting 1 tails is $\frac{2}{4}$.
And because there is 1 microstate (out of 4) that corresponds to getting "0 tails", the odds of getting no tails is $\frac{1}{4}$.
So in summary...

$$P\left(2\ tails\right) = 0.25$$

$$P\left(1\ tails\right) = 0.5$$

$$P\left(0\ tails\right) = 0.25$$

Counting Microstates

In summary to last section, each macrostate can have multiple microstates and we need to know how many microstates there are and to which macrostates they correspond to in order to find probabilities.

Going back to ability scores: if we take, for example, $[9, 10, 11, 12, 13, 16]$ and $[12, 9, 13, 11, 10, 16]$. These sets of scores are two different microstates corresponding to the same macrostate (because, like in D&D, we'll say that the order in which you roll the numbers doesn't matter). Really, each rearranging of these 6 numbers is a seperate microstate for the same macrostate. So, we should ask: how many different ways can we rearrange a set of six numbers. It turns out the answer is the factorial of 6.

Based on this, you might think that the equation for the probability to get a macrostate is

$$"\boldsymbol{P}_{set} = 6! \prod_{i=1}^{6} P(n_i)"$$

but it turns out, that is incorrect. Because there isn't *always* 6! ways to rearrange a set of six number; 6! is just the maximum. It is less if there are repeating numbers.

As an extreme case, take for example the set $[13, 13, 13, 13, 13, 13]$. If we ask: "how many ways are there to roll six 13s?". The answer is, of course, one. There is only one way to roll six 13s: you roll them one after the other until you get six of them. In other words, this macrostate has only 1 microstate.

What about $\{9, 13, 13, 13, 13, 13\}$? Well, there are six ways to roll this set of numbers; one for each position in which the "9" can be in.

It turns out the number of ways you can roll a set with repeating numbers is equal to

$$\frac{6!}{r!}$$

where r is the amount of repeated numbers in your set. You can check with the previous two cases that this now gives the correct answers ($\frac{6!}{6!} = 1$ and $\frac{6!}{5!} = 6$).

The Repeating Factors

In a set of 6 numbers, there can be (at most) 3 subsets of repeating numbers (ie the case where you have 3 repeats of twos, like $\{5, 5, 7, 7, 14, 14\}$.

You need to divide by $r!$ (a "repeating factor") for each of these repeats.

So the "number of ways you can rearrange", which we know is really also the number of microstates for the same macrostate, is equal to

$$\Omega = \begin{cases} \frac{6!}{(r_1)!} & \text{if there is 1 repeat} \\ \frac{6!}{(r_1)!(r_2)!} & \text{if there are 2 repeats} \\ \frac{6!}{(r_1)!(r_2)!(r_3)!} & \text{if there are 3 repeats} \end{cases}$$

In pseudocode, this can be computed like so

```
r1,r2,r3 = 1

for i=3..18 {
        r = Number_Set.count(i) //count the amount of i in the number set
        if r > 1 {
                r3 = r2
                r2 = r1
                r1 = r
        }
}

Omega = 6! / r1! / r2! / r3!
```