A motivic approach to equivariant synthetic spectra

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Background on Synthetic spectra

Let $\mathrm{Pure}(C_1)$ be the full subcategory of Sp on retracts of extensions of even-dimensional spheres. We then define the category of (even, MU-based) synthetic spectra ([Pst23a])

$$\mathrm{Syn} \subset \mathrm{Fun}^{\oplus}(\mathrm{Pure}(C_1)^{\mathrm{op}}, \mathrm{Sp})$$

as the full subcategory on additive functors that preserve (co)fibre sequences that already existed in $\mathrm{Pure}(C_1)$. This category is relevant for the following reasons:

- It carries a sheafy t-structure $\tau_{c\geq\star}$ such that
$$\mathrm{fil}^{\mathrm{ev}}_{\star}\mathbb{1} = \Gamma(\mathbb{1}; \tau_{c\geq\star}\mathrm{map}(-, \mathbb{1}))$$
gives rise to a filtration that approximates $\mathbb{1}$ by ring spectra with even homotopy groups.
- The spectral sequence associated to the aforementioned filtration is precisely the *décalage* of the Adams–Novikov filtration on $\mathbb{1}$, as this filtered spectrum is $\mathrm{Tot}\tau_{\geq 2\star}\mathrm{MU}^{\otimes\bullet+1}$.
- As such, it is intimately related to descent along $\mathbb{1} \to \mathrm{MU}$: MU can be given a cell structure with even dimensional cells coming from the even dimensional cells in BU, and furthermore has homotopy groups concentrated in even degrees.

Background on motivic spectra

Many fundamental invariants of smooth schemes over a base scheme B satisfy the following properties:

1. descent for the Nisnevich topology,
2. $\mathbb{A}^1$-homotopy invariance: their value on X is the same as on $\mathbb{A}^1 \times X$,
3. projective bundle formulas that relate their value on $\mathbb{P}^1 \times X$ and X.

These are therefore naturally represented in the category of motivic spectra over B defined as

$$\mathrm{SH}(B) = \mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_B; \mathrm{Ani})_*[(\mathbb{P}^1)^{\otimes -1}].$$

There exists an equivariant analogue of this construction ([Hoy17]); if G is a sufficiently nice group scheme we may consider the site of smooth B-schemes with G-actions and construction

$$\mathrm{SH}^G(B) = \mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}^G_B; \mathrm{Ani})_*[(\mathbb{P}(\mathcal{E}))^{\otimes -1} \mid \mathcal{E} \in \mathrm{Vect}^G_B]$$

which one can equivalently think of as an extension of SH from schemes to sufficiently nice stacks including $[B/G]$. Note that $\mathrm{SH}(B)$ naturally has a bigraded family of spheres $S^{t,w} = \Sigma^{t-w}\mathbb{G}_m^w$.

Motivic and synthetic spectra

In the nonequivariant setting, there is an equivalence ([Pst23b])

$$\mathrm{Syn}_p \simeq \mathrm{SH}(\mathbb{C})^{\mathrm{cell}}_p$$

for any prime p, where the cellular category is the one generated by bigraded spheres. This allows us to connect computations in the motivic stable homotopy category over $\mathbb{C}$ to the Adams–Novikov spectral sequence and is an essential tool in computations of stable stems. The essential ingredients are the following three results.

1. Let $\mathrm{Pure}(\mu_1)$ denote the full subcategory on retracts of extensions of spheres $S^{2n,n} = (\mathbb{P}^1)^{\otimes n}$, then one can identify
$$\mathrm{SH}(\mathbb{C})^{\mathrm{cell}} \subset \mathrm{Fun}^{\oplus}(\mathrm{Pure}(\mu_1)^{\mathrm{op}}, \mathrm{Sp})$$
as the full subcategory on functors satisfying the same condition as in the definition of Syn.
2. The key to proving this is that there exists a motivic lift MGL of MU such that it also admits a cell structure with cells in $\mathrm{Pure}(\mu_1)$ and furthermore
$$\pi_{2w,w}\mathrm{MGL} \cong \pi_{2w}\mathrm{MU}, \qquad \pi_{2w-k,w}\mathrm{MGL} = 0$$
for all $k > 0$.
3. For an arbitrary prime p, the functor $\mathrm{SH}(\mathbb{C}) \to \mathrm{Sp}$ induced by $X \mapsto X(\mathbb{C})^{\mathrm{an}}$ induces an equivalence
$$\mathrm{Pure}(\mu_1)/p \xrightarrow{\sim} \mathrm{Pure}(C_1)/p.$$

Equivariant algebraic cobordism

We construct $\mathrm{MGL}_G \in \mathrm{SH}^G(\mathbb{C})$ as the universal Thom spectrum, i.e. the colimit the Thom spectra of all rank zero equivariant virtual bundles over all smooth proper $\mathbb{C}$-schemes with G-action (cf. [BH21]):

$$\mathrm{MGL}_G \simeq \varinjlim_{\alpha \in \mathcal{K}^\circ(X)} \mathrm{Th}_X(\alpha) \simeq \varinjlim_{d,i} \Sigma^{-2d,-d}\mathrm{Th}_{\mathrm{Gr}_d(V_i)}(Q^i_d).$$

- This satisfies a sort of pre-global functoriality in the group: if $f\colon [\mathbb{C}/H] \to [\mathbb{C}/G]$ is a map of quotient stacks, there exists a map
$$\alpha_f\colon f^*\mathrm{MGL}_G \to \mathrm{MGL}_H$$
which is an equivalence if f is representable (e.g. corresponds to a subgroup inclusion), this is enough to make the collection
$$\mathrm{Lattices} \cong \mathrm{Tori}^{\mathrm{op}} \to \mathrm{CRing}, T^n \mapsto \pi^{T^n}_{*,*}\mathrm{MGL}_{T^n}$$
into a **global group law** in the sense of [Hau22].
- If G is embeddable, say μ_n or GL_1, then we can construct cell structures on the Graßmannians –and therefore on MGL_G– using only cells of the form $\mathrm{Th}(V)$ for V a vector bundle on $[\mathbb{C}/G]$ (i.e. a complex G-representation).
- The Graßmannian model endows this global group law with strong regularity properties which allow us to conclude
$$\pi^{\mu_n}_{2w,w}\mathrm{MGL}_{\mu_n} \cong \pi^{C_n}_{2w}\mathrm{MU}_{C_n}, \qquad \pi^{\mu_n}_{2w-k,w}\mathrm{MGL}_{\mu_n} = 0$$
for all $k < 0$.

Equivariant synthetic spectra and reconstruction

In the equivariant world, the correct replacement for the notion of having an even cell structure is that of having a cell structure by **complex** representation spheres. We therefore define $\mathrm{Pure}(C_n) \subset \mathrm{Sp}^{C_n}$ to be the full subcategory on retracts of extensions of objects of the form $S^V \otimes (C_n/C_d)_+$ with V a complex representation, and set

$$\mathrm{Syn}^{C_n} \subset \mathrm{Fun}^{\oplus}(\mathrm{Pure}(C_n)^{\mathrm{op}}, \mathrm{Sp})$$

as usual. We can easily deduce the following.

- Syn^{C_n} admits a t-structure $\tau_{c\geq\star}$ coming from Sp giving rise to a filtered spectrum
$$\mathrm{fil}^{\mathrm{ev}}_{\star}\mathbb{1}_{C_n} = \Gamma(\mathbb{1}_{C_n}; \tau_{c\geq\star}\mathrm{map}(-, \mathbb{1}_{C_n})).$$
- More generally, one can evaluate at other orbits and representation spheres to obtain a functor
$$\mathrm{fil}^{\mathrm{ev}}_{\star}\colon \mathrm{Sp}^{C_n} \to (\mathrm{Sp}^{C_n})^{\mathrm{RU}(C_n)^{\leq}}, X \mapsto \{\Gamma((C_n/C_d)_+, \tau_{c\geq V}\mathrm{map}(-, X))\}_{d|n}$$
which recovers the equivariant Adams–Novikov filtration.

On the motivic side, motivated by the cell structure on MGL_{μ_n}, we define $\mathrm{Pure}(\mu_n)$ to be the full subcategory of $\mathrm{SH}^{\mu_n}(\mathbb{C})$ on retracts of extensions of objects of the form $\mathrm{Th}(V) \otimes (\mu_n/\mu_d)_+$ and $\mathrm{SH}^{\mu_n}(\mathbb{C})^{\mathrm{cell}}$ to be the subcategory generated under colimits by these.

1. The vanishing result in MGL_{μ_n} formally gives us an equivalence
$$\mathrm{Mod}(\mathrm{SH}^{\mu_n}(\mathbb{C})^{\mathrm{cell}}; \mathrm{MGL}_{\mu_n}) \simeq \mathrm{Fun}^{\oplus}(\mathrm{Pure}(\mu_n)^{\mathrm{op}} \otimes \mathrm{MGL}_{\mu_n}, \mathrm{Sp})$$
cf. the **weight structure** on MGL-motives ([BKWX22], [Bon10]).
2. The cell structure on MGL_{μ_n} allows us to descend this to an identification
$$\mathrm{SH}^{\mu_n}(\mathbb{C})^{\mathrm{cell}} \subset \mathrm{Fun}^{\oplus}(\mathrm{Pure}(\mu_n)^{\mathrm{op}}, \mathrm{Sp})$$
with the full subcategory on functors satisfying the same condition as in the definition of Syn, cf. the **heart structure** of [HP23].
3. Using isotropy separation and the t-structure obtained from the expression above, one can prove that taking complex points induces an equivalence
$$\mathrm{Pure}(\mu_n)/p \xrightarrow{\sim} \mathrm{Pure}(C_n)/p$$
4. Using our computation of $\pi^{\mu_n}_{2*,*}\mathrm{MGL}_{\mu_n}$ we may deduce from this that both categories of perfect pure objects as well as their classes of cofibre sequences are equivalent mod p, so there is an equivalence
$$\mathrm{Syn}^{C_n}_p \simeq \mathrm{SH}^{\mu_n}(\mathbb{C})^{\mathrm{cell}}_p$$

Why do we care?

- As in the nonequivariant setting, categorifying the equivariant Adams–Novikov spectral has many **computational** applications.
- This reveals a deep connection between the notion of **evenness** and algebrogeometric phenomena over $\mathbb{C}$: the torsion information in the pure Tate spheres in (equivariant) motivic spectra over $\mathbb{C}$ is determined by the complex spheres in (equivariant) spectra.
- The even filtration, which in particular generalises the motivic filtrations of [BMS19] and [BL22], naturally lives in Syn. Cyclotomic spectra can be fruitfully ([DHRY25]) described in terms of objects of $\mathrm{Sp}^{C_{p^\infty}}$, and their **equivariant even filtration** naturally lives in $\mathrm{Syn}^{C_{p^\infty}}$.
- Equivariant motivic spectra admit a notion of **equivariant norms** ([Bac22a]), hence so do equivariant synthetic spectra under this equivalence. The Adams–Novikov filtration on the equivariant sphere spectrum can therefore be equipped with a highly structured equivariant multiplicative refinement.
- At more interesting groups, similar equivalences give rise to chromatic refinements of the **derived geometric Satake equivalence** ([Dev23]).

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