Six-Functor Formalisms

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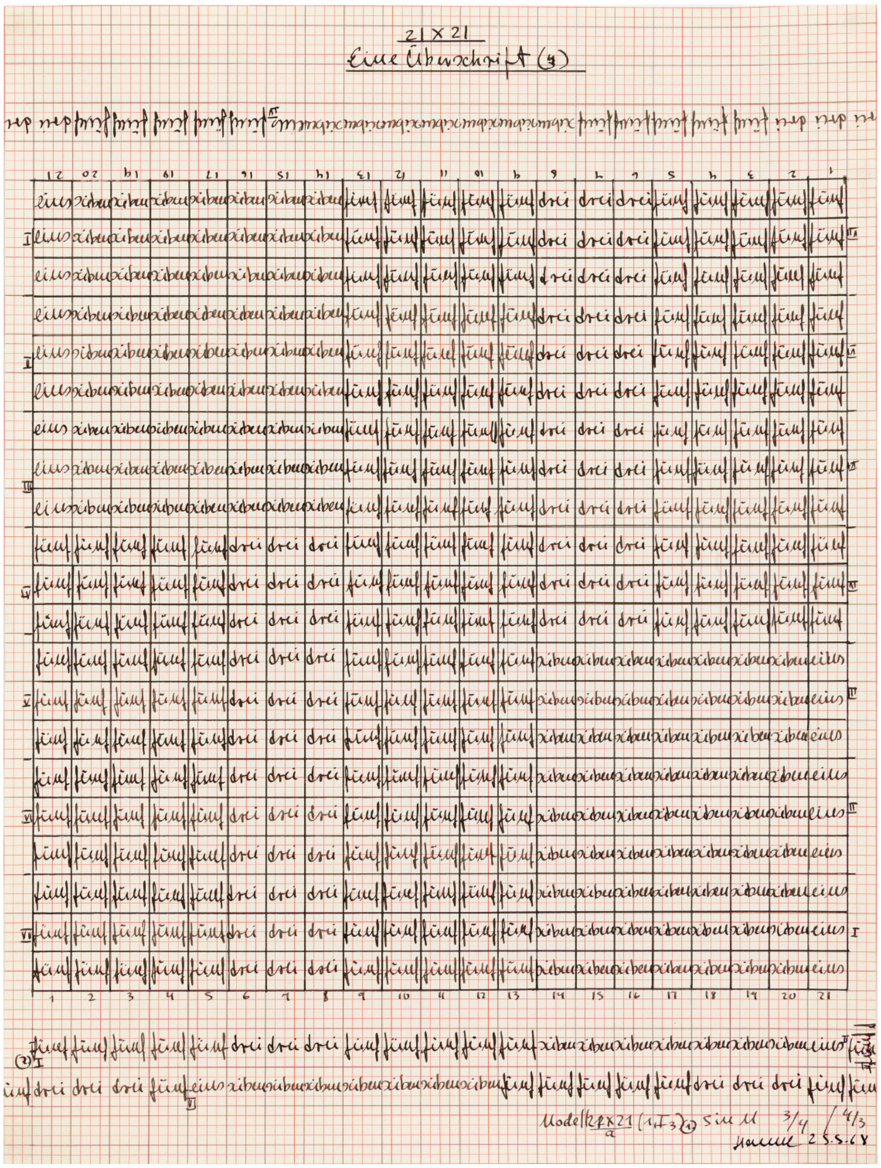

Hanne Darboven, *Eine Überschrift*, 1968

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Schedule

The seminar meets at TBD on TBD in Room TBD.

Date		**Topic**	**Speaker**
TBD	1	Correspondences and Six-Functor Formalisms	TBD
TBD	2	Category of Kernels I	TBD
TBD	3	Category of Kernels II	TBD
TBD	4	Six-Functor Formalism on Condensed Anima	TBD
TBD	5	Applications to Representation Theory	TBD
TBD	6	Sheaves and Cosheaves	TBD
TBD	7	Verdier Duality	TBD
TBD	8	Six-Functor Formalism on Topological Spaces	TBD
TBD	9	Analytic Manifolds	TBD
TBD	10	Étale sheaves on Light Condensed Anima	TBD
TBD	11	Six-Functor Formalism for Étale sheaves	TBD
TBD	12	Duality for p-adic Lie Groups	TBD
TBD	13	Chromatic Applications	TBD

Schedule of talks

Introduction

The theory of six-functor formalisms provides us with a formidable incarnation of the celebrated *yoga of six operations* envisioned by Grothendieck and others. The use of the word *yoga* in this context is subtle and often nebulous, but we hope to highlight in what sense the theory of six-functor formalisms ought to be seen as a culmination of the principle it represents. The word *yoga* is borrowed from the Sanskrit term **योग**, itself a nominal form of the root √**युज्**. Within Sanskrit, the noun *yoga* is almost impossible to translate in a conventional sense: its manifold meanings range from practical matters such as *conjunction* and *connection*[1] to astrological meanings like *asterism*[2] and spiritual connotations like *devotion* or *union of the individual soul with the outer soul*. We hope to convince the participant that this *yoga* of six operations allows the inititate to mentally unify a plethora of seemingly disparate yet fundamental mathematical phenomena by placing them under a common yoke where they become "interlinked in countless ways, both subtle and obvious, akin to how the different themes of a singular and vast counterpoint are interlinked, intertwined in their deployment, while each remains clearly recognizable. [...] together in a harmony that carries them forward and breathes meaning into each in turn, in the form of a movement and plenitude to which all others contribute. [...] issued from this vaster harmony and to be reborn within it moment after moment, rather than the harmony appearing as the "sum" or "result" of preexisting constituent themes."[3] As such, the formal aspects of the theory appear always in harmony with the concrete problems it seeks to unify; neither subjugating the original problems to a cannibalistic *raison d'être* for a hegemonic formal theory, nor casting away the yoke of categorical reasoning for narrow-minded analytic manipulations. Not only will the deep and naturally arising mathematical problems covered in this seminar attain liberation by practicing this esoteric *yoga*, we hope to equip the initiate with the necessary tools and expertise to make this *yoga* into a second nature in which the combined force of all six functors can drive a mathematical problem effortlessly towards its resolution and all that is left to do is to identify and interpret the asterisms produced by this grand celestial mechanism.

[1] In fact, the word *yoga* is etymologically cognate with the English word *yoke*: a wooden tool affixed to the backs of draft animals in order to increase their combined workload and keep them on track.

[2] Perhaps fitting, regarding the occurence of upper and lower stars and the process of divination surrounding their seemingly fortuitous alignments

[3] [Gro21, p. 23]

Overview

The basic goal of this seminar is to learn the basics of six-functor formalisms, how to construct them in practice, and what they might be helpful for. Aside from the formal setup, we will thoroughly discuss some key examples where the need for six-functor formalisms arises naturally, and what kind of geometric or categorical considerations are required to feed these problems into the formal machine of six-functor formalisms. Additionally, some work is required to interpret the output of the formalism and relate it back to the original problem at hand.

- In the first part, we will go over the foundations of how six-functor formalisms can be constructed starting from the datum of a geometric setup and a putative notion of derived category. In practice, it is usually clear how to obtain some basic pullback functoriality, but the technology of suitable decompositions allows us to formalise the process of extending this to three (and six) functors by checking some geometric conditions. Furthermore, imposing suitable descent conditions allows us to formally extend six-functor formalisms from geometric categories to stacks formed on these geometric objects. We will then cover the notion of the category of kernels: this categorical construction turns out to be at the heart of the matter and allows us to make sense of *suave* and *prim* objects in a six-functor formalism which satisfy certain duality properties. We will illustrate this with examples coming from condensed mathematics and relate it to smooth representation theory.
- The second part is concerned with the example of six functors on locally compact Hausdorff spaces with general coefficients. The construction of this six-functor formalism is logically independent of the first section, but reveals a different approach to constructing a fully coherent six-functor formalism using *Verdier duality*. In order to establish the latter, we will introduce the technology of cosheaves and $\mathcal{K}$-sheaves on locally compact Hausdorff spaces. As an application, we can deduce some classical facts such as relative Atiyah duality for manifolds. The lecture notes from Juan Esteban Rodríguez Camargo [Cam25, Part II] might be useful for this part of the seminar.
- In the third part, we turn our attention toward yet another example coming from the theory of p-adic Lie groups. After setting up some geometric properties of analytic manifolds, as well as the theory of étale sheaves on light condensed anima, we can use the formalism of the first part to obtain a six-functor formalism for the latter. Within this six-functor formalism we obtain a notion of smooth and proper morphisms, and obtain formal duality results, e.g. for p-adic Lie groups. Using the notion of *paths* in tandem with the geometric setup we can in fact be more explicit about these dualising objects. We end with a discussion of how this powerful formal setup allows us to recontextualise some fundamental results in chromatic homotopy theory.

Syllabus

Talk 1: Correspondences and Six-Functor Formalisms

Motivate and define six-functor formalisms. Give an overview of some of the fundamental things they can be used for e.g. Poincaré duality, projection formulæ, etc. Discuss how to construct six-functor formalisms from a suitable decomposition (I, P) of the allowed forward maps E and discuss how to extend six functor formalisms to a stacky setup. You should omit the example on condensed anima which will be discussed in Talk 4 but feel free to mention other examples.
References: [HM24, Sections 1–3]

Talk 2: Category of Kernels I

Define the category of kernels and introduce suave and prim maps and objects. Discuss the relation to duality.
References: [HM24, Sections 4.1–4.5]

Talk 3: Category of Kernels II

Discuss the notion of étale and proper maps in a six-functor formalism as a continuation of the discussion on suave and prim maps. Discuss exceptional descent, taking time to cover the notion of descendability following Mathew and illustrate this notion with some classical examples.
References: [HM24, Sections 4.6–4.8], [Mat16].

Talk 4: Six-Functor Formalism on Condensed Anima

Give a crash course introduction to condensed mathematics and then construct a six functor formalism on condensed anima. Discuss this gadget using the material from the previous talks and give applications to topological spaces.
References: [HM24, Sections 3.5, 4.8]

Talk 5: Applications to Representation Theory

Discuss how the six functor formalisms (particularly the one on condensed anima) can be applied to smooth representation theory.
References: [HM24, Section 5]

Talks 6: Sheaves and Cosheaves

Set up the background material on tensor products of cocomplete ∞-categories and the notion of (co)sheaves. Discuss relative shape theory for topoi and how this can be used to state and prove a smooth projection formula for shape submersions.
References: [Vol21, Sections 2–3]

Talks 7: Verdier Duality

Discuss the localisation sequences of categories of sheaves associated to open-closed decompositions, and how these give rise to recollements. Proceed to discuss Verdier duality for sheaves by introducing the notion of $\mathcal{K}$-sheaves.
References: [Vol21, Sections 4–5]

Talks 8: Six-Functor Formalism on Topological Spaces

Bring together the previous results by setting up the full six-functor-formalism of sheaves on topological spaces and discuss the usual formulæ: base change, projection, and Künneth. Use this to set up a J-homomorphism and prove relative Atiyah duality.
References: [Vol21, Sections 6–7]

Talk 9: Analytic Manifolds

Discuss some preliminaries on the theory of analytic manifolds, culminating in the construction of the deformation to the tangent bundle.
References: [Cla25, Sections 2–3]

Talks 10: Étale Sheaves on Light Condensed Anima

Discuss the theory of étale sheaves on light condensed anima, and relate it to the theory of sheaves on analytic manifolds. Discuss (relative) uniform pro-p-groups and their classifying stacks.
References: [Cla25, Sections 4–5]

Talks 11: Six Functors for Étale Sheaves

Set up the six functor formalism for étale sheaves on light condensed anima as an application of [HM24]. Proceed to characterise smooth morphisms of interest to us as well as the theory of *paths*.
References: [Cla25, Sections 6–9]

Talks 12: Duality for p-adic Lie Groups

Reap the fruits of the previous talks by setting up the theory of dualising objects for p-adic Lie groups. Discuss the more general framework for setting up J-homomorphisms in general six functor formalisms.
References: [Cla25, Sections 10–11]

Talks 13: Chromatic Applications

Discuss the applications of the previously established duality and linearisation results in chromatic homotopy theory.
References: [Cla25, Sections 12–14]

References

[Cam25] Juan Esteban Rodríguez Camargo. *Notes on D-modules via derived algebraic stacks*. 2025. URL: `https://drive.google.com/file/d/1HyZ3u7_Zq_nzrcn56BNZSqrdb8FJjzh7/view`.

[Cla25] Dustin Clausen. "Duality and linearization for p-adic lie groups". *arXiv preprint arXiv:2506.18174* (2025).

[Gro21] Alexandre Grothendieck. *Récoltes et semailles, I.* Gallimard TEL, 2021.

[HM24] Claudius Heyer and Lucas Mann. "6-Functor Formalisms and Smooth Representations". *arXiv preprint arXiv:2410.13038* (2024).

[Mat16] Akhil Mathew. "The Galois group of a stable homotopy theory". *Advances in Mathematics* 291 (2016), pp. 403–541.

[Vol21] Marco Volpe. "The six operations in topology". *arXiv preprint arXiv:2110.10212* (2021).