Motivic Precursors to Synthetic Spectra

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0 Conventions

Whenever any mention of τ or the (motivic) Adams(–Novikov) spectral sequence is present, we assume that everything is implicitly p-complete, and η-completed in the motivic case. Only the final comparison with synthetic spectra holds integrally, but we only obtain an equivalence on p-complete subcategories

1 Motivic spectra

Let k be a field. One can then consider the category of finite type smooth k-schemes Sm_k. The goal of the construction of motivic spaces and motivic spectra is to provide a natural home for (sufficiently nice) cohomology theories on smooth k-schemes. The leading examples for these are algebraic K-theory, Chow groups, étale cohomology and others. In fact, the word *motivic* comes from an idea due to Alexander Grothendieck that smooth k-schemes should embed into an abelian category of motives over k such that all sufficiently nice cohomology theories (called Weil cohomology theories) factor through this category of motives. The construction of this category of motives and its central object–motivic cohomology–is highly conjectural and many different variants have been constructed in an attempt to realise it. The variant we propose here is the motivic stable homotopy category $\mathcal{SH}(k)$ developed by Morel and Voevodsky[1]. The prime *motivation* for the category of motives and motivic cohomology is that it ought to contain information about algebraic K-theory as outlined in a host of conjectures, primarily the Beilinson–Lichtenbaum conjecture which predicts the existence of a sort of Atiyah–Hirzebruch spectral sequence from motivic cohomology to algebraic K-theory. The idea behind the construction of $\mathcal{SH}(k)$ is to apply methods from homotopy

[1] $\mathcal{SH}(B)$ can be defined in terms of Sm_B for any quasi-compact quasi-separated (qcqs) base scheme B, and in fact comes equipped with the structure of a six functor formalism with respect to morphisms of base schemes. The construction of motivic cohomology and determination of some motivic stable stems is far more convoluted over arbitrary bases, and we will only be preoccupied with the case $B = \mathbb{C}$, so we restrict to fields of characteristic zero as base schemes

theory to study smooth k-schemes. In particular, we want to have notions of Thom spaces, excision, representability, and homotopy invariance.

1.1 Motivic spaces

Let us begin with a classical story from the theory of smooth manifolds. Let Mfd be the category of smooth manifolds. Cohomology theories on smooth manifolds can be viewed as contravariant functors

$$F : \mathrm{Mfd}^{\mathrm{op}} \to \mathrm{An},$$

i.e. presheaves of anima on Mfd. We let them take values in anima, and can recover cohomology theories valued in graded groups by taking homotopy groups, but animated- unctors are more suited to representability results since phantom maps are less of a concern. We require that cohomology theories satisfy two reasonable axioms

- They ought to be homotopy invariant, i.e. for $F \in \mathscr{P}(\mathrm{Mfd})$ a cohomology theory, we want the projection $X \times I \to X$ to induce an equivalence
$$F(X) \xrightarrow{\sim} F(X \times I).$$
- They ought to satisfy some version of Mayer–Vietoris excision, i.e. descent along open covers.

If we let $\mathrm{Coh}(\mathrm{Mfd})$ denote the full subcategory of $\mathscr{P}(\mathrm{Mfd})$ on the presheaves F satisfying the two axioms above, this is the category of cohomology theories on manifolds we are interested in. Note that there is a functor

$$y\colon \mathrm{An} \to \mathrm{Coh}(\mathrm{Mfd})\colon X \mapsto \mathrm{map}(-, X),$$

where we consider the underlying anima of a manifold to obtain this restricted version of the Yoneda embedding. It turns out that we have defined anima and our notion of cohomology theories well enough for this to actually be an equivalence:

Theorem 1.1. *The map* $y : \mathrm{An} \to \mathrm{Coh}(\mathrm{Mfd})$ *is an equivalence of categories, with inverse given by* $F \mapsto F(*)$.

Proof. This is a nontrivial result, we will not give the proof but note that it follows rather immediately from using homotopy invariance to note that the value of a cohomology theory on (an open ball in) Euclidean space agrees with its value on a point, and using descent along sufficiently nice open covers of manifolds to deduce the value on all manifolds. □

Let us now attempt to replicate this for smooth k-schemes. Consider the presheaf category $\mathscr{P}(\mathrm{Sm}_k)$. Let us single out the presheaves with an appropriate notion of descent. The first idea that comes to mind might be descent with respect to the étale topology, as this is a good replacement for covering space theory in the category of schemes. Unfortunately, this topology is too fine: algebraic K-theory (notoriously) does not satisfy étale descent. There are variants of motivic homotopy theory based on the étale topology, but these will not be discussed.

To find the right topology, coarser than the étale topology, and sufficiently fine to contain the Zariski topology, we can once again look to the theory of smooth manifolds for inspiration. The correct topology is the Nisnevich topology, which we will not describe here but we refer instead to [Bra18] and [Hoy16], while Section 3.3 of [Hoy17] and Appendix C of [Hoy15a] provide a more detailed description.

Remark 1.2. The Nisnevich topology has a host of nice properties with respect to sheafification and cohomological dimension, but we will not preoccupy ourselves with these. The two most important checks are:

- There Nisnevich topology is finer than the Zariski topology and coarser than the étale topology:
$$\mathrm{Zar} \leq \mathrm{Nis} \leq \text{ét}.$$
- Algebraic K-theory satisfies Nisnevich descent ([TT07]).

Let us now turn out attention to the appropriate generalisation of homotopy invariance. It turns out the the appropriate interval object to invert is $\mathbb{A}^1$. One motivation is that algebraic K-theory of smooth schemes is $\mathbb{A}^1$-invariant, i.e. the projection $\mathbb{A}^1_X \to X$ induces

$$K(X) \xrightarrow{\sim} K(\mathbb{A}^1_X).$$

This is a highly nontrivial result due to work of Weibel and Thomason–Trobaugh. Additionally, we know that any smooth k-scheme admits an open subscheme U étale over $\mathbb{A}^m_k$, so that the latter is the appropriate generalisation of Euclidean space hence ought to be contractible. There is a localisation

$$L_{\mathbb{A}^1} : \mathscr{P}(\mathrm{Sm}_k) \to \mathscr{P}^{\mathbb{A}^1}(\mathrm{Sm}_k)$$

onto $\mathbb{A}^1$-invariant sheaves.

We have then assembled all ingredients for our category of cohomology theories on smooth k-schemes, also known as the category of motivic spaces over k. Using some admittedly dated notation, since we view it as an $(\infty, 1)$-category, we denote this by

$$\mathcal{H}(k) = \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_k) \cap \mathscr{P}^{\mathbb{A}^1}(\mathrm{Sm}_k).$$

By straightforward abstract nonsense this is a presentably symmetric monoidal ∞-category rigidly generated by (the Yoneda embeddings of) smooth k-schemes. It naturally admits a pointed variant[2]

$$\mathcal{H}(k)_* = \mathcal{H}(k) \otimes \mathrm{An}_*$$

using the tensor product in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$.

Let us highlight a particularly interesting construction in motivic spaces: namely motivic Thom spaces. It is now possible to entirely mimic the construction of Thom spaces in homotopy theory: for $V \to X$ a vector bundle over a smooth k-scheme X, define its Thom space to be

$$\mathrm{Th}_X(V) = V/(V \setminus X),$$

the colimit being computed in $\mathcal{H}(k)$, and X viewed as a subscheme of V using the zero section.

Lemma 1.3 ([MV99]). *There is an equivalence of motivic spaces*

$$\mathrm{Th}_X(V) \simeq \mathbb{P}(V \oplus \mathcal{O})/\mathbb{P}(V),$$

where $\mathcal{O}$ is a trivial rank one bundle.

As an example, we can compute the Thom space of the affine line over the point $\mathrm{Spec}(k)$

$$\mathrm{Th}_k(\mathbb{A}^1) = \mathbb{P}^1.$$

In analogy with the Thom space of a trivial bundle in homotopy theory this is our motivation for considering $\mathbb{P}^1$ as a motivic sphere. This is confirmed by the observation that $\mathbb{CP}^1 \simeq S^2$ on complex points. This is not exactly minimal, as the affine cover of $\mathbb{P}^1$ by two copies of $\mathbb{A}^1$ and intersection $\mathbb{A}^1 \setminus 0 = \mathbb{G}_m$ gives us a pushout square

$$\begin{tikzcd}
\mathbb{G}_m \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathbb{A}^1 \arrow[d] \\
\mathbb{A}^1 \arrow[r] & \mathbb{P}^1.
\end{tikzcd}$$

Since the affine line has been contracted, $\mathbb{A}^1 \simeq *$, we see that $\mathbb{P}^1$ can be identified with the suspension of $\mathbb{G}_m$, $\mathbb{P}^1 \simeq \Sigma\mathbb{G}_m$. Further, the complex points $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times \simeq S^1$ are the topological 1-sphere, so we define the motivic bigraded spheres using

$$S^{p,q} = S^{p-q} \wedge \mathbb{G}_m^q = \Sigma^{p-q}\mathbb{G}_m^q.$$

In this notation $\mathbb{G}_m = S^{1,0}$ while $\mathbb{P}^1 = S^{2,1}$. The interaction with topological spheres highlighted above can be quantified in terms of a symmetric monoidal left adjoint called the Betti realisation from motivic spaces over the field $\mathbb{C}$. There is a functor

$$\mathrm{Sm}_{\mathbb{C}} \to \mathrm{Mfd} \colon X \mapsto X(\mathbb{C})^{\mathrm{an}},$$

where a smooth $\mathbb{C}$-scheme is sent to its complex points, viewed with the analytic topology induced by embedding into some product of $\mathbb{C}$. This sends the affine line $\mathbb{A}^1_{\mathbb{C}}$ to $\mathbb{C}$ itself, so one can check that it induces a symmetric monoidal left adjoint

$$\mathrm{Re} \colon \mathcal{H}(\mathbb{C}) \to \mathrm{Coh}(\mathrm{Mfd}) \simeq \mathrm{An}$$

called the Betti realisation functor. It is then clear that

[2] Warning: We will largely avoid mentioning basepoints in the latter, even though they are present.

- $\mathrm{Re}(\mathbb{G}_m) = \mathbb{C}^\times \simeq S^1$,
- $\mathrm{Re}(\mathbb{P}^1) = \mathbb{CP}^1 \simeq S^2$,
- and more generally that $\mathrm{Re}(S^{p,q}) \simeq S^{p-q} \wedge S^q \simeq S^p$,

whence our choice of grading convention in the definition of bigraded motivic spheres.

1.2 Motivic spectra

Let us now stabilise our homotopy theory of motivic spaces in an appropriate sense to obtain a bigraded stable homotopy theory. Note that the usual stabilisation procedure will not suffice to obtain a bigraded stable homotopy theory; one must not only invert the topological sphere $S^{1,0}$ but also the geometric sphere $S^{1,1}$. This can be quantified using Thom spaces: recall that the Thom space of a rank n trivial bundle on an anima X recovers its n-th suspension. In fact, spectra are the universal presentable ∞-category in which one can take Thom spectra of virtual bundles i.e. formal differences of bundles. The motivic version of this statement departs from the observation that

$$\mathrm{Th}_k(\mathbb{A}^n) \simeq \mathbb{P}^n \simeq (\mathbb{P}^1)^{\wedge n}$$

Therefore, to extend Thom spaces to virtual bundles we ought to invert $\mathbb{P}^1$. Recall that $\mathbb{P}^1 \simeq S^{1,0} \wedge S^{1,1}$ so that we recover the desideratum above. The process of formally inverting these objects is nontrivial, and in fact Voevodsky was one of the first to formalise stabilisation procedures in which a compact object is inverted in a presentably symmetric monoidal ∞-category. The critical ingredient for this is that the cyclic permutation (123) acts by the identity on $(\mathbb{P}^1)^{\wedge 3}$, cf. Theorem 4.16 in [Rob12] and Theorem 4.3 in [Voe98]. The stabilisation of $\mathcal{H}(k)_*$ with respect to $S^{2,1}$ is then the stable and presentably symmetric monoidal category of motivic spectra over k denoted

$$\mathcal{SH}(k) = \mathcal{H}(k)_*[(\mathbb{P}^1)^{-1}].$$

The unit is the bigraded motivic sphere spectrum $S^{0,0}$. Further,this naturally comes equipped with a symmetric monoidal adjunction $\Sigma^\infty_{\mathbb{P}^1} \dashv \Omega^\infty_{\mathbb{P}^1}$. As usual, we do not denote stabilisation when it is obvious, so that associated to a smooth k-scheme X we obtain a motivic spectrum over k, also denoted X, as the $(\mathbb{P})^1$-suspension spectrum of its Yoneda embedding; this can be though of as *the motive* of the scheme X. Note that in the traditional literature on motivic homotopy (often referred to as $\mathbb{A}^1$-homotopy theory), motivic spectra are aptly referred to as $\mathbb{P}^1$-spectra. For formal reasons, the Betti realisation functor constructed above extends to the stable motivic homotopy category over $\mathbb{C}$, and lands in spectra:

$$\mathrm{Re} : \mathcal{SH}(\mathbb{C}) \to \mathrm{Sp}.$$

Remark 1.4. The programme outlined in these notes is to use motivic homotopy theory over $\mathbb{C}$ to obtain new information about spectra and the stable homotopy groups of spheres. When our base field is $\mathbb{R}$, we can still look at the $\mathbb{C}$-points of a smooth $\mathbb{R}$-scheme, but these will now admit an action of the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \simeq C_2$. In fact, Morel–Voevodsky extend this to an equivariant realisation

$$\mathrm{Re}^{C_2} : \mathcal{H}(\mathbb{R}) \to \mathrm{An}^{C_2}.$$

and this extends stably to a functor into *genuine* C_2-spectra

$$\mathrm{Re}^{C_2} : \mathcal{SH}(\mathbb{R}) \to \mathrm{Sp}^{C_2}$$

as explained in detail in [BS20]. This has been exploited to great effect in computations in C_2-equivariant homotopy theory, cf. [BGI20]

Equipped with these bigraded spheres, we can define a host of bigraded (co)homology theories on smooth k-schemes.

Definition 1.5. Let E and X be motivic spectra, then the bigraded E-homology resp. E-cohomology of X are defined as

$$E_{p,q}X = [S^{p,q}, E \otimes X], \qquad E^{-p,-q}X = [S^{p,q} \otimes X, E].$$

As usual, the homology theory associated to the motivic sphere spectrum defines the bigraded motivic homotopy groups $\pi_{p,q}$ of a motivic spectrum.

While this definition above is convenient for stable homotopy in motivic spectra, it is in fact not sufficient to capture all information about a motivic spectrum. Indeed, for formal reasons we see that $\mathcal{SH}(k)$ is rigidly compactly generated by all smooth k-schemes X and their bigraded suspensions. This means that equivalences in $\mathcal{SH}(k)$ are detected by

$$\pi_{p,q,U}(-) := [S^{p,q} \otimes U_+, -],$$

which–as U varies–defines a homotopy sheaf on the Nisnevich site of k. The definition of motivic homotopy groups above is recovered when one sets $U = \mathrm{Spec}(k)$. As homotopy theorists we can not afford to work with such complicated homotopy groups so we instead compromise on our motivic spectra

Definition 1.6. Let Sp_k be the full subcategory of $\mathcal{SH}(k)$ on objects rigidly generated by the motivic spheres $S^{p,q}$ for all $p, q \in \mathbb{Z}$. The objects of Sp_k are known as cellular spectra.

Remark 1.7. While it is clear that the more naïve homotopy groups $\pi_{p,q}$ detect equivalences on Sp_k, we have compromised on the arithmetic information in motivic spectra: the motives of general k-schemes might not be cellular at all. Fortunately for us, we are not interested in extracting algebraic information in these notes.

2 Motivic Cohomology and the Adams Spectral Sequence

One can construct a version of Eilenberg–MacLane spectra in Sp_k. These are supposed to represent motivic cohomology, which is a very delicate object, hence their construction and description is quite complicated. We take for granted that there exists a motivic spectrum called $\mathrm{M}\mathbb{F}_p$ over k that represents motivic cohomology with $\mathbb{F}_p$-coefficients. Its coefficient ring is denoted $\mathbb{M}_p$, and can be computed explicitly over certain fields.

2.1 Motivic cohomology over algebraically closed fields of characteristic zero

Theorem 2.1 (Voevodsky, [IØ18, Theorem 2.7]). *Suppose that k is of characteristic zero and contains a primitive p-th root of unity. Then one can identify*

$$\mathbb{M}_p(k) \cong \mathrm{K}^{\mathrm{M}}_*(k)/p[\tau],$$

where $\mathrm{K}^{\mathrm{M}}_n$ is the Milnor K-theory of k placed in degrees (n, n), and τ is a polynomial variable of degree $(0, -1)$.

The Milnor K-theory of a field k is defined as the group ring on the abelian group of units $k^\times$, generated by the symbols $[u]$, $u \in k^\times$ in degree one modulo the Steinberg relation

$$[u][1-u] = 0.$$

Note that the addition in this ring corresponds to multiplication in the units, i.e.

$$[a] + [b] = [ab].$$

Remark 2.2. This result is highly nontrivial and arises as a combination of several results, both of which depend heavily on the characteristic of the base field k (assumed zero in this note) and the characteristic of the coefficients (an arbitrary prime p). The motivating conjecture in the construction of motivic cohomology is the Beilinson–Lichtenbaum conjecture, which can be though of as a statement concerning the étale descent of p-complete algebraic K-theory over fields of characteristic zero[3]. It states that if $t \leq w$ are integers, then for all $X \in \mathrm{Sm}_k$,

$$\mathrm{H}^{t,w}(X; \mathbb{F}_p) \cong \mathrm{H}^t_{\text{ét}}(X; \mu_p^{\otimes w}).$$

In this equation, μ_p is the étale sheaf of p-th roots of unity, i.e. the fibre in the Kummer sequence of étale sheaves

$$0 \to \mu_p \to \mathbb{G}_m \xrightarrow{(-)^p} \mathbb{G}_m \to 0.$$

[3]This remark works more generally for k a field in which the prime p is invertible

In the more modern literature this is written as an equivalence of Nisnevich sheaves of complexes[4][5]

$$\mathbb{Z}/p(w) \simeq \tau^{\leq w} R\Gamma_{\text{ét}}(k; \mu_p^{\otimes w})$$

We refer the interested reader to the excellent overviews [MB] and [Hai] for more on this result. Voevodsky's key result was that the Beilinson–Licthenbaum conjecture is equivalent to the Bloch–Kato conjecture also known as the norm residue isomorphism theorem after Voevodsky proved it based on work of Rost. For more on this equivalence of conjectures, see Chapter 2 of [HW19]. The Bloch–Kato conjecture states that the the Galois symbol map

$$\partial : \mathrm{K}_*^{\mathrm{M}}(k)/p \to \mathrm{H}_{\text{ét}}^*(k; \mu_p^{\otimes *})$$

is an isomorphism. Note that the degree of the twist of the sheaf of roots of unity is the same as the topological degree in which we consider the étale cohomology, so this is a statement about the diagonal $t = w$ of the étale cohomology groups in the Beilinson–Lichtenbaum conjecture above. This Galois symbol map can be constructed naturally. Indeed, recall that the Kummer sequence above is an exact sequence of abelian group sheaves on the étale site of k (but not necessarily exact in the Zariski site!). The corresponding long exact sequence in étale cohomology has a connecting homomorphism of the form

$$\partial : \mathrm{H}_{\text{ét}}^0(k; \mathbb{G}_m) \to \mathrm{H}_{\text{ét}}^1(k; \mu_p).$$

The source can be identified with $\mathbb{G}_m(k) = k^\times$ since $\mathbb{G}_m$ is an étale sheaf. Now multiplicativity of étale cohomology allows us to tensor this up to a morphism

$$\partial : (k^\times)^{\otimes n} \to \mathrm{H}_{\text{ét}}^n(k; \mu_p^{\otimes n}).$$

This satisfies a relation called the Steinberg relation, which can be derived from Artin reciprocity in number theory, stating that any pure tensor of the form $\cdots \otimes u \otimes \cdots \otimes 1 - u \otimes \cdots$ in the domain is killed by the Galois symbol, where u is a unit not equal to zero or one. Further, note that the target of this morphims is naturally p-torsion since the additive structure comes from the multiplicative structure on μ_p. Further using the fact that the étale cohomology of k with values in the tensor powers of μ_p forms a ring, we can adjoin this to a map from the free tensor algebra on the source and factor the resulting map through the Steinberg relation and the quotient by p to obtain the map in the Bloch–Kato conjecture:

$$\partial : \mathbb{Z}[k^\times]/([u][1-u], p) = \mathrm{K}_*^{\mathrm{M}}(k)/p \to \mathrm{H}_{\text{ét}}^*(k; \mu_p^{\otimes *}).$$

Using the remark above, we see that the computation of the motivic cohomology of k should be though of as an explicit identification of the diagonal $t = w$ in terms of étale chomology or Milnor K-theory along with a vanishing result for $t > w$ and a periodicity result for $t < w$. The polynomial variable τ is then precisely there to indicate the periodicity in the latter, interpolating between different twists.

Remark 2.3. In the cases of interest to us, the mod p Milnor K-theory of a field is rather simple. If $k = \mathbb{C}$, we see that any unit can be written as a p-th power: $[u] = [(\sqrt[p]{u})^p] = p[\sqrt[p]{u}]$, so that

$$\mathrm{K}_*^{\mathrm{M}}(\mathbb{C})/p \cong \mathbb{Z}/p \cong \mathbb{F}_p.$$

The real numbers are analogous, except in the case $p = 2$ where the unit $[-1]$ is not a square. We denote this unit by ρ and obtain

$$\mathrm{K}_*^{\mathrm{M}}(\mathbb{R})/p \cong \mathbb{F}_p[\rho]$$

where $\rho = 0$ if $p \geq 3$.

Remark 2.4. This is where we first see evidence of our element τ of bidegree $(0, -1)$. In fact, this lies in the Hurewicz image of a class $\tau \in \pi_{0,-1}S$. The latter can be constructed explicitly in terms of a system of p^k-th roots of unity in iterated extensions of k, essentially analogous to Quillen's construction of the Bott element in the p-adic algebraic K-theory of $\overline{\mathbb{F}_\ell}$. This will be outlined below.

[4] This notation reveals that the result is prima facie actually a bit stronger: the motivic cohomology of k vanishes in the range $t > w$. Voevodsky however shows that this is equivalent to the identification above.

[5] Also implicit in this is the result that the bigraded motivic cohomology groups $\mathrm{H}^{t,w}(-:\mathbb{Z}/p)$ arise as the cohomology groups of a Nisnevich sheaf of complexes $\mathbb{Z}/p(w)$, i.e. $\mathrm{H}^{t,w}(-;\mathbb{Z}/p) \cong \mathrm{H}^t(-;\mathbb{Z}/p(w))$. While still conjectural over arbitrary bases, the existence of these complexes was confirmed by Voevodsky's construction over fields of characteristic zero. The second weight index is sometimes called the twist so that motivic complexes can be seen as twisted complexes in analogy with the étale Tate twists appearing in the right hand side of the Beilinson–Lichtenbaum conjecture

Let us now construct τ immeditaly in the stable stems. This construction is copied from [HKO11], Lemma 23. Analogously to Voevodsky's identification of $\mathbb{M}_p$ one can identify the motivic version of the zero stem. The correct analogue of degree zero is on the diagonal $t = w$. In that case, Morel and Hopkins in [Mor04] identify

$$\pi_{n,n} S \cong \mathrm{K}^{\mathrm{MW}}_{-n}(k),$$

where both sides are the degree (n, n) parts of a graded ring. The right hand side is the so-called Milnor–Witt K-theory of the field k, closely related to its Milnor K-theory. Indeed, it is generated by the units $k^\times$ in degree one, as well as a generator η in degree -1. These satisfy the relations

$$\eta[u] = [u]\eta, \qquad [u][1-u] = 0,$$
$$[uv] = [u] + [v] + \eta[u][v], \qquad \eta(2 + \rho\eta) = 0.$$

Remark 2.5. It is clear that $\mathrm{K}^{\mathrm{MW}}_*(k)/\eta \simeq \mathrm{K}^{\mathrm{M}}_*(k)$, so that one recovers an analogue of $(\pi_0 S)/p \simeq \pi_0 \mathbb{F}_p$ in the motivic world when one compares with $\mathbb{M}_p$ as above.

Now let $k = \mathbb{C}$ and consider a system of finite extensions L_k of $\mathbb{Q}$ such that L_k contains a primitive 2^k-th root of unity ζ_{2^k} but not a primitive 2^{k+1}'st root of unity.

It is clear that ζ_{2^k} is a unit in L, hence induces a class[6]

$$[\zeta_{2^k}] \in \mathrm{K}^{\mathrm{MW}}_1(L_k) \cong \pi^{L_k}_{-1,-1}.$$

One can check that not only $\eta(2 + \rho\eta) = 0$ from the definition of the Milnor–Witt K-theory groups, but also that

$$\rho(2 + \rho\eta) \equiv_\eta 0.$$

Indeed, the latter is given by

$$\begin{aligned} \rho(2 + \rho\eta) &= 2\rho + \rho^2\eta, \\ &= \rho + \rho + \eta[\rho][\rho], \\ &= [-1] + [-1] + \eta[-1][-1], \\ &= [(-1)^2], \\ &= [1]. \end{aligned}$$

Now note that per construction, $\zeta_{2^k}^{2^{k-1}} = -1$ in $L_k^\times$ so this gives us a relation in Milnor K-theory of L_k, i.e. Milnor–Witt K-theory of L_k mod η, of the form

$$2^{k-1}[\zeta_{2^k}] \equiv_\eta \rho.$$

Therefore, we obtain the relation

$$2^{k-1}(2 + \rho\eta)[\zeta_{2^k}] \equiv_\eta 0$$

Now $(2 + \rho\eta)$ is already η-torsion so this lifts to

$$2^{k-1}(2 + \rho\eta)[\zeta_{2^k}] = 0.$$

This means that $(2 + \rho\eta)[\zeta_{2^k}]$ in $\pi^{L_k}_{-1,-1}$ is 2^{k-1}-torsion hence lies in the image of the connecting homomorphism in the 2^{k-1}-Bockstein long eqxact sequence of homotopy groups, i.e. lifts to a class $\theta_k \in \pi^{L_k}_{0,-1}/2^{k-1}$. We conclude by using a result of Morel that the limit of these groups can be identified as

$$\lim_{k\to\infty} \pi^{L_k}_{0,-1}/2^{k-1} \simeq \pi_{0,-1},$$

where the stable stems are implicitly 2-complete. We then denote the object in the latter corresponding to the θ_k's by τ.

Remark 2.6. This element τ in $\mathbb{M}_p$ is purely motivic, indeed under Betti realisation $\mathbb{M}_2 \to \mathbb{F}_2$ it maps to one, since the Betti realisation map is described by Voevodsky as the composite

$$\mathrm{H}^{t,w}_{\mathrm{mot}}(\mathrm{Spec}(\mathbb{C}); \mathbb{F}_p) \to \mathrm{H}^t_{\text{ét}}(\mathrm{Spec}(\mathbb{C}) : \mu_p^{\otimes w}) \to \mathrm{H}^t_{\mathrm{sing}}(\mathrm{pt}; \mathbb{F}_p)$$

but the second map is an isomorphism by Riemann existence, while the first is an isomorphism for $t \leq w$ e.g. for $0 \leq 1$ in the case of τ so that τ must be sent to the generator 1 in $\mathrm{H}^0(\mathrm{pt}; \mathbb{F}_2) \cong \mathbb{F}_2$

[6] We temporarily decorate motivic stable stems with the base field they are computed over to avoid any confusion.

2.2 The motivic dual steenrod algebra and Adams spectral sequence

Having determined our coefficients, we now want to determine the dual motivic Steenrod algebra A_* (which as usual is easier to describe than the Steenrod algebra). It is once again computed explicitly by Voevodsky and given by

$$A_* \cong \begin{cases} \mathbb{M}_p[\tau_i, \xi_j \mid i \geq 0, j \geq 1]/(\tau_i^2 = 0) & p \geq 3, \\ \mathbb{M}_2[\tau_i, \xi_j \mid i \geq 0, j \geq 1]/(\tau_i^2 = \tau\xi_{i+1} + \rho(\tau_{i+1} + \tau_0\xi_{i+1})) & p = 2. \end{cases}$$

We see that this closely resembles the usual dual Steenrod algebra. Indeed, over $\mathbb{C}$ (which we now use throughout) we saw that $\mathbb{M}_p \cong \mathbb{F}_p[\tau]$ and $\rho = 0$ so that at odd primes clearly

$$A_* \cong A_*^{\mathrm{c}}[\tau],$$

while at even primes the ρ term in the relation dies and one can set $\xi_{i+1} = \tau_i^2/\tau$ to see that the mod 2 dual motivic Steenrod algebra is also just a polynomial algebra on the classical one in the variable τ.

Remark 2.7. The cohomology of the motivic Steenrod algebra is then a trigraded object

$$\mathrm{Ext}_A^{s,f,w}(\mathbb{M}_p, \mathbb{M}_p),$$

graded by stem, filtration, and weight.

Just as in spectra, one can construct an Adams resolution for the motivic sphere by $\mathrm{M}\mathbb{F}_p$, and obtain the motivic Adams spectral sequence (motAsseq)

$$E_2^{s,f,w} = \mathrm{Ext}_A^{s,f,w}(\mathbb{M}_p, \mathbb{M}_p) \implies \pi_{s,w}S.$$

In the motivic world we have to be a bit more careful with convergence. The $\mathrm{M}\mathbb{F}_p$-completion of the motivic sphere spectrum is equivalent to its (p, η)-completion where η is the algebraic Hopf map. Note that for degree reasons the class τ must survive to an element in the motivic stable stems.

Remark 2.8. This object η is a lift of the classic Hopf map $S^3 \to S^2$ to the motivic realm, namely as the canonical projection $\mathbb{A}^2 \setminus 0 \to \mathbb{P}^1$, i.e. a map $S^{3,2} \to S^{2,1}$. Beware that this is not a nilpotent element in the motivic stable stems; Nishida's nilpotence theorem fails motivically! In fact, (spoiler) since τ-inversion ought to recover the usual stable stems we will see that higher powers of η, while nonzero in the motivic stable stems, are τ-torsion. This leads to interesting complications when one attempts to extend chromatic phenomena to the motivic realm. The periodicity theorem does not hold motivically, but just as the quotient of S by its only non-nilpotent self map p admits a v_1-self map whose cofibre admits a v_2-self map etc, Miller conjectured that S/η admits a self map called w_1 which extends to a w-family of Smith–Toda complexes along with Morava K-theories $K(w_n)$. This is proven and worked out by Andrews and Gheorghe in [And18] and [Ghe17a].

Just as before, we see that over $\mathbb{C}$,

$$\mathrm{Ext}_A(\mathbb{M}_p, \mathbb{M}_p)[\tau^{-1}] \cong \mathrm{Ext}_{A^{\mathrm{c}}}(\mathbb{F}_p, \mathbb{F}_p)[\tau^{\pm}].$$

This tells us that the $\mathbb{C}$-motivic Adams spectral sequence can be largely determined from the classical one, and new information can be extracted from the classical one by looking at τ-torsion in the motivic one. This leads to a whole host of computational results, the methods of which are certainly familiar to us.

In fact, this computational observation above can be categorified: since τ is an element of the motivic stable stems, we can consider the τ-invertible subcategory of synthetic spectra. This τ-inversion is a smashing localisation such that $S^{p,q}[\tau^{-1}] \simeq S^{p,q'}[\tau^{-1}]$ for any integers p, q, q'. Its image must then be rigidly generated by the spheres $S^p = S^{p,q}[\tau^{-1}]$, hence equivalent to Sp.

Since τ-inversion was precisely the map on motivic cohomology inducing Betti realisation, we see that this categorifies to a factorisation

$$\mathrm{Re} : \mathrm{Sp}_{\mathbb{C}} \xrightarrow{\tau^{-1}} \mathrm{Sp}_{\mathbb{C}}[\tau^{-1}] \xrightarrow{\mathrm{Re}} \mathrm{Sp},$$

where the last arrow is now an equivalence. Diagramatically, as this will be useful later, we represent this identification on the level of categories and spectral sequences by a square

$$\begin{tikzcd} \mathrm{Sp}_{\mathbb{C}} \arrow[r, "\tau=1"] \arrow[d, dotted] & \mathrm{Sp} \arrow[d, dotted] \\ \mathrm{Ext}_{A_*}(\mathbb{M}_p, \mathbb{M}_p) \arrow[r, "\tau=1"] \arrow[d, Rightarrow, "\text{motAsseq}"] & \mathrm{Ext}_{A_*^c}(\mathbb{F}_p, \mathbb{F}_p) \arrow[d, Rightarrow, "\text{Asseq}"] \\ \pi_{*,*} \arrow[r, "\tau=1"] & \pi_*. \end{tikzcd}$$

This commutative diagram has given rise to many advances in the Adams spectral sequence, as outlined in [DI09]. Conceptually, it begs the question of what happens on the other side, i.e. $\tau = 0$. Note that the subcategory at $\tau = 0$ of $\mathrm{Sp}_{\mathbb{C}}$ corresponds to $\mathrm{Mod}(\mathbb{S}/\tau, \mathrm{Sp}_{\mathbb{C}})$. Indeed, by a result of Gheorghe in [Ghe17b], we see that $C\tau := S/\tau$ admits the structure of an $\mathbb{E}_\infty$-algebra, and in a unique way such that the projection $S^{0,0} \to C\tau$ is the unit map. This allows us to view the category of modules over it as stable presentably symmetric monoidal and rigidly generated by the suspensions of $C\tau$. To describe this special fibre of $\mathrm{Sp}_{\mathbb{C}}$, we will need the machinery of the motivic Adams–Novikov spectral sequence.

3 Algebraic Cobordism and the motivic Adams–Novikov spectral sequence.

There exist analogues of MU and BP in $\mathrm{Sp}_{\mathbb{C}}$ given by the motivic spectra MGL and BPL. These satisfy a variety of properties analogous to their classical versions, and give rise to a BPL-based Adams spectral sequence that we call the motivic Adams–Novikov spectral sequence.

Definition 3.1. Let $\mathrm{Gr}(k, n)$ be the Grassmanian of k-planes in $\mathbb{C}^n$, viewed as an object of Sm_k, hence a motivic space. The tautological bundle over this spaces is denoted $E(k, n)$. We then define a motivic spectrum MGL by

$$\mathrm{MGL} := \operatorname*{colim}_{n\to\infty} \operatorname*{colim}_{k\to\infty} \Sigma^{-2k,-k}\mathrm{Th}_{\mathrm{Gr}(k,n)}(E(k, n)).$$

This is called the motivic spectrum of algebraic cobordism

Note that this definition mimics the definition of MU in a straightforward way. Indeed, since it is clear that the complex points of $\mathrm{Gr}(k, n)$ with the analytic topology are just the usual Grassmannians, and the suspension $\Sigma^{-2k,-k}$ realises to a Σ^{-2k} suspension, the left adjoint $\mathrm{Re} : \mathrm{Sp}_{\mathbb{C}} \to \mathrm{Sp}$ must take MGL to MU. The algebraic cobordism spectrum satisfies a host of useful properties likening it to MU.

- It is a cellular spectrum, hence lies in $\mathrm{Sp}_{\mathbb{C}}$ (as was implied above).
- It is oriented in a motivic sense, and is in fact the universal oriented cohomology theory.
- It is connective and even with respect to the Chow degree $c = t - 2w$, i.e. it is concentrated in even and nonnegative Chow degrees.

Beware that chromatic homotopy theory can not be immediately generalised to the motivic context as hinted at by the failure of Nishida's nilpotence theorem above: it turns out that MGL does not detect nilpotence. For example, it does not detect the nonnilpotence of η in the motivic stable stems.

Remark 3.2. Algebraic cobordism first arose as a cohmology theory Ω^* on smooth k-schemes with values in graded rings, such that it would be the universal cohomology theory as such that is oriented in a precise sense. This relates to a motivic version of Landweber exactness and an identification of the formal groups associated to algebraic K-theory and higher Chow groups. This programme is outlined in the work of Levine–Morel, [LM07]. The motivic spectrum MGL was constructed by Morel and Voevodsky, and it was Levine in [Lev08] who showed that the orientation $\Omega^* \to \mathrm{MGL}^{*,*}$ could be identified with the inclusion of the Chow degree zero part, i.e. $\Omega^n = \mathrm{MGL}^{2n,n}$.

Since MGL satisfies a motivic version of being oriented, we can compute $\mathrm{MGL}_{**}\mathrm{MGL}$ in a similar way to $\mathrm{MU}_*\mathrm{MU}$, by reducing it to a computation of $\mathrm{MGL}_{**}\mathrm{Gr}(k, n)$ and observing that the latter is free on motivic Chern classes, whence

$$\mathrm{MGL}_{**}\mathrm{MGL} \cong \mathrm{MGL}_{**}[b_1, b_2, \cdots],$$

with the motivic Chern classes b_i in degree $(2i, i)$. The central result about MGL is the Hopkins–Morel–Hoyois theorem, announced by Hopkins and Morel as a motivic analogue of Quillen's computation of MU_*, but only written down and generalised to nonzero characteristics by Hoyois in [Hoy15b].

Theorem 3.3 (Hopkins–Morel–Hoyois). *The Chow degree zero part of* MGL *is given by*

$$\mathrm{MGL}_{2t,t} \cong \mathrm{MU}_t,$$

with the isomorphism being induced by Betti realisation. Further, this is multiplicative and can be lifted to an equivalence

$$\mathrm{MGL}/(a_1, a_2, \cdots) \simeq \mathrm{M}\mathbb{Z},$$

where the a_i are the generators of the Lazard ring.

In particular, this tells us that motivic cohomology is oriented, since it admits a map of homotopy motivic ring spectra from MGL.

When working p-locally, one can split MGL into copies of BPL just as one does with MU and BP by means of the Quillen idempotent. The cohomology of BPL is computed as

$$\mathrm{H}^{**}\mathrm{BPL} = \mathbb{M}_p[v_1, v_2, \cdots],$$

where the generators v_i are in degree $(2(2^i - 1), 2^i - 1)$, and realise to the usual v_i's in BP. Further, the homology $\mathrm{H}_{**}\mathrm{BPL}$ maps to $\mathrm{H}_{**}\mathrm{M}\mathbb{F}_p = \mathcal{A}_*$ by the Hopkins–Morel–Hoyois theorem and corresponds to the inclusion of the polynomial part into the latter. Due to this identification and the evenness of the homology of BPL, the motivic Adams spectral sequence for BPL will collapse, giving rise to the formula

$$\pi_{**}\mathrm{BPL} \cong \mathbb{Z}_{(2)}[\tau, v_1, v_2, \cdots].$$

Since the v_i's are concentrated in Chow degree zero, we see that

$$\mathrm{BPL}_{**} \cong (\mathrm{BPL}_{**})_{c=0}[\tau] \cong \mathrm{BP}_*[\tau].$$

One can construct the motivic Adams–Novikov spectral sequence in the usual way, e.g. using the cobar complex of BPL to obtain

$$\mathrm{Ext}^{s,f,w}_{\mathrm{BPL}_{**}\mathrm{BPL}}(\mathrm{BPL}_{**}, \mathrm{BPL}_{**}) \Rightarrow \pi_{s,w}.$$

The target consists of the BPL-nilpotent complete stable stems, which are equivalent to the $\mathrm{M}\mathbb{F}_p$-complete stable stems. Once again, the motivic Adams–Novikov E_2-page is just the classical one tensored with the additional variable τ in Chow degree two, so that one can identify

$$\mathrm{Ext}_{\mathrm{BPL}_{**}\mathrm{BPL}}(\mathrm{BPL}_{**}, \mathrm{BPL}_{**}) \cong \mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*) \otimes \mathbb{Z}[\tau],$$

where the degree (s, f) part of the classical Adams–Novikov E_2-page sits in the trigraded motivic E_2-page in degree $(s, f, f/2)$.

4 The Miller square and the algebraic fibre of $\mathbb{C}$-motivic spectra

Let us now bring together the descriptions of the motivic Adams and Adams–Novikov spectral sequences above into a structural framework for $\mathbb{C}$-motivic homotopy theory and how it helps compute stable stems. We begin with a classical construction of Miller, which is that of a certain commutative diagram of spectral sequences which we refer to as the Miller square.

$$\begin{tikzcd}
 & \mathrm{Ext}_{P_*}(\mathbb{F}_p, I^*/I^{*+1}) \arrow[dl, Rightarrow, "\text{CEsseq}"'] \arrow[dr, Rightarrow, "\text{Nsseq}"] & \\
\mathrm{Ext}_{\mathcal{A}^c_*}(\mathbb{F}_p, \mathbb{F}_p) \arrow[dr, Rightarrow, "\text{Asseq}"'] & & \mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*). \arrow[dl, Rightarrow, "\text{ANsseq}"] \\
 & \pi_* &
\end{tikzcd}$$

The spectral sequences in this commutative diagram are the Cartan–Eilenberg spectral sequence (CEsseq), algebraic Novikov spectral sequences (Nsseq), Adams spectral sequence (Asseq), and Adams–Novikov spectral sequence (ANsseq) respectively. Each of them are depicted with their E_2-pages and the groups they converge to. Note that the Cartan–Eilenberg spectral sequence and algebraic Novikov spectral sequence hence share an E_2-page but have different differentials since they converge to two different things. This square can be obtained formally by intertwining the $\mathbb{F}_p$-Adams and Adams–Novikov filtrations into a bifiltration on the stable stems to obtain this commutative square. This can be used to exchange information between the Adams and Adams–Novikov spectral sequences to great effect, as Miller used it in [Mil81] to prove the telescope conjecture at height one for odd primes.

However, it does not reach far enough to tell us information about differentials past the E_2 page. To probe higher differentials, one hopes to obtain a different square (or perhaps span) in which information about higher differentials can be obtained. This turns out to be precisely the niche that $\mathbb{C}$-motivic spectra fulfill.

To observe this, note that the algebraic Novikov spectral sequence can be seen as an $\mathbb{F}_p$-Adams spectral sequence in the category $\mathcal{D}(\mathrm{BP}_*\mathrm{BP})$ of $\mathrm{BP}_*\mathrm{BP}$-comodules. Indeed, the Adams–Novikov E_2-page simply consists of the endomorphisms of the unit in this category, and giving it an appropriate $\mathbb{F}_p$-Adams filtration recovers the algebraic Novikov spectral sequence. This shows that the top right and bottom left legs of the Miller square are Adams spectral sequences intrinsic to some homotopy theory–$\mathrm{BP}_*\mathrm{BP}$-comodules and spectra respectively. We might therefore hope to extend it to a span of the form

$$\begin{tikzcd}
\mathrm{Ext}_{P_*}(\mathbb{F}_p, I^*/I^{*+1}) \arrow[d, Rightarrow] & ? \arrow[l] \arrow[r] & \mathrm{Ext}_{A_*^c}(\mathbb{F}_p, \mathbb{F}_p) \arrow[d, Rightarrow] \\
\mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*) & ? \arrow[l] \arrow[r] & \pi_*.
\end{tikzcd}$$

In other words, is there a deformation of spectra Sp with algebraic fibre $\mathcal{D}(\mathrm{BP}_*\mathrm{BP})$ recovering such a span? The answer is given precisely by $\mathrm{Sp}_{\mathbb{C}}$. Indeed, recall that we constructed the $\mathbb{E}_\infty$-ring spectrum $C\tau$. In fact, using the motivic Adams–Novikov spectral sequence we can easily compute its motivic homotopy groups. Indeed, the cofibre sequence

$$S^{0,-1} \to S^{0,0} \to C\tau$$

induces a long exact sequence on BPL-homology, hence on motivic Adams–Novikov E_2-pages. Now the motivic Adams–Novikov E_2-pages of spheres are just given by (shifts of) the regular Adams–Novikov E_2-page with an additional polynomial variable τ as established above. Therefore, the τ-mulitplication on these in the long exact sequence acts injectively, and we can identify

$$\mathrm{Ext}_{\mathrm{BPL}_{**}}(\mathrm{BPL}_{**}, \mathrm{BPL}_{**}C\tau) \cong \mathrm{Ext}_{\mathrm{BPL}_{**}}(\mathrm{BPL}_{**}, \mathrm{BPL}_{**})/\tau \cong \mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*).$$

Due to the evenness in the latter, one sees that the motivic Adams–Novikov spectral sequence for $C\tau$ actually collapses, so that

$$\pi_{**}C\tau \cong \mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*).$$

Further, a computation reveals that the Cartan–Eilenberg E_2-page can be rewritten as

$$\mathrm{Ext}_{P_*}(\mathbb{F}_p, I^*/I^{*+1}) \cong \mathrm{Ext}_{\mathbb{A}_*}(\mathbb{M}_p, \mathbb{F}_p).$$

The latter can then be identified with the motivic Adams E_2-page of $C\tau$, since the homology of the latter is given by $\mathbb{M}_{p/\tau \cong \mathbb{F}_p}$.

We conclude that the Miller square is part of a span

$$\begin{tikzcd}
\mathrm{Mod}(C\tau; \mathrm{Sp}_{\mathbb{C}}) \arrow[d, dotted] & \mathrm{Sp}_{\mathbb{C}} \arrow[l, "\tau=0"'] \arrow[r, "\tau=1"] \arrow[d, dotted] & \mathrm{Sp} \arrow[d, dotted] \\
\mathrm{Ext}_{A_*}(\mathbb{F}_p[\tau], \mathbb{F}_p) \arrow[d, Rightarrow] & \mathrm{Ext}_{A_*}(\mathbb{F}_p[\tau], \mathbb{F}_p[\tau]) \arrow[l, "\tau=0"'] \arrow[r, "\tau=1"] \arrow[d, Rightarrow] & \mathrm{Ext}_{A_*^c}(\mathbb{F}_p, \mathbb{F}_p) \arrow[d, Rightarrow] \\
\pi_{**}C\tau & \pi_{**} \arrow[l, "\tau=0"'] \arrow[r, "\tau=1"] & \pi_*.
\end{tikzcd}$$

This tells us that the motivic Adams spectral sequence recovers both the algebraic Adams spectral sequence, i.e. the spectral sequence computing the Adams–Novikov E_2-page $\pi_{**}C\tau$, and the usual Adams spectral sequence. This observation has lead to great advances in the computation of these objects, and is the central tool in e.g. [DI09].

4.1 The algebraic fibre

We noted above that the category of $\mathrm{BP}_*\mathrm{BP}$-comodules should sit on the left $\tau = 0$ side of the improved Miller square. Now note that

$$\pi_{**}C\tau = [S^{0,0}, C\tau] \cong [C\tau, C\tau]_{C\tau},$$

so that we have identified the endomorphisms of the unit in $\mathrm{BP}_*\mathrm{BP}$-comodules and $C\tau$-modules respectively. This begs the auestion of whether this admits a categorification akin to $\mathrm{Sp}_{\mathbb{C}}[\tau^{-1}] \simeq \mathrm{Sp}$. The answer lies in a theorem of Gheorghe–Wang–Xu in [GWX20].

Theorem 4.1 ([GWX20]). *There is an equivalence of categories*

$$\mathrm{Mod}(C\tau; \mathrm{Sp}_{\mathbb{C}})^{\omega} \simeq \mathcal{D}(\mathrm{BP}_*\mathrm{BP})^{\omega,\mathrm{ev}}$$

between compact $C\tau$-modules and compact objects of the derived category of BP_*BP-comodules generated in even degrees.

The proof of this result largely follows from some abstract nonsense and we will present a slick proof from Achim Krause's thesis. First, note the corollary

Corollary 4.2. *The theorem of [GWX20] induces an equivalence*

$$\mathrm{Mod}(C\tau; \mathrm{Sp}_{\mathbb{C}}) \simeq \mathcal{S}\mathrm{table}^{\mathrm{ev}}_{\mathrm{BP}_*\mathrm{BP}},$$

where the right hand side is an even variant of Hovey's stable derived category of comodule, defined as

$$\mathcal{S}\mathrm{table}^{\mathrm{ev}}_{\mathrm{BP}_*\mathrm{BP}} = \mathrm{IndCoh}(\mathrm{BP}_*\mathrm{BP})^{\mathrm{ev}} := \mathrm{Ind}(\mathcal{D}(\mathrm{BP}_*\mathrm{BP})^{\omega,\mathrm{ev}}).$$

This corollary obviously follows from taking Ind on both sides of the main theorem and noting that $C\tau$ is rigidly generated by the suspensions of $C\tau$ itself. While the right hand side does not recover the derived category of (even) $\mathrm{BP}_*\mathrm{BP}$-comodules, we should in fact strive to work with Hovey's variant instead since this resolves certain pathologies inherent to the derived category of comodules over a Hopf algebroid such as the lack of rigid generation. Compare the discussion in [Pst22].

Let us now present the proof of the Gheorghe–Wang–Xu theorem in section 6 of [Kra].

Proof. Consider the quotient

$$\mathrm{BPL}/\tau \simeq \mathrm{BPL} \otimes C\tau$$

since $C\tau$ is an $\mathbb{E}_\infty$-algebra by Gheorghe's theorem, the quotient BPL/τ is an $\mathbb{E}_\infty$ algebra over $C\tau$ and the unit induces a symmetric monoidal basechange adjunction

$$\mathrm{Mod}(C\tau; \mathrm{Sp}_{\mathbb{C}}) \rightleftarrows \mathrm{Mod}(\mathrm{BPL}/\tau; \mathrm{Sp}_{\mathbb{C}}).$$

The comonad associated to this adjunction, i.e. the composite of the left adjoint and the right adjoint with the counit as a comonad structure map, can then be identified with an algebra object in BPL/τ-modules by monoidality; this object arises as the left adjoint of the right adjoint of the unit in BPL/τ-modules, i.e.

$$\mathrm{BPL}/\tau \otimes_{C\tau} \mathrm{BPL}/\tau \simeq (\mathrm{BPL} \otimes \mathrm{BPL})/\tau.$$

It is clear per construction that this adjunction factors through the free-forgetful adjunction with comodules over this comonad, i.e.

$$\mathrm{Mod}(C\tau; \mathrm{Sp}_{\mathbb{C}}) \rightleftarrows \mathrm{Comod}((\mathrm{BPL} \otimes \mathrm{BPL})/\tau; \mathrm{Mod}(\mathrm{BPL}/\tau; \mathrm{Sp}_{\mathbb{C}})) \rightleftarrows \mathrm{Mod}(\mathrm{BPL}/\tau; \mathrm{Sp}_{\mathbb{C}}).$$

Now we want to apply some recognition theorem for comonadic adjunctions to deduce that the first adjunction is an equivalence. This will not be true on the nose; it is only valid on objects that are complete with respect to the comonad $(\mathrm{BPL} \otimes \mathrm{BPL})/\tau$, i.e. such that the appropriate conservativity condition of the comonadic Barr–Beck theorem are satisfied on these objects. As described in Definition 2.24 and Theorem 2.44 of op. cit. one can describe completeness with respect to the comonad $(\mathrm{BPL} \otimes \mathrm{BPL})/\tau$ as convergence of the associated motivic Adams–Novikov spectral sequence, which in particular is true for all compact modules over $C\tau$, as these are bounded below. We conclude that the adjunction is comonadic on compact objects and induces an equivalence

$$\mathrm{Mod}(C\tau; \mathrm{Sp}_{\mathbb{C}})^{\omega} \simeq \mathrm{Comod}((\mathrm{BPL} \otimes \mathrm{BPL})/\tau; \mathrm{Mod}(\mathrm{BPL}/\tau; \mathrm{Sp}_{\mathbb{C}}))^{\omega}.$$

The next step is then to reduce the right hand side to purely algebraic information. Since the homotopical information contained in BPL and the structure maps of the comodule are sufficiently sparse in weight degrees, one can use a shearing operation to bring them all into degree zero and obtain an equivalent object. For this, we view the latter as a graded object where every grading degree corresponds to the weight, and shear this down to weight zero.

Let $\mathrm{Sp}^{\mathrm{Gr}}$ denote the category of graded spectra, obtained as the functor category $\mathcal{F}\mathrm{un}(\mathbb{Z}^{\delta}, \mathrm{Sp})$ out of the integers considered as a discrete anima, and equipped with the Day convolution with respect to addition of integers and smash product of spectra. Define a functor

$$\Gamma_*\colon \mathrm{Mod}(\mathrm{BPL}/\tau; \mathrm{Sp}_{\mathbb{C}}) \to \mathrm{Sp}^{\mathrm{Gr}} : X \mapsto \mathrm{Map}_{\mathrm{BPL}/\tau}(\Sigma^{0,*}\mathrm{BPL}/\tau, X)$$

sending a BP/τ-module to the BPL/τ-linear mapping spectra of BPL/τ into it, shifted by weight. Note that a BPL/τ-linear map from $\Sigma^{0,w}\mathrm{BPL}/\tau$ into a BPL/τ-module X is equivalent to a map of motivic spectra from $S^{0,w}$ to X so that this is precisely recording the motivic homotopy groups of X in weight w. Now note that the subset $[\Sigma^{0,*}\mathrm{BPL}/\tau] \subset \pi_0\mathrm{Pic}(\mathrm{BPL}/\tau)$ forms a subgroup, so that this defines a lax symmetric monoidal functor, hence sends the unit BPL/τ to an $\mathbb{E}_\infty$-algebra $\Gamma_*\mathrm{BPL}/\tau$ in graded spectra. As mentioned before, the functor Γ_* precisely records motivic homotopy groups in all weights, hence is conservative. In fact, it is the right adjoint of a symmetric monoidal functor going the other way, and it is then easily checked that this adjunction satisfies the monadic Barr–Beck theorem. We conclude that there is an equivalence

$$\mathrm{Mod}(\mathrm{BPL}/\tau; \mathrm{Sp}_{\mathbb{C}}) \simeq \mathrm{Mod}(\Gamma_*\mathrm{BPL}/\tau; \mathrm{Sp}^{\mathrm{Gr}}).$$

Let us now apply the shearing endomorphism to graded spectra defined by

$$\sigma = \Sigma^{-2*}\colon \mathrm{Sp}^{\mathrm{Gr}} \to \mathrm{Sp}^{\mathrm{Gr}}.$$

Its value on the unit of is given by

$$\sigma(\Gamma\mathrm{BPL}/\tau)_w = \Sigma^{-2w}\mathrm{Map}_{\mathrm{BPL}/\tau}(\Sigma^{0,w}\mathrm{BPL}/\tau, \mathrm{BPL}/\tau)$$

The homotopy groups of this spectrum are

$$\pi_r\sigma(\Gamma\mathrm{BPL}/\tau)_w = \pi_{r+2w}\mathrm{Map}_{\mathrm{BPL}/\tau}(\Sigma^{0,w}\mathrm{BPL}/\tau, \mathrm{BPL}/\tau) \cong$$

Now BPL/τ, just like BPL is concentrated in chow degree zero, so that the weight w part in the right hand side is concentrated in the topological degree corresponding to Chow degree zero, i.e. $t = 2w$ or $r = 0$. We conclude that this is a discrete spectrum in every degree, and in this case, we recover the degree $(0, 0)$ motivic homotopy groups of BPL/τ which are precisely BP_*. Now it turns out that this entire module category can be described in terms of this Eilenberg–macLane object. Indeed, since these are all modules over the unit $\Gamma_*\mathrm{BPL}/\tau$ and the Day convolution monoidal structure is such that every spectrum appearing in a graded spectrum is a module over $\Gamma_0\mathrm{BPL}/\tau$, it suffices to note that the latter is computed by

$$\pi_r\Gamma_0\mathrm{BPL}/\tau \simeq \pi_r\mathrm{Map}(S^{0,0}, \mathrm{BPL}/\tau)$$

and the latter is concentrated in weight zero, hence Chow degree zero and topological degree zero, where it is given by

$$\pi_{0,0}\mathrm{BPL}/\tau \simeq \mathbb{Z}.$$

We conclude that every graded part of an object in $\mathrm{Mod}(\Gamma\mathrm{BPL}/\tau, \mathrm{Sp}^{\mathrm{Gr}})$ is a $\mathbb{Z}$-module, hence can be formed in $\mathcal{D}(\mathbb{Z})^{\mathrm{Gr}}$. The shear transformation then carries this to

$$\mathrm{Mod}(\Gamma\mathrm{BPL}/\tau; \mathcal{D}(\mathbb{Z})^{\mathrm{Gr}}) \simeq \mathrm{Mod}(\sigma\Gamma\mathrm{BPL}/\tau; \mathcal{D}(\mathbb{Z})^{\mathrm{Gr}}) \simeq \mathrm{Mod}(\mathrm{BP}_*; \mathcal{D}(\mathbb{Z}))^{\mathrm{ev}}.$$

The last step follows from the observation that all sheared objects are generated in Chow degree zero, hence even topological degree.

By some more careful but essentially similar evenness arguments, we see that the structure of the comonad $(\mathrm{BPL}{\otimes}\mathrm{BPL})/\tau$ also collapses such that all coherence is already present in its graded-and-sheared version $\mathrm{BP}_*\mathrm{BP}$. We conclude that there is a chain of equivalences

$$\mathrm{Mod}(C\tau; \mathrm{Sp}_{\mathbb{C}})^{\omega} \simeq \mathrm{Comod}((\mathrm{BPL} \otimes \mathrm{BPL}/\tau); \mathrm{Mod}(\mathrm{BPL}/\tau; \mathrm{Sp}_{\mathbb{C}}))^{\omega} \simeq \mathrm{Comod}(\mathrm{BP}_*\mathrm{BP}; \mathrm{Mod}(\mathrm{BP}_*; \mathcal{D}(\mathbb{Z})))^{\omega,\mathrm{ev}}.$$

□

We conclude that $\mathrm{Sp}_{\mathbb{C}}$ sits in a span

$$\mathcal{S}\mathrm{table}^{\mathrm{ev}}_{\mathrm{BP}_*\mathrm{BP}} \xleftarrow{\tau=0} \mathrm{Sp}_{\mathbb{C}} \xrightarrow{\tau=1} \mathrm{Sp}$$

recovering the algebraic Adams spectral sequence for the Adams–Novikov E_2-page and the Adams spectral sequence respectively. In fact, using our description of the algebraic fibre in terms of graded spectra, we get the span

$$\mathrm{Mod}(\Gamma_*\mathrm{BPL}/\tau; \mathrm{Sp}^{\mathrm{Gr}}) \xleftarrow{\tau=0} \mathrm{Sp}_{\mathbb{C}} \xrightarrow{\tau=1} \mathrm{Mod}(\mathbb{S}\tau^{-1}; \mathrm{Sp}).$$

While this is not precisely a universal property of the central object for some delicate reason involving recollements of quasi-coherent sheaves, one can compare this with the span

$$\mathrm{Sp}^{\mathrm{Gr}} \xleftarrow{\tau=0} \mathrm{Sp}^{\mathrm{Fil}} \xrightarrow{\tau=1} \mathrm{Sp}$$

between filtered spectra $\mathrm{Sp}^{\mathrm{Fil}} = \mathcal{F}\mathrm{un}(\mathbb{Z}^{\mathrm{op}}, \mathrm{Sp})$, where τ is the map that shifts a filtration up by one, so that its cofibre computes the associated graded, while its inversion computes the colimit of a filtration. In fact, $\mathrm{Sp}_{\mathbb{C}}$ fits in this framework, as described in Gheorghe–Isaksen–Krause–Ricka's paper [Ghe+21], in which they provide a filtered model for $\mathrm{Sp}_{\mathbb{C}}$:

$$\mathrm{Sp}_{\mathbb{C}} \simeq \mathrm{Mod}(\mathrm{Tot}(\tau_{\geq 2\star}\mathrm{BP}^{\otimes \bullet}); \mathrm{Sp}^{\mathrm{Fil}}).$$

5 Comparison with Synthetic spectra

We conclude by discussing the comparison between even synthetic spectra based on MU and cellular motivic spectra over $\mathbb{C}$. This section is merely a summary of Sections 6 and 7 in [Pst22]. First, note that there is a grading shift in the version of synthetic spectra we use:

Definition 5.1. Let $\mathrm{Sp}^{\mathrm{fpe}}_{\mathrm{MU}}$ be the full subcategory of Sp on finite spectra X such that MU_*X is a projective MU_*-module concentrated in even degrees. Define the category of even synthetic spectra based on MU as

$$\mathcal{S}\mathrm{yn}^{\mathrm{ev}}_{\mathrm{MU}} = \mathrm{Sh}^{\mathrm{Sp}}_{\Sigma}(\mathrm{Sp}^{\mathrm{fpe}}_{\mathrm{MU}}),$$

where the site of even finite MU-projective spectra are equipped with the MU_*-surjection pretopology.

Remark 5.2. Note that MU is Adams type in an even way: it can be expressed as a filtered colimit of even finite MU-projective spectra.

Remark 5.3. This has all the same formal properties as synthetic spectra based on MU. One can show that MU-based synthetic spectra are cellular, i.e. rigidly generated by the bigraded spheres by an argument using the Hurewicz theorem (see [Pst22] Section 6.1), and then even synthetic spectra correspond to the full subcategory generated by the spheres $S^{t,w}$ with w even.

The comparison between $\mathcal{S}\mathrm{yn}^{\mathrm{ev}}_{\mathrm{MU}}$ and $\mathrm{Sp}_{\mathbb{C}}$ proceeds in two steps: first one identifies $\mathrm{Sp}_{\mathbb{C}}$ with a motivic variant of the synthetic construction, namely as product preserving sheaves of spectra on the site of even finite projective MGL-modules. One then compares the latter with even MU-based synthetic spectra by the Betti realisation functor. the induced adjunction will then induce an equivalence on subcategories of p-complete objects.

5.1 The synthetic construction based on MGL

Consider the full subcategory of $\mathrm{Sp}_{\mathbb{C}}$ on motivic spectra that are finite and such that their MGL-homology is projective over MGL_* and generated in Chow degree zero. Denote this subcategory by $\mathrm{Sp}^{\mathrm{fp}}_{\mathrm{MGL}}$, where the subscript $\mathbb{C}$ is omitted because it is clear that MGL is a motivic spectrum. By the Hopkins–Morel–Hoyois theorem it is clear that an element of this subcategory realises to an even finite MU-projective spectrum, so that this restricts to

$$\mathrm{Re} : \mathrm{Sp}^{\mathrm{fp}}_{\mathrm{MGL}} \to \mathrm{Sp}^{\mathrm{fpe}}_{\mathrm{MU}}.$$

This is a morphism of excellent ∞-sites if one considers the source with the MGL_{**}-surjection pretopology.

Now define an analogue of the spectral Yoneda embedding by

$$\Upsilon\colon \mathrm{Sp}_{\mathbb{C}} \to \mathrm{Sh}_{\Sigma}^{\mathrm{Sp}}(\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}}), X \mapsto \Upsilon X = \mathrm{Map}(-, X),$$

using the spectral enrichment of $\mathrm{Sp}_{\mathbb{C}}$ and the inclusion of $\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}}$ into it.

this functor clearly preserves limits, as well as filtered colimits since a filtered colimit diagram gives rise to a levelwise filtered colimit diagram and every object of $\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}}$ is finite hence compact. By stability it is therefore continuous and cocontinuous.

To prove that Υ is fully faithful, i.e. that the map

$$\mathrm{map}_{\mathrm{Sp}_{\mathbb{C}}}(X, Y) \to \mathrm{map}_{\mathrm{Sh}_{\Sigma}^{\mathrm{Sp}}(\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}})}(\Upsilon X, \Upsilon Y)$$

is an equivalence for all X, Y, we can then restrict to the case where X is a generator of $\mathrm{Sp}_{\mathbb{C}}$, e.g. a bigraded sphere of the form $S^{2t,t}$ which is in particular even finite MGL-projective.

To apply a Yoneda lemma argument we now show that this spectral Yoneda embedding is equivalent to the stabilisation of the usual (restricted) Yoneda embedding. Note that this is certainly not true in the usual construction of synthetic spectra, where the difference between Y and ν encapsulated the deformation and t-structure inherent to synthetic spectra.

In the motivic case however, we can compute the t-structure homotopy groups of $\Upsilon S^{2t,t}$ in the t-structure on $\mathrm{Sh}_{\Sigma}^{\mathrm{Sp}}(\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}})$ and see that these are given by the MGL-homology of $S^{2t,t}$ according to a standard argument in loc. cit. Lemma 7.18. Now $S^{2t,t}$ is even finite MGL-projective hence its MGL-homology is generated in Chow degree zero, so that this vanishes in negative Chow degree and we conclude that $\Upsilon S^{2t,t}$ is connective in $\mathrm{Sh}_{\Sigma}^{\mathrm{Sp}}(\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}})$, whence its values are all connective spectra. Since the suspension was computed levelwise, this tells us that

$$\Upsilon S^{2t,t} \simeq \Sigma^{\infty}_{+} y(S^{2t,t})$$

as desired. We can then simply apply the Yoneda lemma:

$$\begin{aligned} \mathrm{map}(\Upsilon S^{2t,t}, \Upsilon Y) &\simeq \mathrm{map}(y(S^{2t,t}), \Omega^{\infty}\Upsilon Y), \\ &\simeq \mathrm{map}(y(S^{2t,t}), y(Y)), \\ &\simeq \mathrm{map}(S^{2t,t}, Y). \end{aligned}$$

Essential surjectivity is immediate from the cocontinuity of Υ combined with the observation that Υ acts as the suspension of the Yoneda embedding on the bigraded spheres in Chow degree zero, and the latter are precisely the generators of $\mathrm{Sh}_{\Sigma}^{\mathrm{Sp}}(\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}})$.

Finally, let us consider the adjunction

$$\mathrm{Sh}_{\Sigma}^{\mathrm{Sp}}(\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}}) \rightleftarrows \mathcal{S}\mathrm{yn}_{\mathrm{MU}}^{\mathrm{ev}}$$

induced by Betti realisation on the level of sites. We will not prove here that it induces an equivalence on p-complete objects but rather refer the reader to Theorem 7.34 in [Pst22]. The key part of the theorem there, known as the Gheorghe–Isaksen theorem in loc. cit. is that the Betti realisation functor induces an equivalence

$$\pi_{t,w} M \to \pi_t \mathrm{Re}(M)$$

in nonnegative Chow degree on the homotopy of a p-complete finite MGL-projective spectrum M. The p-completion assumption is now indispensable because we work with the motivic τ, which only exists in the p-complete setting.

Bringing together the equivalences established above, we then obtain a p-complete equivalence

$$\mathrm{Sp}_{\mathbb{C}} \simeq \mathrm{Sh}_{\Sigma}^{\mathrm{Sp}}(\mathrm{Sp}_{\mathrm{MGL}}^{\mathrm{fp}}) \simeq \mathcal{S}\mathrm{yn}_{\mathrm{MU}}^{\mathrm{ev}}.$$

Remark 5.4. Note that the equivalence between p-complete even MU-synthetic spectra and cellular p-complete motivic spectra is not particularly clarifying; it goes through a sort of synthetic construction based on MGL. However, there is another, albeit more roundabout way to prove this using filtered spectra. Indeed, as hinted at earlier, work of Gheorghe–Isaksen–Krause–Ricka in [Ghe+18] provides a model for $\mathrm{Sp}_{\mathbb{C}}$ in terms of filtered spectra. Further, work of [BHS20] as well as the author's thesis building on op. cit. and work of Jacob Hegna provide a filtered model for synthetic spectra (which works over a more general homology theory, and works integrally). Comparing the two filtered models then recovers the equivalence above. In this filtered model, it slightly more obvious what the universal properties of $\mathrm{Sp}_{\mathbb{C}}$ and $\mathcal{S}\mathrm{yn}_{\mathrm{MU}}^{\mathrm{ev}}$ are with regards to the span between filtered spectra, graded spectra, and spectra viewed as the algebraic (special) and topological (generic) fibres of a deformation respectively. In this regard, it can be seen as stating that synthetic spectra have the universal property of sitting in this span , hence must be equivalent to $\mathrm{Sp}_{\mathbb{C}}$.

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