The Cobordism Hypothesis

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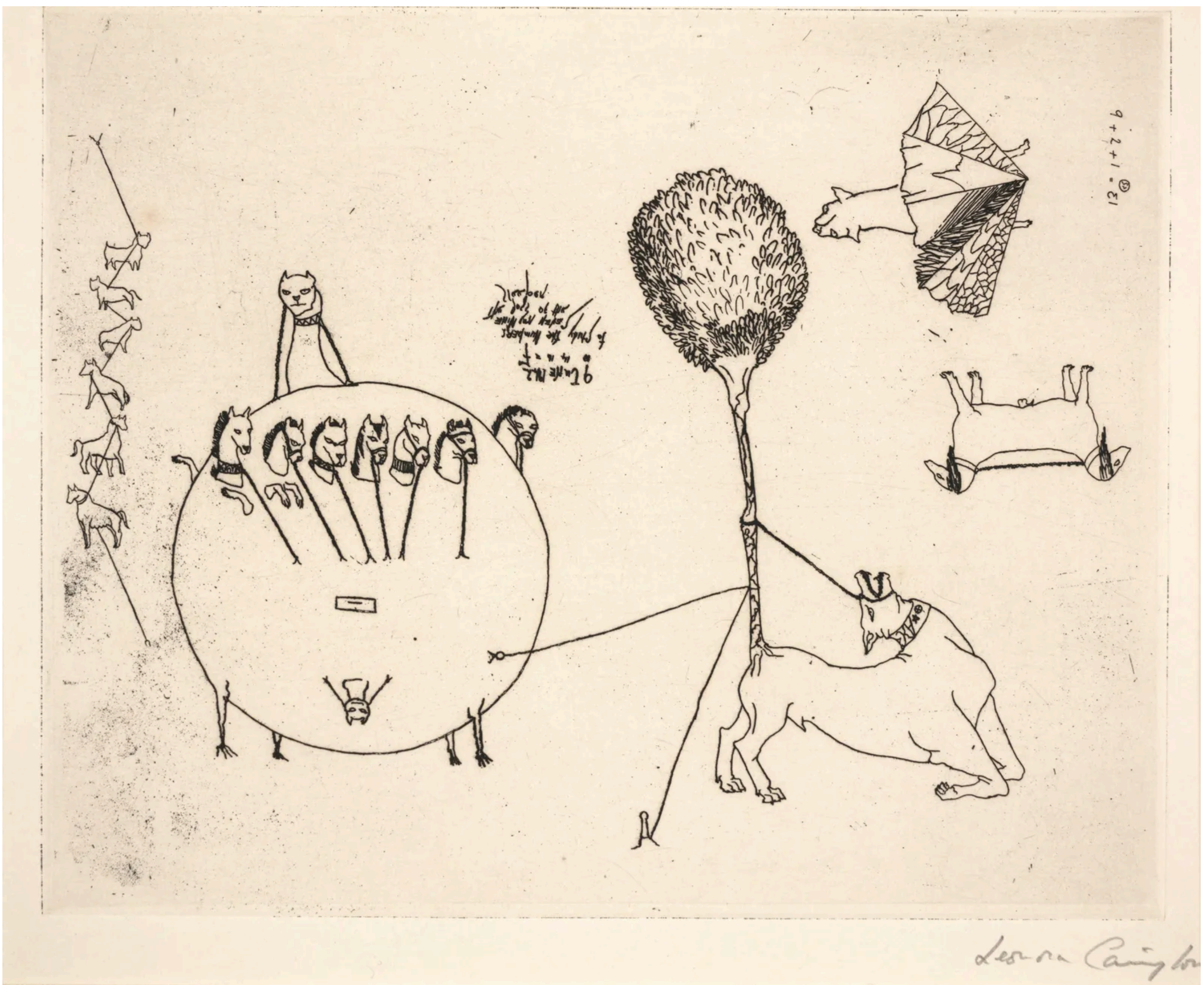

Leonora Carrington, *To Study the Numbers* from *Surrealist Portofolio VVV*, 1941

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Schedule

The seminar meets at 14:15-16:00 on Thursdays in the TBA.

Date		Topic	Speaker
16/04	1	Complete Segal spaces and (∞, n)-categories	
23/04	2	Dualisablility	
30/04	3	The one-dimensional case	
07/05	4	The h-principle	
04/06	5	Embedded submanifolds & scanning	
11/06	6	Classifying spaces of diffeomorphism groups	
18/06	7	Galatius–Madsen–Tilmann–Weiss & invertible TQFTs	
25/06	8	Morse & Cerf functions	
02/07	9	Framed Morse functions	
TBD	10	Lurie's proof strategy	
TBD	11	Categorical chain complexes	
TBD	12	Geometry of the proof sketch	

Schedule of talks

Preliminaries

- Understanding of basic geometric topology is essential, e.g. smooth manifolds, tangent bundles, tangent structures, cobordisms, critical points etc.
- Some basic understanding of higher category theory is essential. At least you should know about ∞-categories, the ∞-categories of spaces, spectra, commutative monoids and commutative groups, symmetric monoidal ∞-categories.
- Lurie gave a series of talks on "Topological Quantum Field Theory and the Cobordism Hypothesis", and we recommend to watch the first talk to get a feeling for the subject, see [Lur].

Overview

The cobordism hypothesis is a deep conjecture, suggesting a connection between geometric topology and higher category theory with applications to mathematical physics and higher algebra.

Cobordism Categories: A central motivating problem in geometric topology is the classification of (smooth) manifolds. While a general classification up to diffeomorphism is out of reach, it is more feasible to classify them up to a weaker notion, called *cobordance*. In his celebrated paper [Tho54], Thom proves that the set of cobordism classes of manifolds of a fixed dimension d may be computed as the d-th homotopy group of a *Thom spectrum* denoted MO. This result laid the groundwork for a fruitful connection between geometric classifications and computational stable homotopy theory.

A further generalisation of Thom's result was proved by Galatius–Madsen–Tilmann–Weiss in [Gal+09]. In their work, manifolds and cobordisms between them are viewed not as a set but as a fully fledged *cobordism category*. Classically, the d-dimensional cobordism category is a category internal to topological spaces which may be informally described as follows.

- Objects are $(d-1)$-dimensional manifolds.
- Morphisms are d-dimensional cobordisms.

This category is topologised using spaces of self-diffeomorphisms of manifolds. Their striking result says that the homotopy type of this category is still given by a Thom spectrum. This refines Thom's result in that it coherently encodes information about *higher morphisms.*

The *cobordism hypothesis* is yet another generalisation of this result. While the theorem by Galatius–Madsen–Tilmann–Weiss provides a description of the homotopy type of the cobordism category, the cobordism hypothesis tries to assert a universal property of the category itself. In order to achieve this, one has to study the *fully extended bordism category* Bord_n. This is an (∞, n)-category in which

- objects are 0-dimensional manifolds (a.k.a. points),
- morphisms are 1-dimensional bordisms,
- ...
- n-morphisms are n-dimensional bordisms.

This category can further be equipped with a symmetric monoidal structure, using the disjoint union of manifolds.

In their 1995 paper [BD95], Baez and Dolan put forward a conjecture dubbed the *cobordism hypothesis*. The conjecture states that this aforementioned symmetric monoidal category is the *free symmetric monoidal (∞, n)-category on a single fully dualizable generator*. Informally speaking, this means that any symmetric monoidal functor $\mathrm{Bord}_n \to \mathcal{C}$ into another symmetric monoidal (∞, n)-category $\mathcal{C}$ is uniquely determined by its value on the point, and moreover the point has to be sent to an object fulfilling certain dualisability (a.k.a. finiteness) conditions. The cobordism hypothesis hence predicts a purely "algebraic" or "categorical" universal property for the geometrically defined cobordism category.

Topological Quantum Field Theories: While the cobordism hypothesis is interesting through the eyes of geometric topology, it actually originates from mathematical physics.

In mathematical physics, *topological quantum field theories* (TQFTs for short) –or more accurately, their fully extended variants– are assignments that prescribe objects of various categorical depth (e.g. scalars, vector spaces, categories, etc.) to geometric objects of various dimensions, which are further assumed to be compatible under cobordisms between geometric objects and decategorification procedures on the output. These are conveniently encoded in terms of symmetric monoidal functors from the fully extended bordism (∞, n)-category Bord_n to a target symmetric monoidal (∞, n)-category of algebraic nature. From this point of view, the cobordism hypothesis predicts that a fully extended topological quantum field theory is uniquely determined by its value at the point. In these terms, one can think of it as an analogue of the Eilenberg–Steenrod axioms in algebraic topology: a classical homology theory on spaces is uniquely determined by the value at a point.

The topological quantum field theory perspective on the cobordism hypothesis relates it to other interesting topics in geometric topology, such as factorization homology.

Dualisability: The value of a point under a symmetric monoidal functor out of the fully extended bordism (∞, n)-category a fortiori satisfies a stringent collection of finiteness or dualisability conditions called *full dualisability*. This entirely categorical notion has sparked attention in higher algebra. That is, independently of the original geometric definition, we are interested in understanding the free fully dualisable symmetric monoidal (∞, n)-category on a single generator. The cobordism hypothesis predicts that the answer to this problem, even though purely algebraic in nature, relates to geometric topology, e.g. to spaces of self-diffeomorphism of manifolds.

Lurie's Proof Sketch: In 2009, Lurie posted a sketch of a proof of the cobordism hypothesis [Lur08]. The goal of this seminar will be to work through this proof sketch, in order to understand the basic strategy.

Syllabus

Talk I: Complete Segal spaces and (∞, n)-categories

Introduce the Segal condition for simplicial spaces. Define mapping spaces and the completeness condition. Introduce the notion of a Dwyer-Kan equivalence (the condition appearing in [HS25, Lem. 2.3]). Define the Rezk nerve. The goal is to prove the following.

- The Rezk nerve is fully faithful with essential image the subcategory of complete Segal spaces.
- The initial map from an arbitrary Segal space into a complete Segal space is a Dwyer-Kan equivalence.

Building up on this, give the definition of an n-category using n-fold Segal spaces, see e.g. [CS19]. Show that a functor is an equivalence if it is essentially surjective on k-morphisms for $k < n$ and fully faithful on n-mapping spaces.

Talk II: Dualisability in higher category theory

Discuss the notion of dualisability in higher categories following [Lur08, Section 2.3]. Present the explicit criteria for dualisability in an $(\infty, 2)$-category following [Lur08, Proposition 4.2.3] and use this to construct the Serre automorphism. If time permits, illustrate with some concrete examples as in [Lur08, Examples 1.1.9, 1.1.11].

Talk III: The one-dimensional cobordism hypothesis

Describe the one-dimensional cobordism category, i.e. its objects and morphism spaces[1]. Present a full proof [2] of the cobordism hypothesis in dimension one following [Har12].

Talk IV: The h-principle

Introduce the h-principle as a technique to show that certain sheaves of topological spaces on the category of smooth manifolds give rise to sheaves of spaces, following [Kup19], possibly also consulting [KS77]. Prove the h-principle for flexible sheaves, and state its analog for microflexible sheaves. Sketch the h-principle for immersions in positive codimension. If time permits: Apply to show that one can "turn the sphere inside out": the inclusion of S^2 into $\mathbf{R}^3$ is regularly homotopic to the inclusion postcomposed with the reflection through the origin, as in [LM24, Ex. 4.31].

Talk V: Embedded submanifolds and Segal's scanning methods

Define spaces of embedded submanifolds, following [GR10]. Compute the space of zero-dimensional submanifolds of euclidean space as a baby case for intuition. Building on that, prove that the space of embedded submanifolds into $\mathbf{R}^n$ is equivalent to a Thom space [GR10, Thm. 3.22] (the key word here is *scanning*). See also [Mye] for an expository note.

Talk VI: Classifiyng spaces of diffeomorphism groups of manifolds and the fully extended bordism category

Prove that the space of embedded submanifolds into a disk is a model for the classifying space of manifold bundles [Sch24, Appendix B]. Define the bordism category following [Sch24]. Give a description of 0-objects, 1-objects, ..., and the space of n-morphisms. State that the space of n-morphisms is given by a disjoint union of classifying spaces of diffeomorphisms of cobordisms, a special case (for the empty manifold) being the statement from the beginning.

Talk VII: Galatius–Madsen–Tilmann–Weiss and invertible TQFTs

State the cobordism hypothesis [Lur08]. State the special case where the target is a groupoid, classifying *invertible field theories*. This is (a slight extension of) the Galatius-Madsen-Tilmann-Weiss theorem. Sketch the proof following [Sch24] (or [GR10] and [Mye] for the classical Galatius-Madsen-Tilmann-Weiss).

Talk VIII: Morse and Cerf functions

Following [Kup19], introduce the space of generalised Morse functions on a smooth manifold. Compute its value on Euclidean space. Sketch Kupers' method to verify h-principles on closed manifolds, and give an idea why it can be applied here.

[1] A formal definition can be skipped, as we will discuss this later anyway.
[2] This talk requires some basic knowledge of the theory of ∞-operads.

Talk IX: Framed Morse functions

Following [Kup19], introduce the space of framed generalised Morse functions on a smooth manifold. Give intuition on how a framed Morse function gives rise to a handlebody decomposition with coordinates for descending discs. Building on the previous talk, verify the h-principle for the space of framed Morse functions.

Talk X: Lurie's proof strategy

Present Lurie's proof strategy of the cobordism hypothesis following [Lur08, Sections 3.1-3.2]. Emphasise the inductive formulation given as [Lur08, Theorem 3.1.8].

Talk XI: Categorical chain complexes

Present Lurie's approach to reducing the cobordism hypothesis to an $(\infty, 1)$-categorical statement using categorical chain complexes following [Lur08, Section 3.3].

Talk XII: The geometric argument in Lurie's proof sketch

Discuss [Lur08, Section 3.4].

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