

PCA, CCA & Systems Identification

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Outline

- ▶ SVD & Eigendecomposition
- ▶ Principal components analysis
- ▶ Primer: canonical correlation analysis
- ▶ Systems identification

Refresher: SVD and PCA

Singular Value Decomposition

Any complex matrix M of size $m \times n$ can be decomposed as:

$$M = U\Sigma V^* \quad (\text{or } V^T \text{ if } M \text{ is real})$$

$m \times n$

$m \times m$
 $U^*U = UU^* = I$
 columns of U :
left singular vectors
 (orthogonal basis)

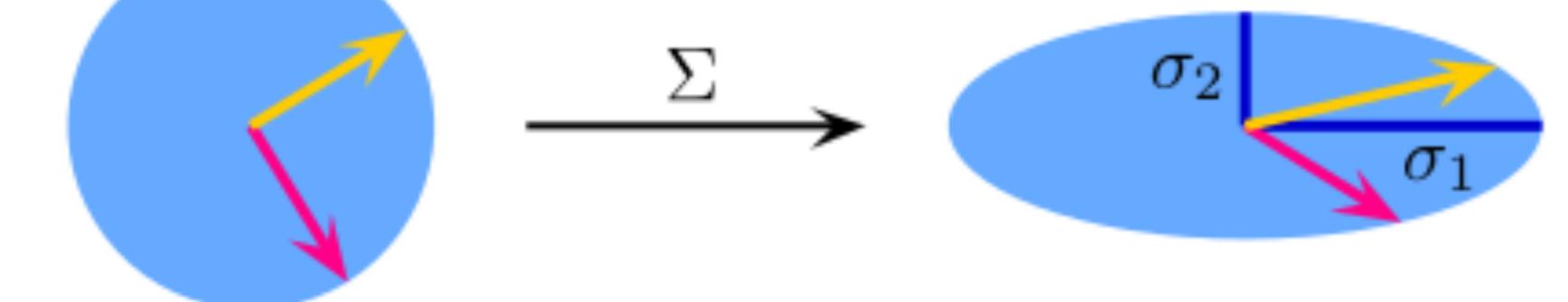
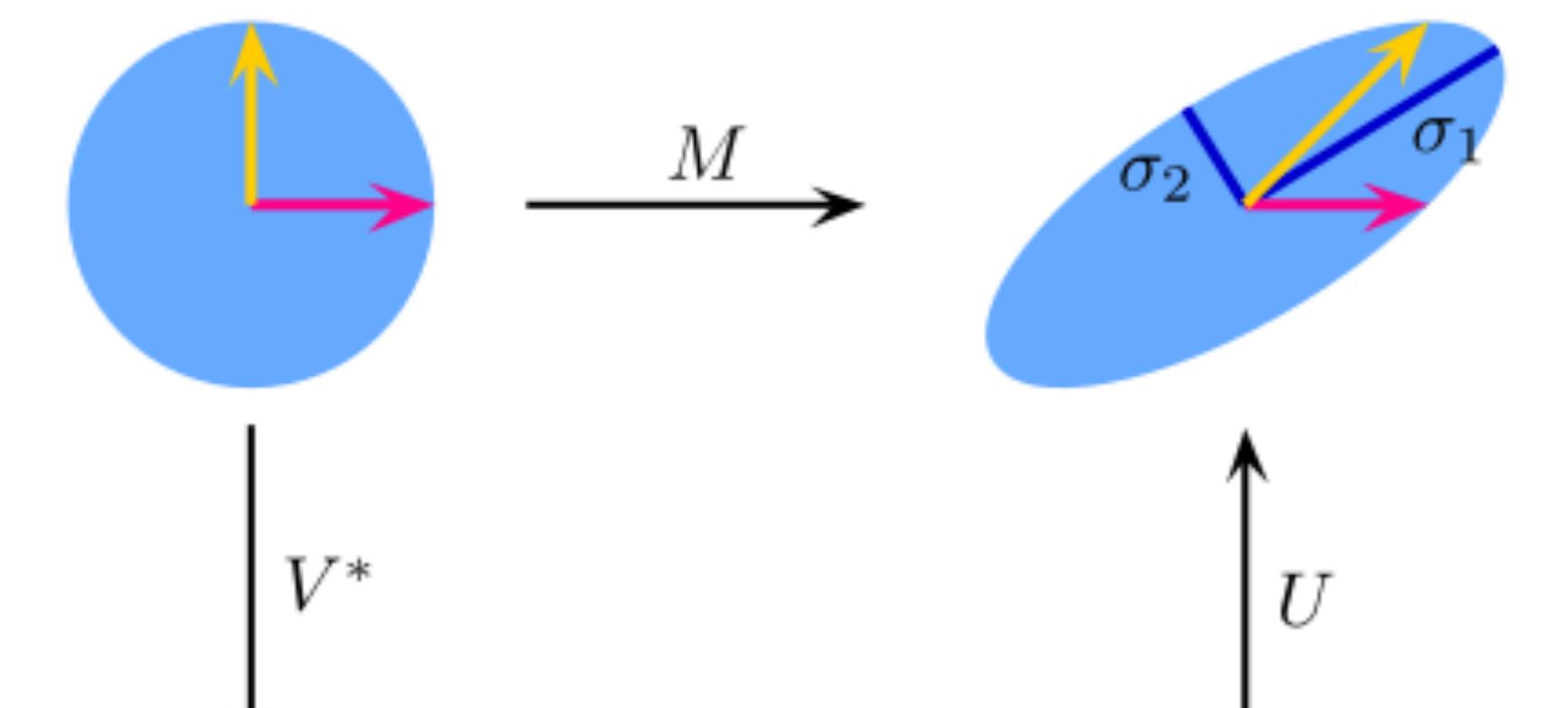
$n \times n$
 $V^*V = VV^* = I$
 columns of V :
right singular vectors
 (orthogonal basis)

$\Sigma = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & \sigma_n \\ \vdots & \vdots & \vdots \end{pmatrix}$
 $m \times n$

Σ is uniquely determined by $M \rightarrow \sigma_i$ are the **singular values**

The number of $\sigma_i > 0$ is the rank of M

Geometric interpretation



Eigendecomposition

If M is square ($n \times n$) and diagonalizable

$$M = Q\Lambda Q^{-1}$$

eigenvectors

$$Q = (\vec{q}_1 \ \vec{q}_2 \ \dots \ \vec{q}_n)$$

(orthogonal)

eigenvalues

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & \lambda_n \end{pmatrix}$$

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$$

A vector \mathbf{q} is said to be an eigenvector of M if

$$M\vec{q} = \lambda\vec{q}$$

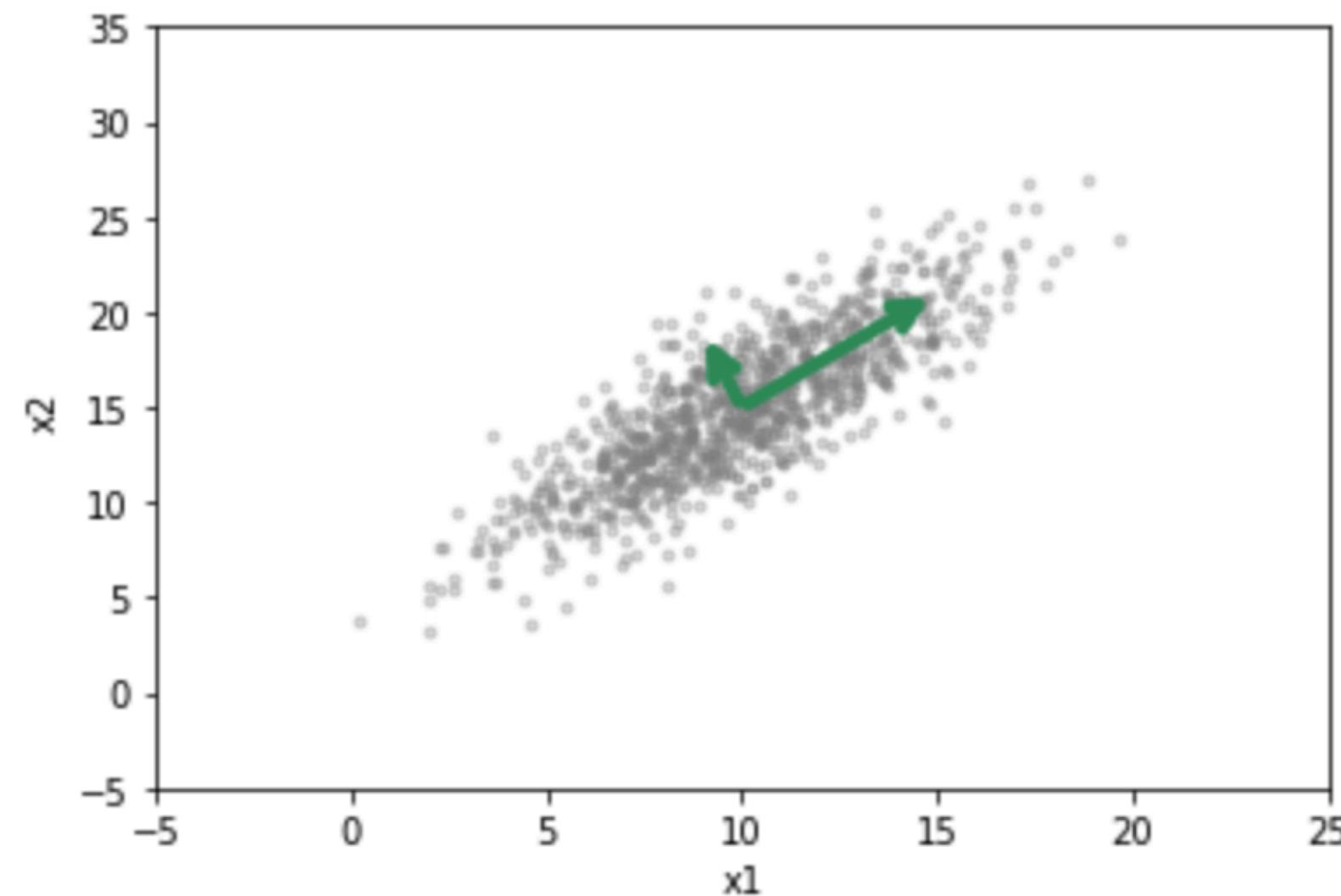
Equivalently, for a matrix of \mathbf{q} 's

$$MQ = Q\Lambda$$

$$M = Q\Lambda Q^{-1}$$

PCA: overview

We want to linearly transform (rotate) the coordinate system into a set of orthogonal dimensions.



We do this by successively projecting the data along the axis of highest variance, given by the vector z .

$$z = Xw$$

Assuming the data area centered, we want to maximize the variance along z

$$\underset{\|w\|=1}{\operatorname{argmax}} \{ \|Xw\|^2\}$$

$$\underset{\|w\|=1}{\operatorname{argmax}} \{(Xw)^T Xw\}$$

$$\underset{\|w\|=1}{\operatorname{argmax}} \{w^T \boxed{X^T X} w\}$$

- ▶ w is the largest eigenvector of the covariance matrix $X^T X$ (captures most variance). It is the first PC of the data

The next PC is found by repeating the procedure on a matrix with PC1 subtracted (and so forth)

$$X_2 = X - z w^T$$

- ▶ Note that the centering means that we rotate x about the mean
- ▶ Maximizing variance \equiv minimizing reconstruction error

PCA via eigendecomposition of $X^T X$

More generally,

$$Z = XW \quad n \times d$$
$$Z = (\vec{z}_1 \quad \vec{z}_2 \quad \dots \quad \vec{z}_n) \quad (a.k.a. \text{ PC scores})$$

$d \times d$
Columns of W are the PCs, i.e. the eigenvectors of the data covariance matrix

$$X^T X = C = W\Lambda W^{-1} = W\Lambda W^T \quad (\text{because } C \text{ is real symmetric})$$

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & \lambda_n \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & \sigma_n^2 \end{pmatrix}$$

The loadings of the datapoints onto each PC are given by $\vec{w}_i \times \sqrt{\lambda_i}$

So we can simply compute C and perform eigendecomposition.
But computationally more expensive than the alternative

PCA via SVD of X

A much more common implementation is via SVD, bypassing the expensive computation of $X^T X$

$$X = U \Sigma W^T$$

$$X^T X = (U \Sigma W^T)^T U \Sigma W^T$$

$$= W \Sigma^T U^T U \Sigma W^T$$

$$= W \Sigma^T I \Sigma W^T$$

$$= W \Sigma^T \Sigma W^T$$

$$= W \hat{\Sigma}^2 W^T$$

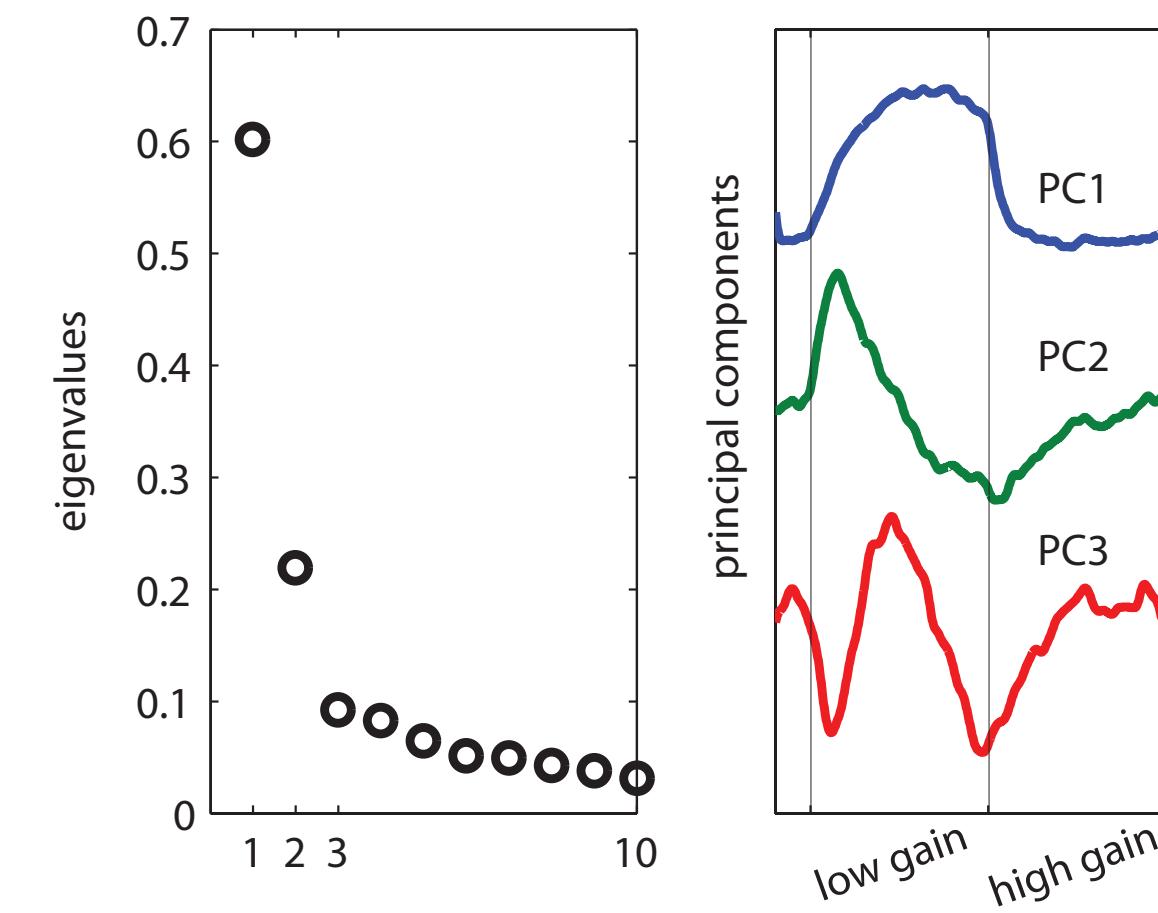
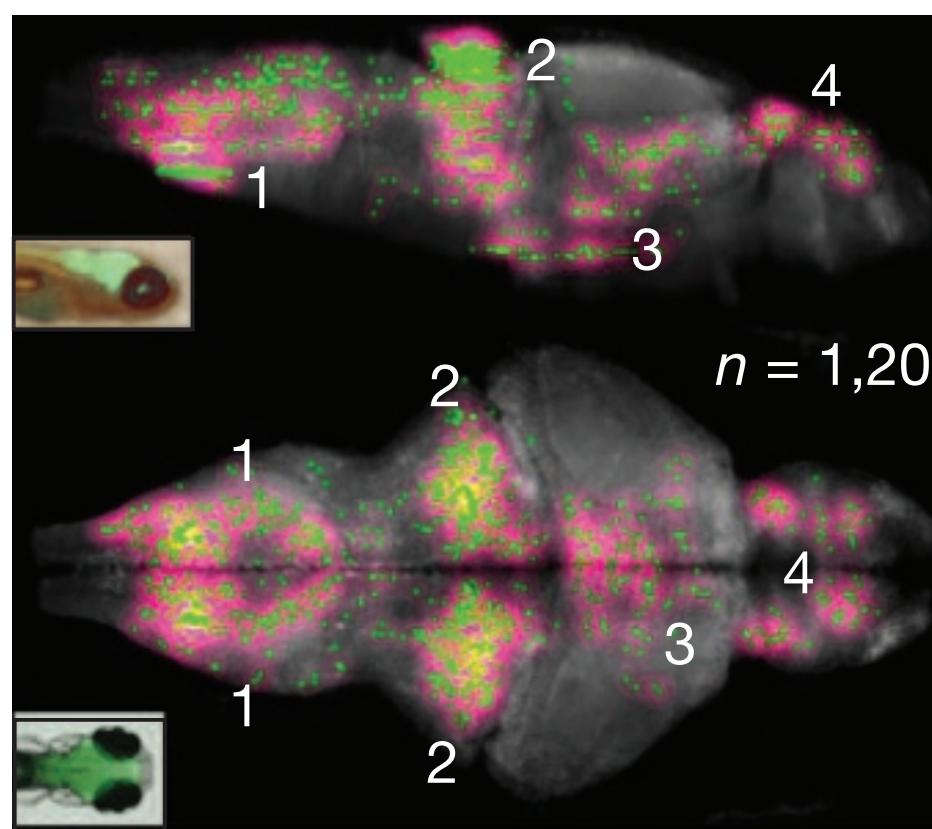
- ▶ $\hat{\Sigma}^2$ is a square diagonal matrix without the bottom rows of 0's, with the singular values of X , equal to the `sqrt()` of the eigenvalues of $X^T X$

- ▶ W is equivalent to eigenvectors of $X^T X$

- ▶ And the PC scores: $Z = XW = U \Sigma W^T W = U \Sigma$

PCA as dimensionality reduction

Neural dynamics are often much lower-dimensional than full activity space (but that needs to be interpreted w.r.t. behavior)



Ahrens et al., 2012, *Nature*

Because we transform the data into components of progressively smaller variance, we can discard the trailing PCs to reduce the dimensionality of the data

Similarly, we can project the data down to k dimensions to reconstruct, denoise etc.

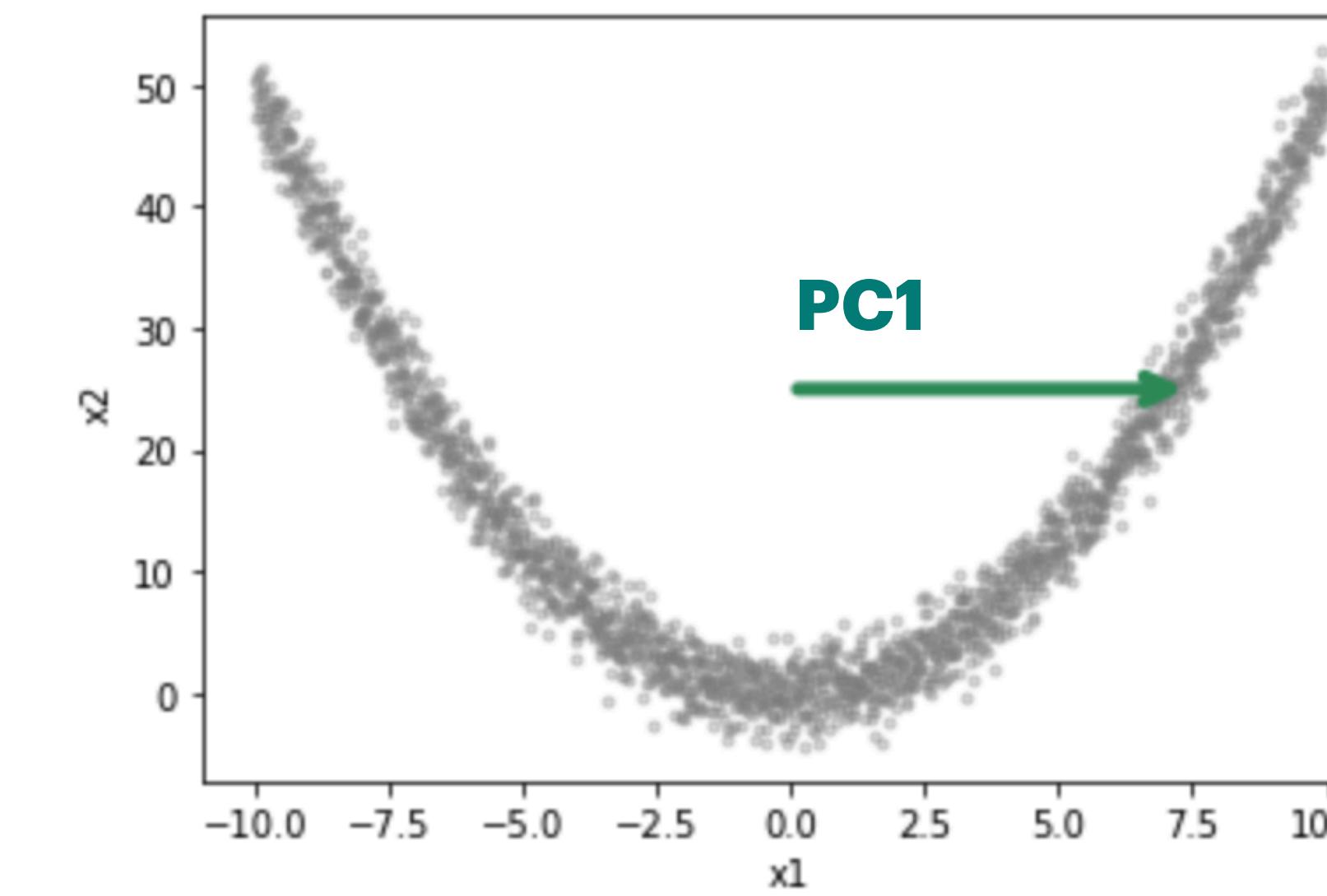
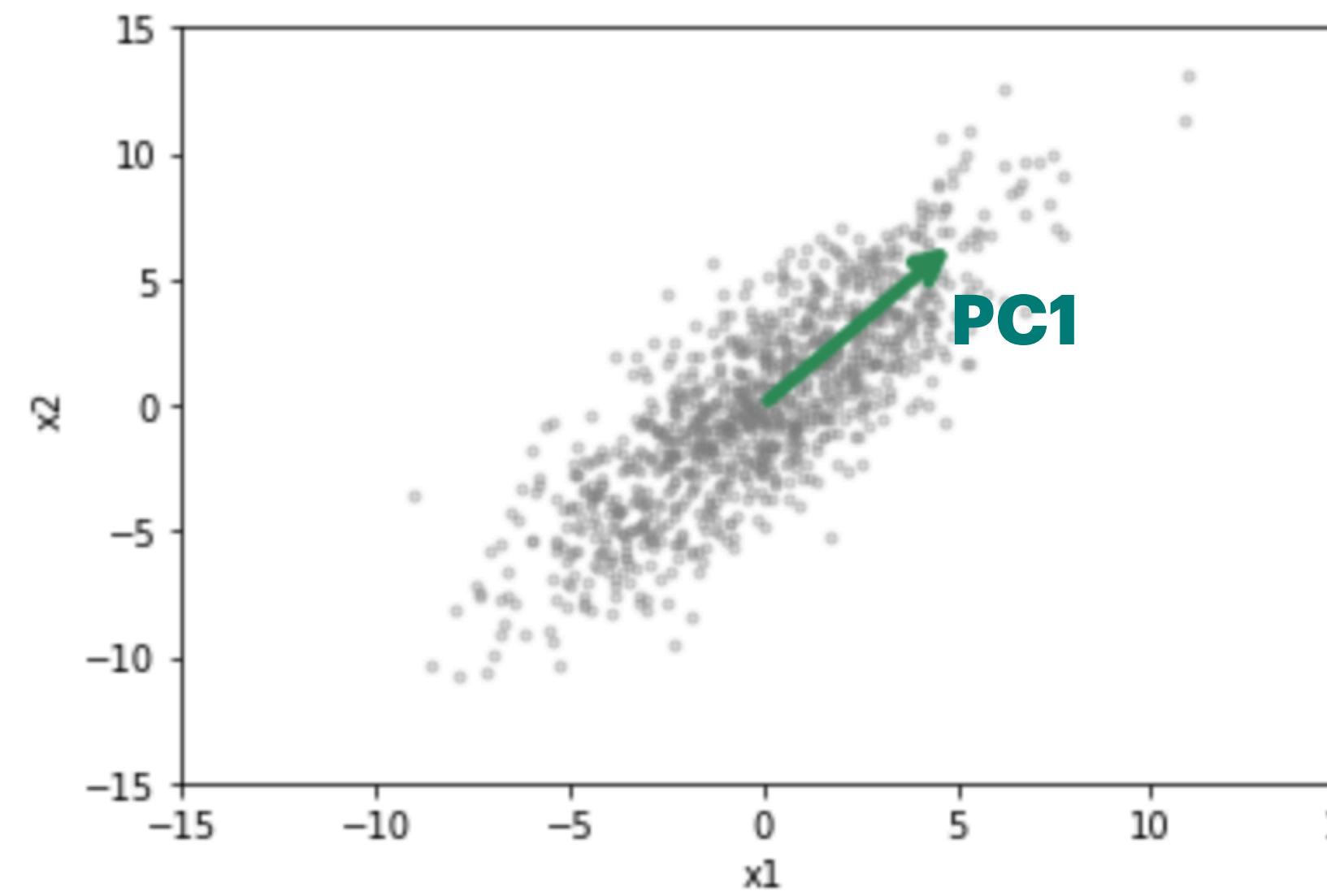
$$Z = XW$$
$$n \times d \quad n \times d \quad d \times d$$

$$\tilde{X} = Z_k W_k^T$$
$$n \times d \quad n \times k \quad k \times d$$

Some applications in neuroscience

- ▶ PCA: dimensionality reduction for clustering, e.g. spike sorting
- ▶ PCA/SVD: dimensionality reduction for feature selecting of high-D datasets going into a GLM (e.g. movies)
- ▶ PCA: visualizing / analyzing high-D neural dynamics projected onto a low-D space
- ▶ SVD/Eigen: analyzing connectivity matrices of RNNs
- ▶ SVD/Eigen: Systems identification (next)
- ▶ And many others

Reminder: these are linear methods!

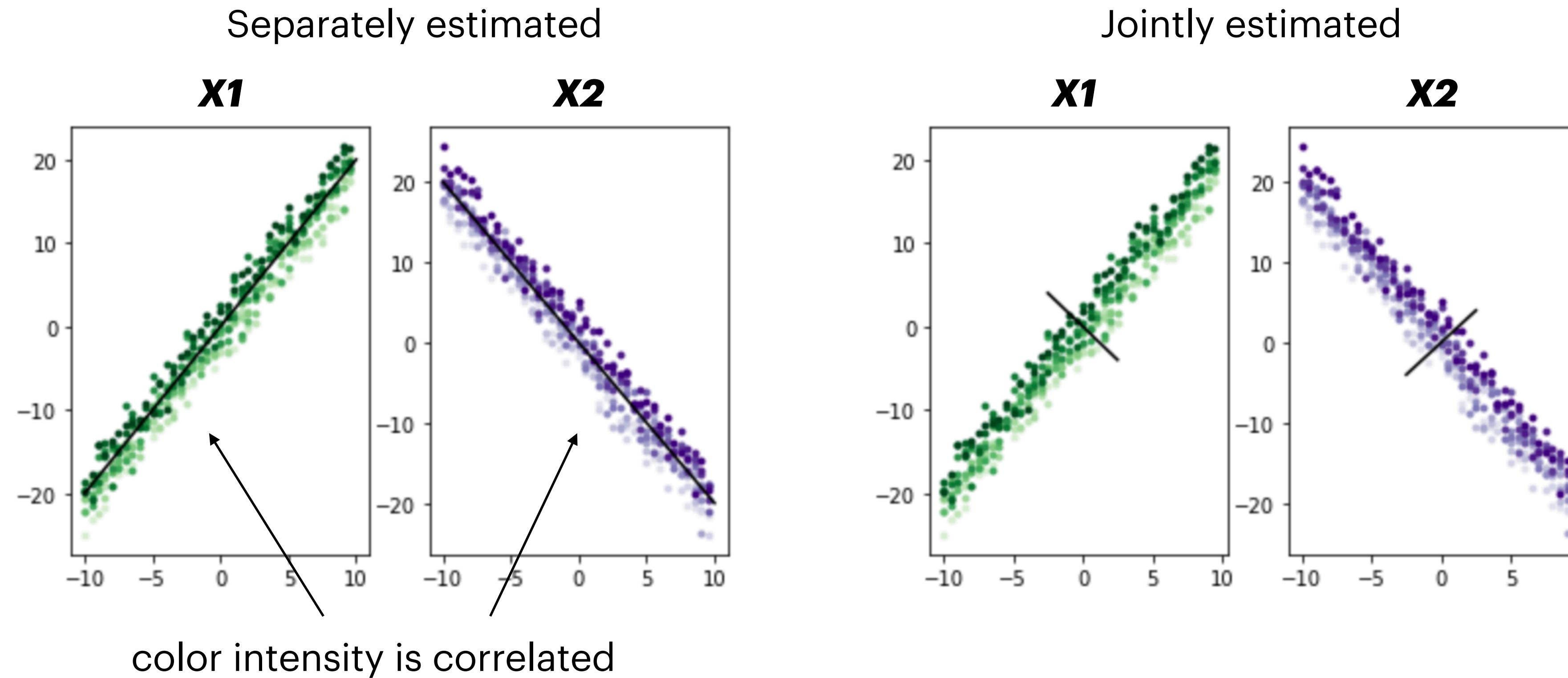


- ▶ Josh Glaser will discuss some non-linear dimensionality reduction techniques

CCA primer

CCA: Intuition

CCA is an extension of PCA for two variables that aims to find the axes along which these variables are maximally correlated



In other words, CCA jointly reduces the dimensionality of X_1 and X_2

CCA: Computation

The separate PCAs would be given by maximizing the covariance

$$\operatorname{argmax} \left\{ \frac{w_1^T X_1^T X_1 w_1}{w_1^T w_1} \right\}$$

$$\operatorname{argmax} \left\{ \frac{w_2^T X_2^T X_2 w_2}{w_2^T w_2} \right\}$$

Maximizing the cross-covariance

$$\operatorname{argmax} \left\{ \frac{w_1^T X_1^T X_2 w_2}{\sqrt{w_1^T w_1} \sqrt{w_2^T w_2}} \right\}$$

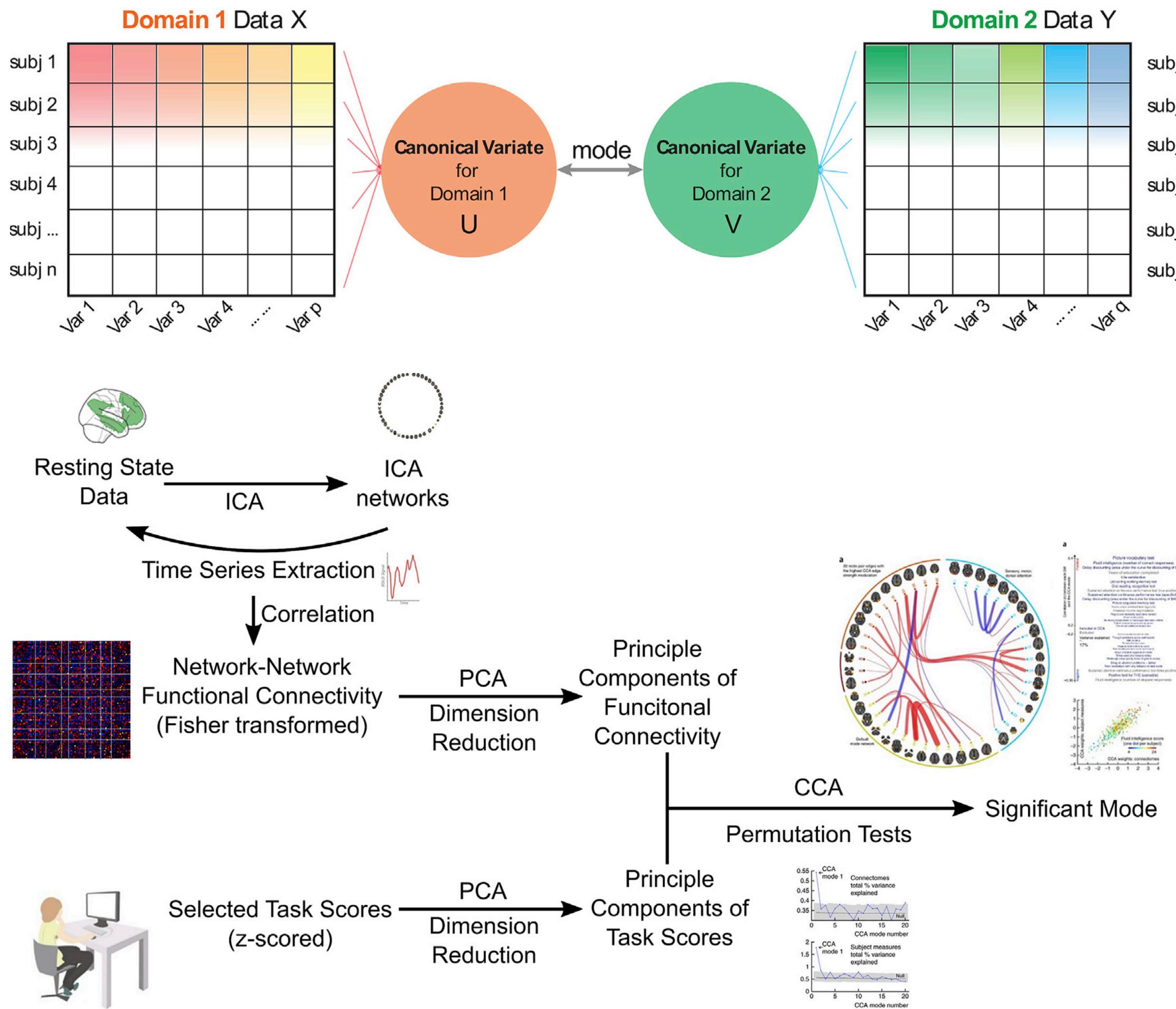
And the correlation (CCA)

$$\operatorname{argmax} \left\{ \frac{w_1^T X_1^T X_2 w_2}{\sqrt{w_1^T X_1^T X_1 w_1} + \sqrt{w_2^T X_2^T X_2 w_2}} \right\}$$

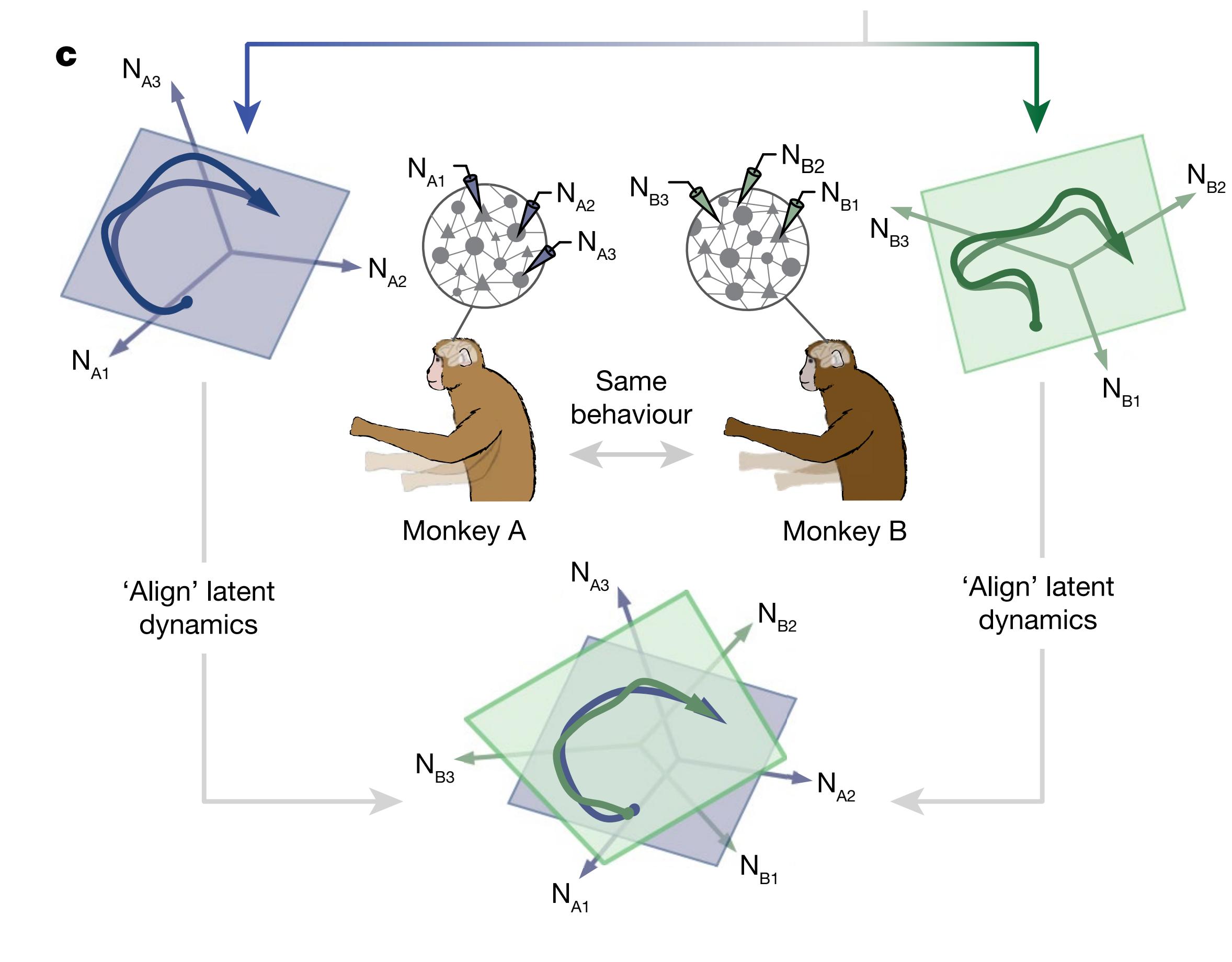
- ▶ Similarly solved w/ eigendecomposition of correlation matrix

CCA: application examples

- Identify common sources of variation in multimodal data (e.g. non-simultaneous imaging <-> behavior)

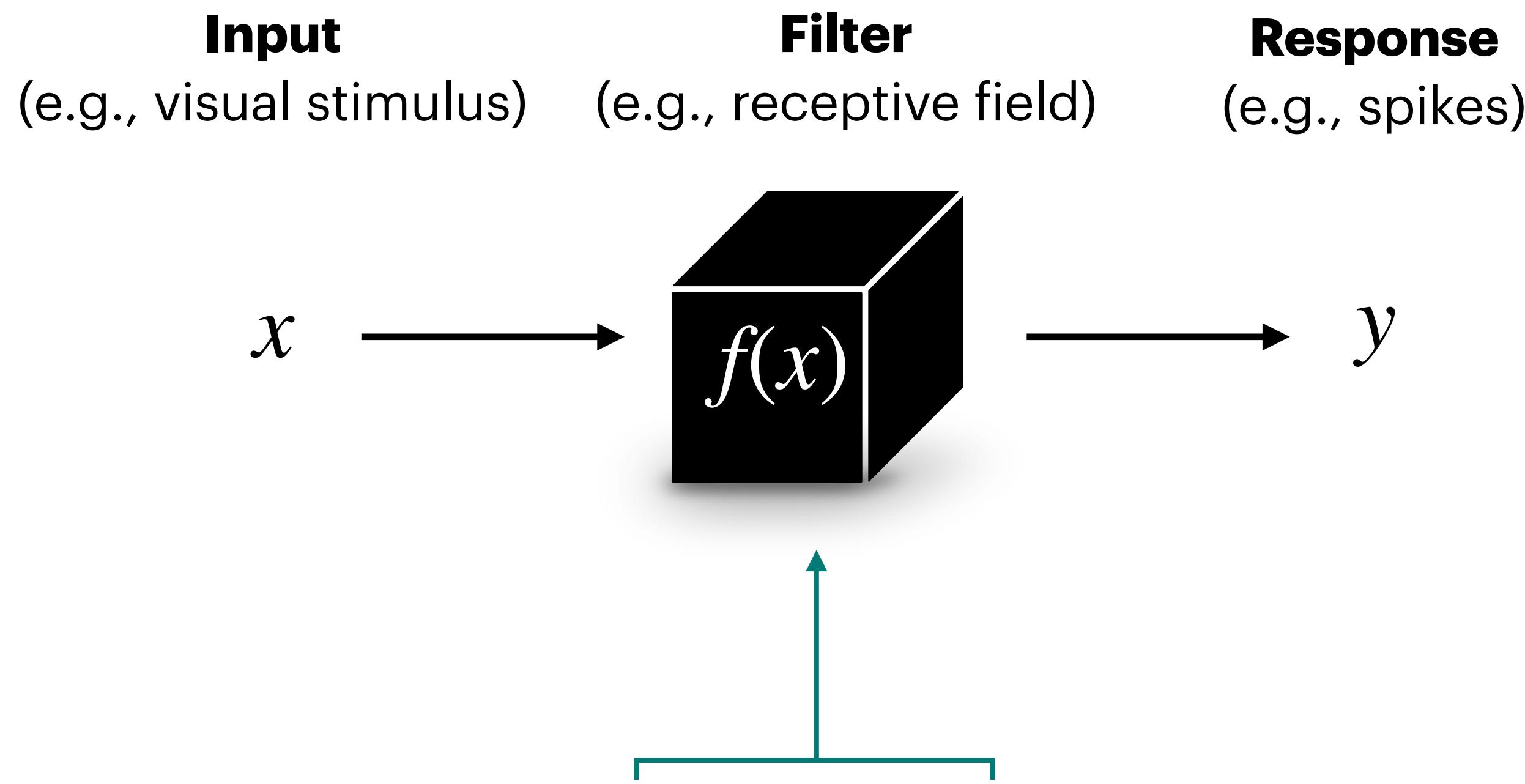


- Align dynamics across subjects, species, areas



Systems Identification

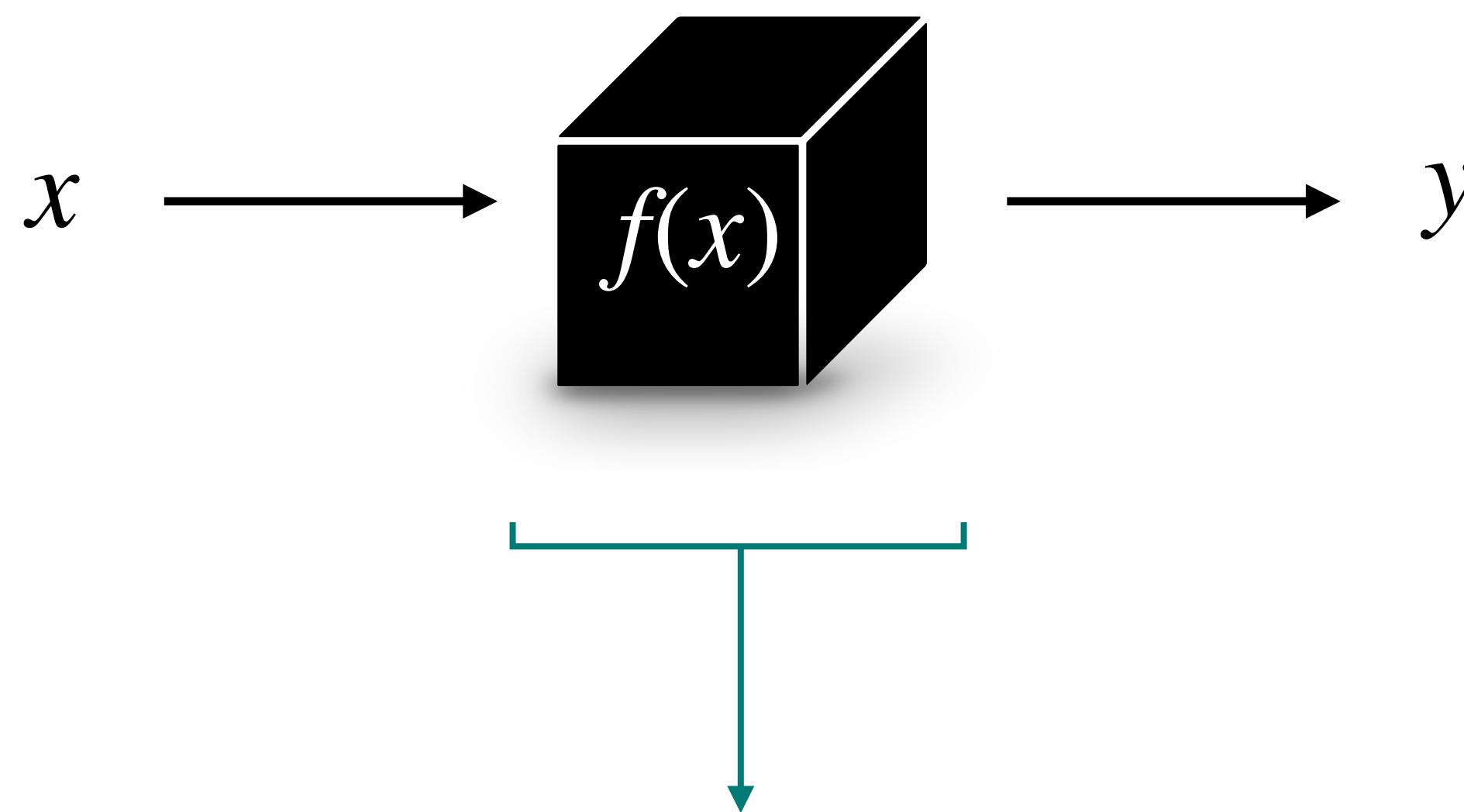
A common approach



We design stimuli based on
a hypothesis about $f(x)$ (e.g., oriented bars),
and average y over many presentations of the same
set of stimuli

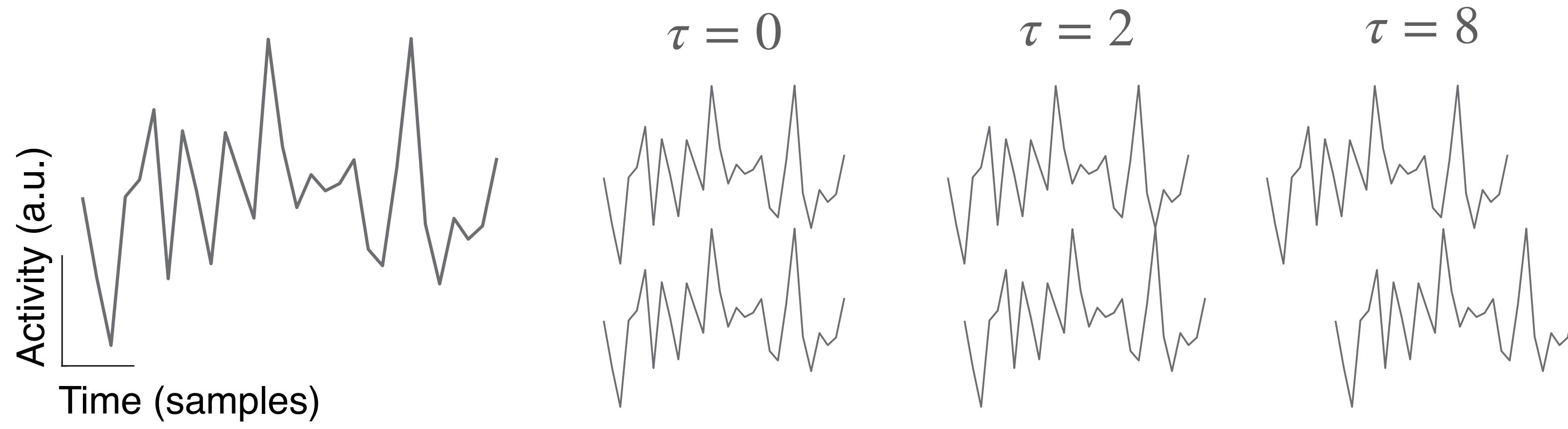
Goal of systems identification

Input
(e.g., visual stimulus) **Filter**
(e.g., receptive field) **Response**
(e.g., spikes)



We want to estimate $f(x)$ in a data-driven
way, without opening the black box

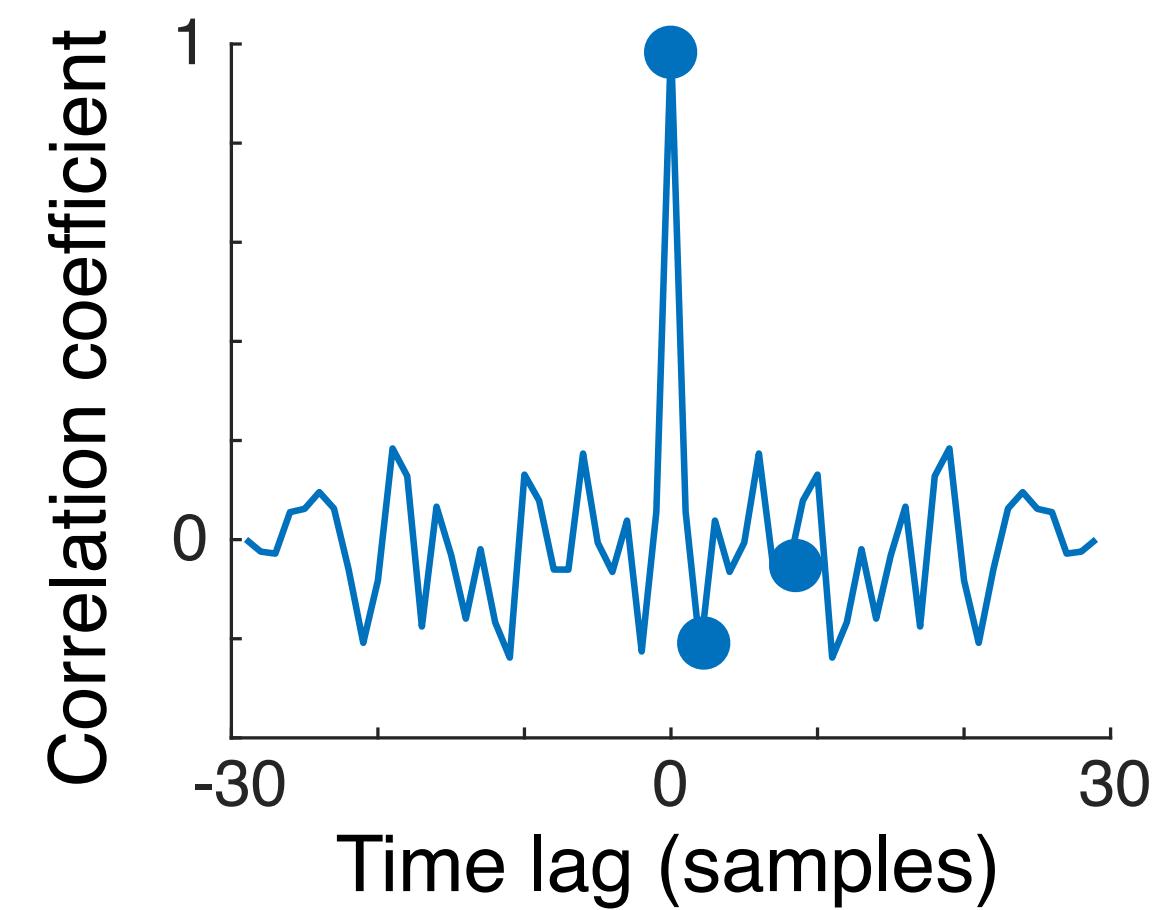
Auto- and cross-correlation



More formally,

Auto-correlation $\phi_{xx}(\tau) = \int_0^{\infty} x(t)x(t - \tau)dt$

Cross-correlation $\phi_{xy}(\tau) = \int_0^{\infty} y(t)x(t - \tau)dt$

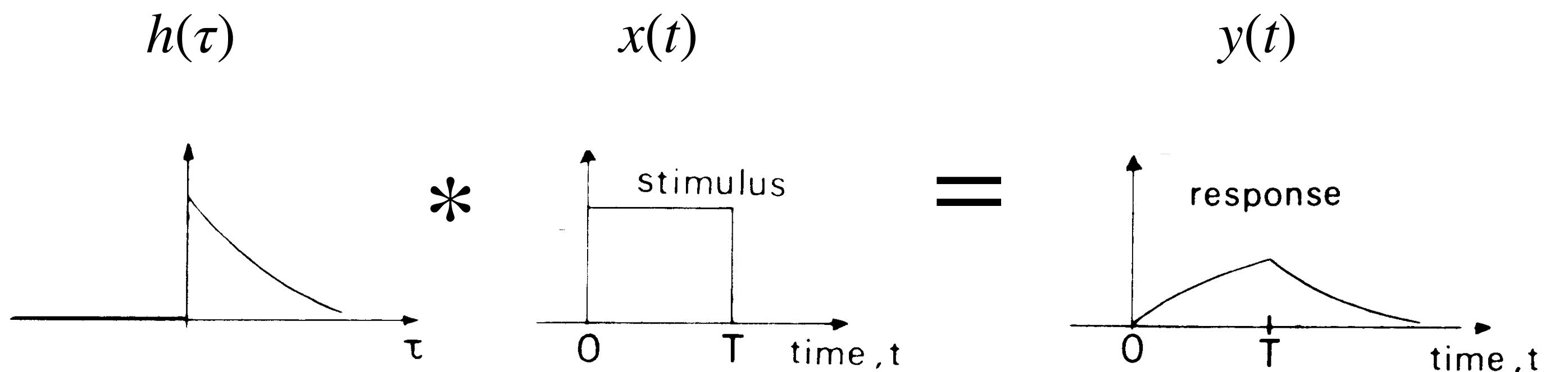


- ▶ function is symmetrical around $x=0$ but we usually just want the causal delays for systems identification
- ▶ Notice that the autocorrelation for gaussian (white) noise is a delta function

Impulse response function (time domain)

The response y of a filter to an input x is given by the convolution of its kernel h and x .

$$y(t) = \int_0^\infty h(\tau)x(t - \tau)d\tau$$

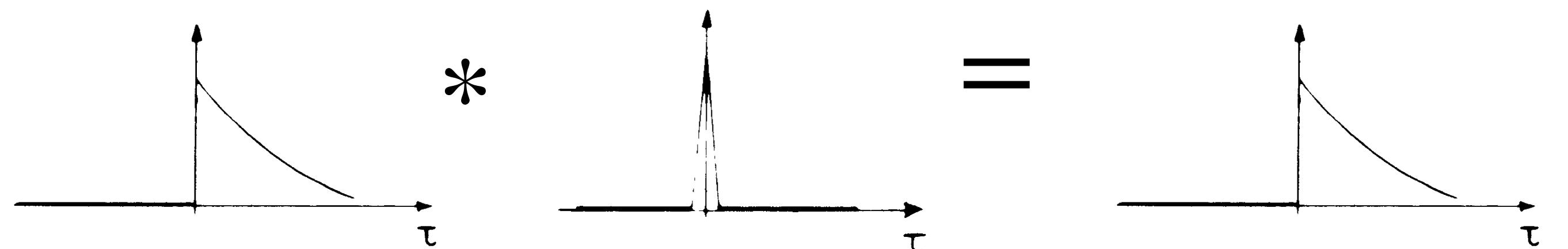


The response y of a filter to a delta input x is equal to its kernel h .

$$\delta(t) = \begin{cases} 1 & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$y(t) = \int_0^\infty h(\tau)\delta(t - \tau)d\tau$$

$$y(t) = h(t)$$



Marmarelis & Marmarelis, 1978

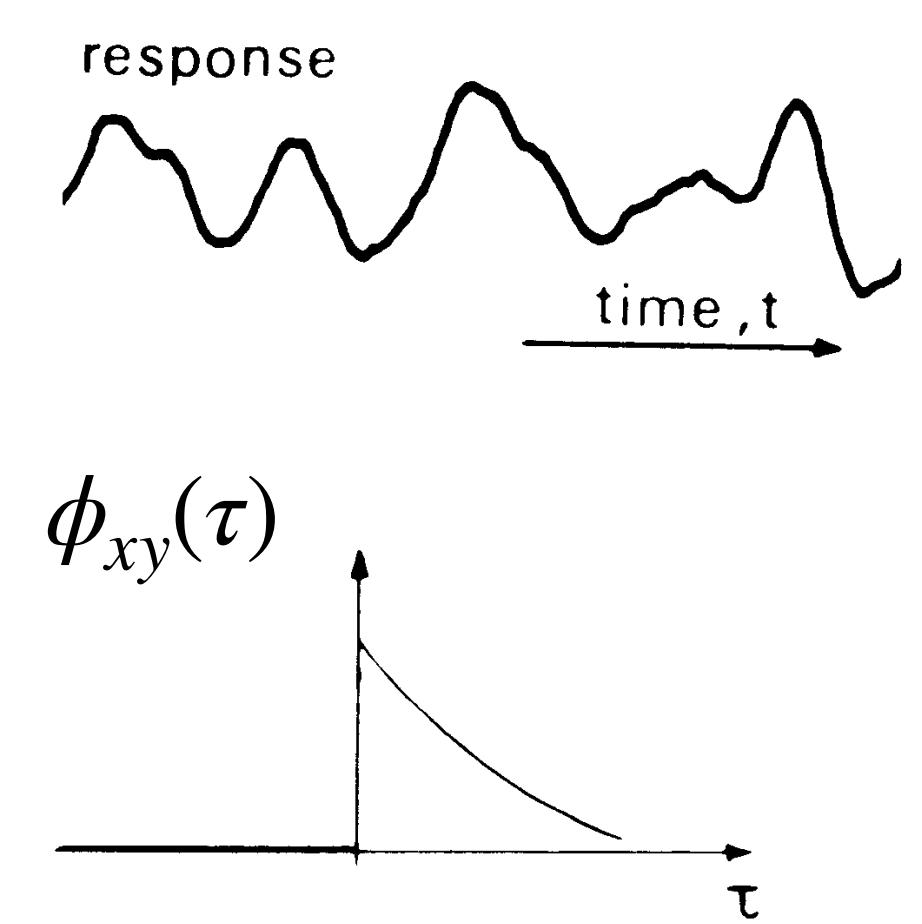
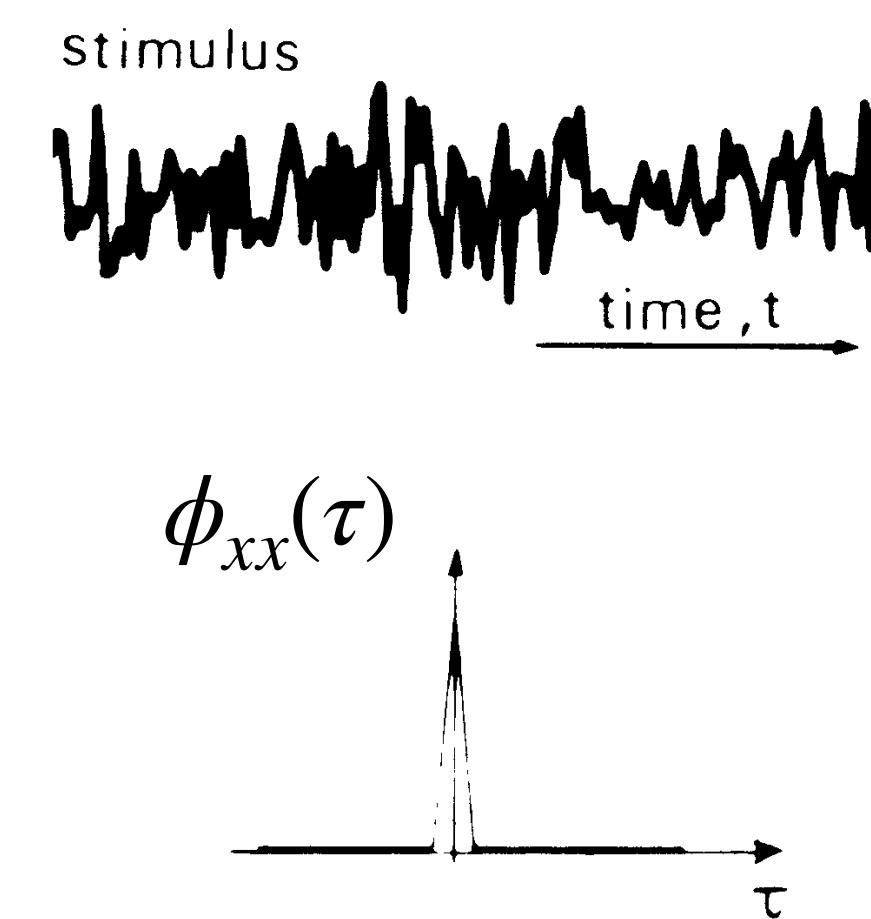
Relationship to auto- (cross-) correlation

A useful property: if this system is linear, time invariant, and has finite memory, the same relationship is true for the auto (cross-) correlation functions

$$\begin{aligned}\phi_{xy}(\tau) &= \int_0^\infty y(t)x(t - \tau)dt \\ &= \int_0^\infty \int_0^\infty h(\lambda)x(t - \lambda)x(\tau - t)dtd\lambda \\ &= \int_0^\infty h(\lambda)\phi_{xx}(\tau - \lambda)d\lambda\end{aligned}$$



If the autocorrelation is a delta function (e.g., white noise), $h()$ is just a scaled version of the cross-correlation function ϕ_{xy}



Similarly:

$$\begin{aligned}\phi_{yy}(\tau) &= \int_0^\infty y(t)y(t - \tau)dt \\ &= \int_0^\infty h(\mu) \left[\int_0^\infty h(\lambda)\phi_{xx}(\tau - \mu - \lambda)d\lambda \right] d\mu\end{aligned}$$

Marmarelis & Marmarelis, 1978

Frequency domain

It is often more convenient to do these operations in frequency domain, where all the integrals just become simpler algebra

The power spectrum of a signal is equivalent to the Fourier Transform of its autocorrelation function

$$\phi_{xx}(\tau) = \int_0^\infty x(t)x(t - \tau)dt$$

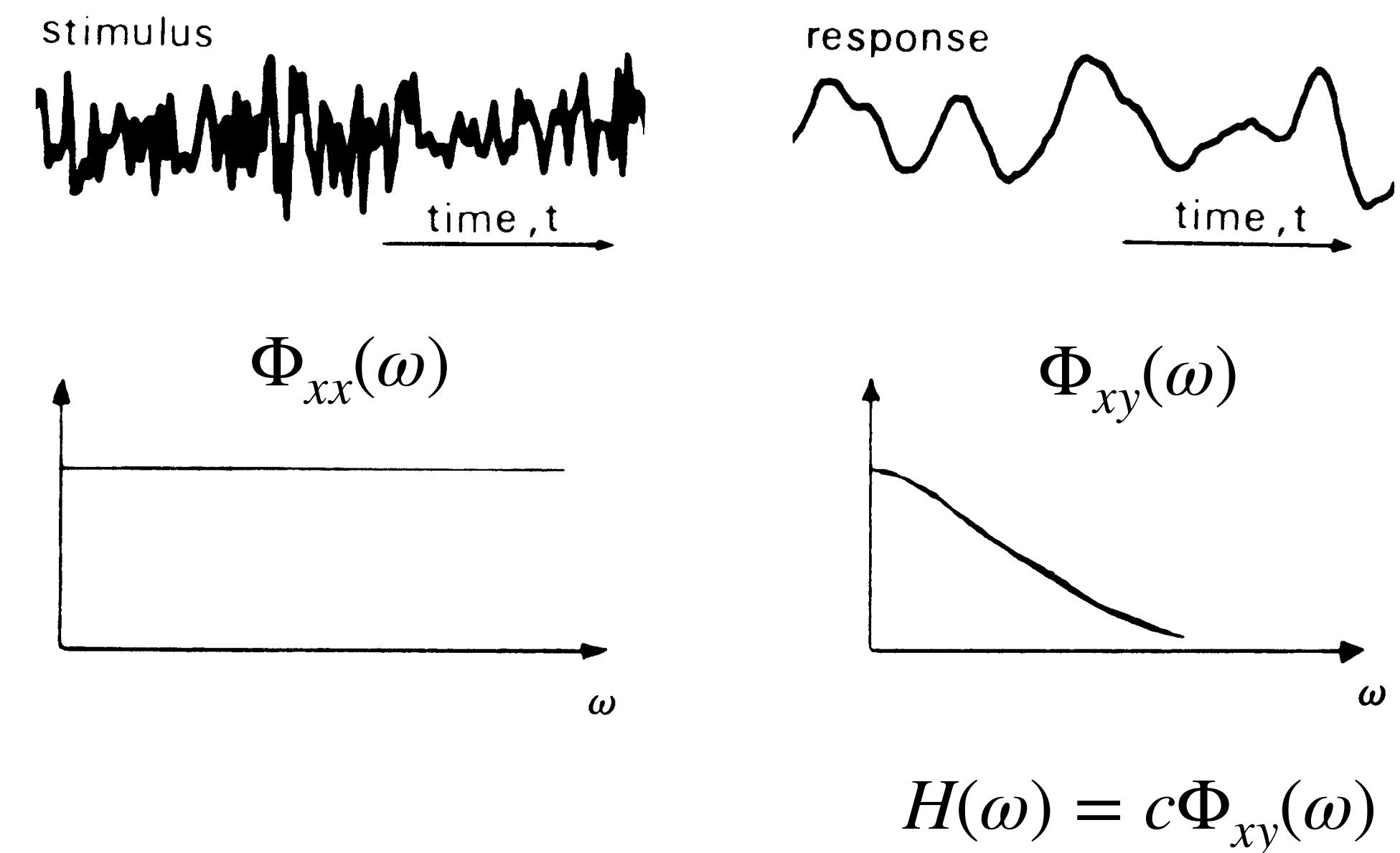
$$\Phi_{xx}(\omega) = \int_0^\infty \phi_{xx}(\tau)e^{-i\omega\tau}d\tau$$

Equivalently (from the previous slide)

$$\Phi_{xy}(\omega) = H(\omega)\Phi_{xx}(\omega) \longrightarrow H(\omega) = \frac{\Phi_{xy}(\omega)}{\Phi_{xx}(\omega)}$$

$$\Phi_{yy}(\omega) = |H(\omega)|^2 \Phi_{xx}(\omega)$$

White noise has a flat power spectrum, so you just multiply $H(\omega)$ by a constant

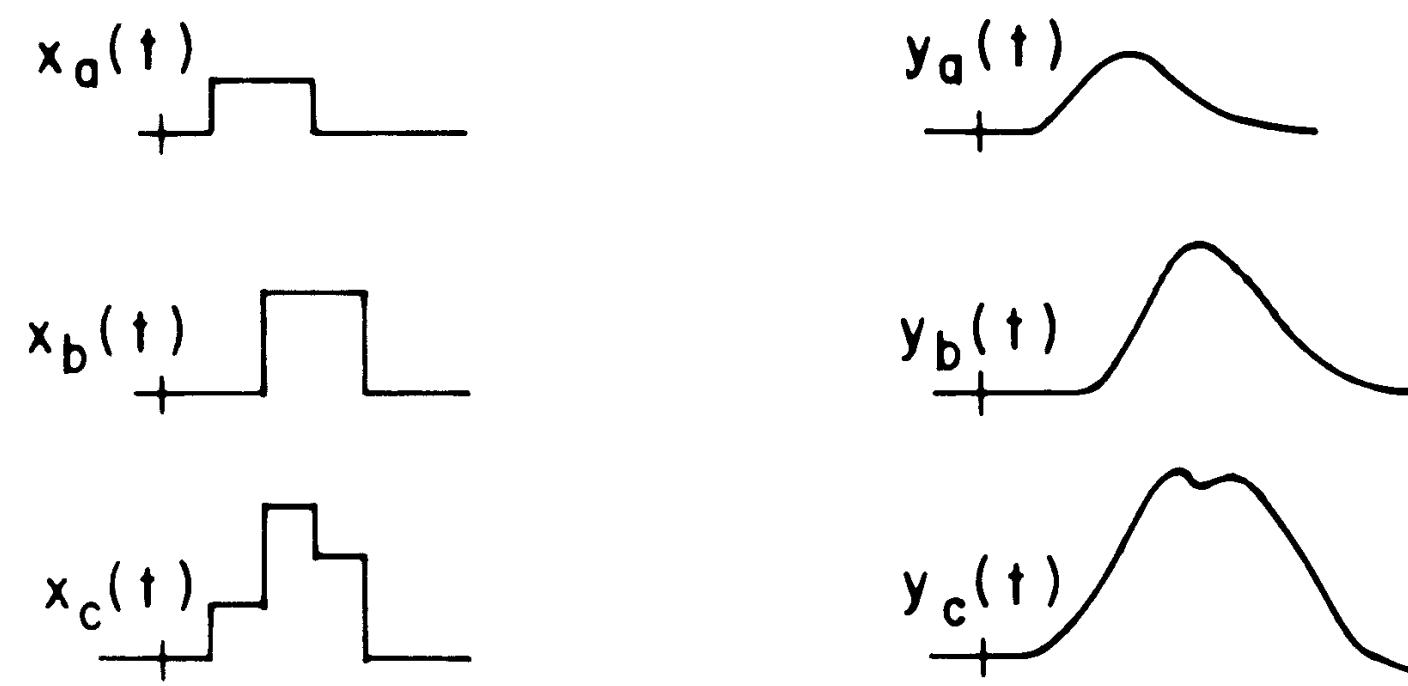


$$H(\omega) = c\Phi_{xy}(\omega)$$

Marmarelis & Marmarelis, 1978

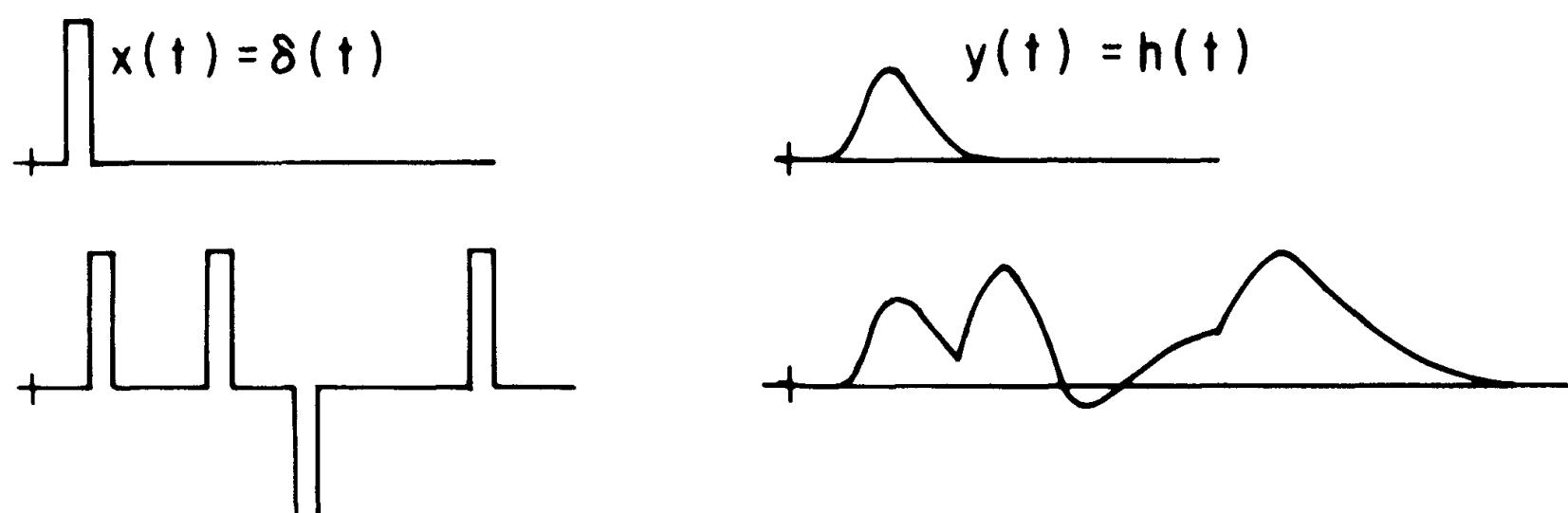
Estimating a linear system

Principle of superposition: A linear system's response to a combined input is the a linear combination of the response to individual inputs



If your input is a train of deltas, the output is just the sum of kernels over N pulses, scaled by their amplitude

$$x(t) = \sum_{k=1}^N a_k \delta(t - t_k) \quad y(t) = \sum_{k=1}^N a_k h(t - t_k)$$



In practice, the crosscorrelation ϕ_{xy} is usually estimated with averaging, indicated by $\langle \rangle$, using the time of the input as reference

$$\phi_{xy}(\tau) = \langle x(t)y(t + \tau) \rangle, \tau > 0$$

An equivalent way of doing it is going in the other direction, i.e. picking the response time as a reference and asking what average stimulus preceded that response

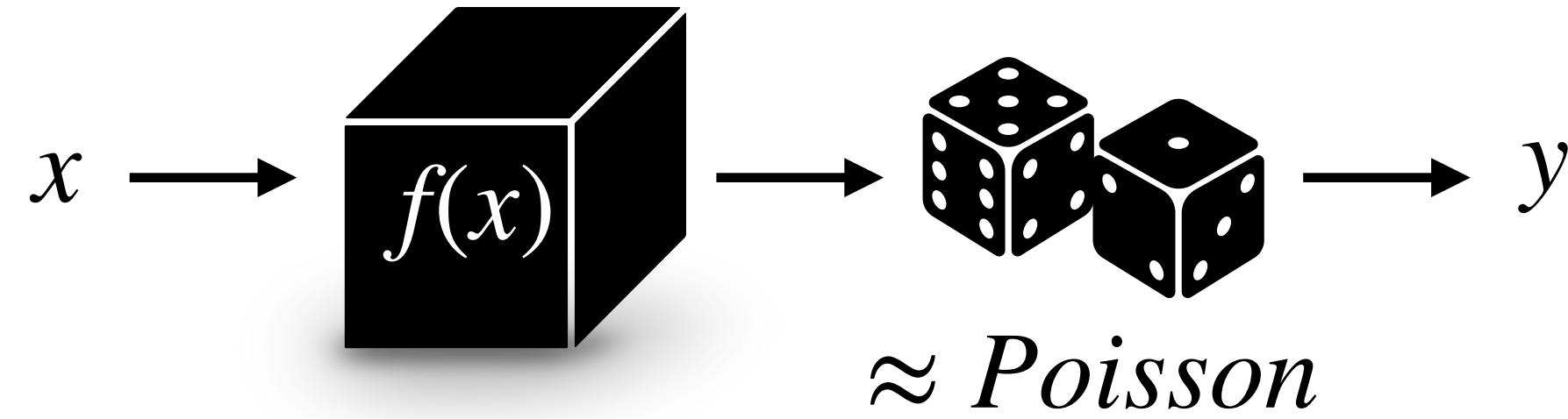
$$\mu = t + \tau$$

$$\phi_{xy}(\tau) = \langle x(\mu - \tau)y(\mu) \rangle = \langle x(t)y(t + \tau) \rangle$$

- In neuro, this is often referred to as **reverse correlation** (but note we're still estimating the same ϕ_{xy})

Estimating a linear neural system

Neurons are not linear because of the spiking non-linearity, but in some cases that can be thought of as a non-linearity on top of a linear operation



So, when x is a delta function, $h()$ is obtained by computing the average stimulus that preceded each spike occurring at time s

$$\phi_{xy}(\tau) \approx \frac{1}{N} \sum_{i=1}^N x(s_i - \tau)$$

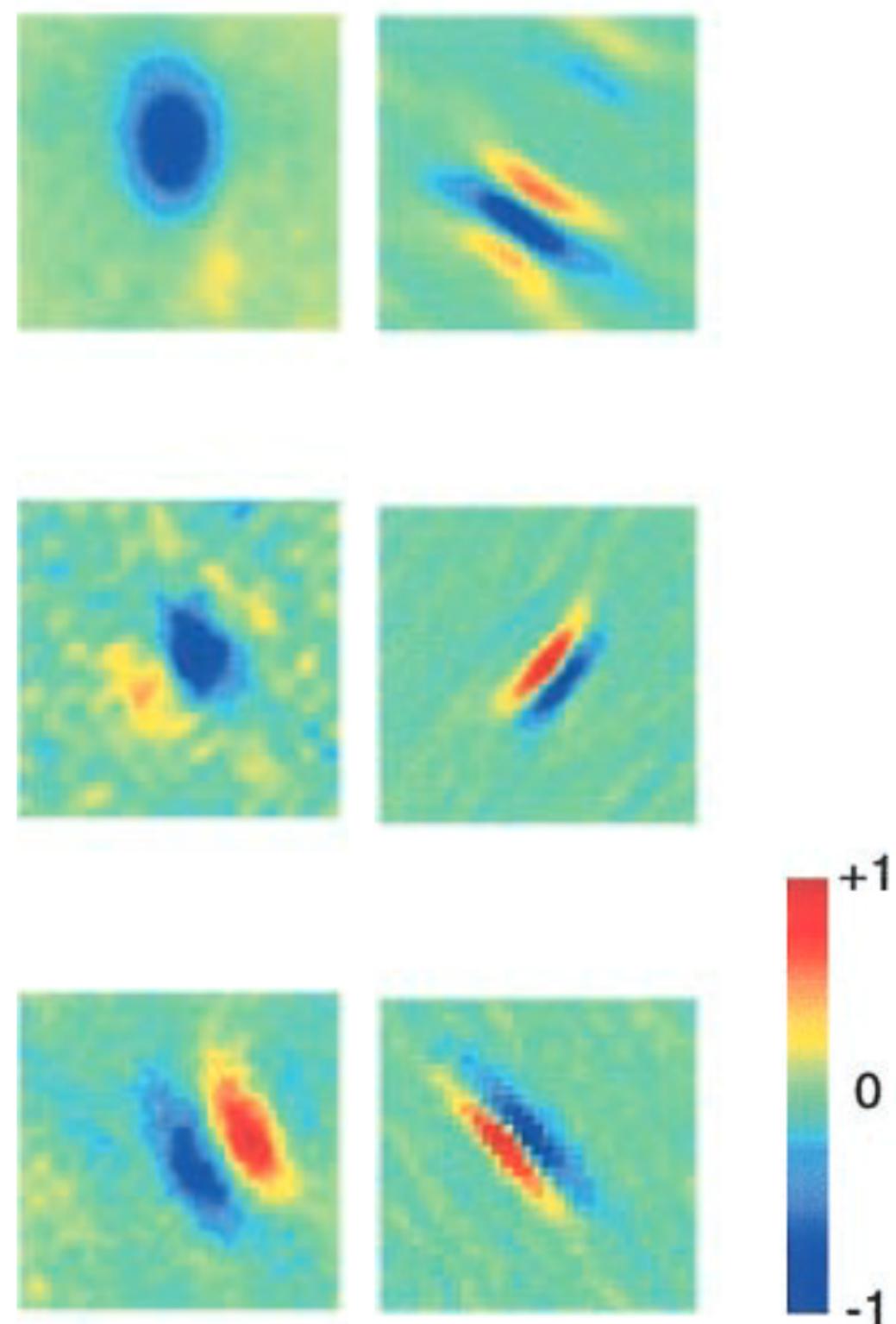
- ▶ This is known as the *spike-triggered average* (STA)
- ▶ x does not need to be unidimensional. E.g., for a visual receptive field, what we typically want to estimate is $h(x,y,t)$. But everything still applies

In this case it can be shown that the cross-correlation is still proportional to $h()$:

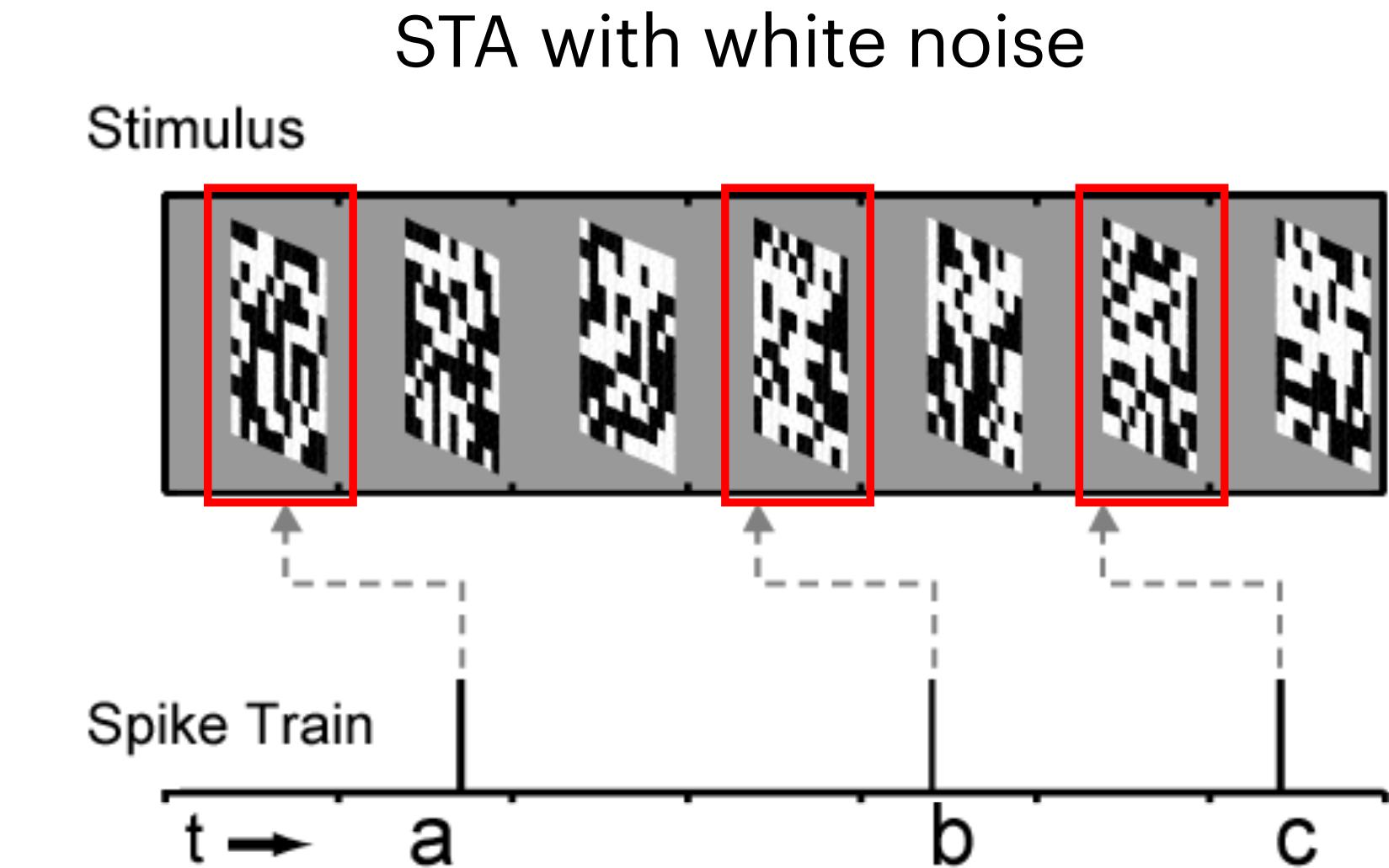
$$\phi_{xy}(\tau) = k \langle x(\mu - \tau)y(\mu) \rangle$$

Estimating a linear neural system: a V1 simple cell

A simple cell has separate excitatory and inhibitory subregions that combine linearly to determine its response to a stimulus inside its receptive field
(Responses are phase dependent)



Ringach, 2005, J Neurosci



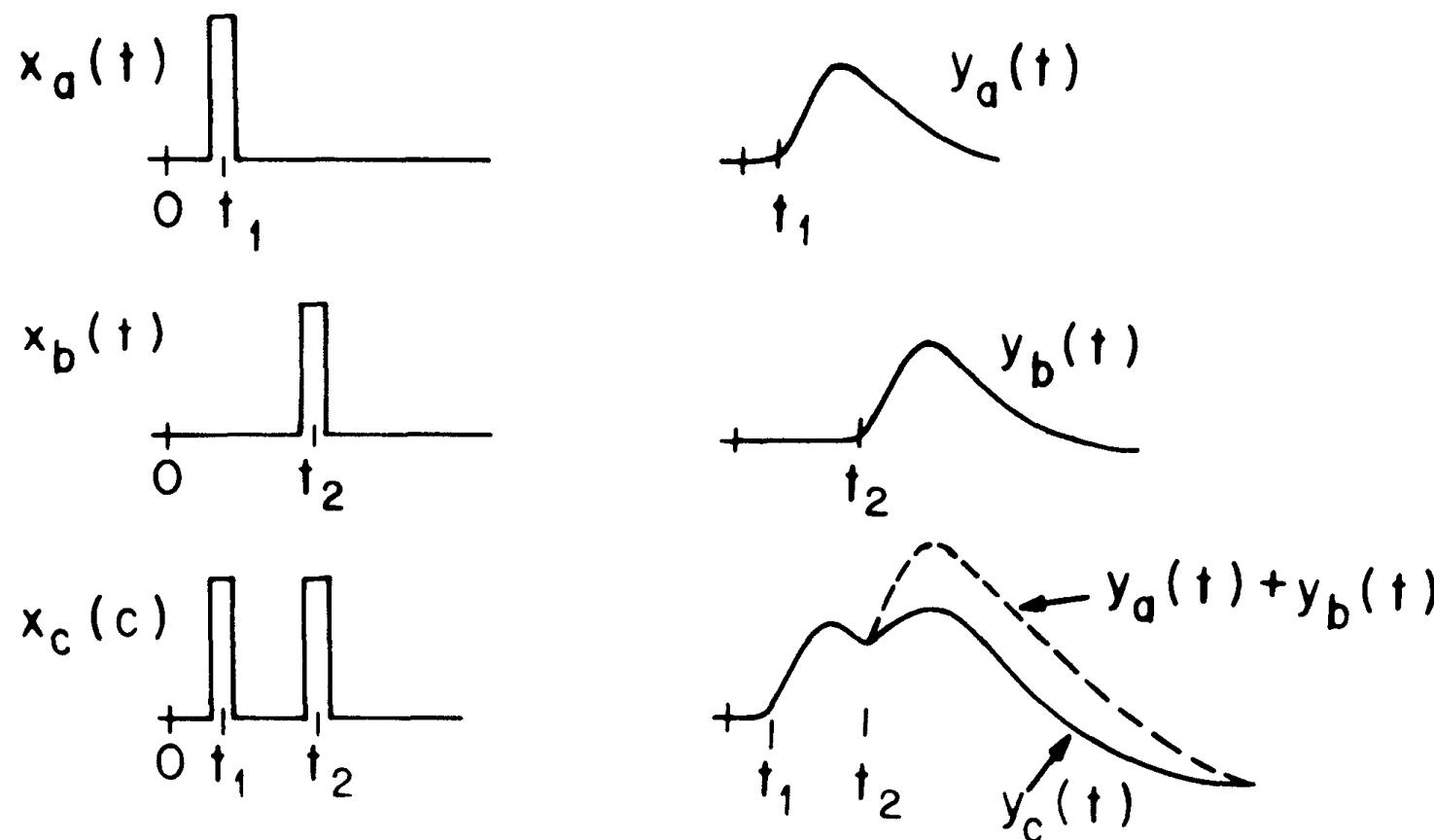
$$a + b + c + \dots = \text{STA}$$

From Yang Dan

(sparse or dense binary, grayscale white noise, basis sets of sinusoidal gratings, all work)

Estimating a non-linear system

A non-linear system does not follow the principle of superposition



Linear:

$$x_c(t) = x_a(t) + x_b(t)$$

$$y_c(t) = y_a(t) + y_b(t)$$

Non-linear:

$$x_c(t) = x_a(t) + x_b(t)$$

$$y_c(t) \neq y_a(t) + y_b(t)$$

$$y_c(t) = y_a(t) + y_b(t) + \theta_{t_1 t_2}(t)$$

interaction term

More generally, a time-invariant system with finite memory can be approximated by the sum of its nth-order kernels (interaction terms)

$$\begin{aligned} y(t) = & \int_0^{\infty} h_1(\tau)x(t - \tau)d\tau \\ & + \int_0^{\infty} \int_0^{\infty} h_2(\tau_1, \tau_2)x(t - \tau_1)x(t - \tau_2)d\tau_1 d\tau_2 \\ & + \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} h_3(\tau_1, \tau_2, \tau_3)x(t - \tau_1)x(t - \tau_2)x(t - \tau_3)d\tau_1 d\tau_2 d\tau_3 \dots \end{aligned}$$

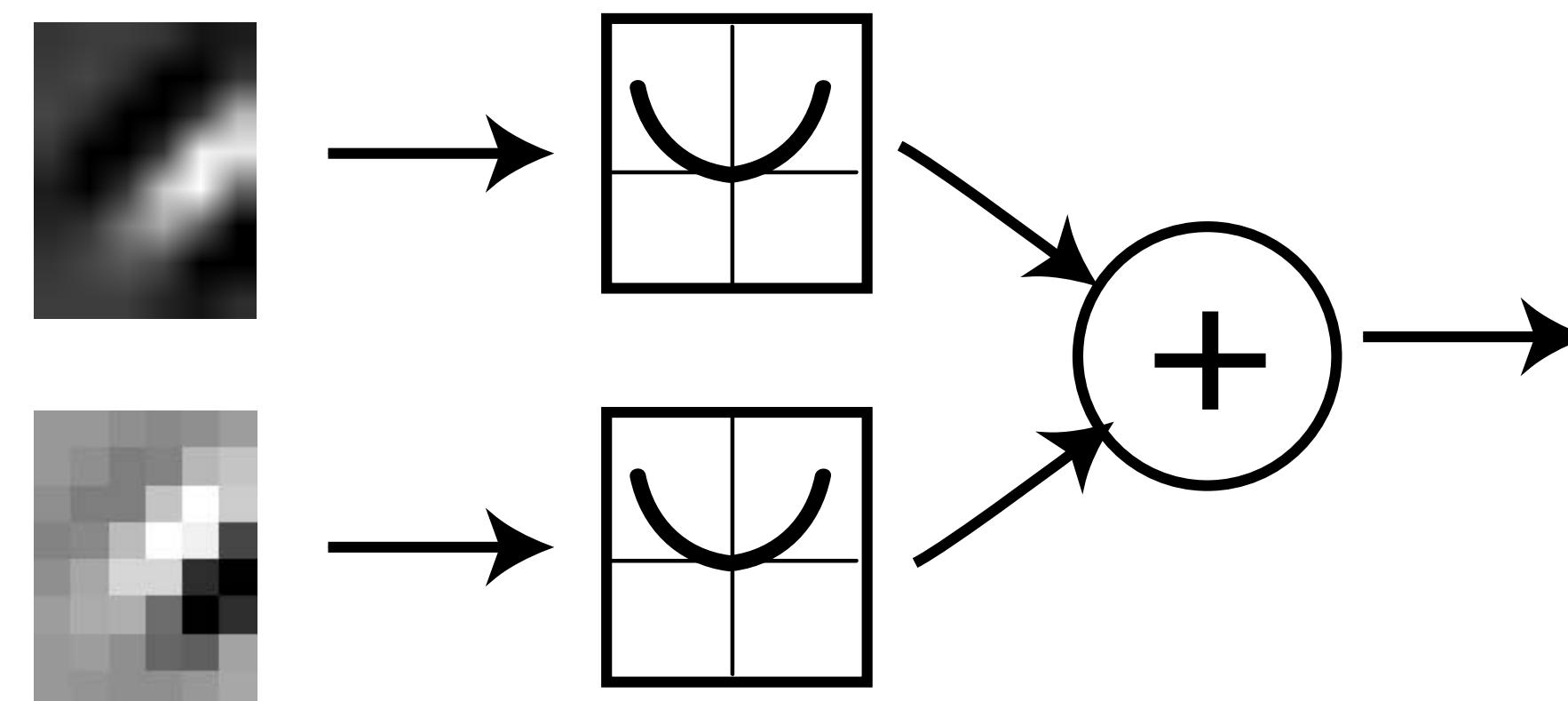
This is known as the [Volterra Series](#)

A more helpful formulation for white noise is given by [Wiener Theory](#), which is very related, but in which the nth order terms are orthogonal to each other w.r.t. white noise. You can find an extended treatment of this in Chapter 4 of Marmarelis and Marmarelis (1978).

Estimating a non-linear neural system: a V1 complex cell

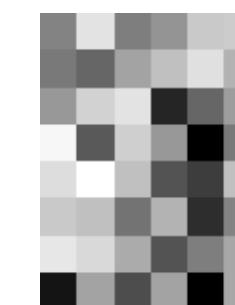
Directly estimating Volterra/Wiener kernels is often impractical and very data hungry. But *eigendecomposition* can come to the rescue as an alternative estimate of the second-order term

E.g. take a complex cell, whose responses to visual stimuli are phase invariant and therefore non-linear. A sum of quadratic non-linearities is often used to model these cells



Schwartz et al., 2006, J. Vis.

Now we do a white-noise experiment as before and compute the STA, which returns noise



The second mode of the distribution can be estimated from the covariance matrix C of the spike-triggered stimuli:

$$\hat{C}(t) = \frac{1}{N} \sum_{n=1}^N (\vec{s}_n - \overrightarrow{\text{STA}})^T (\vec{s}_n - \overrightarrow{\text{STA}})$$

Number of spikes

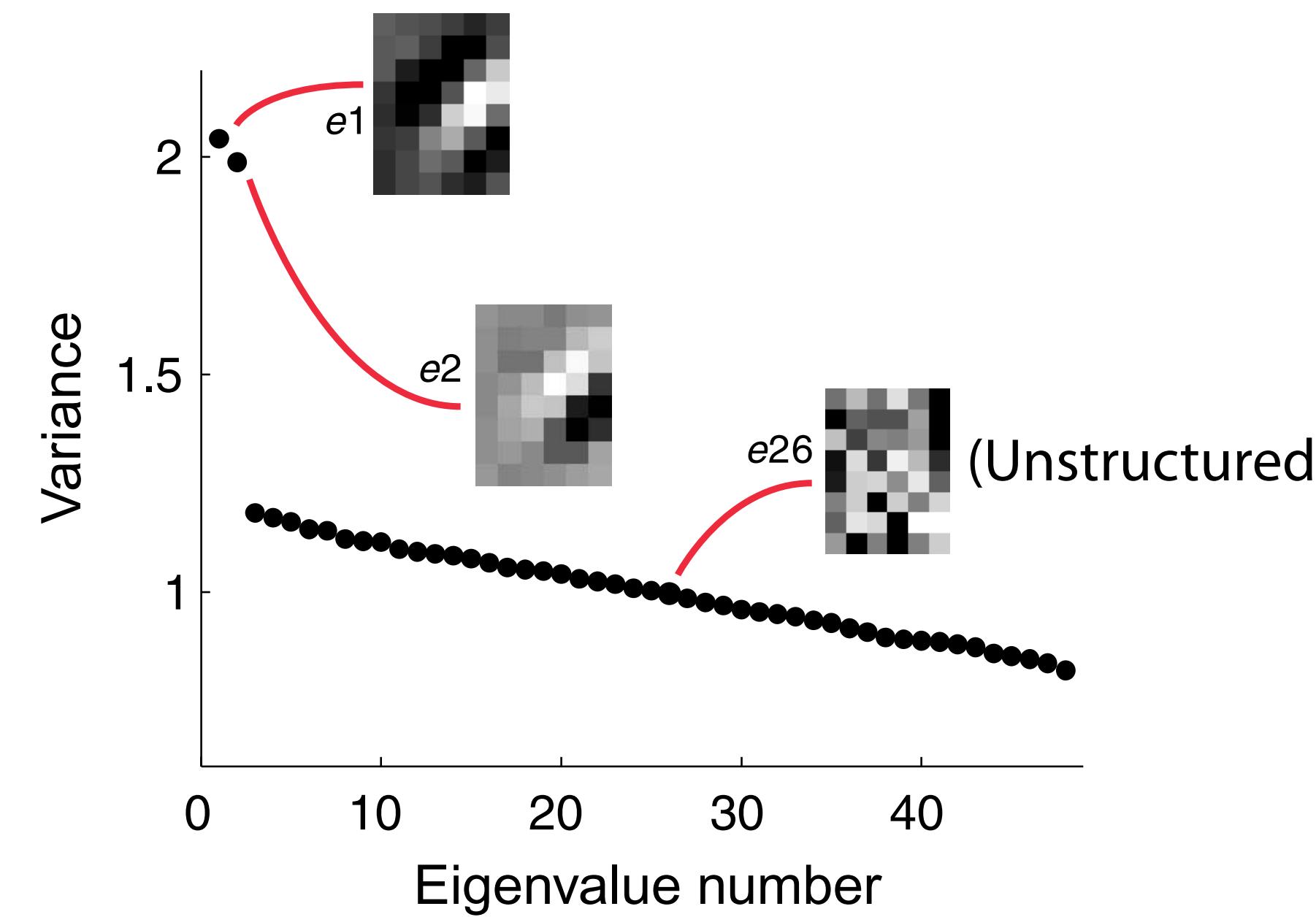
Vector of stimulus
values triggered
by spike n

Subtract the
average (STA) as
defined previously

- ▶ In practice we bin spike data and multiply each term by spike counts

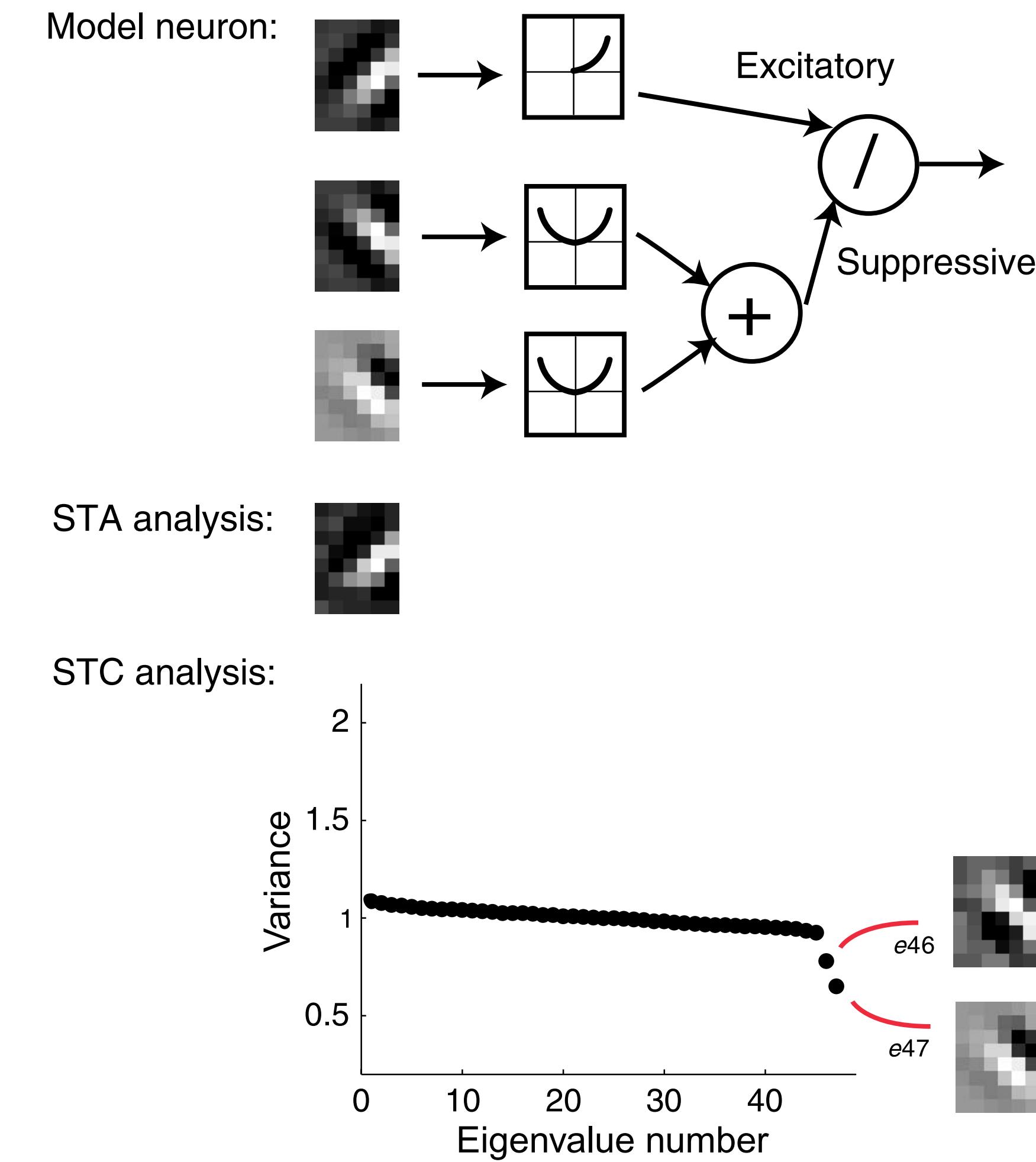
Estimating a non-linear neural system: a V1 complex cell

Finally, we can do eigendecomposition of C to obtain its significant orthogonal components as estimates of receptive field subunits



Eigenvector significance is usually estimated with shuffling / bootstrapping procedures

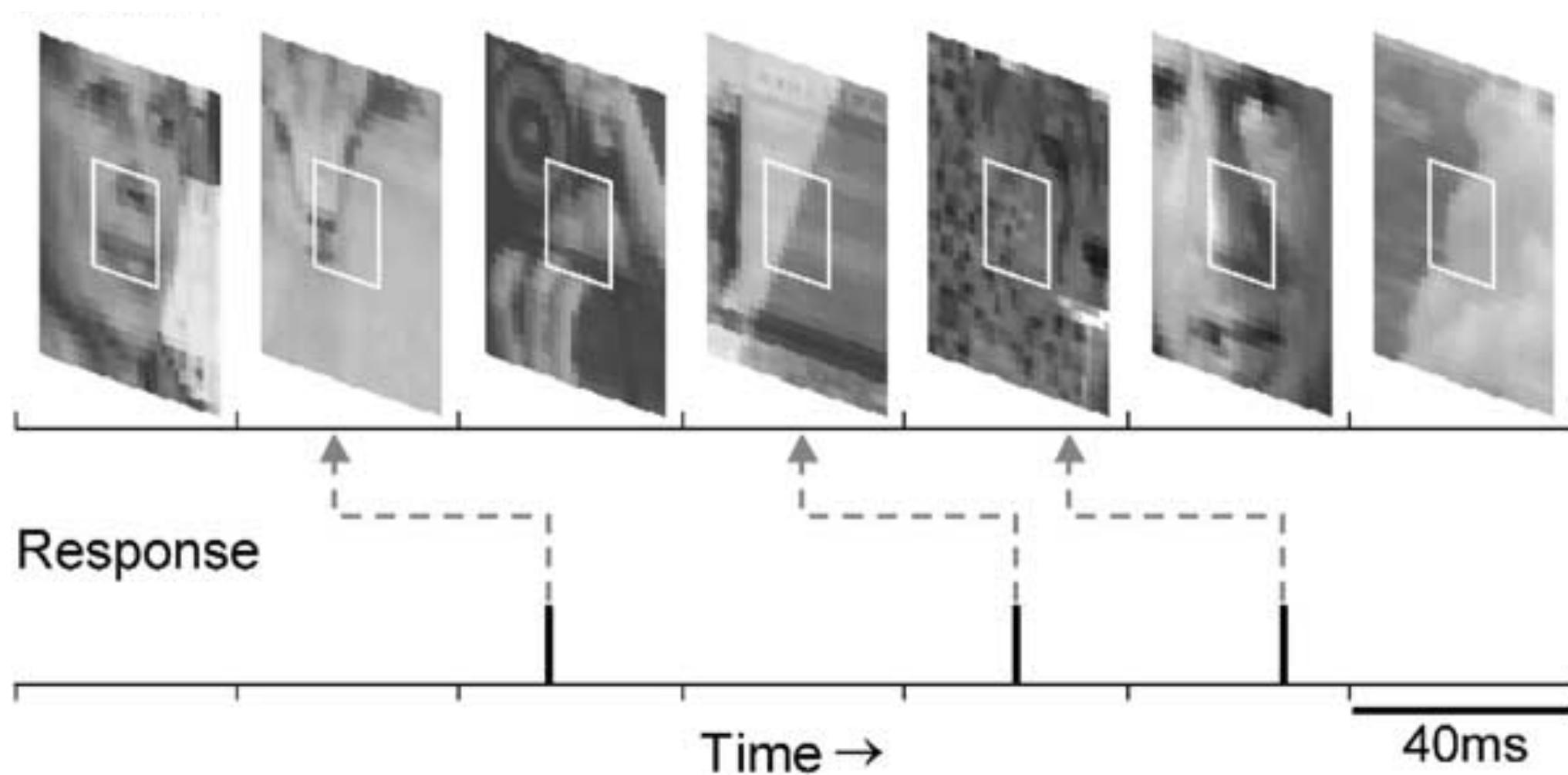
And it can get more complicated, too!



Beyond white noise

White noise has many convenient properties, but systems identification can also generalize beyond the Gaussian noise distribution

For example, V1 response properties differ when probed with natural stimuli, which have different statistics than white noise or gratings (pink noise, $1/f^2$, because nearby pixels are correlated)



But remember that the kernel depends on both the stimulus-response cross-correlation and the stimulus auto-correlation. So we can still use non-white stimuli if we correct for its statistical (spectral) properties:

$$H(\omega) = \frac{\Phi_{xy}(\omega)}{\Phi_{xx}(\omega)}$$

One of our journal club papers (Touryan et al., 2005) will go over this in detail so I won't give any more spoilers

Also, for an interesting statistical (Bayesian) interpretation of systems identification, refer to the Wu et al. (2006) paper in the suggested reading list

Summary

- ▶ Eigendecomposition decomposes a square matrices into an orthogonal basis set
- ▶ PCA is the eigendecomposition of the covariance matrix, rotates the data around the mean to obtain orthogonal dimensions
- ▶ PCA (& eigenvalue decomposition) has countless uses in neuroscience: linear dimensionality reduction, systems identification, regression, visualization
- ▶ CCA is a PCA extension that finds the most correlated axes between two variables
- ▶ CCA is very useful for e.g., multi-area, multi-day, multi-animal comparisons
- ▶ Systems identification estimates the operations a black box performs on inputs without opening the box
- ▶ White noise has many convenient properties that allow systems identification directly from auto- and cross-correlation functions
- ▶ STA and STC are common implementations of systems identification in sensory receptive field mapping

Suggested reading

- ▶ Hastie, Tibshirani & Friedman (2017). The Elements of Statistical Learning, 2nd edition. Chapter 14 (PCA section).
- ▶ Wang et al. (2020). Finding the needle in a high-dimensional haystack: Canonical correlation analysis for neuroscientists. *Neuroimage* 216: 166745.
- ▶ Marmarelis & Marmarelis (1978). Analysis of Physiological Systems: the white-noise approach. Springer, New York NY <https://link.springer.com/book/10.1007/978-1-4613-3970-0>, esp. chapters 2-4
- ▶ Ringach & Shapley (2004). Reverse correlation in neurophysiology. *Cog. Sci.* 28:147-166
- ▶ Schwartz et al. (2006). Spike-triggered neural characterization. *J. Vis.* 6:484-507
- ▶ Wu et al. (2006). Complete functional characterization of sensory neurons by systems identification. *Annu. Rev. Neurosci.* 29:477-505
- ▶ Journal club papers on canvas

The end