Models of decision making

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NUIN 443, 04/08/2024

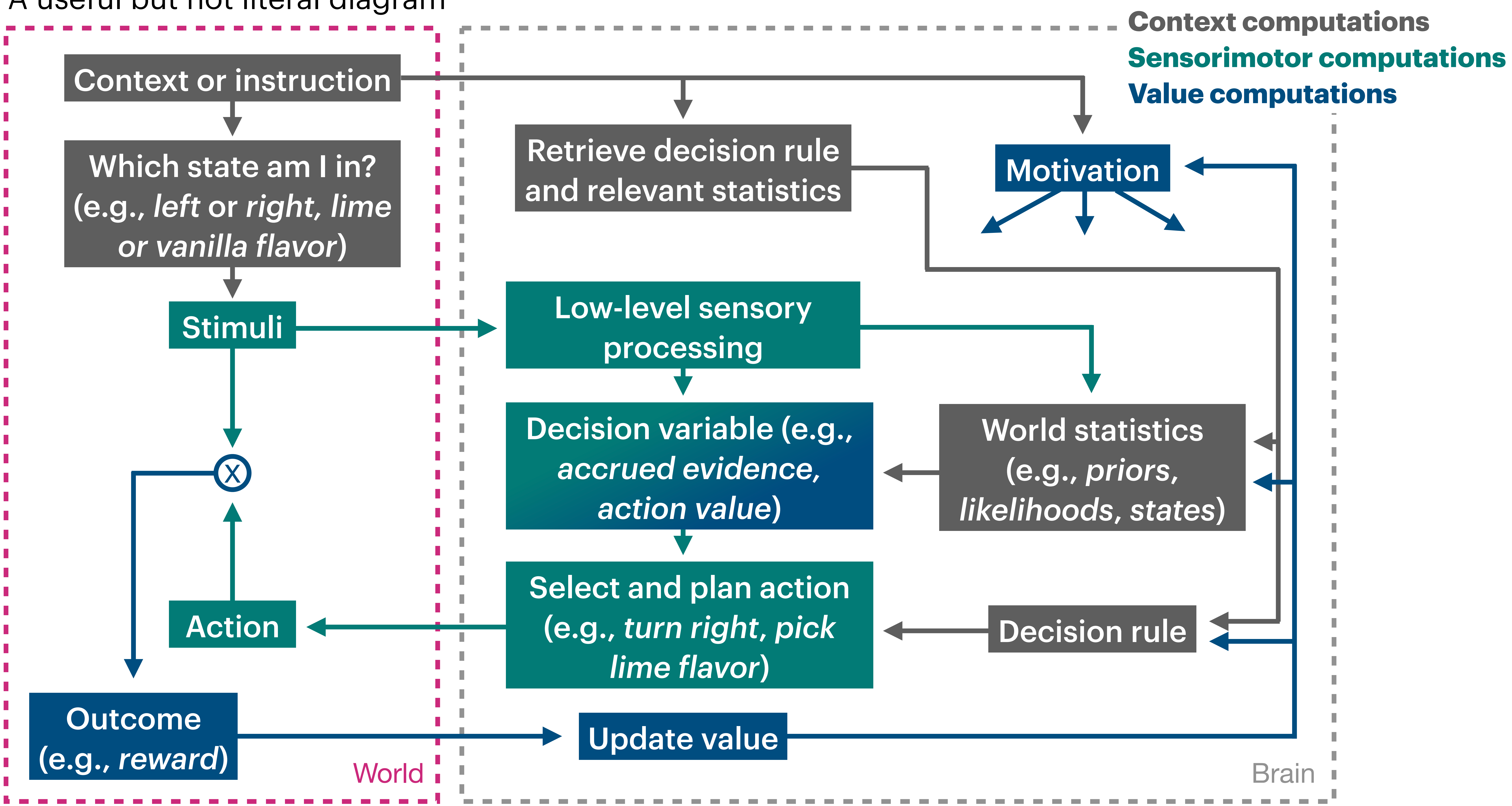
Outline

- Intro: decision making
- Perceptual decision making
  - Signal detection theory (SDT)
  - Sequential sampling: DDMs and friends
- Value-based decision making
  - Value, utility
  - RL

Intro: decision making

Introduction

A useful but not literal diagram



after Gold & Shadlen, 2007

Decision making in the lab: unofficial taxonomy

Decision making

Perceptual

Faster / simpler
few computational steps between sensory processing and action
- *E.g.*: grating orientation discrimination
- *Normative computational framework*: SDT

Review: Parker & Newsome, 1998

Slower / more complex
time extended / muti-computation
- *E.g.*: evidence accumulation, working memory (delayed match to sample)
- *Normative computational frameworks*: DDMs & friends

Review: Gold & Shadlen, 2007

Value based

Reward learning
learn reward structure through trial and error
- *E.g.*: two-arm bandit
- *Normative computational framework*: RL

Reviews: Dayan & Niv, 2008; Lee et al, 2012

Economic
make decisions based on learned cost and benefit
- *E.g.*: foraging
- *Normative computational frameworks*: Optimal foraging theory, Prospect theory etc.

Reviews: Padoa-Schioppa, 2011; Lee et al, 2012; Calhoun & Hayden, 2015

Common task types by response:
Go/no-go: make or omit a response
Two- (n-) alternative forced choice: choose one of n alternatives

Common task types by timing:
Trial start: Self-paced vs. **Enforced trials**
Decision: Reaction time vs. **Fixed-time interrogation**

Decisions are noisy

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Most formal decision-making frameworks start from the same assumption: *decisions are noisy.*

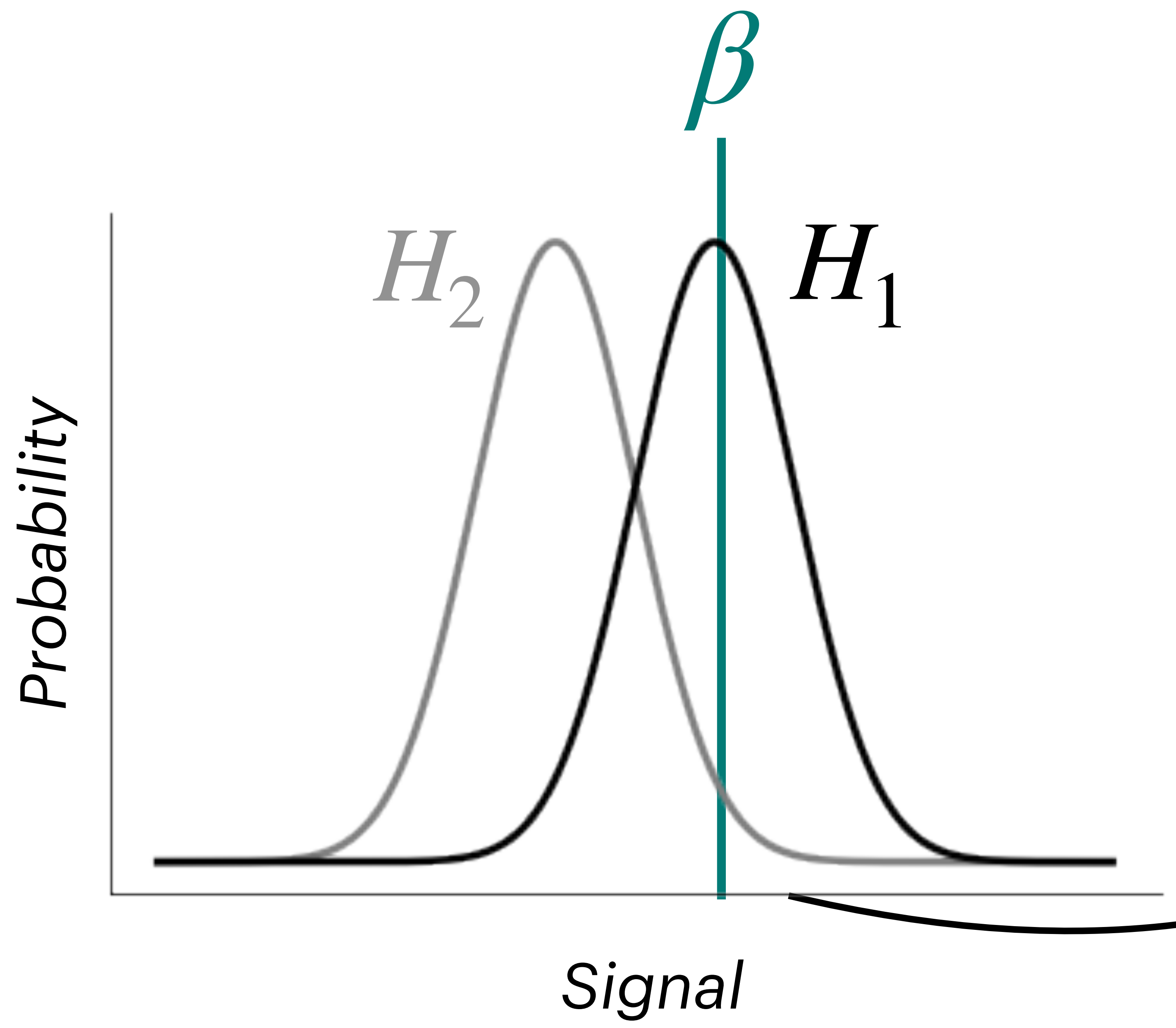
Take a visual decision:

- External noise:
  - Photon emission is Poisson
  - Environmental conditions vary (rain water is scattering)
- Internal noise
  - Eye optics scatter photons
  - Synaptic release, transduction, temperature
  - Ongoing brain dynamics (may not be noise, but still leads to behavioral variability)

Perceptual decision making: Signal detection theory (SDT)

SDT: decisions from single samples

We have two distributions, H_1 and H_2, for the magnitude of a signal (stimuli, spike counts etc)



- H_1 and H_2 can represent different things depending on the task. E.g., for detection tasks H_2 is a noise distribution, but for discrimination H_1 and H_2 each correspond to a stimulus class

Now we draw a single piece of noisy evidence, E. How do we decide which distribution it came from?

We first compute the likelihood ratio L_{12} between the two conditional probabilities

$$L_{12}(E) = \frac{P(E \mid H_1)}{P(E \mid H_2)}$$

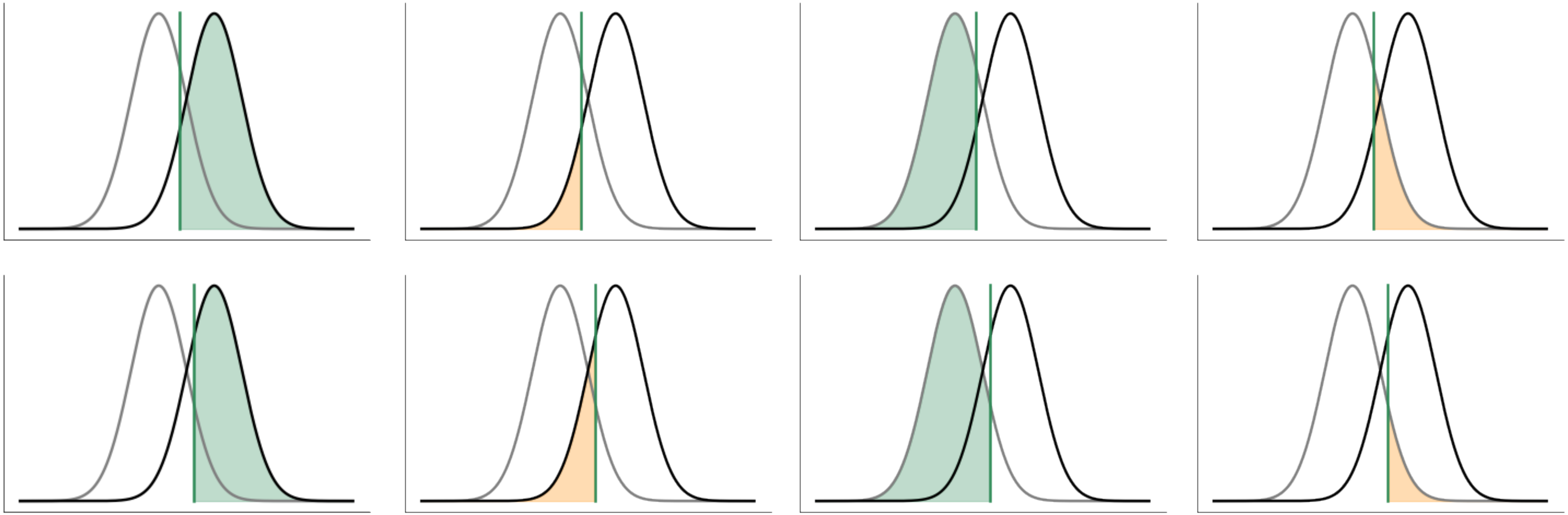
We choose H_1 if

$L_{12}(E) > \beta$ (criterion, i.e. threshold)

Choosing beta

Take a go/no-go decision

Hit rate | **Miss rate** | **Correct rejection rate** | **False alarm rate**



- If the goal is overall accuracy and H_1 and H_2 are equally likely, it can be shown that the optimal is $\beta = 1$
- If the goal is overall accuracy and H_1 and H_2 have different prior probability P, the the optimal is $\beta = P(H_1)/P(H_2)$

The receiver operating characteristic (ROC) curve

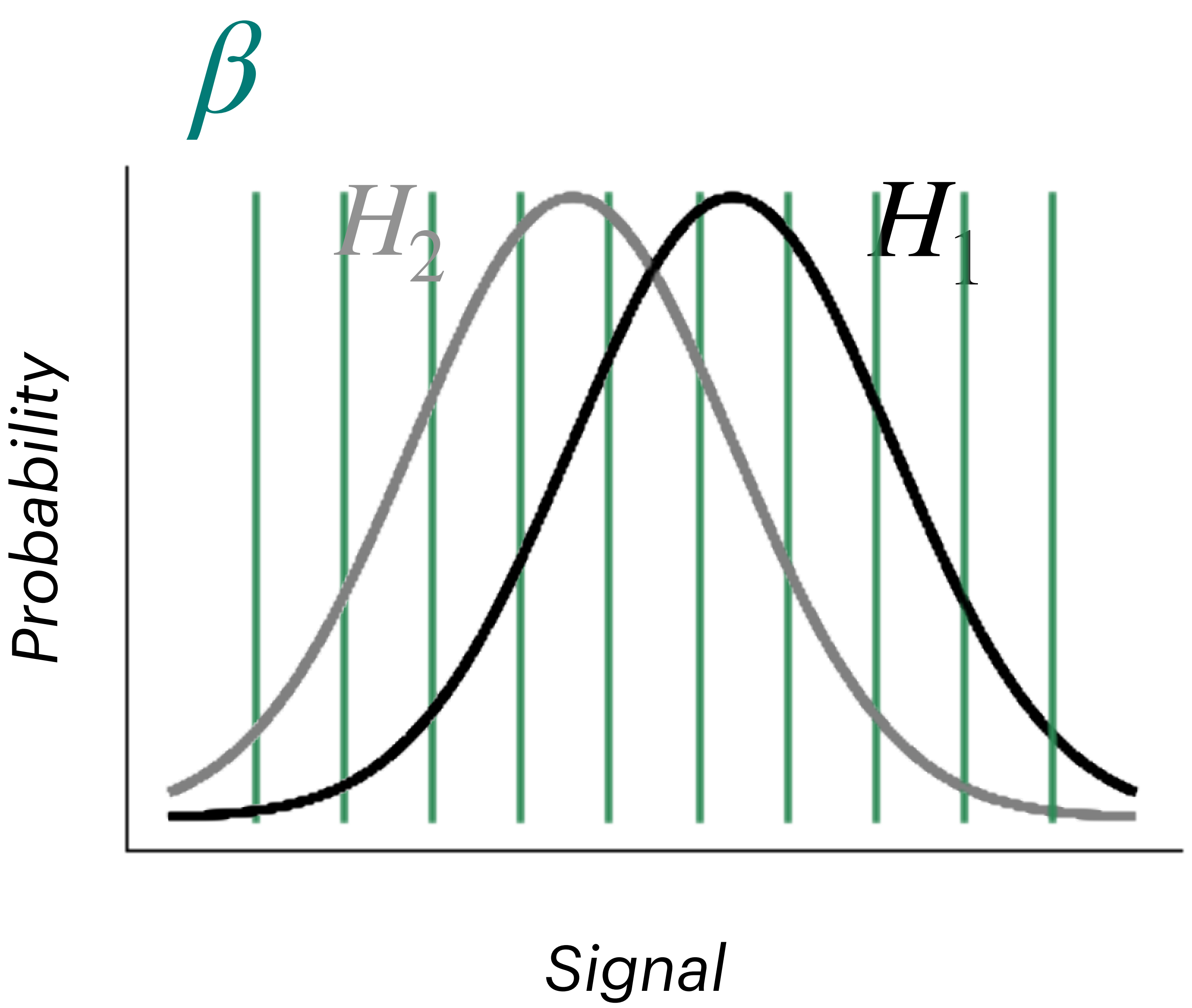
The ROC curve captures the full H_1 and H_2 by systematically shifting \beta and asking how the classification probabilities change

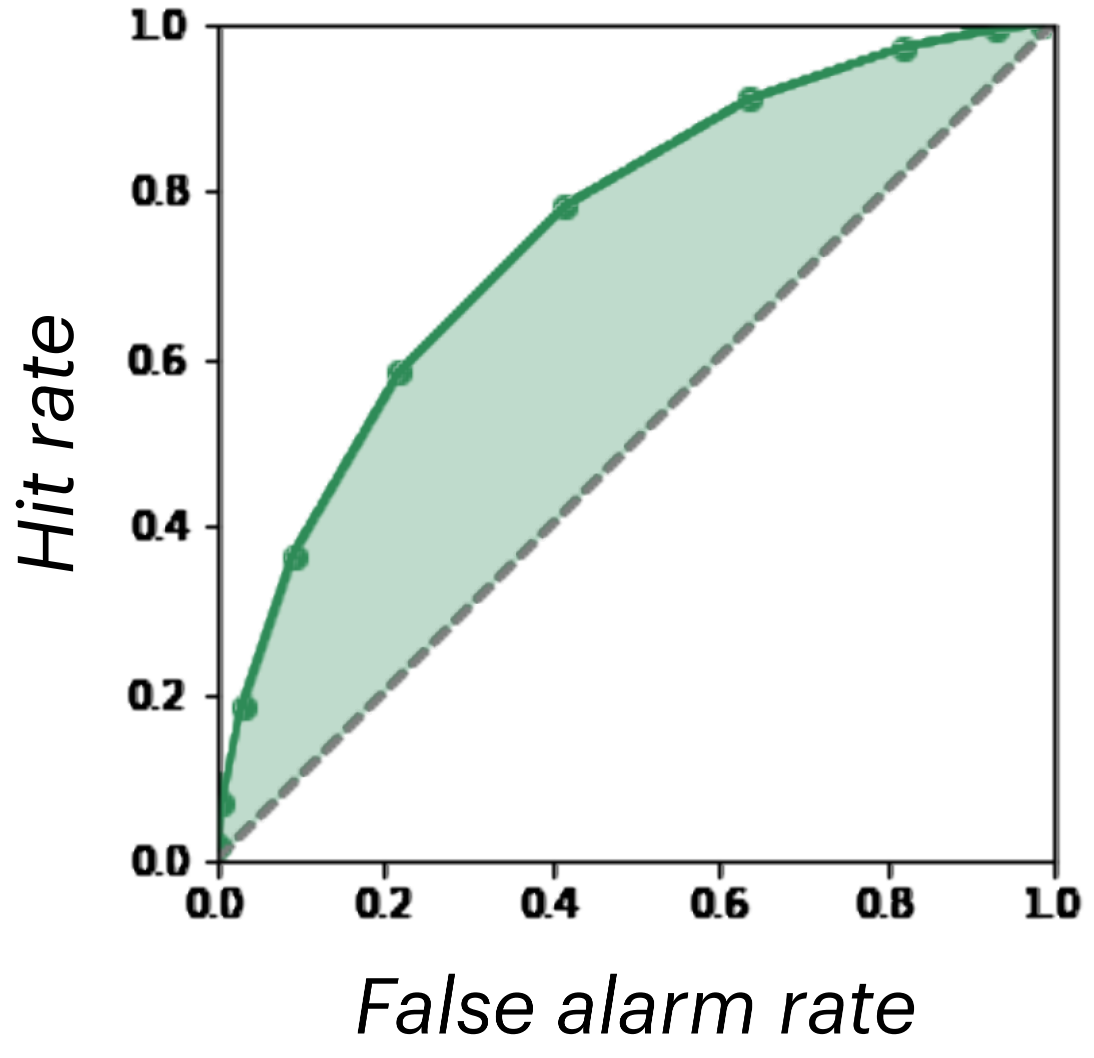
$$P(E \in H_1 | \beta)\ ,\ P(E \in H_2 | \beta)$$

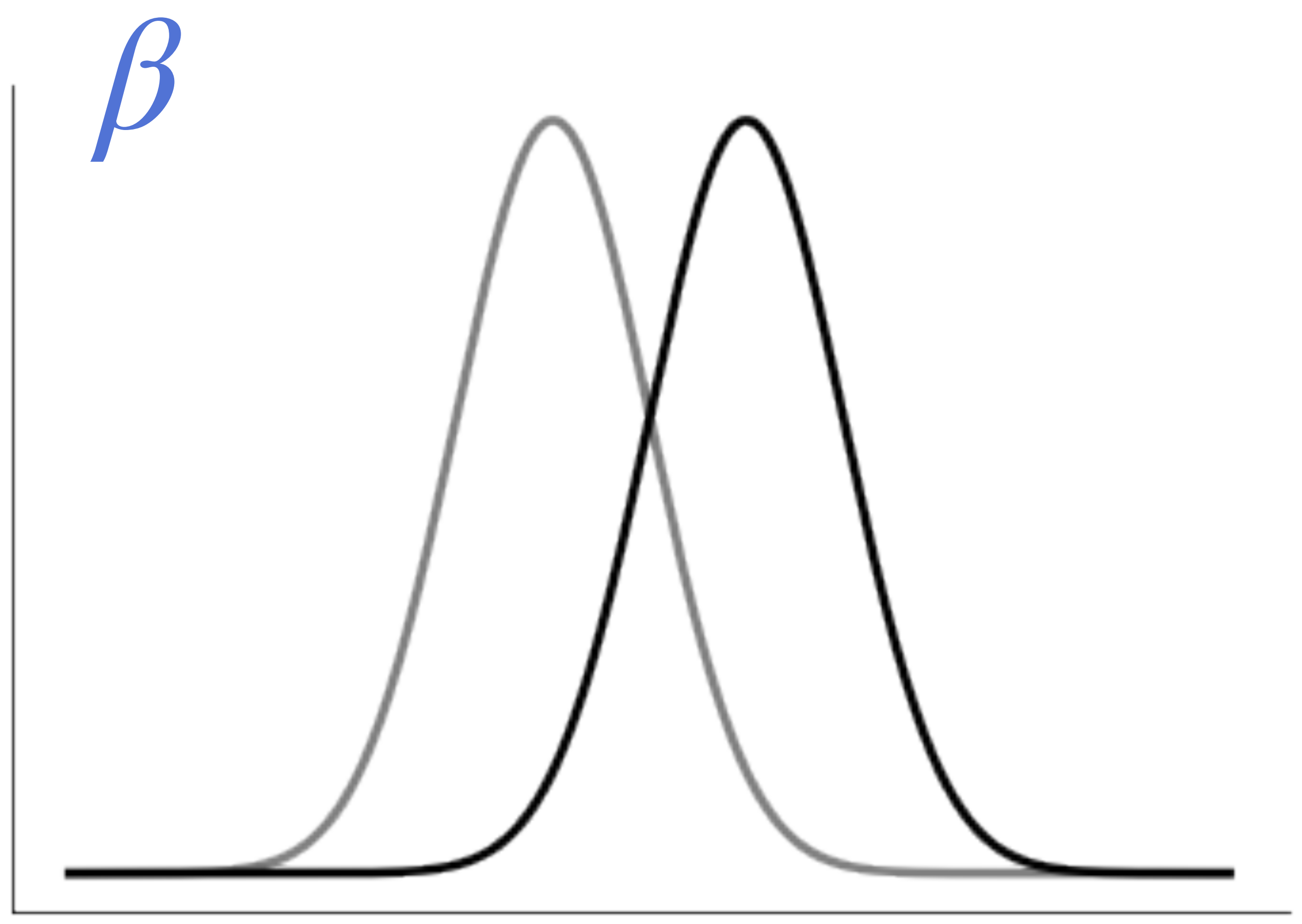
The area under the ROC curve corresponds to the classification accuracy

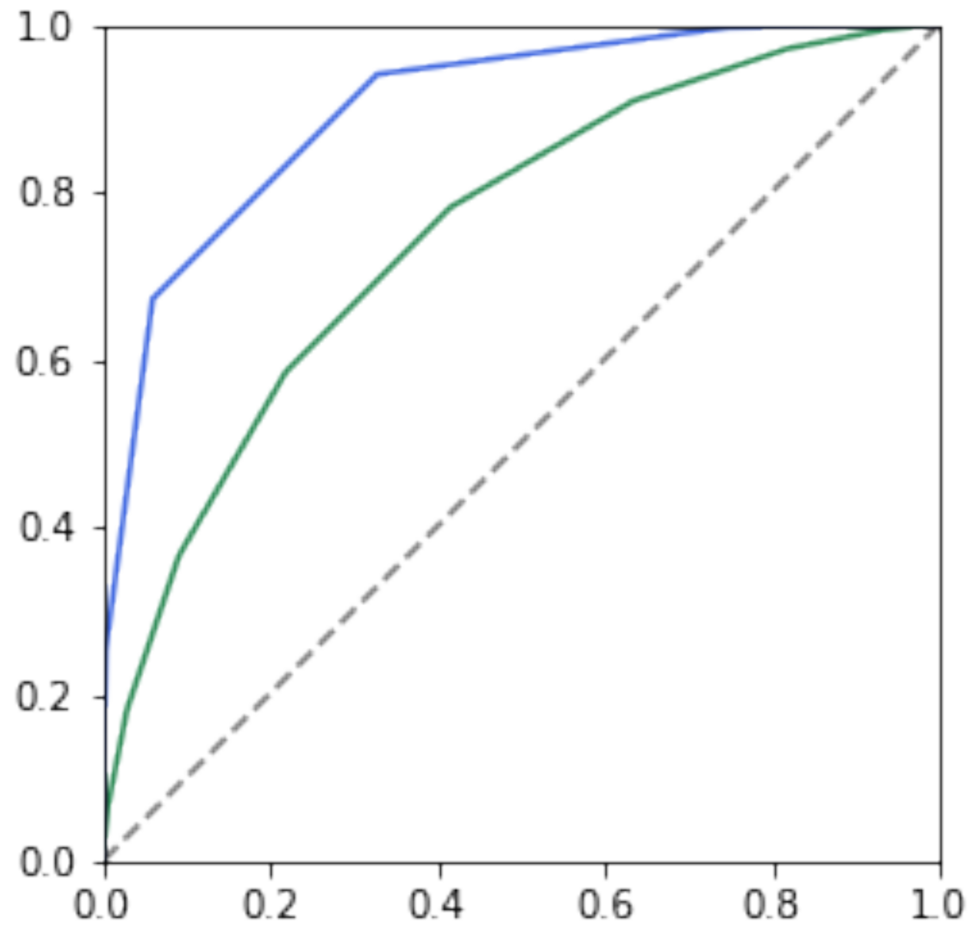
E.g., distributions with the same mean but smaller \sigma will be pushed towards the top left corner, and indistinguishable distributions will yield curves on the unity line

Note that this is a non-parametric measure (does not assume any shape of the distribution), so it's broadly useful for different types of data









Discriminability: *d′*

d′ is a commonly used summary metric of how discriminable H_1 and H_2 are

$$d' = \frac{separation}{spread}$$

Which is given by the difference in mean divided by the pooled standard deviation

$$d' = \frac{\mu_2 - \mu_1}{\sqrt{(\sigma_1^2 + \sigma_2^2)/2}}$$

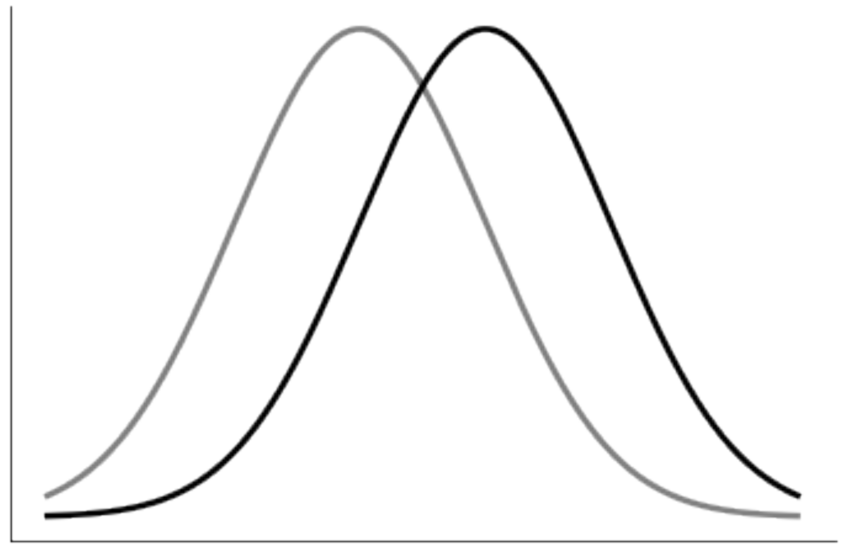
Which reduces to

$$d' = \frac{\mu_2 - \mu_1}{\sigma}\ , \text{for } \sigma_1 = \sigma_2$$

In practice, we usually assume normal data and unit \sigma and compute the inverse Gaussian (*z* score)

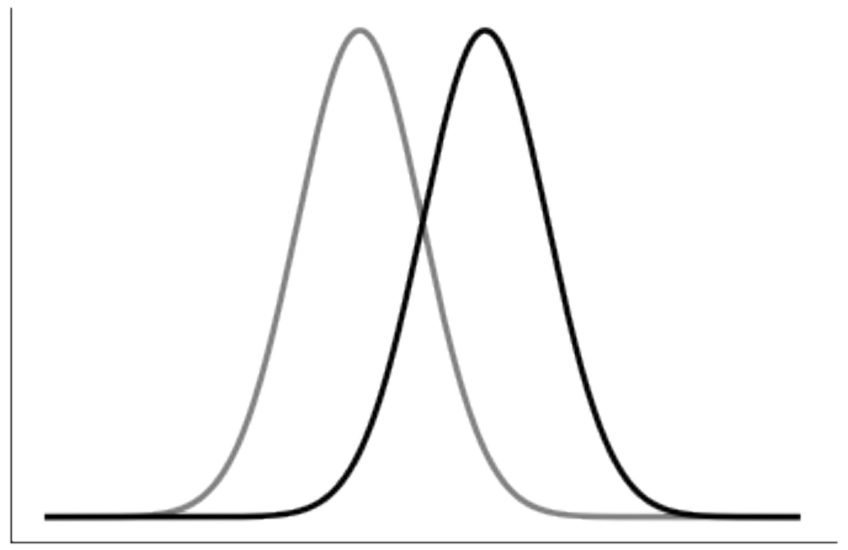
$$d' = z(HR) - z(FA)$$

$d' = 1$



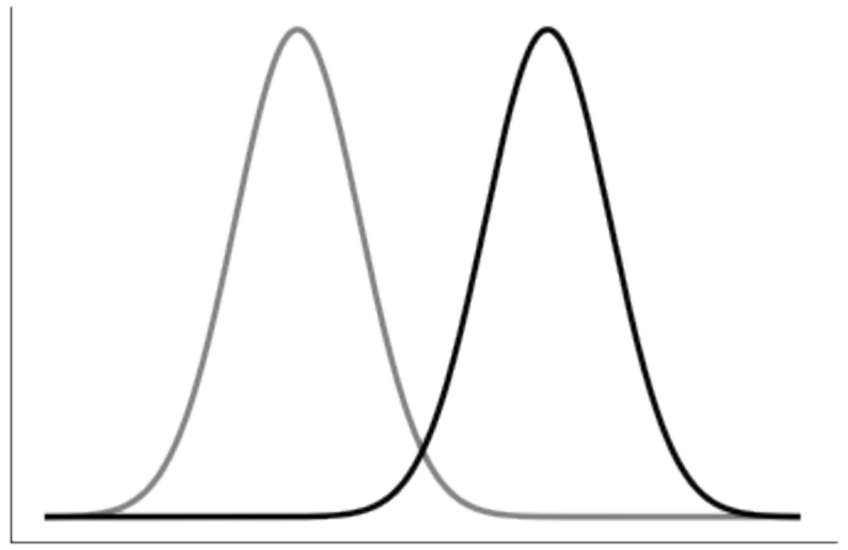
$\mu_2 = 1,\ \mu_1 = -1,\ \sigma = 2$

$d' = 2$



$\mu_2 = 1,\ \mu_1 = -1,\ \sigma = 1$

$d' = 4$



$\mu_2 = 2,\ \mu_1 = -2,\ \sigma = 1$

Perceptual decision making: Sequential sampling models

Sequential sampling

Sequential sampling is a generalization of SDT where we take multiple (independent) discrete samples E_i to decide if they come from H_1 or H_2. In this case, the likelihood ratio becomes

$$L_{12}(E_1, E_2, \ldots, E_n) = \frac{P(E_1, E_2, \ldots, E_n \,|\, H_1)}{P(E_1, E_2, \ldots, E_n \,|\, H_2)} = \prod_{i=1}^{n} \frac{P(E_i \,|\, H_1)}{P(E_i \,|\, H_2)}$$

Or, for mathematical convenience, the log likelihood ratio

$$\ell_{12} = \sum_{i=1}^{n} \log \frac{P(E_i \,|\, H_1)}{P(E_i \,|\, H_2)}$$

The log likelihood ratio is bound by two thresholds

$$\beta_2 \leq \ell_{12} \leq \beta_1$$

This is such that with the addition of each sequential sample we approach one of them, and make a decision when we hit either threshold

$l_{12}^{i} \longrightarrow$ stop if $l_{12}^{i} = \beta_1$ or $l_{12}^{i} = \beta_2$

otherwise

$$l_{12}^{i+1} = l_{12}^{i} + \log \frac{P(E_i \,|\, H_1)}{P(E_i \,|\, H_2)}$$

stop if $l_{12}^{i+1} = \beta_1$ or $l_{12}^{i+1} = \beta_2$

otherwise

$\vdots$

This is known as the *sequential probability ratio test*. It can be shown to be optimal in the sense that, for a given threshold, it gives the best accuracy. But with *speed-accuracy tradeoff* (more on this soon).

DDMs: continuous sequential sampling

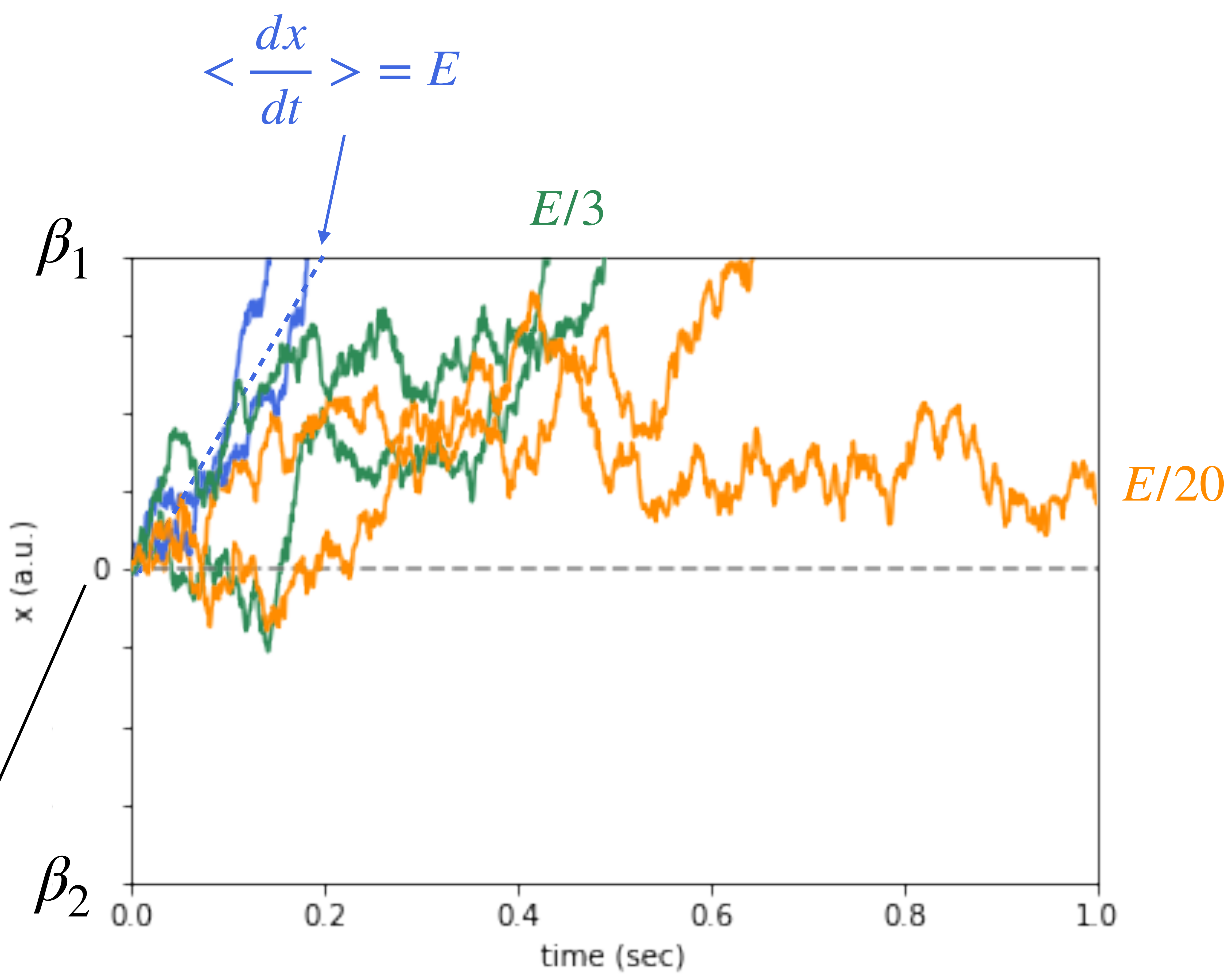
Drift diffusion models are essentially the continuous-time limit of sequential sampling

$$dx = Edt + cdW$$

latent decision variable (accumulated evidence)

Wiener process, a.k.a. brownian motion, a.k.a. "**diffusion**": Independent gaussian noise ~ $\mathcal{N}(0, cdt)$

"**Drift**": *E* dictates how quickly *x* moves towards the decision threshold (a.k.a. bound). Can be thought of as the strength of sensory evidence



x starts at 0 in vanilla DDMs

Decisions with DDMs

Again, a vanilla DDM is

$$dx = Edt + cdW \ , \ x_{t=0} = 0$$

Ignoring decision thresholds for now, the probability density of *x* at time *t* is

$$p(x,t) = \mathcal{N}(Et, c\sqrt{t})$$

If I need to make a decision between H_1 and H_2 at time *T*, if *x* > 0 I decide H_1

The average error rate, *ER*, at time *T* is the probability that *x* < 0, which we get by integrating the probability density

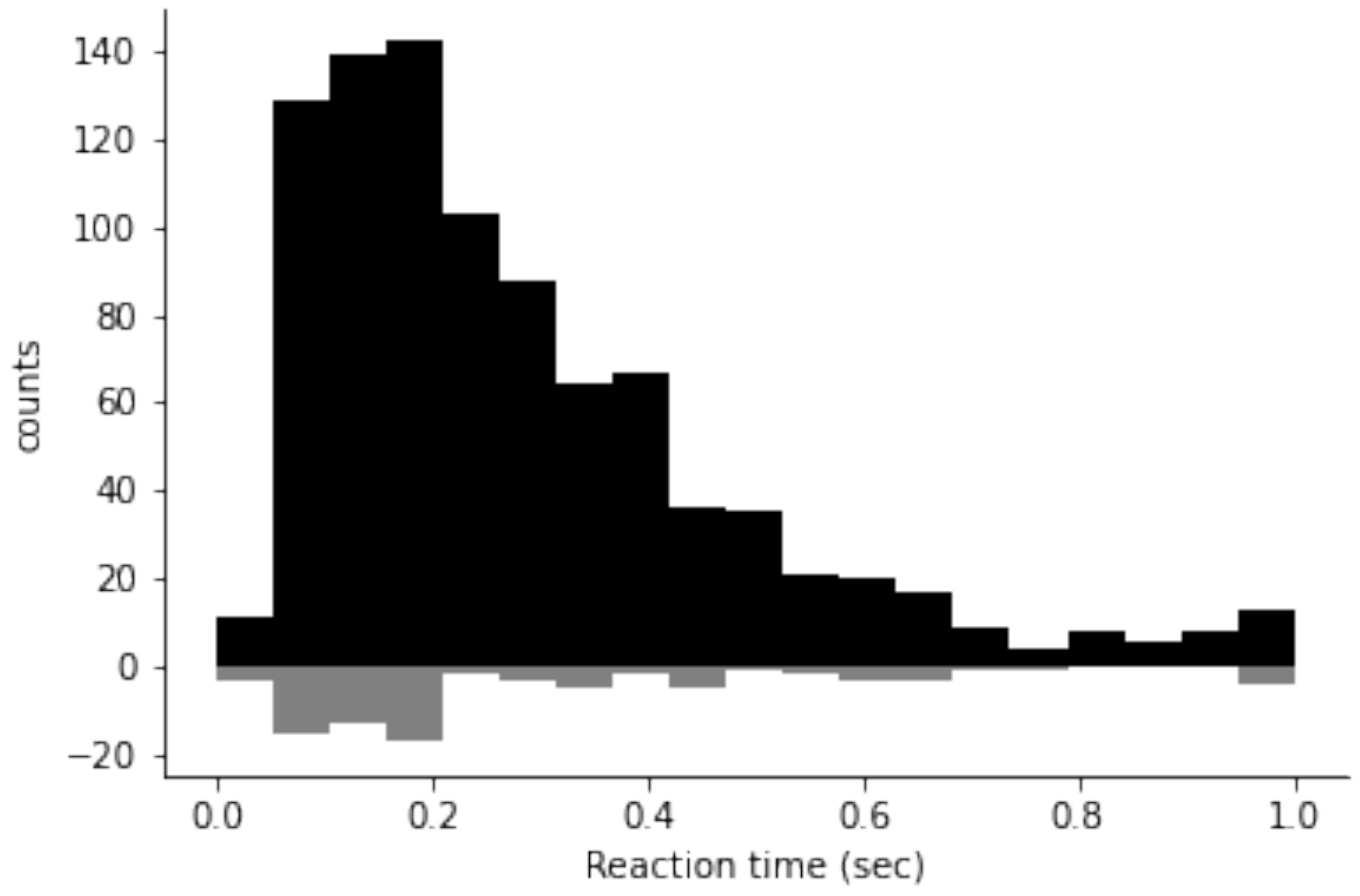
$$ER = \Phi(-\frac{E}{c}\sqrt{T}) \ , \ \Phi(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-(x^2/2)} dx$$

For full derivation see Bogacz et al, 2006

In a free response task, we stop deciding when x equals one of the boundaries \beta, with decision (reaction) time *RT*. In this case

$$ER = \frac{1}{1 + e^{\frac{2E\beta}{c^2}}} \qquad RT = \frac{\beta}{E} \tanh\left(\frac{E\beta}{c^2}\right)$$

This probabilistic nature of the model captures well reaction time distributions during perceptual decision making, including in error trials



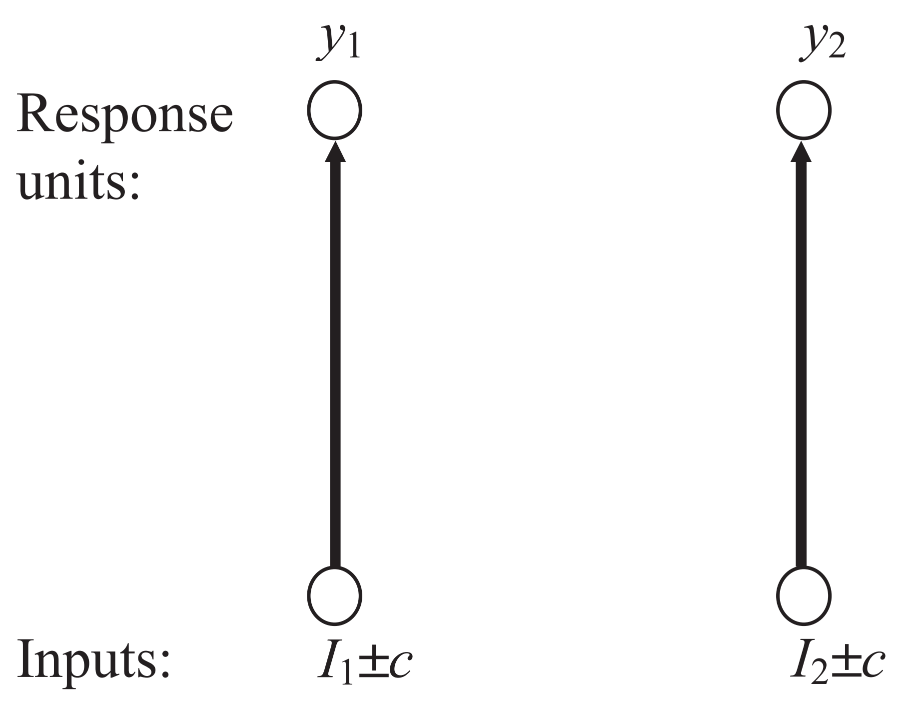
DDM variations

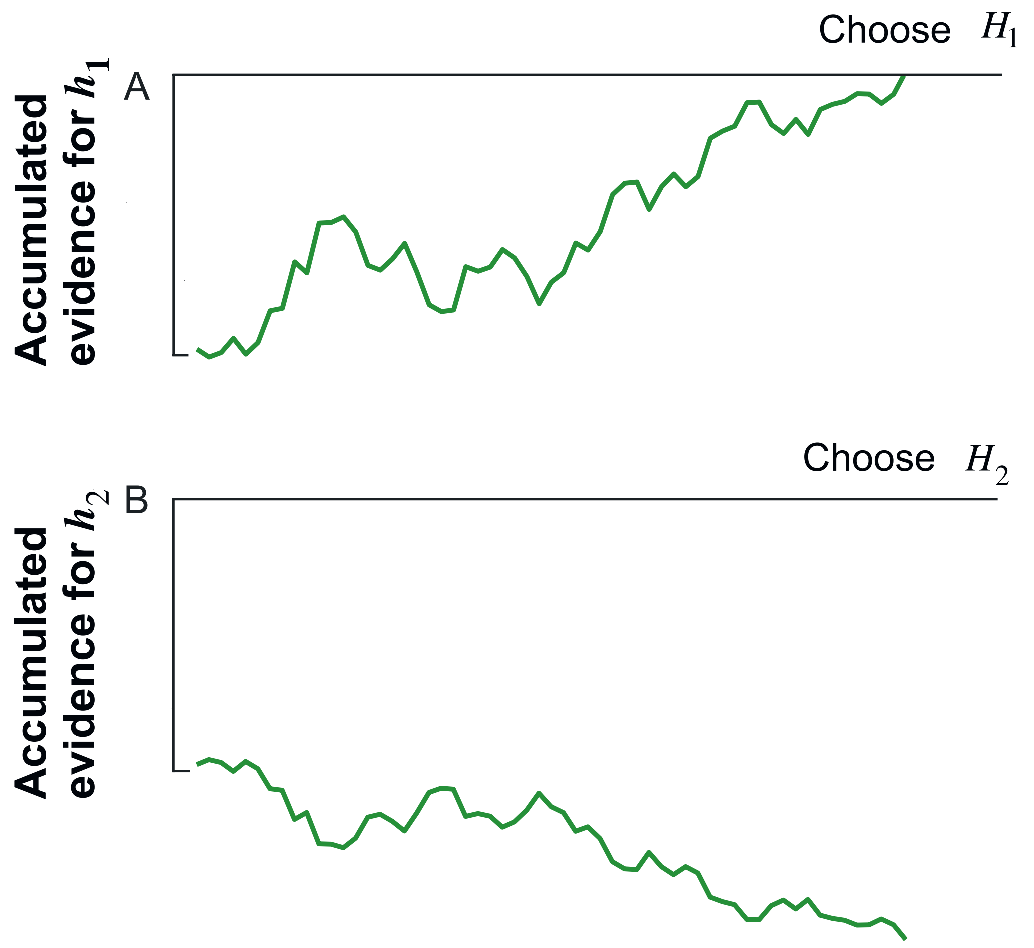
1. Vanilla DDM modifications.

- Starting point $x_{t=0}$ can become a parameter, e.g. to capture bias
- Making *E* stochastic (i.e. the mean of a Gaussian, drawn on each trial) better captures RTs
- Time-varying bounds can capture different strategies, e.g. primacy/recency

2. Race models.

- Two separate accumulators for H_1 or H_2, decision is which one finishes first
- Do not reduce to DDMs
- Struggle to capture RT distributions, esp. in error trials

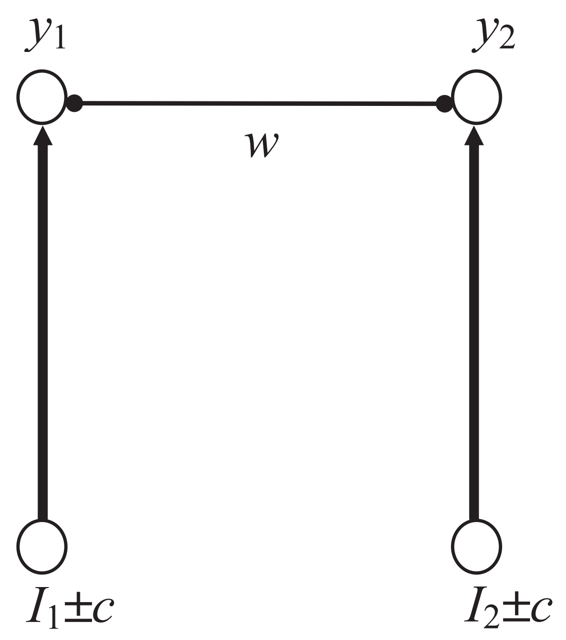




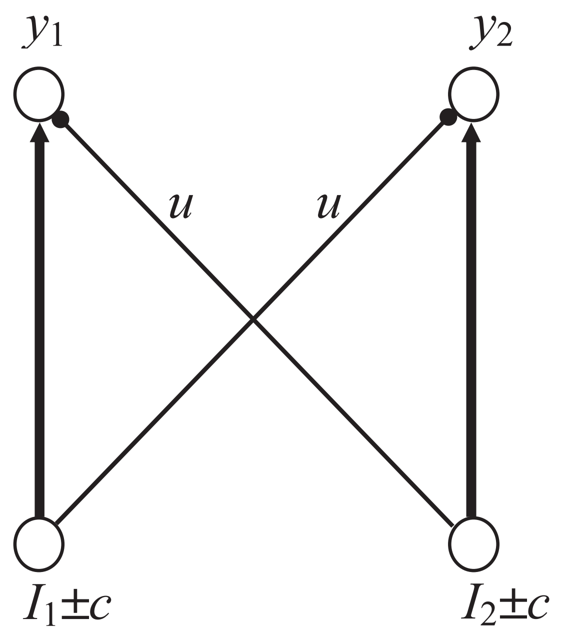
3. Leaky integrators with inhibition.

- Related to the Ornstein-Uhlenbeck model (next)
- Connectionist; inspired by circuit wiring diagrams
- Reduce to DDMs under certain parameter regimes

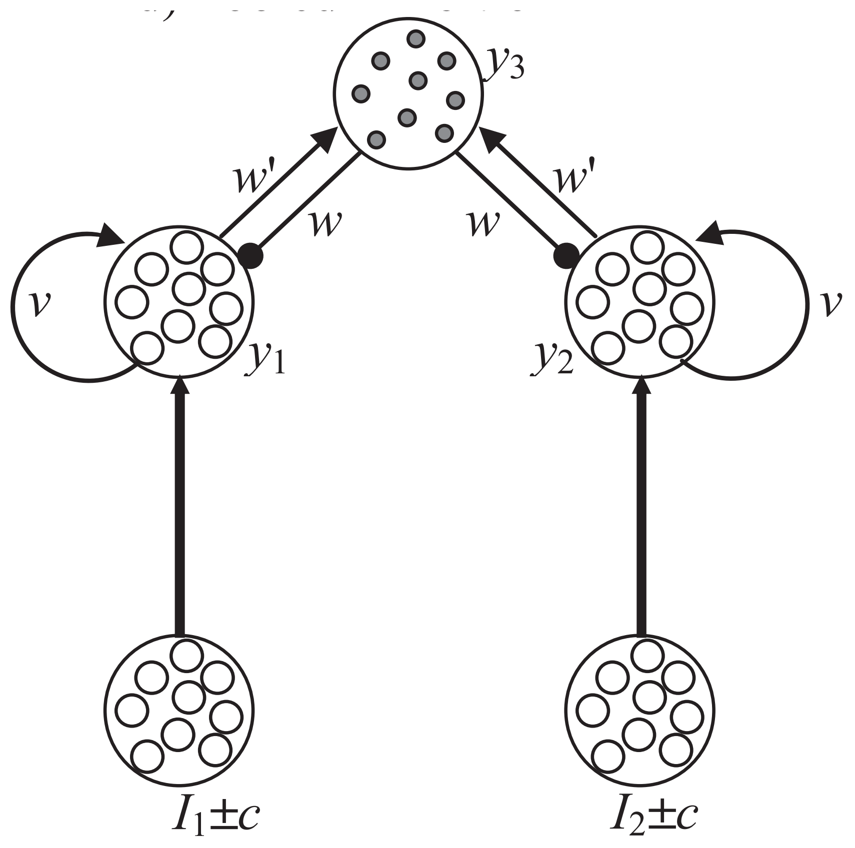
Mutual inhibition
(Usher & McClelland, 2001)



Feedforward inhibition
(Shadlen & Newsome, 2001)



Pooled inhibition
(Wang, 2002)



Bogacz et al, 2006
Gold & Shadlen, 2007

The Ornstein-Uhlenbeck (O-U) model

O-U models have an additional term that make *dx* also depend on its current value, with magnitude controlled by \lambda

$$dx = (\lambda x + E)dt + cdW \ , \ x_{t=0} = 0$$

Depending on the sign of *\lambda, x* accelerates or decelerates towards the bound

$\lambda < 0$: growing x leads to decreasing dx/dt

The time-dependent probability density converges to

$$p(x) = \mathcal{N}\left(-\frac{E}{\lambda}, \frac{c}{\sqrt{-2\lambda}}\right)$$

This is said to be a *leaky integrator* (i.e. evidence is lost over time)

$\lambda > 0$: growing x leads to increasing dx/dt

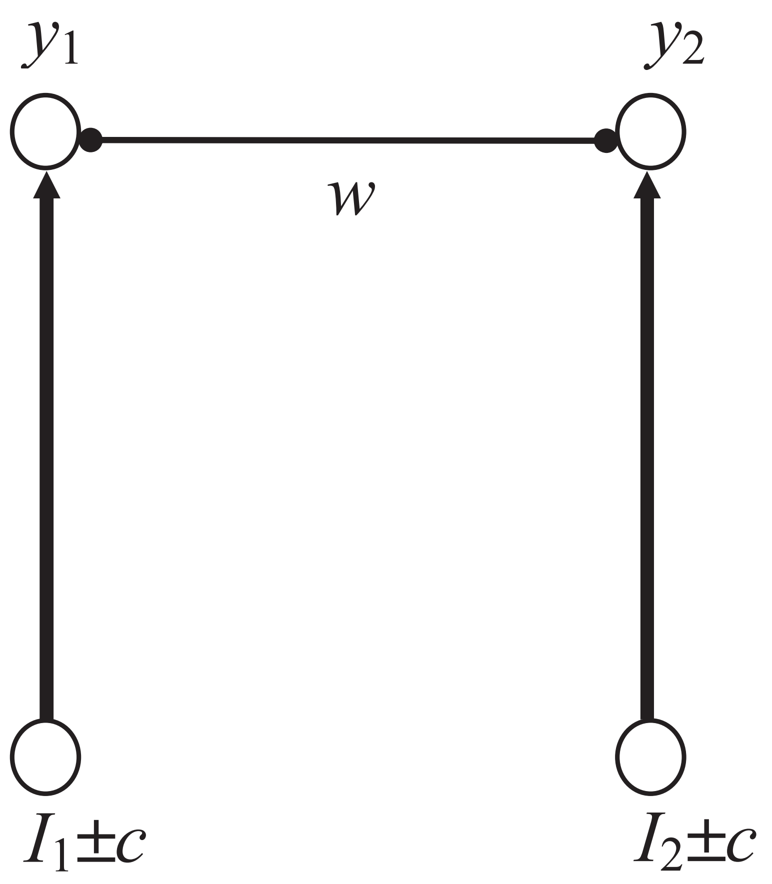
This is said to be an *unstable integrator* (i.e. evidence progressively leads to “impulsive” decisions)

$\lambda = 0$: vanilla DDM

For full derivation see Bogacz et al, 2006

Mutual inhibition model

Mutually inhibiting units:



Input:

$$\begin{cases} dy_1 = (-ky_1 - wy_2 + I_1)dt + c\,dW_1 \\ dy_2 = (-ky_2 - wy_1 + I_2)dt + c\,dW_2 \end{cases}, \; y_1^{t=0} = y_2^{t=0} = 0$$

Activity decay

Inhibition from other unit

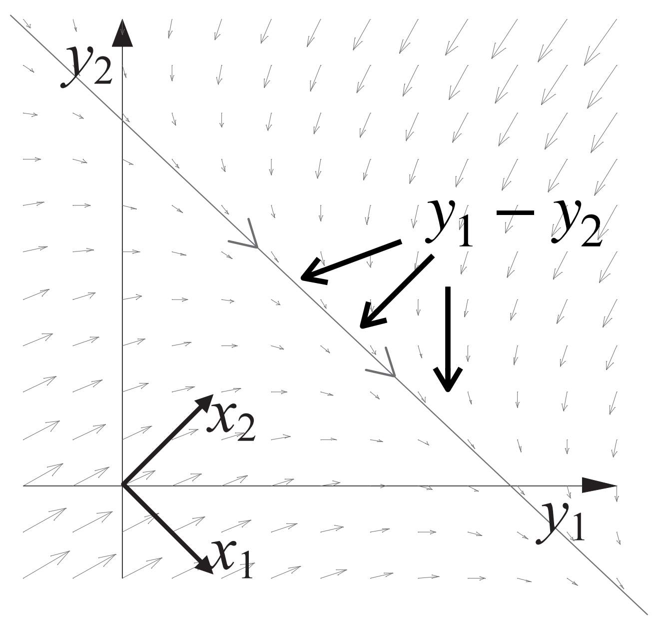
Input (evidence)

- Note that this is a linear approximation of the original model, where activity is rectified

Model state space $(I_1 > I_2)$

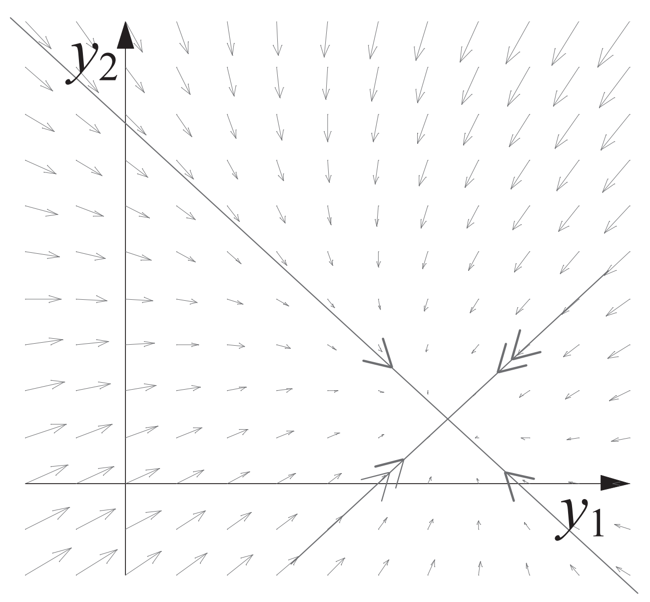
It can be shown that this behaves like like an O-U model with $\lambda = w - k$

$k = w \equiv \lambda = 0$



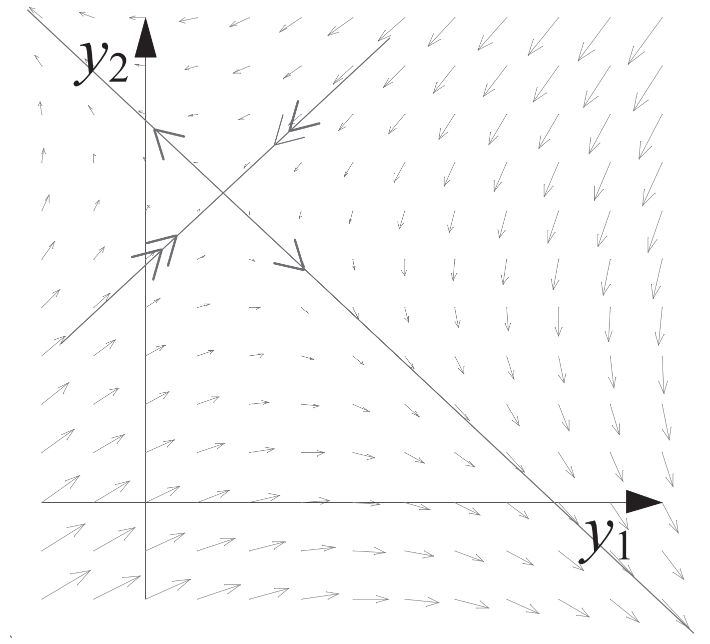
Line attractor, Equivalent to DDM

$k > w \equiv \lambda < 0$



Attracts towards fixed point, Equivalent to leaky integrator

$k < w \equiv \lambda > 0$



Repelled from fixed point, Equivalent to unstable integrator

Usher & McClelland, 2001 cf. Bogacz et al, 2006

Fitting DDMs

Vanilla DDMs have known analytical solutions

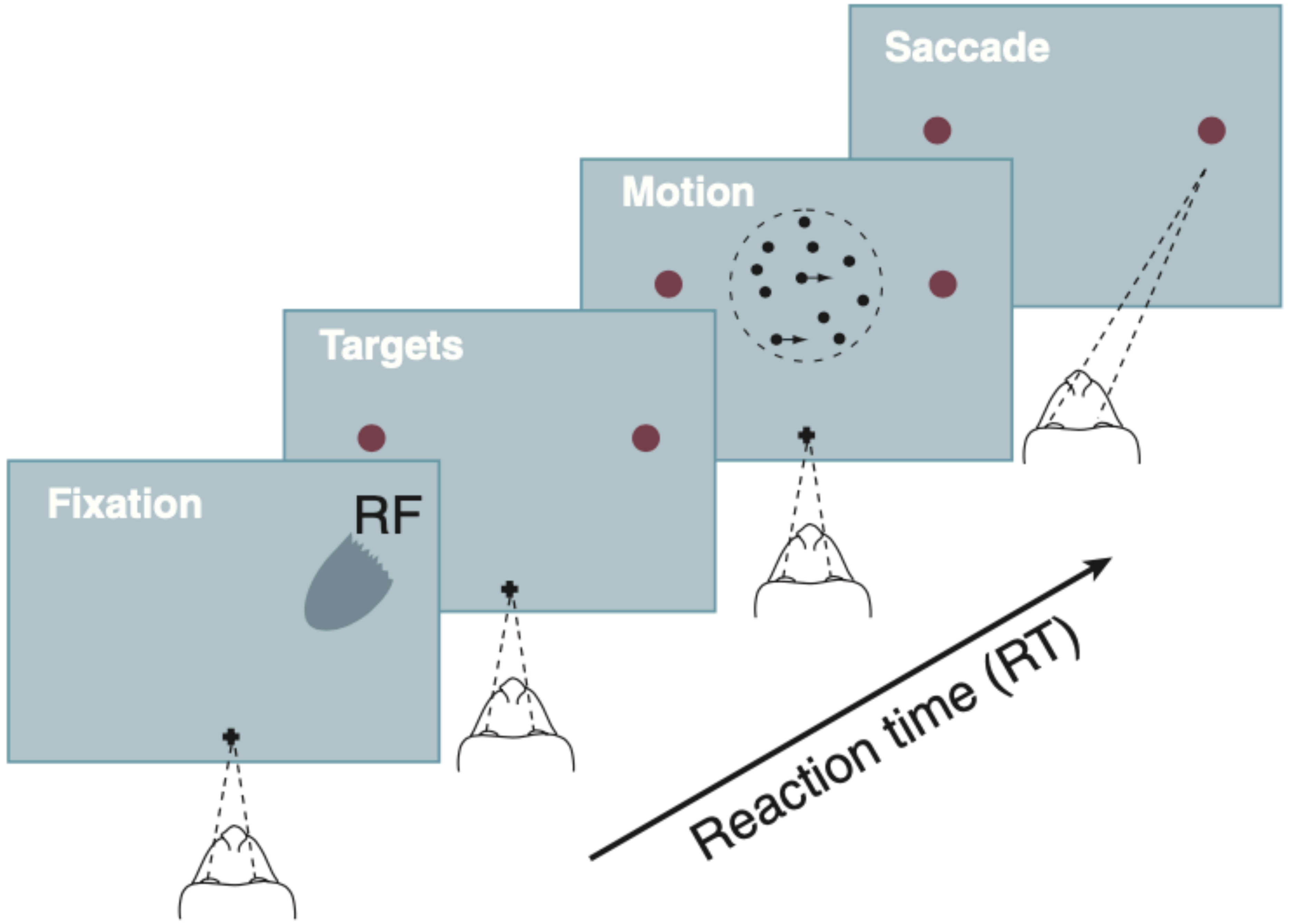
Extended DDMs and other variants don't, we need to iteratively propagate the distributions forward in time, usually by solving the Fokker-Planck equation, which describes the temporal evolution of the probability density function *p*

$$\frac{\partial}{\partial t} p(x,t) = -\frac{\partial}{\partial x}[E(t)p(x,t)] + \frac{\partial^2}{\partial x^2}\left[\frac{c^2(t)}{2}p(x,t)\right]$$

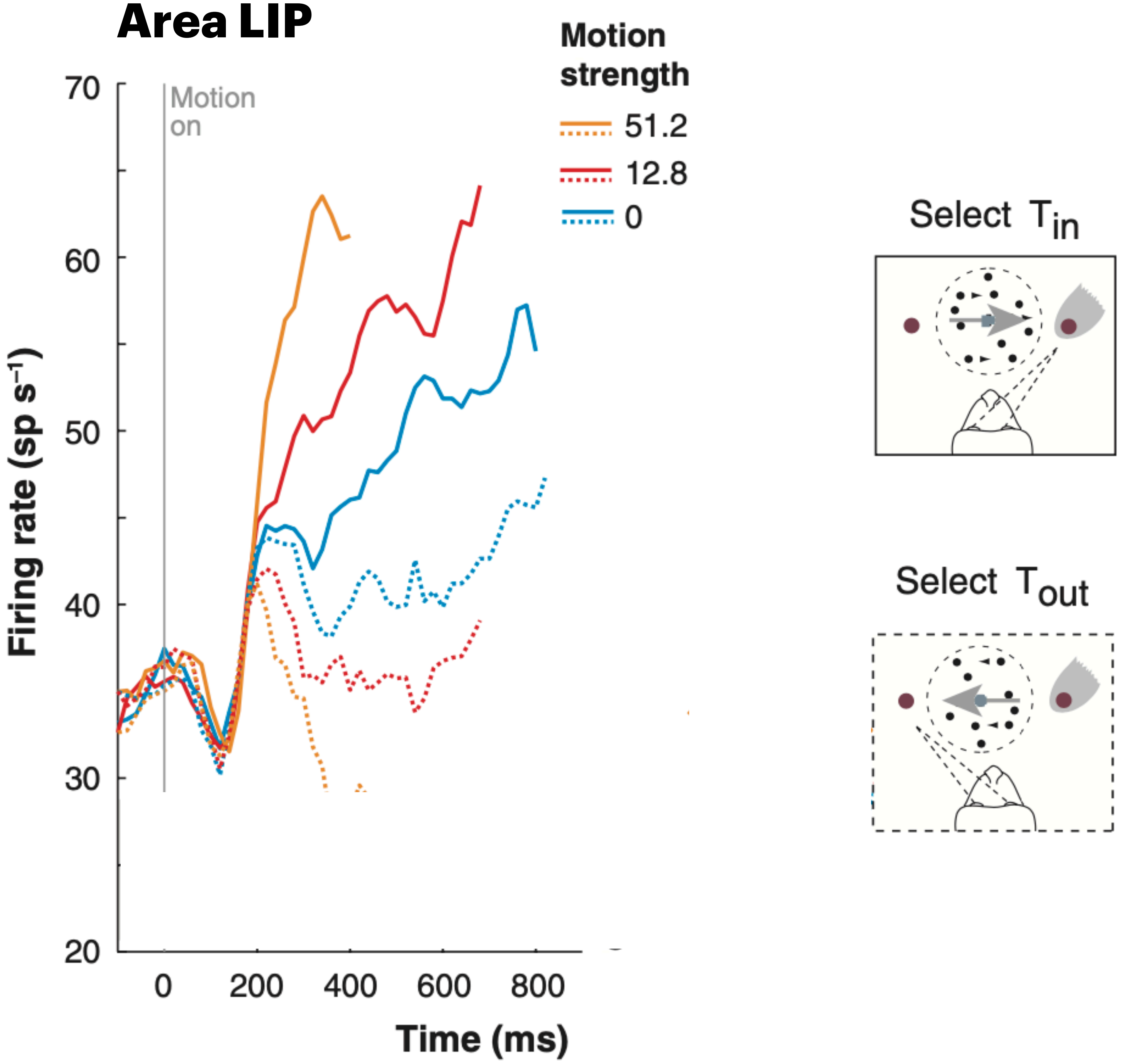
For details see Shinn et al, 2020

DDM and neural dynamics

Random dot motion task



Area LIP

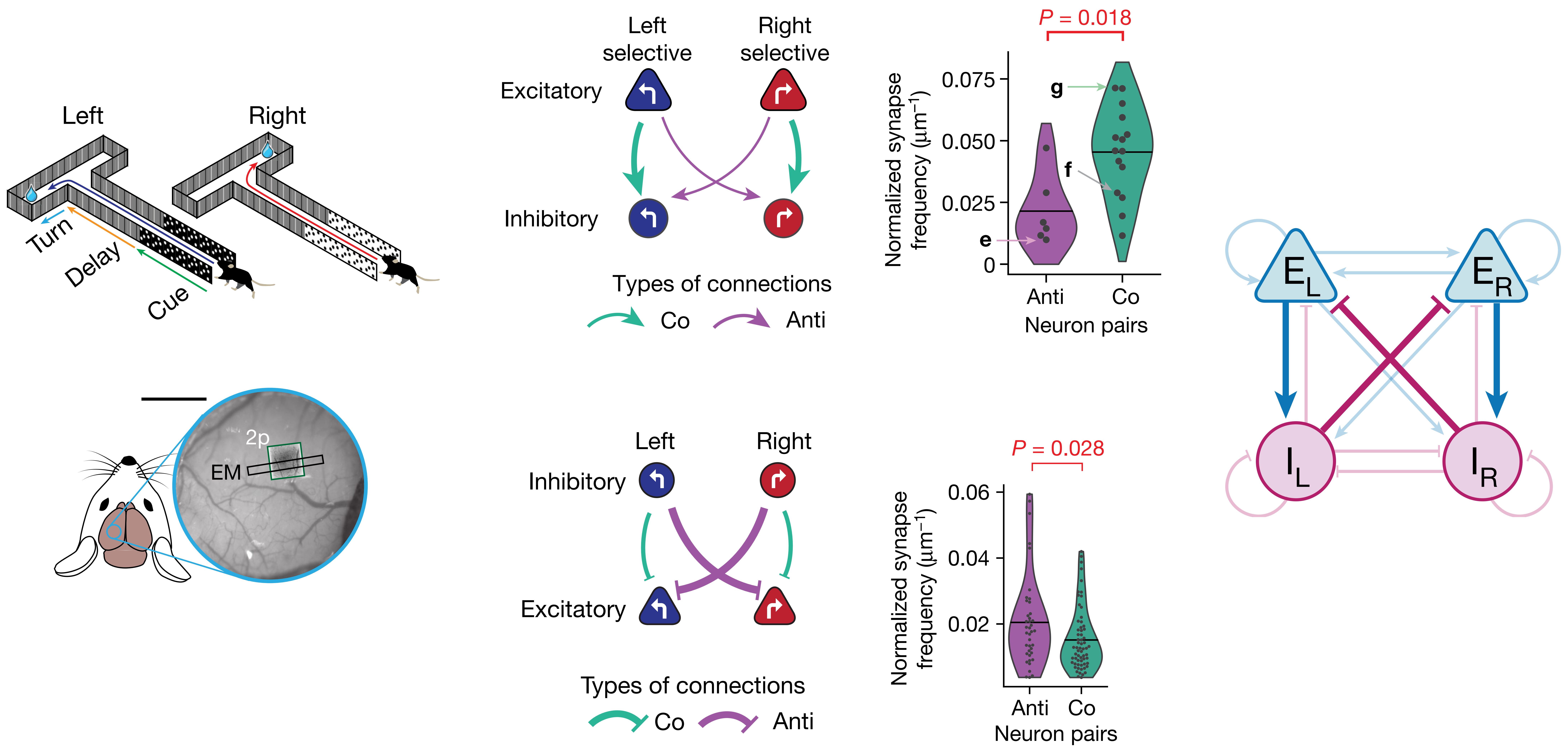


- These average ramping responses have been also observed in a number of other cortical and subcortical structures, suggesting distributed computation
- Single-trial population dynamics appear more complicated than DDM-like ramps

Shadlen, Newsome, Britten, Gold … c.f. Gold & Shadlen, 2007, *Annu Rev Neurosci*

<u>See also</u>: Hanks & Brody, 2016, *Curr Opin Neurobiol*
<u>But see, e.g.</u>: Latimer … Pillow, 2015, *Science*

Mutual inhibition and cortical wiring



Kuan et al., 2024, *Nature*

Value-based decision making

A few key concepts

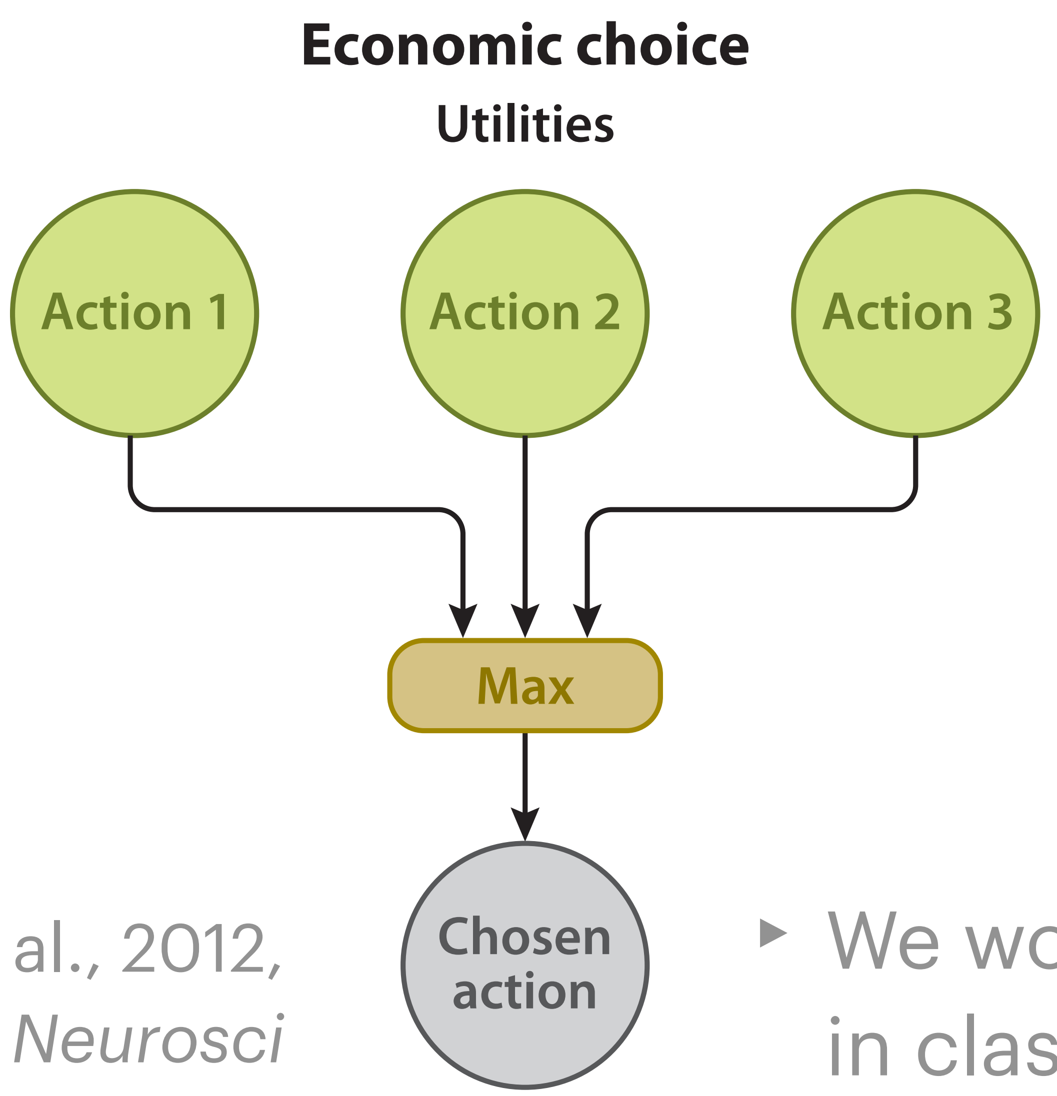
A *reward* is an event that satisfies a motivational drive
- It produces a positive affective experience
- It is salient; attracts attention
- It drives learning:
  - Pavlovian: *prediction*. Learn about events that predict reward
  - Operant: *positive reinforcement*. Increase the frequency of actions that lead to reward

A *punishment* is an event that generates withdrawal or avoidance
- It produces a negative affective experience
- It is salient; attracts attention
- It drives learning:
  - Pavlovian: *prediction*. Learn about events that predict punishment
  - Operant: *negative reinforcement*. Decrease the frequency of actions that lead to punishment

Cost is a loss or expenditure associated with an action, e.g.: metabolic (effort), opportunity (time, forgone alternatives)

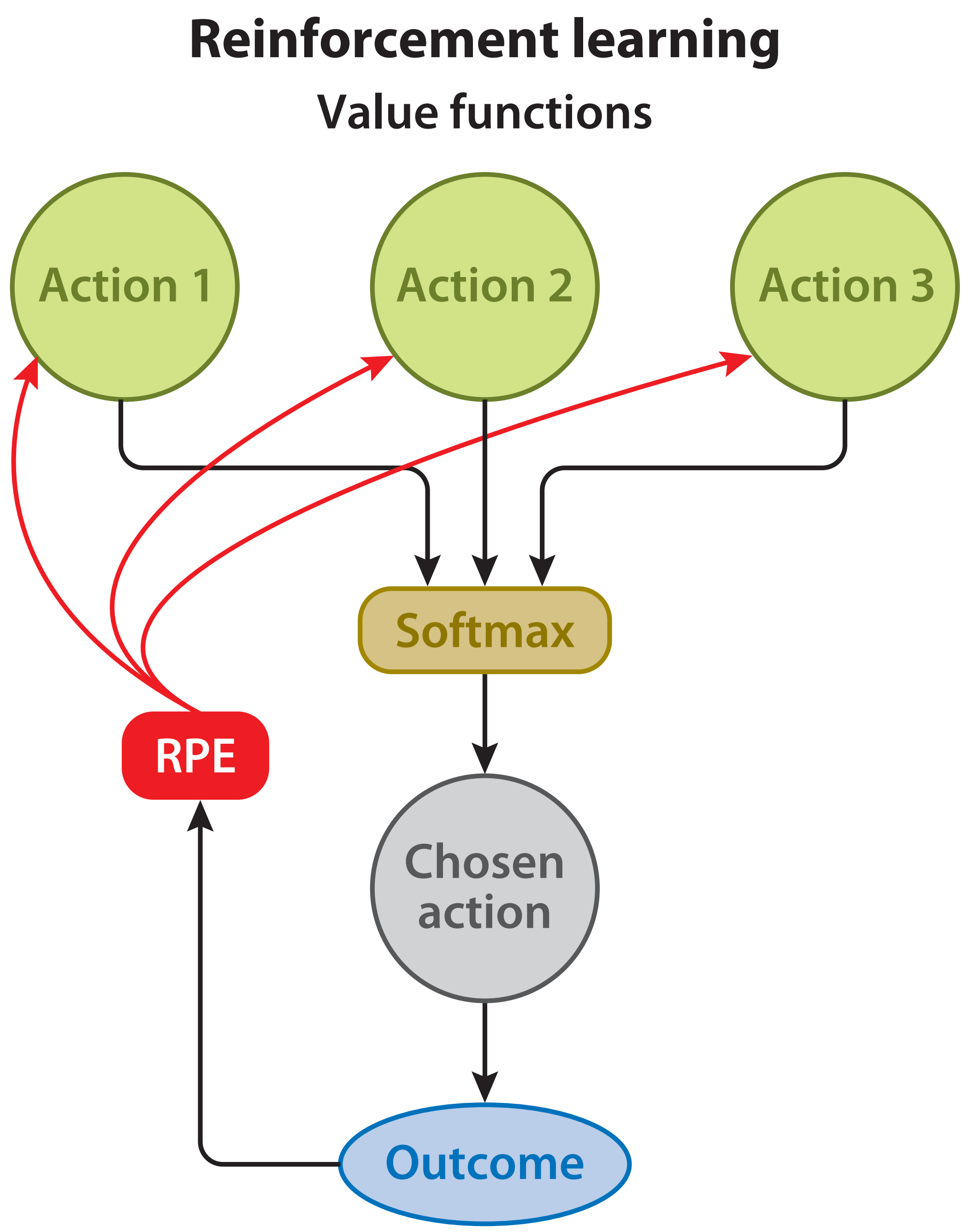
Loosely speaking, value-based decision making refers to decisions that are based on expected reward, punishment and cost

Economic decisions vs. RL



Lee et al., 2012, *Annu Rev Neurosci*

- We won't cover these in class any further



Economic decisions are made to maximize *utility*, defined as the total satisfaction an animal derives from a state (action). Utility is therefore *subjective*.

Prospect theory has been successful in describing these kinds of decisions

For details see Kahneman & Tversky, 1979; Padoa-Schioppa, 2011

Foraging is a ubiquitous form of economic decision in which animals sequentially decide to accept or reject offers based on utility (value) and cost. *Optimal foraging theory* is a normative framework describing this.

For details see Pulliam, 1974; Calhound & Hayden, 2015

In RL, decisions are made to maximize *value*, *V*, defined as the expected sum of future rewards *r* for a given *policy* \pi, which maps a *state* *S* to an *action* *A*

$$V_\pi = E\left[\sum_{t=0}^{\infty} r_t\right], \ \pi = S \times A = P(A = a \,|\, S = s)$$

The policy is updated by a ***R**eward **P**rediction **E**rror*

RL: the framework

Classic RL is a Markovian process where an agent transitions between different states *s* depending on their action *a* in *s*

$$S = \{s_0, s_1, \ldots, s_n\}$$

$$A = \{a_0, a_1, \ldots, a_n\}$$

$$P_a(s, s') = P(S_{t+1} = s' \,|\, S_t = s, A_t = a)$$

where *a* depends on a policy

$$\pi(s, a) = P(A_t = a \,|\, S_t = s)$$

and retrieve some reward *r* immediately after this transition

$$r_a(s, s') = P(A_t = a \,|\, S_t = s)$$

Each state is associated with a value function *V*

$$V_\pi(s) = E[G_0 \,|\, s_0 = s] = E\left[\sum_{t=0}^{\infty} \gamma^t r_t \,|\, s_0 = s\right],\ 0 \leq \gamma^t \leq 1$$

G is thus the discounted reward function given that we started in state s, where typically *gamma* decreases with *t*, i.e. distant future rewards have less weight on *V*.

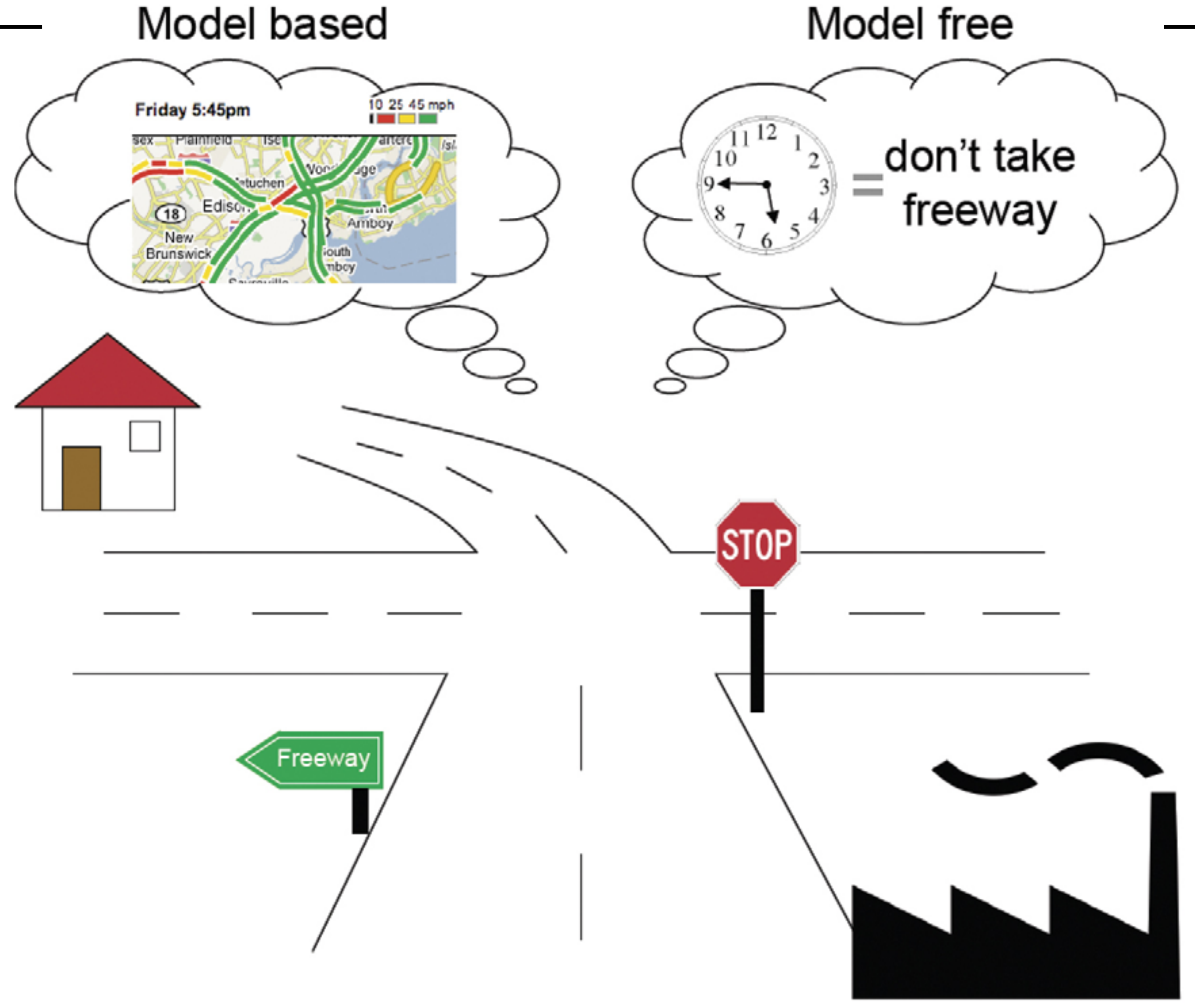
Similarly, each action within a state is also associated with a value function *Q*

$$Q_\pi(s, a) = E\left[\sum_{t=0}^{\infty} \gamma^t r_t \,|\, s_0 = s, a_0 = a\right]$$

Our goal is to optimize our policy to maximize action value

$$\pi_*(s) = \arg\max_a Q_*(s, a)$$

Model-based vs. model-free RL



Dayan & Niv, 2008, *Curr Opin Neurobiol*

- We have a model of the environment, i.e. $P(s,s')$ and $r(s,s')$
- Updating one policy can update others via this model
- More efficient / generalizable
- Lots of evidence that animals do this (e.g. reward devaluation, Tollman's cognitive maps)

- We need to learn every $Q(s,a)$ by trial and error, using RPE
- Updating one policy has no bearing on another
- Thus, very inefficient
- Still thought to describe some types of decision making well, e.g. habitual actions

The Q-learning algorithm

Recall that

$$Q(s,a) = E\left[\sum_{t=0}^{\infty} \gamma^t r_t \,|\, s_0 = s, a_0 = a\right]$$

From Bellman's optimality equations, this is equivalent to the sum of current reward and discounted Q for future states

$$Q(s,a) = r_t + \gamma E\left[\sum_{t=1}^{\infty} \gamma^t r_t \,|\, s_1 = s', a_1 = a'\right]$$

For simplicity, in the neuroscience context we can often think of states as time or behavioral trial

$$Q_t = r_t + \gamma Q_{t+1}$$

To learn Q, we define a *prediction error*, *delta*, to update our estimate of Q

$$\delta_t = r_t + \gamma \hat{Q}_{t+1} - \hat{Q}_t$$

Now we can iteratively update our estimate of Q with a learning rate *alpha*

$$\hat{Q}_t \leftarrow \hat{Q}_t + \alpha\delta_t \ , \ 0 \leq \alpha \leq 1$$

$$\hat{Q}_t \leftarrow \hat{Q}_t + \alpha(r_t + \gamma \hat{Q}_{t+1} - \hat{Q}_t)$$

$$\hat{Q}_t \leftarrow \hat{Q}_t + \alpha\gamma \hat{Q}_{t+1} + \alpha \underbrace{(r_t - \hat{Q}_t)}_{\textbf{RPE}}$$

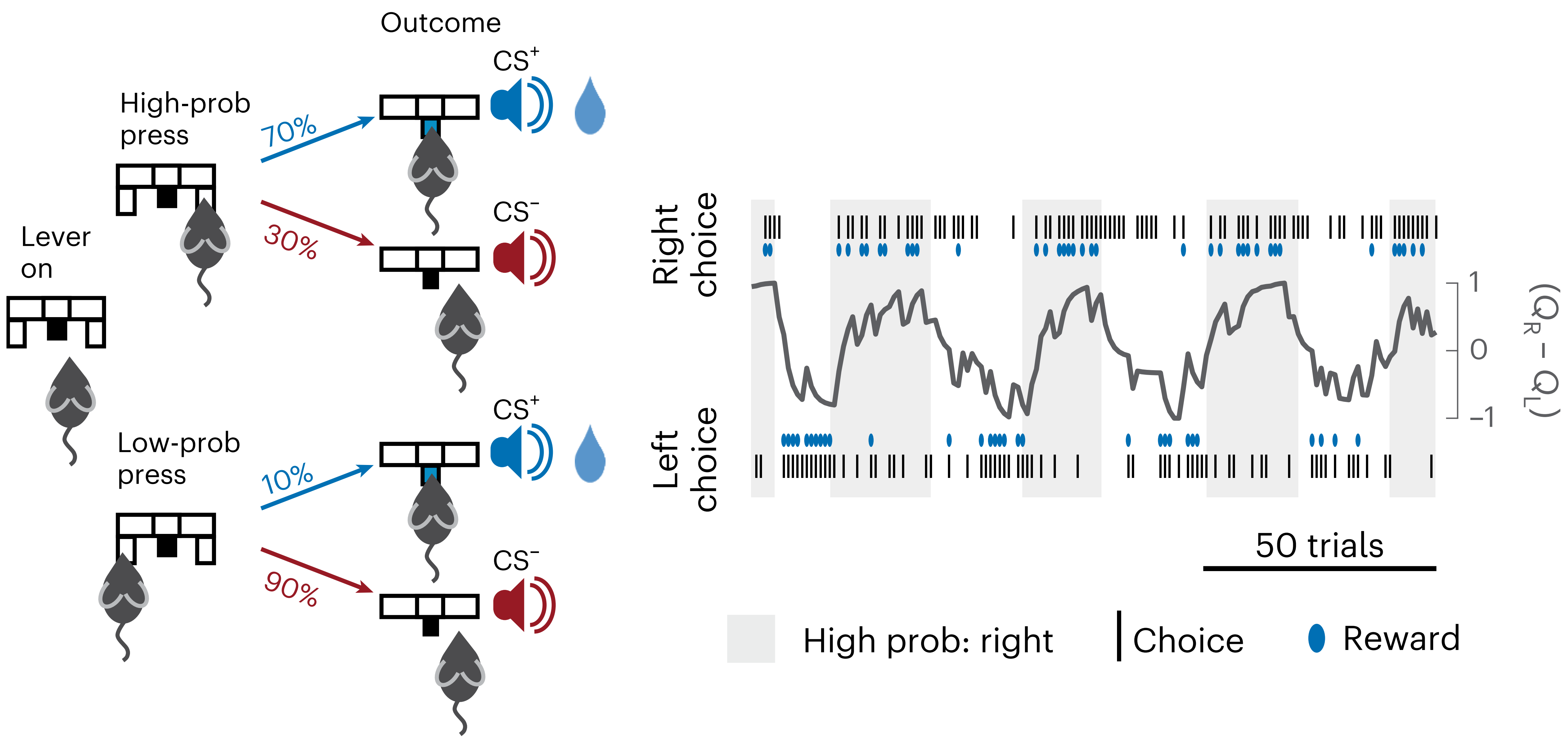
Rearranging this we get another common way of expressing the Q-learning rule

$$\hat{Q}_t \leftarrow (1-\alpha)\hat{Q}_t + \alpha(r_t + \gamma \hat{Q}_{t+1})$$

See Murphy 2023, Chapter 34, for more on Bellman's equations

Fitting Q-learning to behavioral data

In many value-based decision making (i.e. operant) tasks, animals need to constantly update estimates of the value of their actions. E.g., *two-arm bandit*:



Cox et al., 2023, *Nature Neurosci*

Equivalently from the previous slide, the outcome of an action, e.g. getting a reward when choosing left, triggers an update in the value of that action

$$Q_{t+1} = Q_t + \alpha\delta_t$$

For example, if the animal chose left

$$Q^L_{t+1} = Q^L_t + \alpha(r_t - Q^L_t)$$

$$Q^R_{t+1} = Q^R_t$$

We then predict the probability of a left choice with a logistic function

$$P(L) = \frac{1}{1 + e^{\beta_Q(Q_L - Q_R)}}$$

(Additional behavioral parameters can also go in the exponential term)

- Our parameters are *alpha* and *beta*. Because P(L) has a defined likelihood function, we can fit this with MLE, given some initial value of *Q*'s
- Some authors alternatively use Markov chain Monte Carlo methods, which Josh Glaser will cover

Primer: temporal difference (TD) learning

In Pavlovian conditioning, there is no action, so we typically want to predict reward from a state. In this case, instead of estimating Q, we want to estimate the state value *V*. The math holds

$$\hat{V}_t \leftarrow \hat{V}_t + \alpha(r_t + \gamma \hat{V}_{t+1} - \hat{V}_t)$$

A useful extension of regular TD (and Q-learning) is the addition of *eligibility traces*, since in the real world reward is often delayed with respect to a state or action.

- The difficulty in assigning a reward to a past events is known as the *credit assignment problem*.

This is done by estimating the value function over a window of *n* time points.

$TD(\lambda)$ algorithm :

$$G_{t:t+1} = r_t + \gamma V_{t+1}$$

$$G_{t:t+n} = r_t + \sum_{i=1}^{n} \gamma^i V_{t+i}$$

$$\hat{V}_t \leftarrow \hat{V}_t + \alpha(G_{t:t+n} - \hat{V}_t)$$

But rather than picking a specific n, we can take a weighted average of all possible n's with a single parameter \lambda

$$G_t^\lambda = (1-\lambda) \sum_{n=1}^{\infty} \lambda^{n-1} G_{t:t+n}$$

$$\hat{V}_t \leftarrow \hat{V}_t + \alpha(G_t^\lambda - \hat{V}_t)$$

See Murphy 2023, Chapter 35, for details

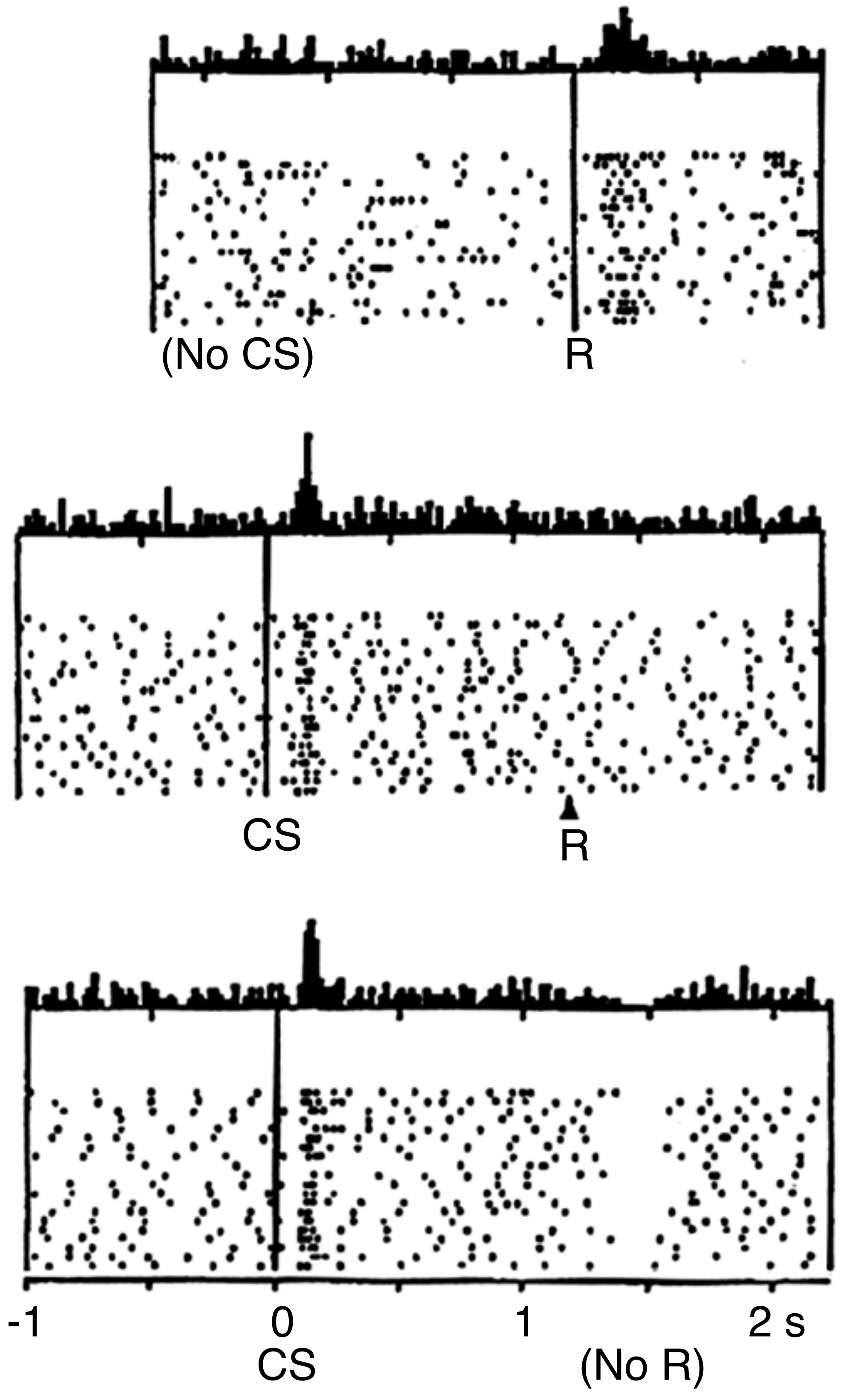
Value-base decisions in the brain

- In Pavlovian conditioning, the activity of dopamine neurons looks like RPE

Before learning
Reward response

After learning
Reward prediction response

After learning
Reward prediction + RPE responses



Schultz et al, 1997, *Science*

- Things get messier in more complex tasks, but many results are still compatible with the RL framework (though still lots of debate)
- Like perceptual decisions, neural correlates of action value, stimulus value, effort, motivation, reward etc seem to be distributed (see reviews in suggested reading)

Summary

- Decisions can be broadly divided into perceptual and value based
- SDT is the optimal way to decide based on a single sample from noisy data
- Sequential sampling is a generalization of SDT for multiple independent samples
- DDMs are the continuous-time limit of sequential sampling, and explain well behavioral data in perceptual tasks (choice and reaction time)
- Other models related to DDMs implement leaky integrators and inhibitory interactions inspired by neural-circuit architecture (e.g. mutual inhibition)
- Value-based decisions can be divided into economic (deterministically maximize utility) or reinforcement learning (probabilistically learn value)
- RL algorithms iteratively update value estimates based on future expected reward

Suggested reading

- Parker & Newsome (1998). Sense and the Single Neuron: Probing the Physiology of Perception. *Annu Rev Neurosci* 21:227-277
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- Shinn et al (2020). A Flexible Framework for Simulating and Fitting Generalized Drift-Diffusion Models. *eLIFE* 9:e56938
- Kuan et al (2024). Synaptic Wiring Motifs in Posterior Parietal Cortex Support Decision-Making. *Nature,* in press
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- Pulliam (1974). On the Theory of Optimal Diets. *The American Naturalist* 108:59-74
- Calhoun & Hayden (2015). The Foraging Brain. *Curr Opin Behav Sci* 5:24-31
- Murphy (2023). Probabilistic Machine Learning: Advanced Topics. Chapters 34 and 35.
- Journal club papers on canvas

The end