Lattice-Boltzmann Method in Acoustics – Prof. Andrey R. Silva

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Exercise List I
Cylindrical wave propagation in LGBK model

PROBLEM STATEMENT

In this work, a 2DQ9 lattice-Boltzmann scheme will be used to simulate the cross section of a cylindrical acoustic wave propagating in a viscous, compressible and isothermal medium. The scheme adopts the BGK collision operator with an arbitrary relaxation frequency $\omega = 1.7$ and it runs in a 2D lattice space of 300 by 300 cells. The acoustic source is a monopole implemented as a harmonic density disturbance at the center of the domain.

The simulated results will be compared with the problem's analytical solution obtained via the viscous wave equation in cylindrical coordinates. The solution provided is given by

$$p(r,t) = AH_0^{(2)}(kr)\, e^{j\omega t}, \qquad \text{Eq. 1}$$

where A is the acoustic pressure amplitude of the disturbance, $H_0^{(2)}$ is the Hankel function of the second kind and order zero, and r is the radial distance from the sound source. $k = 2\pi f/c_0 - j\alpha$ is the complex wavenumber and α is the spatial absorption coefficient, given by

$$\alpha = \frac{\omega}{c_0\sqrt{2}}\sqrt{\frac{\sqrt{1+(\omega\tau_s)^2}-1}{1+(\omega\tau_s)^2}}. \qquad \text{Eq. 2}$$

PROCEDURE 1

Objective: Define NTS.

To define the necessary number of timesteps we can start by calculating the time taken by the first wavefront to reach the boundaries of the numerical domain, which is a 2D lattice space of 300 by 300 cells. The acoustic source is located roughly at the center, composed by 9 cells; 149 through 151 in the x direction and the same coordinates in the y direction, given in lattice length. In this case, the first wavefront will travel 149 cells at the speed of sound c_s (in lattice units) until it reaches the nearest boundary, thus the time requires is given by

$$NTS = \left\lfloor \frac{149}{c_s} \right\rfloor, \qquad \text{Eq. 3}$$

in lattice units of time, which corresponds to a timestep.

Since the sound speed in lattice units is $1/\sqrt{3}$, the required number of timesteps is $NTS = \lfloor 149\sqrt{3} \rfloor = 258\ timesteps$.

PROCEDURE 2

Objective: Create a harmonic acoustic source in the center of the lattice.

The harmonic acoustic source that we want to simulate can be formulated in terms of density disturbance as

$$\rho^*(t) = \rho_0^* + A^* \sin(2\pi f^* t^*), \qquad \text{Eq. 4}$$

where ρ_0^* is the average density of the medium[1], A^* is the disturbance amplitude, f^* is the disturbance frequency and t^* is the time. In the simulation code this is implemented inside the iteration loop, right after the streaming step. The average density $\rho_0 = 1$ (in lattice units) is added to the acoustic fluctuation with amplitude $A = 0.001$ (in lattice units of density). The frequency is adjusted for each simulation case, as will be discussed in the next section, and the time is set as multiple of the fundamental timestep Δt.

PROCEDURE 3

Objective: Obtain the numerical result for the acoustic pressure field $p_{an}(x,t)$ for three different wavelengths discretization.

Three discretization levels will be simulated changing the number of cells across a wavelength λ using the same domain: $CPW = [8, 16, 25]$ (Cells per Wavelength). Initially, the 300x300 cells computational domain represented a 0.5x0.5 m physical space, that means that the resolutions Δx is $0.0017\ m/LatticeLength$ and a wavelength of 8 latticeLenghts represented $\lambda = 8\Delta x = 0.0133\ m$, or a source frequency of $25500\ Hz$. To keep the scaling consistent across the simulations, the computational domain will remain untouched, but the physical length will be adjusted accordingly to represent the same physical acoustic frequency. Table 1 shows the relations between $\Delta x, L_x$ and λ and how they maintain the physical parameters λ^* and f^* of the acoustic source.

Table 1 - Scaling in different discretization levels.

λ	λ^*	f^*	f	L_x	Δx
8	0.0133 m	25500 Hz	0.0722	0.50 m	1.7e-3 m/latticeLength
16	0.0133 m	25500 Hz	0.0361	0.25 m	8.3e-4 m/latticeLength
25	0.0133 m	25500 Hz	0.0231	0.16 m	5.3e-4 m/latticeLength

The relation between the frequency in lattice units and in the physical units is given by

$$f^* = \frac{1}{T^*} = \frac{1}{T\Delta t} = f\frac{1}{\Delta t} \quad \rightarrow \quad f^* = f\frac{\zeta}{\Delta x}, \qquad \text{Eq. 5}$$

where $\zeta = c_s^*/c_s$, $c_s = 1/\sqrt{3}$ and $c_s^* = 340\ m/s$.

The results were gathered at $ta = NTS$ along the lattice coordinate vector $x = (150{:}\,300;\ 150)$.

[1] The symbol * is used to denote physical variables.

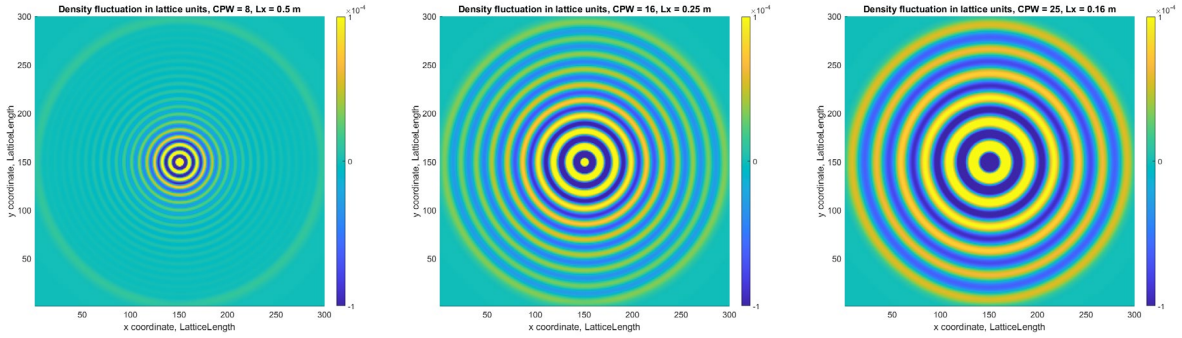

Figure 1 - Density contour plot for each discretization level

PROCEDURE 4

Objective: Obtain the analytical pressure field $p_a(x,t)$

To obtain the analytical pressure field, we are going to use the equations 1 and 2 implemented in the function *cylin_wave.m*, provided by the Professor. But first, we must find the corresponding fluid kinematic viscosity when the relaxation frequency is set to the arbitrary value of $\omega = 1/\tau = 1.7$, and we must also find the disturbance acoustic pressure corresponding to the imposed density fluctuation A.

The kinematic viscosity ν and the BGK relaxation time τ are related through

$$\nu = c_s^2\left(\tau - \frac{1}{2}\right) = c_s^2\left(\frac{1}{\omega} - \frac{1}{2}\right). \quad \text{Eq. 6}$$

In lattice units, this resulted in $\nu = 0.0294$. To translate into physical units, we may use the relation

$$\nu^* = \zeta \Delta x \nu, \quad \text{Eq. 7}$$

of which the values are listed in Table 2.

The model being simulated is isothermal, so the relation between pressure and density is given by

$$p = \rho c_s^2. \quad \text{Eq. 8}$$

This way, we can find the pressure oscillation $A_p = Ac_s^2 = \frac{0.001}{3} = 3.3 \cdot 10^{-4}$, in lattice units of pressure. Now we can plug these values in the function and compare the results with the simulation, which can be seen in Figure 2.

Now we repeat this process using the kinematic viscosity of air in Standard Temperature and Pressure (STP) $\nu^*_{STP} = 1.5 \cdot 10^{-5}\ m^2/s$. Using Eq. 7, we can calculate the viscosity ν_{STP} in lattice units to be plugged in the analytical equation (listed in Table 2). Results are shown in Figure 3.

Table 2 – Arbitrary and STP densities in lattice and physical units.

CPW	$\nu\ (\omega = 1.7)$	ν^*	ν_{STP}	ν^*_{STP}
8	0.0294	0.0289 m²/s	1.52e-05	1.5e-05 m²/s
16	0.0294	0.0144 m²/s	3.06e-05	1.5e-05 m²/s
25	0.0294	0.0092 m²/s	4.78e-05	1.5e-05 m²/s

PROCEDURE 5

Objective: Compare $p_{an}(x)$ for each discretization scheme, as well as their respective analytical solution $p_a(x)$.

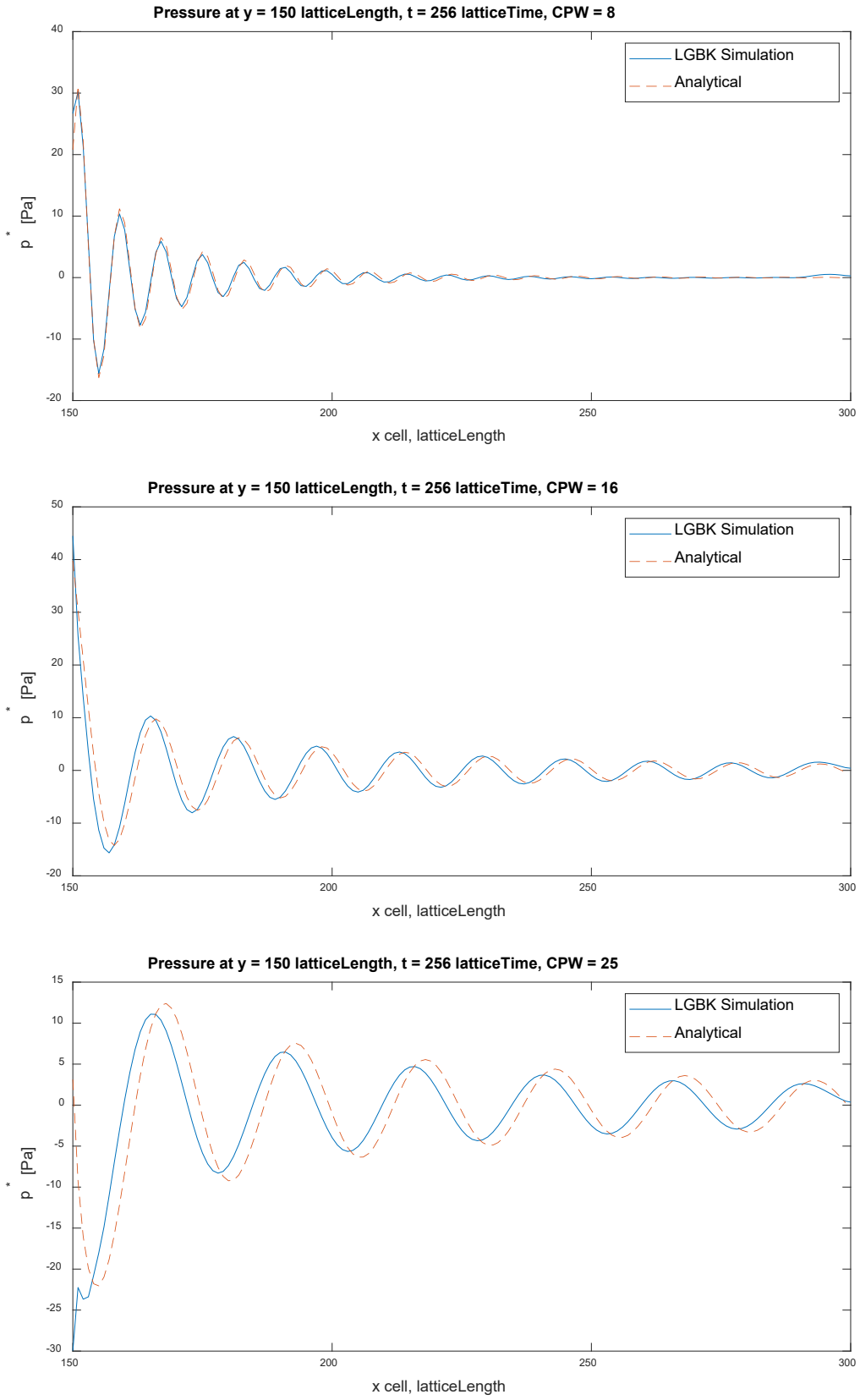

Figure 2 - Physical acoustic pressure in different discretization

PROCEDURE 6

Objective: compare the different results obtained for $p_a(x)$ (numerical) and $p_{air}(x)$ (analytical with ν_{STP}).

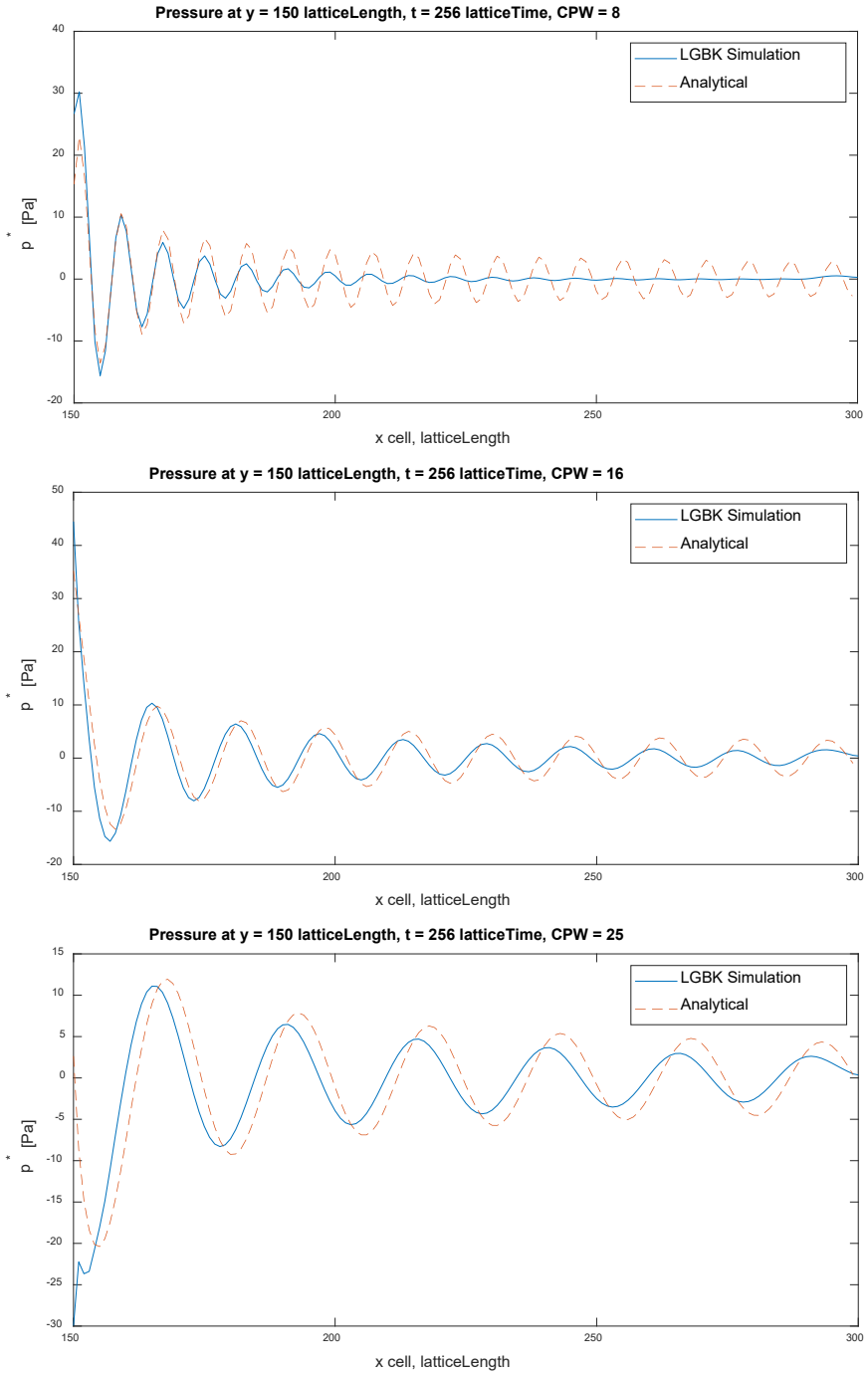

Figure 3 - Acoustic pressure spatial distribution with $\nu = 0.0294$ (numerical) and ν_{STP} (analytical)

DISCUSSION

Through Figure 2 it becomes clear that the LGBK method is very robust and adequate for acoustic problems. There it can be seen that even with low discretization, the results are very close to well-known analytical solutions.

QUESTION 1

For high discretization schemes (16, 25 cells per wavelength) a slight disagreement between $p_{an}(x)$ and $p_a(x)$ should be noticeable when the wave approaches the lattice boundary. Can you explain why?

The case simulated with the LGBK scheme includes transient effects, as the domain was at rest in the beginning. However, the analytical model solves the pressure field for a steady condition, and this could be the source of the difference between the solutions in the first wavefront region.

QUESTION 2

Due to the limitations of the LBGK model, the physical viscosity used in the simulations is $O(2)$ higher than the kinematic viscosity of air in normal conditions. Even so, the error between $p_{an}(x)$, $p_a(x)$ and $p_{air}(x)$ is reasonably small. Can you draw any conclusions over this fact?

From Equations 1 and 2, it can be seen that $p_{an}(x,t)$ depends on the complex wavenumber k, which is a function of α, that is a non-linear monotonically decreasing function of kinematic viscosity, but monotonically increasing function of frequency. This implies that α, a function of viscosity, is more relevant for high frequency waves, and it can be seen in Figure 3, where there are bigger differences between the analytical and simulated results for the lower resolution results, which have a higher lattice frequency.

Even for the coarse case (CPW=8) the relations between the imaginary and real parts of k is

$$\frac{Re(k)}{Im(k)} = \frac{0.7854}{0.0313} = 25.0935, \qquad \text{Eq. 9}$$

what shows that the viscous damping of the acoustic wave is much smaller than is propagating part.

However, the tendency in Figure 3 is very clear, the higher the lattice frequency of the wave, the grater the effects of viscosity.