Modal Analysis – Prof. Arcanjo Lenzi

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Exercise List - 4

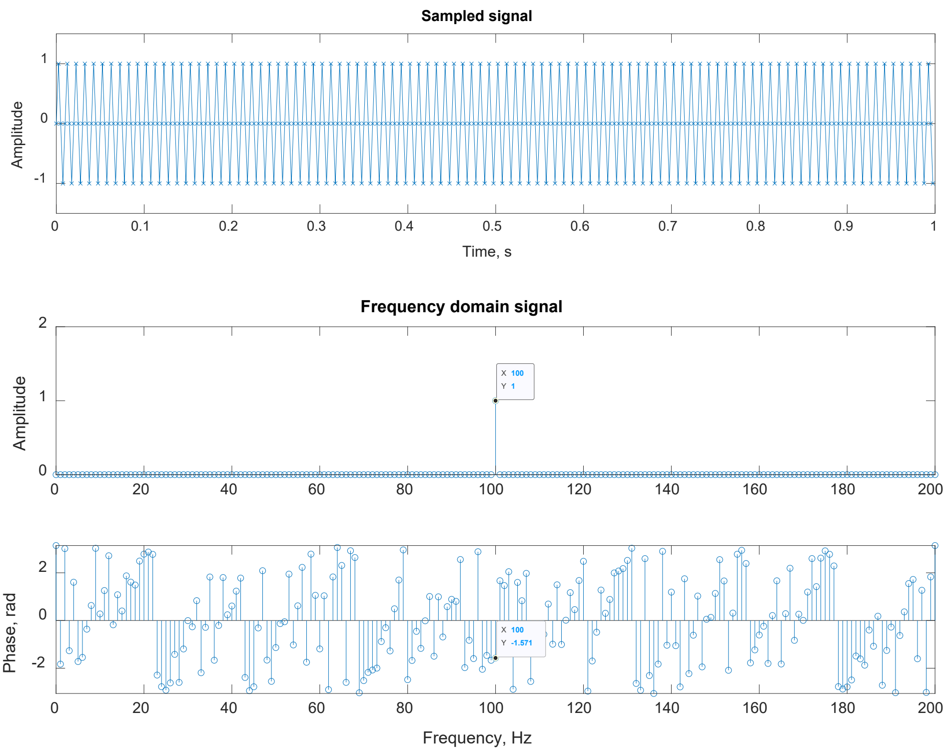
DSP – Digital Signal Processing

The objective of this work is to study a few aspects of digital signal processing, namely:

- Aliasing
- Windowing
- Zero padding

1

Plot the spectrum of a sinusoidal signal $f(t) = \sin(\omega t)$ that was sampled for $T = 1\ s$ with a sampling rate of $\Delta f = 400\ Hz$.



a) What is the sampled signal peak value at $f = 100\ Hz$?

The DFT of a periodic signal sampled with o whole number of periods provides a clear spectrum. As the signal is a perfect sine wave, the spectrum provides amplitude 1 at $f = 100\ Hz$, with a phase of $-\pi/2$, which represents a sinusoid.

b) How many points there are in the digital half-power bandwidth?

As there is only one no-zero spectral line, there is no points in the digital half-power bandwidth.

c) Find the damping using the half-power bandwidth.

As there are no points in the digital half-power bandwidth, the damping must be zero.

d) Calculate the sum of the squares of the spectral components and compare with the RMS value of the real signal.

The RMS value of the original analog signal (sine wave) is given by its amplitude scaled by a factor of $1/\sqrt{2}$. The original amplitude is 1, so $x_{RMS} = 1/\sqrt{2} = 0.7071$. Performing the summation of all spectral components squared of the FFT of the sampled signal results in 0.7071, numerically the same as the theoretical value. The equation used was

$$x_{rms} = \sqrt{\sum \frac{|X_k(f_k)|^2}{2}}, \qquad \text{Eq. 1}$$

and the code used to compute is shown below

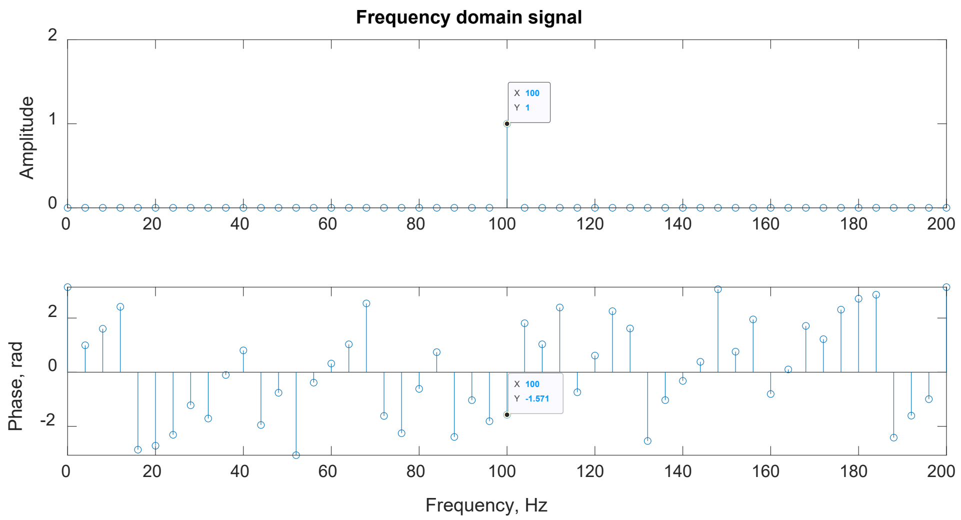
```
% FFT
F1 = fft(f1(t))/Lf;           % scale for number of sampling points
f  = linspace(0,Fs/2,Lf/2+1); % frequency vector, cut mirrored part, up to Nyquist freq
F1 = 2*F1(1:length(f));       % double the power of spectrum

% RMS
rms = sqrt(sum(abs(F1).^2)/2);
```

2

Repeat item 1 using sampling time of $T = 0.25\ s$. How the reduction in frequency resolution affects the half-power bandwidth? The sampling time is still a multiple of the periods of the signal.

Having a shorter sampling time reduced the frequency resolution, as can be seen in the figure below. However, the sampling time is still a whole number of periods of the original signal, and as the DFT (and FFT) consider the sampled signal to be periodic, the transformation still results in a perfect sine spectrum. Therefore, the digital half-power bandwidth is still zero, as expected.



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c) Find the damping using the half-power bandwidth.

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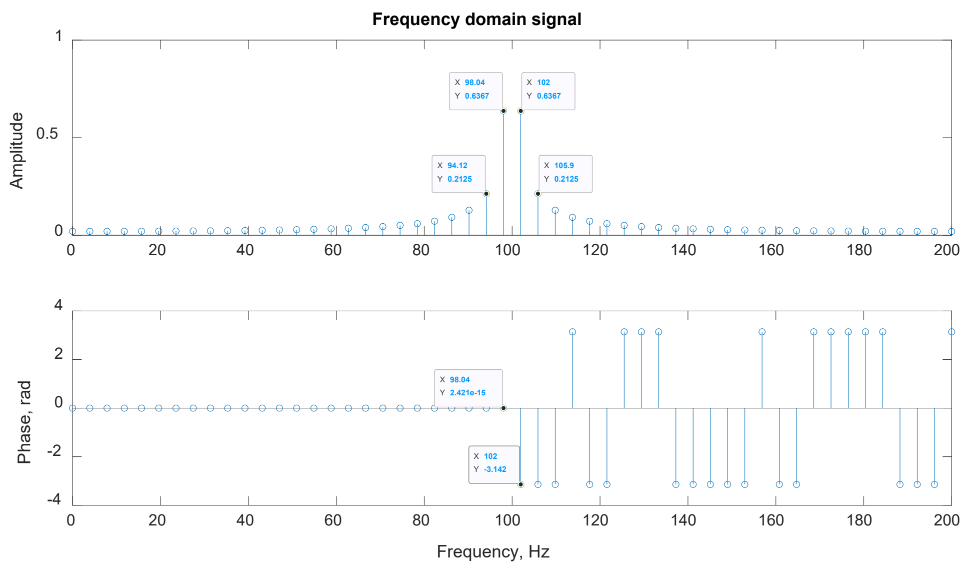
d) Calculate the sum of the squares of the spectral components and compare with the RMS value of the real signal.

Performing the summation of all spectral components squared of the FFT of the sampled signal results in 0.7071, numerically the same as the theoretical value.

3

Repeat item 2 using 102 sampling points, or sampling time of $T = 0.255\ s$. The sampling time is not a multiple of the signal's period anymore, how does this affect the spectrum?



a) What is the sampled signal peak value at $f = 100\ Hz$?

Due to frequency discretization, there is no data available at $f = 100\ Hz$, but the maximum values are 0.6367 in the frequencies $f = 98.04\ Hz$ and $f = 102\ Hz$.

b) How many points there are in the digital half-power bandwidth?

Due to frequency discretization, there is no data that hits exactly the half-power values of the spectrum. However, interpolating the results give bandwidth $bw = 3.5\ Hz$.

c) Find the damping using the half-power bandwidth.

Using the bandwidth $bw = 3.5\ Hz$ we get $\eta = 3.4\%$ using the following equation:

$$\eta = \frac{bw}{f_0},$$

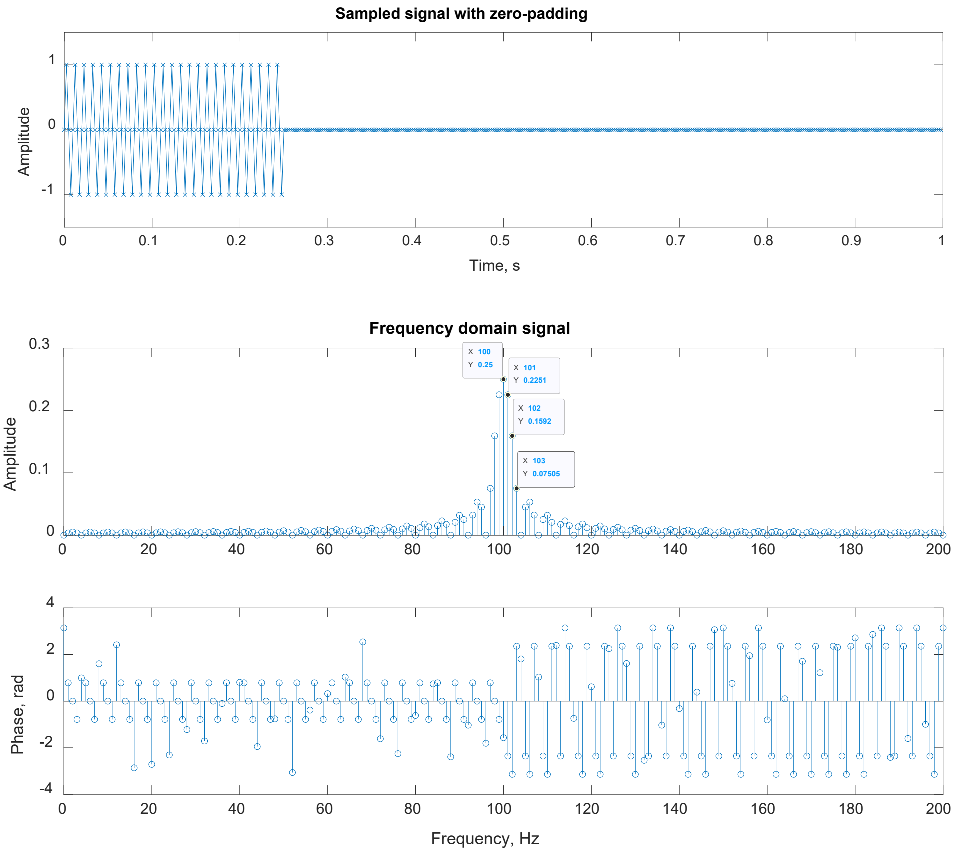
where $bw = f_2 - f_1$ and f_0 is the peak frequency.

d) Calculate the sum of the squares of the spectral components and compare with the RMS value of the real signal.

Performing the summation of all spectral components squared of the FFT of the sampled signal results in 0.70724, numerically very close to the theoretical value.

4

Repeat item 2 with a zero-padding of 300 elements. How does this change the digital and spectral resolution?



a) What is the sampled signal peak value at $f = 100\ Hz$?

At the peak we get $X(100\ Hz) = 0.25$

b) How many points there are in the digital half-power bandwidth?

Due to frequency discretization, there is no data that hits exactly the half-power values of the spectrum, but there is one data point between peak and half-power frequency. Using interpolation results in bandwidth $bw = 3.46\ Hz$.

c) Find the damping using the half-power bandwidth.

Using the bandwidth $bw = 3.46\ Hz$ we get $\eta = 3.5\%$.

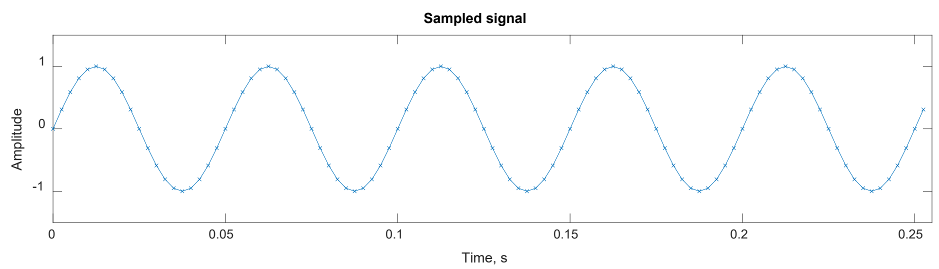
d) Calculate the sum of the squares of the spectral components and compare with the RMS value of the real signal.

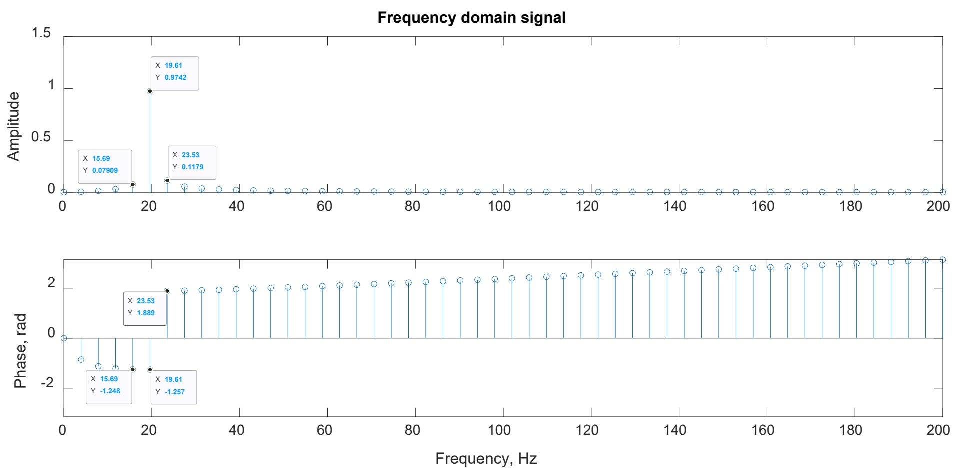
Performing the summation of all spectral components squared of the FFT of the sampled signal results in 0.3536. This low value is explained by the fact that the signal was modified, introducing a region with no energy, the zero-padding zone, therefore the rms value of the signal is expected to decrease. The exact same result for the rms is obtained both using Eq. 1 and computing in the time domain using

$$x_{rms} = \sqrt{\frac{1}{T}\int_0^T x^2(t)dt.} \qquad \text{Eq. 2}$$

5

Repeat item 3 with $f(t) = \sin(2\pi \cdot 20t)$, a 20 Hz sinusoidal signal. Use 102 sampling points, or sampling time of $T = 0.255\ s$.





a) What is the sampled signal peak value at $f = 20\ Hz$?

At the peak we get $X(19.61\ Hz) = 0.9742$

b) How many points there are in the digital half-power bandwidth?

Due to frequency discretization, there is no data that hits exactly the half-power values of the spectrum, but there is one data point between peak and half-power frequency. Using interpolation results in bandwidth $bw = 2.61\ Hz$.

c) What is the new half-power bandwidth?

Using the bandwidth $bw = 2.61\ Hz$ we get $\eta = 13.3\%$.

d) Calculate the sum of the squares of the spectral components and compare with the RMS value of the real signal.

Performing the summation of all spectral components squared of the FFT of the sampled signal results in 0.70082, very close to the theoretical value. The error is due to the low sampling time, tests with $T = 5.1\ s$ (roughly 100 periods) results in rms equal 0.707.

6

Repeat item 3 with a Hanning window and compare both spectra.

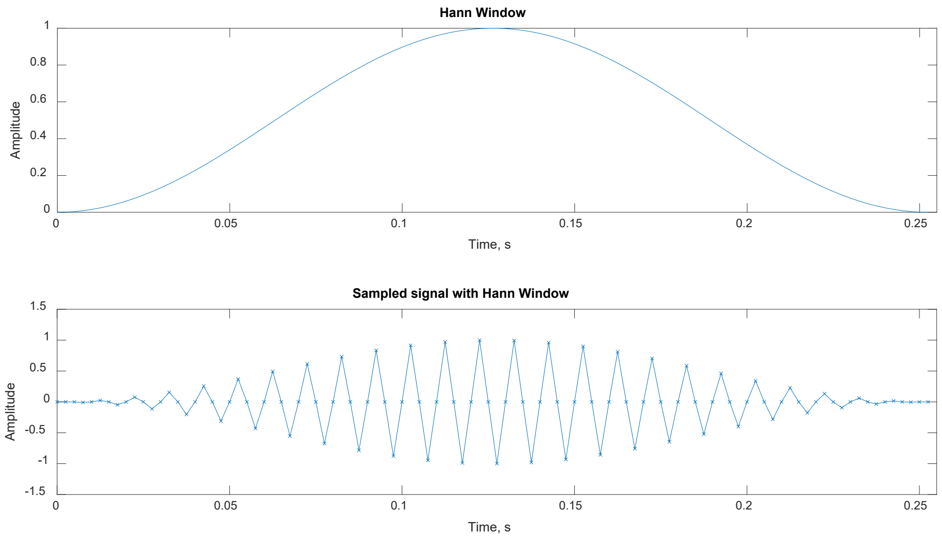
When windowing a signal, we must compensate for energy loss. The Energy Correction Factor (ECF) that we must scale the spectrum is calculated via

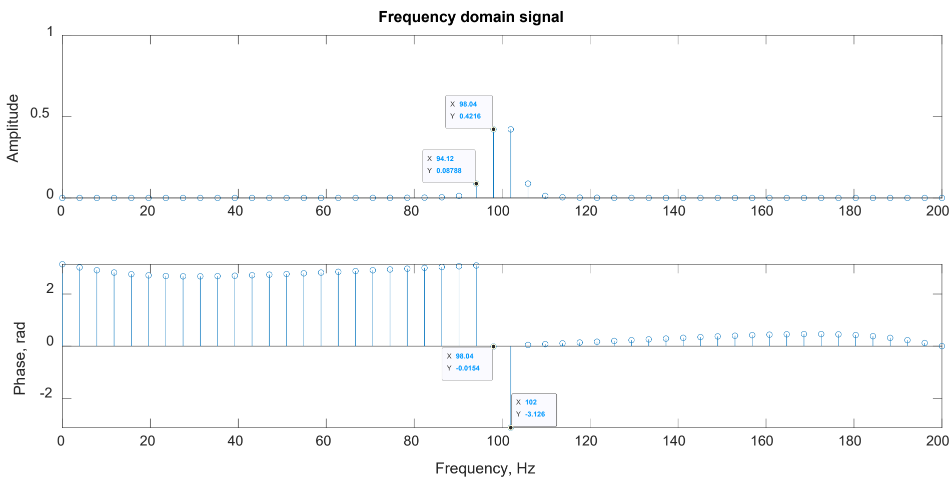
$$ECF = \frac{1}{\sqrt{\frac{1}{T}\int_0^T w^2(t)dt}}, \quad \text{Eq. 3}$$

where $w(t)$ is the window profile and T is the sampling time. The discrete version of the above equation is

$$ECF = \frac{1}{\sqrt{\frac{1}{N}\sum_{n=1}^{N}|w_n|^2}}, \quad \text{Eq. 4}$$

which resulted in a scaling factor of 1.6411.





a) What is the sampled signal peak value at $f = 100\ Hz$?

At the peak we get $X(98.04\ Hz) = 0.42$.

b) How many points there are in the digital half-power bandwidth?

Due to frequency discretization, there is no data that hits exactly the half-power values of the spectrum, but there is one data point between peak and half-power frequency. Using interpolation results in bandwidth $bw = 2.90\ Hz$.

c) Find the damping using the half-power bandwidth.

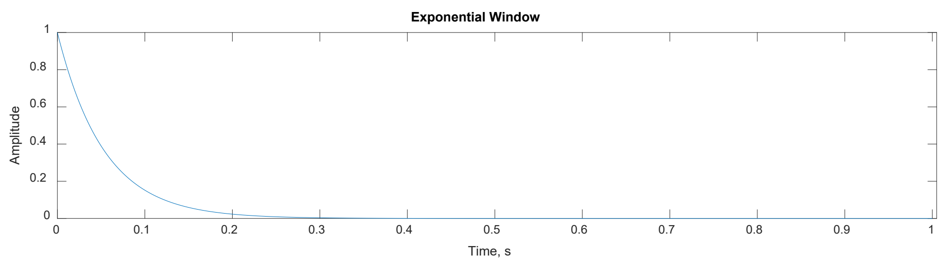
Using the bandwidth $bw = 2.90\ Hz$ we get $\eta = 2.8\%$.

d) Calculate the sum of the squares of the spectral components and compare with the RMS value of the real signal.

Performing the summation of all spectral components squared of the FFT of the sampled signal results in 0.7071, numerically very close to the theoretical value. We can see here that the ECF really did its job by correcting the energy level.

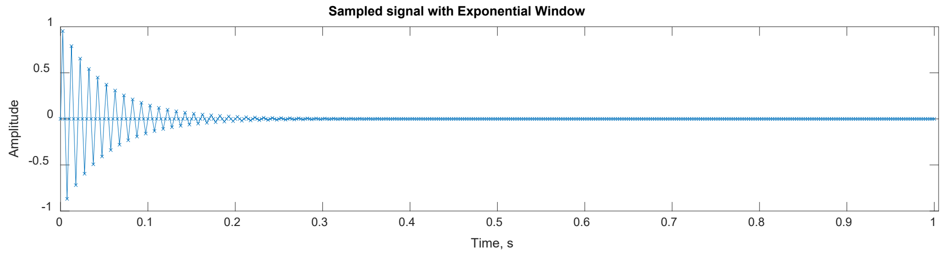
7

Repeat item 1 using 402 sampling points and an exponential window $w(t) = e^{-\zeta\omega t}$, with $\zeta = 0.03$ ($\eta = 0.06$).

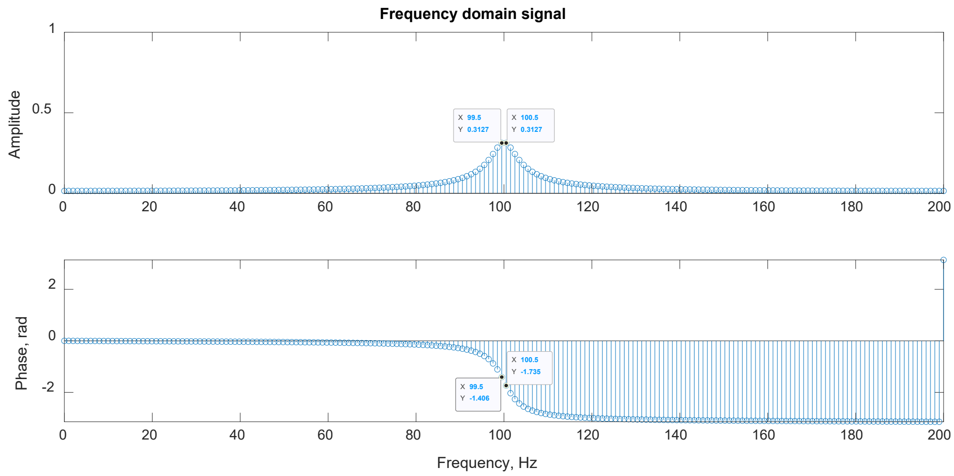


Using Eq. 4 we get $ECF = 6.0131$, which was used to scale the spectrum

a) Plot the signal in the time domain



b) What is the sampled signal peak value at $f = 100\ Hz$?



At the peak we get $X(100.5\ Hz) = 0.3127$

c) How many points there are in the digital half-power bandwidth?

Due to frequency discretization, there is no data that hits exactly the half-power values of the spectrum, but there is one data point between peak and half-power frequency. Using interpolation results in bandwidth $bw = 5.2\ Hz$.

d) Find the damping using the half-power bandwidth.

Using the bandwidth $bw = 5.2\ Hz$ we get $\eta = 5.2\%$, very close to $\eta_{window} = 6\%$.

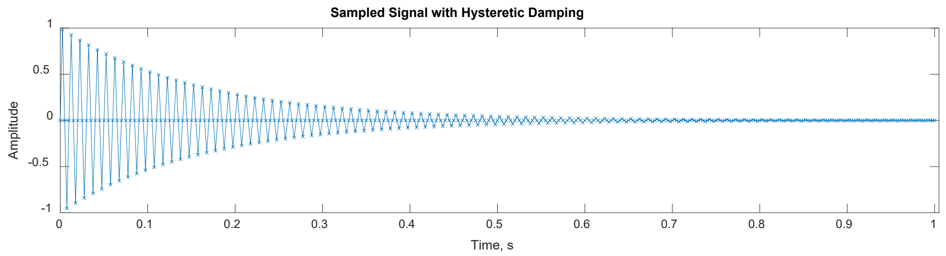
e) Calculate the sum of the squares of the spectral components and compare with the RMS value of the real signal.

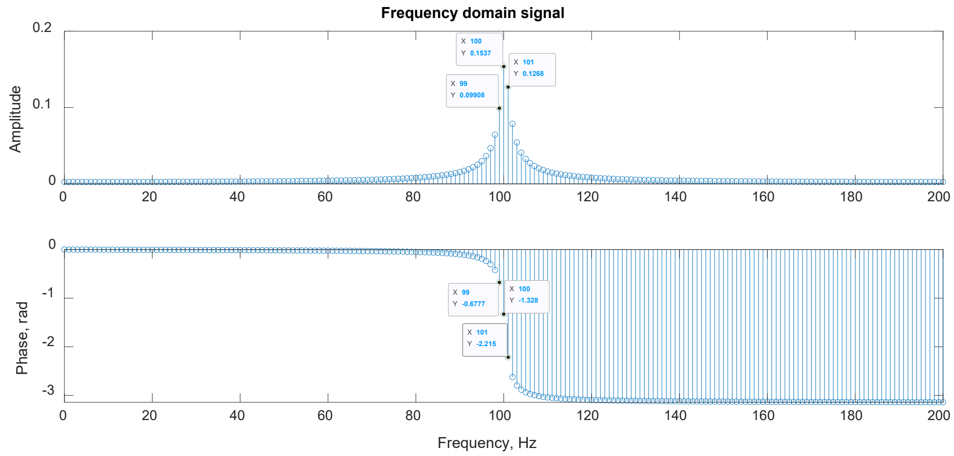
Performing the summation of all spectral components squared of the FFT of the sampled signal results in 0.6897, only 2.5% below the theoretical value. We can see here that the ECF really did its job by correcting the energy level.

8

Repeat item 1 using 402 sampling points and considering that the original signal has a damping factor of $\eta = 0.01$ due to hysteretic damping. Next, apply an exponential window ($w(t) = e^{-\zeta\omega t}$, with $\zeta = 0.03$) to the signal and repeat the process.

a) Plot the spectrum of this new signal

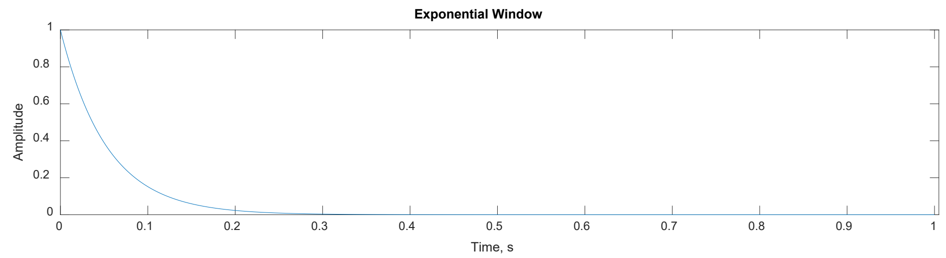


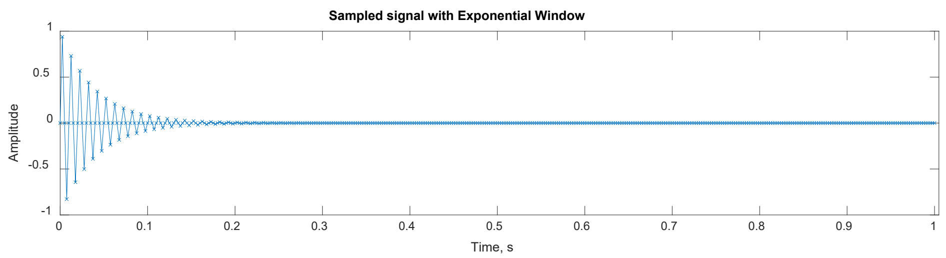


b) Find the damping using the half-power bandwidth

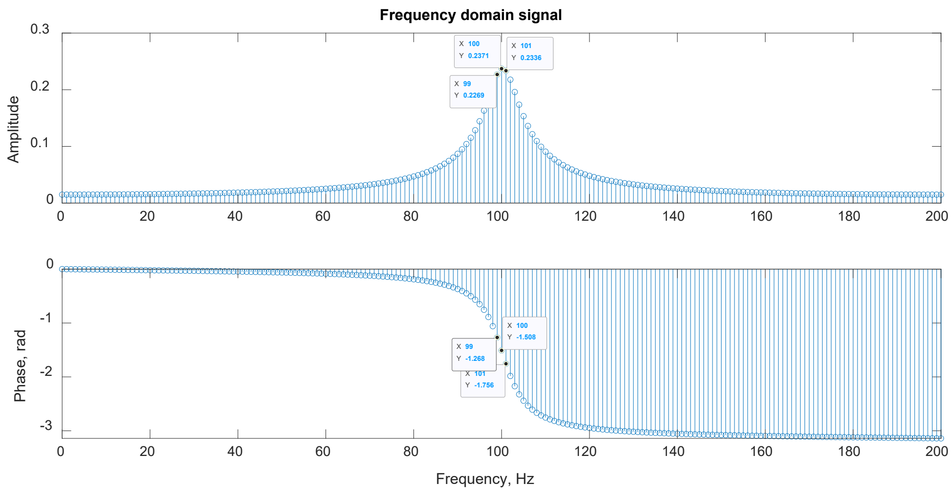
Due to frequency discretization, there is no data that hits exactly the half-power values of the spectrum, but there is one data point between peak and half-power frequency. Using interpolation results in bandwidth $bw = 2.75\ Hz$ and we get $\eta = 2.76\%$, a little below the expected value.

NOW WITH A WINDOW





a) What is the sampled signal peak value at $f = 100\ Hz$?



At the peak we get $X(100\ Hz) = 0.2371$.

b) How many points there are in the digital half-power bandwidth?

Due to frequency discretization, there is no data that hits exactly the half-power values of the spectrum, but there is one data point between peak and half-power frequency. Using interpolation results in bandwidth $bw = 8.58\ Hz$.

c) Find the damping using the half-power bandwidth.

Using the bandwidth $bw = 8.58\ Hz$ we get $\eta = 8.6\%$.

d) Calculate the sum of the squares of the spectral components and compare with the RMS value of the real signal.

Performing the summation of all spectral components squared of the FFT of the sampled signal results in 0.5960, which is 3 times bigger than the value we get calculating the rms in the time domain. Here the ECF did not work properly.