Modal Analysis – Prof. Arcanjo Lenzi

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Exercise List 2.3

1 SDOF – Hysteretic Damping

The objective of this work is investigate how FRF signals measured from real continuous systems can be used to construct a MDOF vibration model using modal analysis.

The system used in this particular work is a simply supported beam, illustrated in Figure 1. This beam is 1 m long, 30 mm wide and 5 mm thick, with a density of $\rho = 7850\ kg/m^3$, totaling $m = 1.977\ kg$. The structural damping assumed is $\eta = 0.05$.

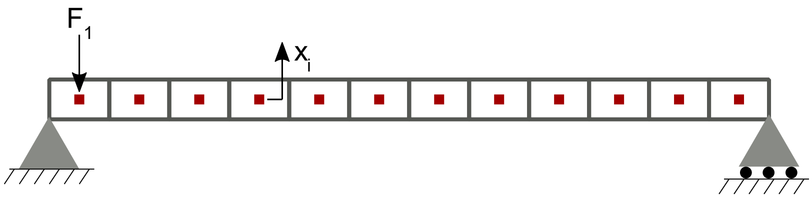


Figure 1 - System sketch.

To simulate the acquired data, an analytical model is used to find the natural frequencies and mode shapes. The structure is instrumented with 12 accelerometers, evenly spaced, as illustrated in Figure 1, resulting in an experimental mesh with spatial resolution of $\Delta L = 83.3\ mm$. The analytical model gives the receptance by

$$\alpha_{jk}(\omega) = \frac{X_i}{F_k} = \sum_{n=1}^{N} \frac{{}_n\Phi_j\ {}_n\Phi_k}{\omega_n^2 - \omega^2 + j\eta_n\omega_n^2}, \qquad \text{Eq. 1}$$

where

$${}_n\Phi_k = \sqrt{\frac{2}{m}} \sin\frac{n\pi x_k}{L} \qquad \text{Eq. 2}$$

is the k^{th} element of the mass-normalized eigenvector $\{\Phi\}_n$ associated to the eigenvalue ω_n^2, n is the mode number, and x_k is the position of the k^{th} experimental DoF. The n^{th} natural frequency is given by

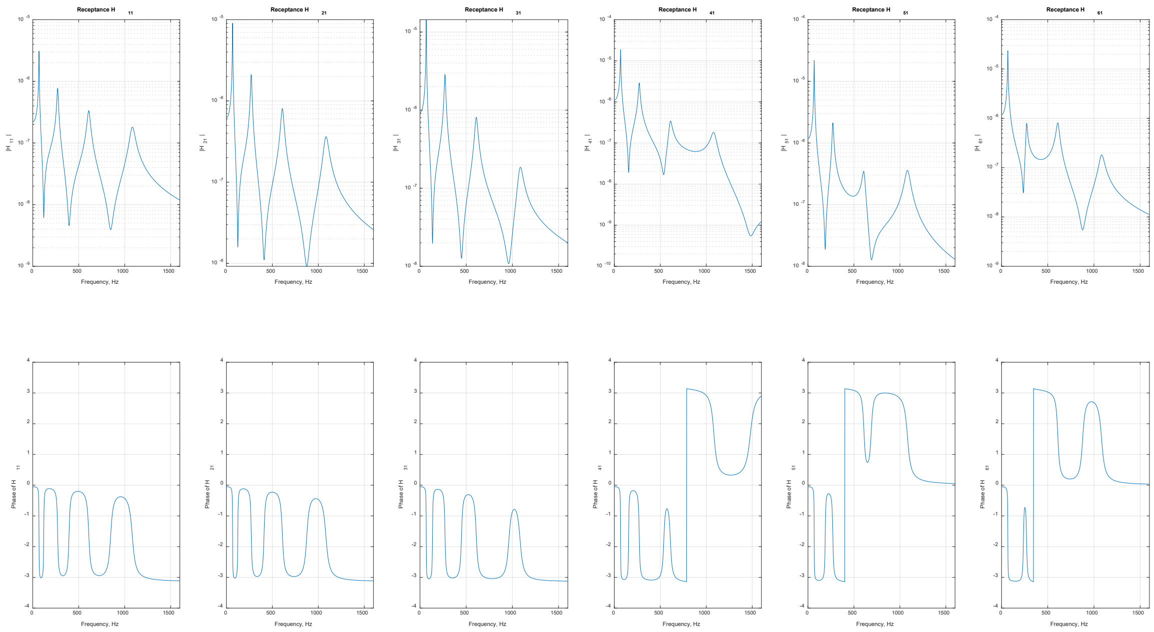
$$\omega_n = \frac{n\pi}{L}\sqrt{\frac{E}{\rho}}. \qquad \text{Eq. 3}$$

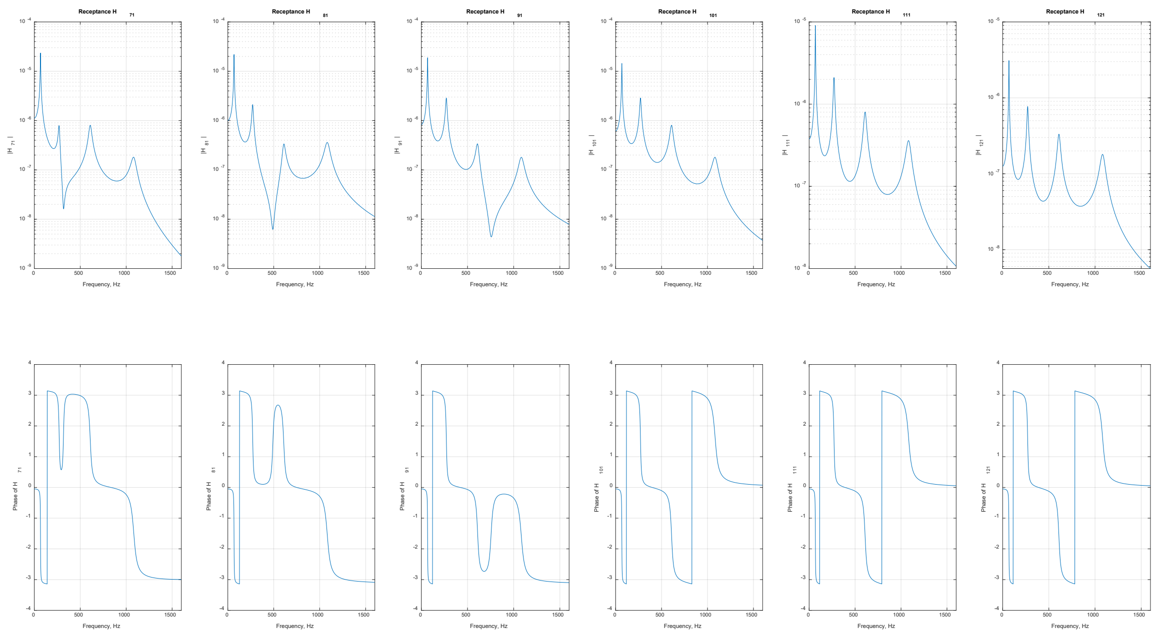
The analytical model requires that we consider all modes, which, for a continuous systema is infinite. However, we shall consider only the contribution of the first four modes to obtain the FRFs, this give us the four natural frequencies to be considered:

$$f_1 = 67.70\ Hz, f_2 = 270.60\ Hz, f_3 = 608.86, f_4 = 1082.4\ Hz$$

1.1

Objective: Plot the analytical receptance for each DoF in relation to an excitation in x_1. Use only the first for modes ($N = 4$ in Eq. 1), with a frequency resolution of $\Delta f = 1\ Hz$.





1.2

Objective: Plot the Nyquist circle for $0 \leq f \leq 400\ Hz$. Do the modes interfere? Are the shapes circular?

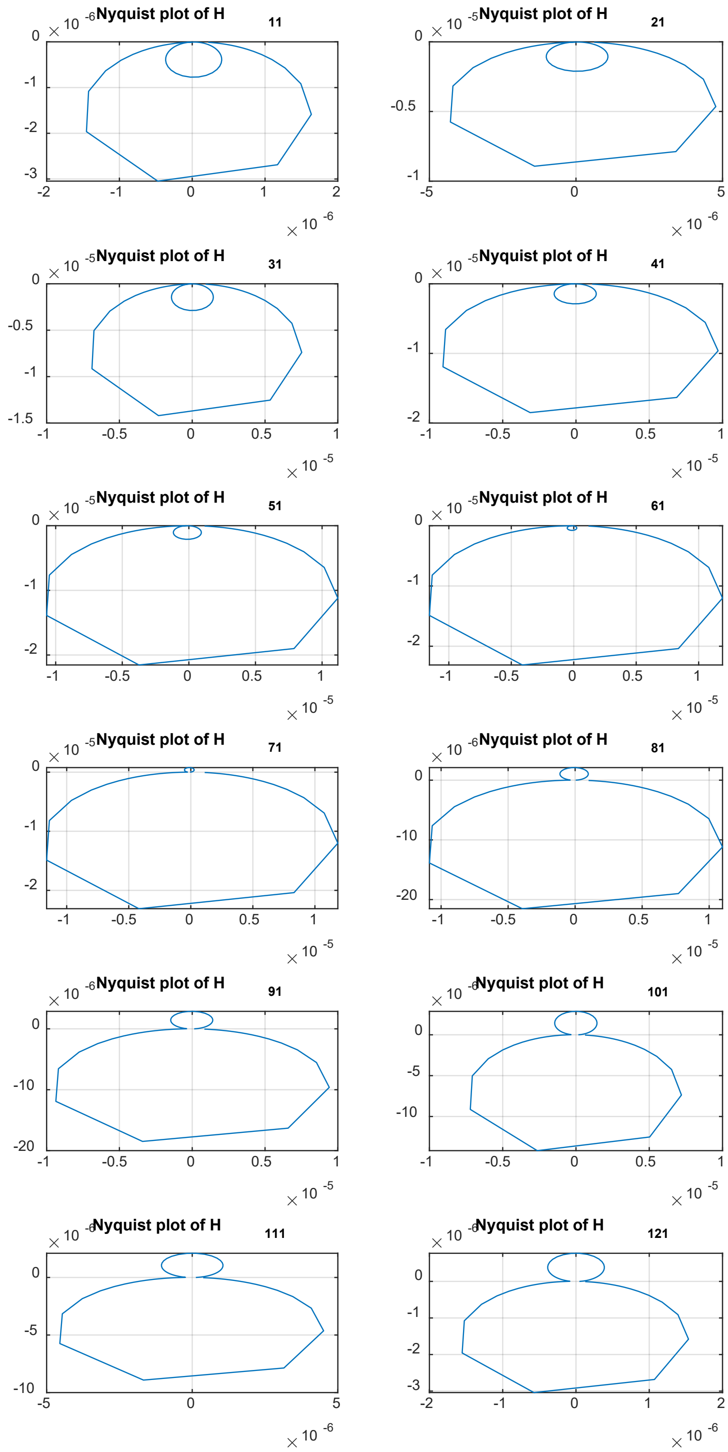


Figure 2 - Nyquist plot for the receptance H_{jk} for $k = 1$ and $j = 1,..,12$

The resonant frequencies are sufficiently spaced so that the shapes are very close to a perfect circle.

1.3

Objective: find the resonant frequencies $\{\omega_n\}$, modal damping $\{\eta_n\}$, and modal constants ${}_rA_{jk}$ from the properties of Nyquist circles.

First, we found the peaks and their respective frequencies ω_n. Then, we found the diameter ${}_n\emptyset_{jk} = (|MX| + |MN|)$ of the Nyquist plot using the maximum MX and minimum MN of the real part of receptance $\alpha_{jk}(\omega)$ around ω_n. We also note the frequencies the maximum and minimum occurs, so we can calculate the half-power bandwidth and the damping parameter η_n associated to that mode n. Knowing this, we can finally calculate the modal constant using

$$ {}_nA_{jk} = (|MX| + |MN|) f_n^2 \eta_n. \qquad \text{Eq. 4} $$

The following code was developed to find the peaks, resonant frequencies f_n, half-power bandwidth, modal damping η_n and modal constants.

```
%% 7 Find Ressonant Frequencies, Modal Constants and Modal Damping

for j=1:size(H,2) % DoF of response (H_jk)
[peak_at_ressonance,res_frq] = findpeaks(abs(H(:,j)),f, 'MinPeakHeight',1e-7,...
                                                         'MinPeakDistance',100);

eta_guess=0.1;
for r=1:length(res_frq) % mode
    bw = eta_guess*res_frq(r);
    around_f = and(f>=(res_frq(r)-bw/2),f<=(res_frq(r)+bw/2));    % only near ressonance
    around_pk = abs(H(:,j))>=peak_at_ressonance(r)/2;             % amp grater than half power (1.414)
    around = and(around_f,around_pk');
%     figure(r+10)
%     plot(f(around),abs(real(H(around,j))))
    [amp_at_ressonance,halfPowerFreq] = findpeaks(abs(real(H(around,j))),f(around), 'MinPeakHeight',1e-8,...
                                                                                     'MinPeakDistance',2);

    HalfPowerBandwith = abs(halfPowerFreq(2)-halfPowerFreq(1));
    eta_rec_aux(j,r) = HalfPowerBandwith/res_frq(r);
    diameter(j,r) = sum(amp_at_ressonance);

    % We know the diameter, but dont konw the signal of rPhi_j.rPhi_k
    if imag(H((f==res_frq(r)),j))>0
        diameter(j,r) = - diameter(j,r);
    end
end
end
eta_rec = mean(eta_rec_aux,1); % mean of each column, each mode has a damping parameter for all DoF
Ar_j1 = diameter.*(2*pi*res_frq).^2.*eta_rec;
```

The results recuperated from the FRFs (index rec) are:

$$\{f_{rec}\} = [68 \quad 271 \quad 609 \quad 1082]^T\ Hz$$

$$\{\eta_{rec}\} = [0.0441 \quad 0.0480 \quad 0.0504 \quad 0.0498]^T$$

Now we are able to reconstruct the FRF from the parameter measured using

$$\alpha_{jk}(\omega) = \frac{X_i}{F_k} = \sum_{n=1}^{N} \frac{{}_nA_{jk}}{\omega_n^2 - \omega^2 + j\eta_n\omega_n^2}. \qquad \text{Eq. 5}$$

The results are shown in Figure 3.

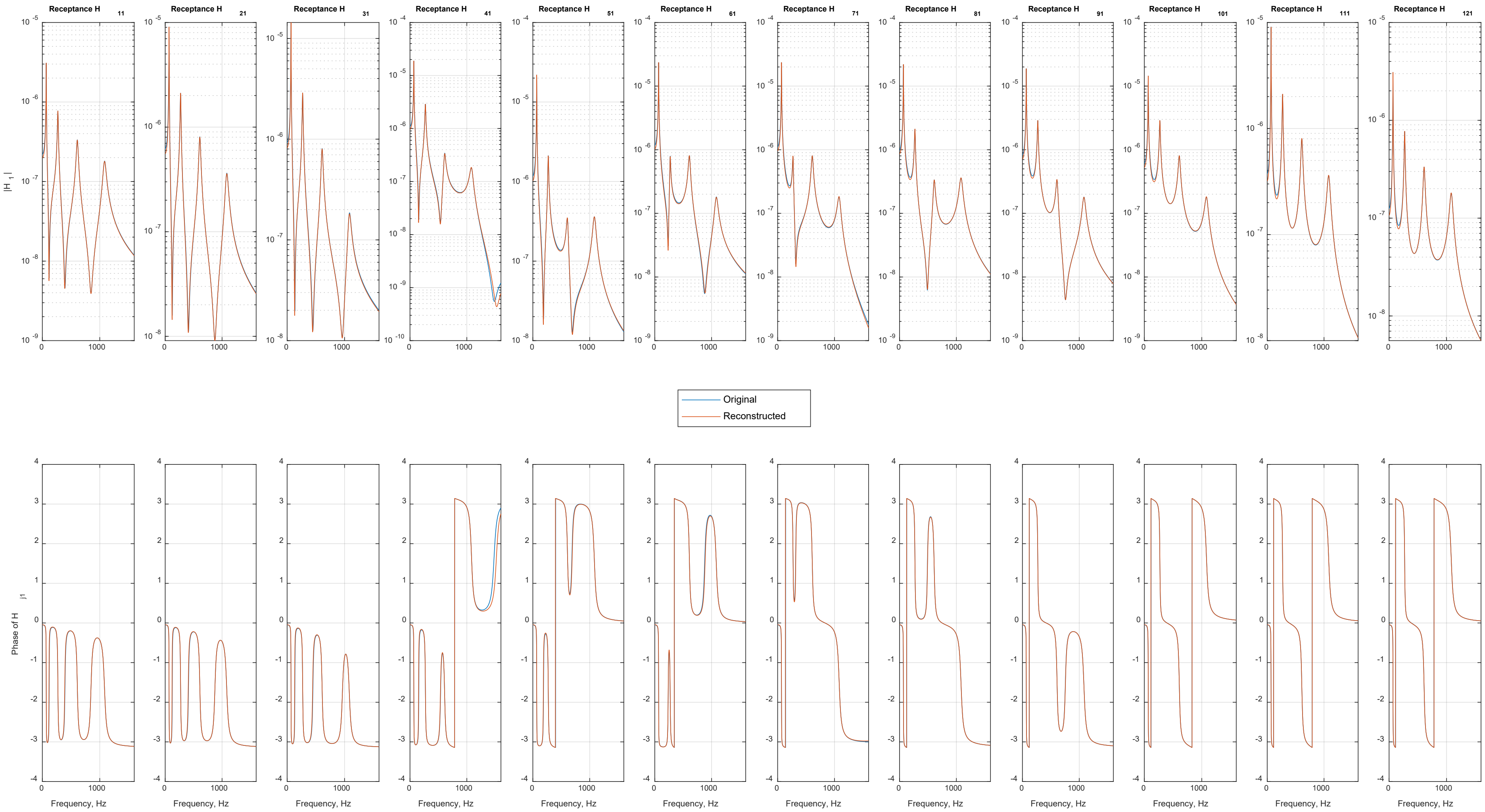


Figure 3 - Reconstructed FRFs

1.4

Objective: From modal constants ${}_nA_{jk}$, extract mode shapes Φ_{rec}.

By definition, modal constants are the product of two mass-normalized components of the mode shapes

$$ {}_nA_{jk} = {}_n\Phi_j \, {}_n\Phi_k. \qquad \text{Eq. 6} $$

So, for the point response of DoF 1, the first component of the n^{th} mode shape is

$$ \left| {}_n\Phi_1 \right| = \sqrt{{}_nA_{11}}. \qquad \text{Eq. 7} $$

From Eq. 4, we can write the components of the n^{th} mode shape, for an input force at DoF 1 and response at DoF 1 through j

$$ \begin{Bmatrix} {}_n\Phi_1 \\ \vdots \\ {}_n\Phi_j \end{Bmatrix} = \frac{1}{{}_n\Phi_1} \begin{Bmatrix} {}_nA_{11} \\ \vdots \\ {}_nA_{j1} \end{Bmatrix}. \qquad \text{Eq. 8} $$

The following piece of code was used to recuperate the mode shape vectors $\{\Phi_{rec}\}$.

```
%% 9 Reconstruct Eigenvectors from modal constants

% Ar_j1(j,r) -> % rAjk, modal contant for input k=1, output j and mode r
N_rec = size(Ar_j1,2); % number of modes
N_dof = size(Ar_j1,1); % number of degrees of fredom j
Phi2 = zeros(N_dof, N_rec);

for n=1:N_rec
Phi2(1,n) = sqrt(abs(Ar_j1(1,n))); % we dont know the signal of 1_Phi_1
Phi2(:,n) = (1/Phi2(1,n)).*Ar_j1(:,n);
end
```

The recuperated mode shape vectors are plotted against the original ones in Figure 4.

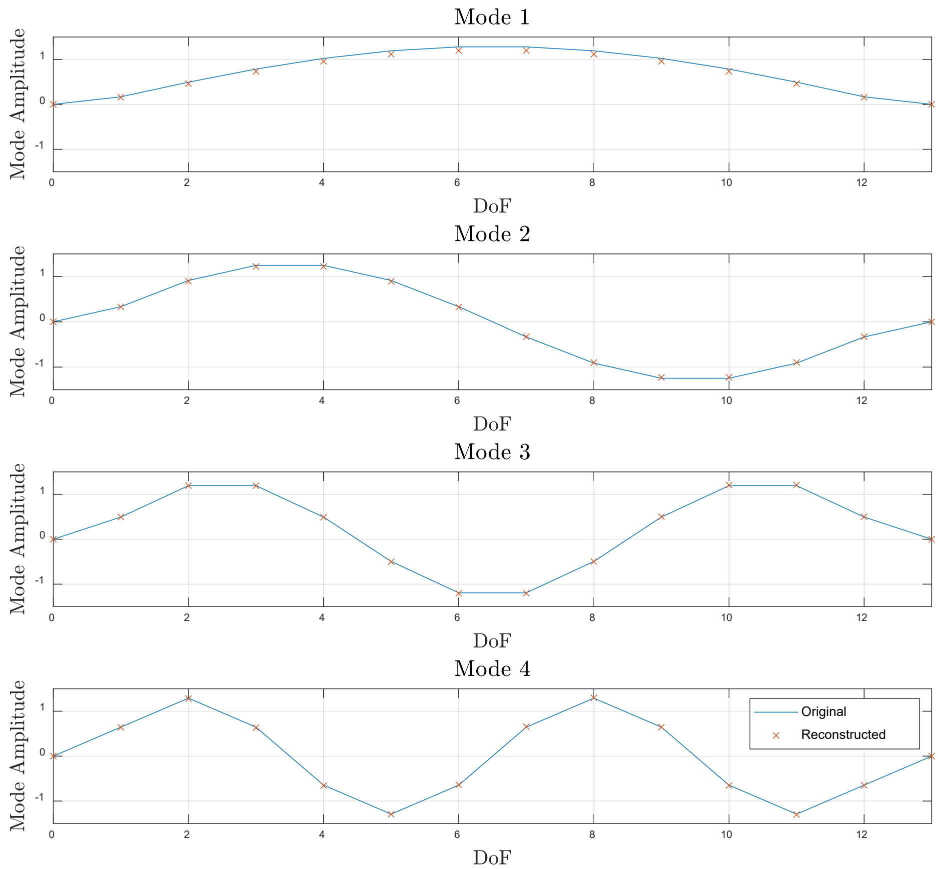


Figure 4 - Mode shapes.

General Observations

The method implemented is known as Peak-Find, and it worked really well. This is mainly because the FRF used as input were analytically calculates, that means there were no noise in the signal. Furthermore, the resonant frequencies were very far apart considering the damping parameter and the frequency resolution. This scenario, almost ideal and far from real cases, made the results obtained very close to the target.

There are other FRF processing methods to extract the modal parameters f_n, η_n and $\{\Phi_n\}$, such as circle fitting, that may present better results for less than perfect data.