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Homework n°1
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Convex Optimization

Homework n°1

Exercise 1

1) $\boxed{\text{A rectangle is convex.}}$

Indeed, it can be written $\bigcap_{i=1}^{n} \left(\{x \in \mathbb{R}^n \mid x \cdot (-e_i) \leq -\alpha_i\} \cap \{x \in \mathbb{R}^n \mid x \cdot e_i \leq \beta_i\} \right)$ where $(e_1, \ldots, e_n)$ is the canonical basis of $\mathbb{R}^n$.
Therefore a rectangle is convex as intersection of (closed) halfspaces, which are convex.

2) $\boxed{\text{The hyperbolic set is convex.}}$

Indeed, let $x, y \in \mathbb{R}_+^2$ be such that $x_1 x_2 \geq 1$ and $y_1 y_2 \geq 1$, and let $\lambda \in [0, 1]$.
Then for $z = \lambda x + (1 - \lambda) y$:

$$\begin{aligned} z_1 z_2 &= (\lambda x_1 + (1-\lambda) y_1)(\lambda x_2 + (1-\lambda) y_2) \\ &= \lambda^2 \underbrace{x_1 x_2}_{\geq 1} + (1-\lambda)^2 \underbrace{y_1 y_2}_{\geq 1} + \lambda(1-\lambda)(x_1 \underbrace{y_2}_{\geq 1/y_1} + \underbrace{x_2}_{\geq 1/x_1} y_1) \text{ (notice that necessarily } x_1 \neq 0, y_1 \neq 0) \\ &\geq \lambda^2 + (1-\lambda)^2 + \lambda(1-\lambda)\left(\frac{x_1}{y_1} + \frac{y_1}{x_1}\right). \end{aligned}$$

And $\frac{x_1}{y_1} + \frac{y_1}{x_1} - 2 = \frac{x_1^2 + y_1^2 - 2x_1 y_1}{x_1 y_1} = \frac{(x_1 - y_1)^2}{x_1 y_1} \geq 0$, so $\frac{x_1}{y_1} + \frac{y_1}{x_1} \geq 2$, and then $z_1 z_2 \geq \lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda) = 1$, which proves that z is in the hyperbolic set, which is then convex.

3) $\boxed{\text{This set is convex.}}$

$\{x \in \mathbb{R}^n \mid \|x - x_0\|_2 \leq \|x - y\|_2 \text{ for all } y \in S\} = \bigcap_{y \in S} \{x \in \mathbb{R}^n \mid \|x - x_0\|_2 \leq \|x - y\|_2\}$
And for a given $y \in S$,

$$\begin{aligned} \|x - x_0\|_2 \leq \|x - y\|_2 &\iff \|x - x_0\|_2^2 \leq \|x - y\|_2^2 \\ &\iff x^T(y - x_0) \leq \frac{\|y\|^2 - \|x_0\|^2}{2}, \end{aligned}$$

so $\{x \in \mathbb{R}^n \mid \|x - x_0\|_2 \leq \|x - y\|_2\}$ is actually a halfspace (except if $y = x_0$, but in this case it is $\mathbb{R}^n$, so it is convex) and is then convex.
And the intersection of convex sets is convex, so the initial set is indeed convex.

4) $\boxed{\text{This set is not necessarily convex.}}$

Consider the case $n = 1$, $S = \{-1, 1\}$ and $T = \{0\}$.
Then $\{x \in \mathbb{R} \mid \mathbf{dist}(x, S) \leq \mathbf{dist}(x, T)\} =]-\infty, -1/2] \cup [1/2, +\infty[$ is not convex.

5) $\boxed{\text{This set is convex.}}$

$\{x \mid x + S_2 \subseteq S_1\} = \bigcap_{y \in S_2} \{x \mid x + y \in S_1\} = \bigcap_{y \in S_2} \{x \mid x \in S_1 - y\} = \bigcap_{y \in S_2} (S_1 - y).$

For each $y \in S_2$, $S_1 - y$ is convex (as image of the convex set S_1 by an affine function), and the intersection of convex sets is convex, hence the conclusion.

Exercise 2

First of all, notice that for each question, $\mathbf{dom}\, f$ is a convex set.

1) $f : (x_1, x_2) \mapsto x_1 x_2$ on $\mathbb{R}^2_{++}$.

We easily compute, for $(x_1, x_2) \in \mathbb{R}^2_{++}$:

$$\frac{\partial f}{\partial x_1}(x_1, x_2) = x_2, \quad \frac{\partial f}{\partial x_2}(x_1, x_2) = x_1$$

$$\frac{\partial^2 f}{\partial x_1^2}(x_1, x_2) = 0, \quad \frac{\partial^2 f}{\partial x_2^2}(x_1, x_2) = 0, \quad \frac{\partial^2 f}{\partial x_1 \partial x_2}(x_1, x_2) = \frac{\partial^2 f}{\partial x_2 \partial x_1}(x_1, x_2) = 1,$$

so the Hessian of f at any $x = (x_1, x_2) \in \mathbb{R}^2_{++}$ is $\nabla^2 f(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, wich satisfies $\det(\nabla^2 f(x)) = -1$. So the eigenvalues of $\nabla^2 f(x)$ have different signs, and $\nabla^2 f(x)$ is not positive semidefinite nor negative semidefinite.

So $\boxed{f \text{ is } \textbf{not convex nor concave}}$.

The superlevet set $E_\alpha = \{(x_1, x_2) \in \mathbb{R}^2_{++} \mid x_1 x_2 \geq \alpha\}$ is convex as a consequence of question 2) of Exercise 1.
Indeed, the proof is the same, or we can say that for $\alpha > 0$, $\{(x_1, x_2) \in \mathbb{R}^2_{++} \mid x_1 x_2 \geq \alpha\} = \{(\sqrt{\alpha} x_1, \sqrt{\alpha} x_2) \in \mathbb{R}^2_{++} \mid x_1 x_2 \geq 1\}$ is the image of the hyperbolic set by an affine function, and is therefore convex (which is also clearly the case when $\alpha \leq 0$ because then $E_\alpha = \mathbb{R}^2_{++}$).

So $\boxed{f \text{ is } \textbf{quasiconcave}\text{, but } \textbf{not quasiconvex}}$ (because for instance $f(1,1) \leq 1$, $f(2, \frac{1}{2}) \leq 1$, but $f(\frac{1}{2}((1,1) + (2, \frac{1}{2}))) = f((\frac{3}{2}, \frac{3}{4})) = \frac{9}{8} > 1$).

2) $f : (x_1, x_2) \mapsto 1/(x_1 x_2)$ on $\mathbb{R}^2_{++}$.

We have $\nabla f(x) = \begin{pmatrix} -1/(x_1^2 x_2) \\ -1/(x_1 x_2^2) \end{pmatrix}$ and then $\nabla^2 f(x) = \begin{pmatrix} 2/(x_1^3 x_2) & 1/(x_1^2 x_2^2) \\ 1/(x_1^2 x_2^2) & 2/(x_1 x_2^3) \end{pmatrix}$.

$\det(\nabla^2 f(x)) = \dfrac{3}{x_1^4 x_2^4} > 0$ so both eigenvalues of $\nabla^2 f(x)$ have the same sign, and
$\mathbf{Tr}(\nabla^2 f(x)) = \dfrac{2}{x_1 x_2}\left(\dfrac{1}{x_1^2} + \dfrac{1}{x_2^2}\right) > 0$, so the eigenvalues of $\nabla^2 f(x)$ are positive, and $\nabla^2 f(x)$ is a positive definite matrix.

So $\boxed{f \text{ is } \textbf{strictly convex, not concave}}$.

Since it is convex, it is also $\boxed{\textbf{quasiconvex}}$. It is $\boxed{\textbf{not quasiconcave}}$ (because for instance $f(1,1) \geq 1$, $f(2, \frac{1}{2}) \geq 1$, but $f(\frac{1}{2}((1,1) + (2, \frac{1}{2}))) = f((\frac{3}{2}, \frac{3}{4})) = \frac{8}{9} < 1$).

3) $f : (x_1, x_2) \mapsto x_1 / x_2$ on $\mathbb{R}^2_{++}$.

We have $\nabla f(x) = \begin{pmatrix} 1/x_2 \\ -x_1/x_2^2 \end{pmatrix}$, and then $\nabla^2 f(x) = \begin{pmatrix} 0 & -1/x_2^2 \\ -1/x_2^2 & 2x_1/x_2^3 \end{pmatrix}$.

$\det(\nabla^2 f(x)) = -1/x_2^4 < 0$, so the eigenvalues of $\nabla^2 f(x)$ have different signs, and $\nabla^2 f(x)$ is not positive nor negative semidefinite.
Therefore $\boxed{f \text{ is } \textbf{not convex nor concave}}$.

For $\alpha \in \mathbb{R}$, set $a = (1, -\alpha)^T$, then $\{(x_1, x_2) \in \mathbb{R}^2_{++} \mid f(x_1, x_2) \geq \alpha\} = \{x \in \mathbb{R}^2_{++} \mid a^T x \geq 0\}$ is a halfspace, so it is convex. Similarly, we also have that $\{(x_1, x_2) \in \mathbb{R}^2_{++} \mid f(x_1, x_2) \leq \alpha\}$ is convex. So $\boxed{f \text{ is } \textbf{quasiconcave and quasiconvex, i.e. quasilinear}}$.

4) $f : (x_1, x_2) \mapsto x_1^\alpha x_2^{1-\alpha}$ on $\mathbb{R}^2_{++}$, where $0 \leq \alpha \leq 1$.

We have $\nabla f(x) = \begin{pmatrix} \alpha x_1^{\alpha-1} x_2^{1-\alpha} \\ (1-\alpha) x_1^\alpha x_2^{-\alpha} \end{pmatrix}$ and then $\nabla^2 f(x) = \alpha(1-\alpha) x_1^{\alpha-2} x_2^{-1-\alpha} \begin{pmatrix} -x_2^2 & x_1 x_2 \\ x_1 x_2 & -x_1^2 \end{pmatrix}$.

$\det\left(\nabla^2 f(x)\right) = \left(\alpha(1-\alpha) x_1^{\alpha-2} x_2^{-1-\alpha}\right)^2 \left(x_1^2 x_2^2 - x_1^2 x_2^2\right) = 0$, so at least one of the eigenvalues of $\nabla^2 f(x)$ is 0.
And $\mathbf{Tr}\left(\nabla^2 f(x)\right) = -\alpha(1-\alpha) x_1^{\alpha-2} x_2^{-1-\alpha} \left(x_1^2 + x_2^2\right) \leq 0$.
So the eigenvalues of $\nabla^2 f(x)$ are nonpositive, so $\nabla^2 f(x)$ is negative semidefinite.
f is **concave** and therefore **quasiconcave**.

If $\alpha \in \{0, 1\}$ then f is linear, so f is also convex and quasiconvex.

Assume from now on that $\alpha \notin \{0, 1\}$.
Then $\mathbf{Tr}\left(\nabla^2 f(x)\right) < 0$ for $x \neq 0$, so $\nabla^2 f(x)$ is not positive semidefinite, and f is **not convex**.

For $b \in \mathbb{R}$, set $E_b = \left\{(x_1, x_2) \in \mathbb{R}^2_{++} \mid f(x_1, x_2) \leq b\right\} = \{(x_1, x_2) \in \mathbb{R}^2_{++} \mid x_1^\alpha x_2^{1-\alpha} \leq b\}$.
If $b > 0$, $E_b = \{(x_1, x_2) \in \mathbb{R}^2_{++} \mid x_2 \leq g_\alpha(x_1)\}$ where $g_\alpha : x \mapsto b^{\frac{1}{1-\alpha}} x^{-\frac{\alpha}{1-\alpha}}$ with $\mathbf{dom}\, g_\alpha = \mathbb{R}_{++}$, i.e., E_b is the hypograph of g_α.
And $g_\alpha(x) = \text{cst} \times x^{-\frac{\alpha}{1-\alpha}}$, so $g''_\alpha(x) = \underbrace{\text{cst}}_{>0} \times \underbrace{\left(-\frac{\alpha}{1-\alpha}\right)}_{<0} \underbrace{\left(-\frac{\alpha}{1-\alpha} - 1\right)}_{<0} \underbrace{x^{-\frac{\alpha}{1-\alpha}-2}}_{>0} > 0$. So g_α is not concave, so its hypograph E_b is not convex, so f is not quasi-convex.

Conclusion:

$$\boxed{f \text{ is } \begin{cases} \text{concave, convex, quasilinear} & \text{if } \alpha \in \{0, 1\} \\ \text{concave, not convex, quasiconcave, not quasiconvex} & \text{if } \alpha \in \,]0, 1[\end{cases}}$$

Exercise 3

Here again, notice that for each of the questions, $\mathbf{dom}\, f$ is a convex set.

1) $f(X) = \mathbf{Tr}(X^{-1})$ on $\mathbf{dom}\, f = \mathbf{S}^n_{++}$.

It is enough to check that for $X \in \mathbf{S}^n_{++}, V \in \mathbf{S}^n$, the function $g : t \mapsto f(X + tV)$ is convex on $\mathbf{dom}\, g = \left\{t \in \mathbb{R} \mid X + tV \in \mathbf{S}^n_{++}\right\}$. For $t \in \mathbf{dom}\, g$:

$$\begin{aligned} g(t) &= \mathbf{Tr}((X + tV)^{-1}) \\ &= \mathbf{Tr}\left(X^{-1/2}(I_n + tX^{-1/2}VX^{-1/2})^{-1}X^{-1/2}\right) \\ &= \mathbf{Tr}\left((I_n + tX^{-1/2}VX^{-1/2})^{-1}X^{-1}\right). \end{aligned}$$

The matrix $X^{-1/2}VX^{-1/2}$ is symmetric (because $X^{-1/2}$ and V are) and real, so there exist D diagonal and Q orthogonal such that $X^{-1/2}VX^{-1/2} = QDQ^T$. Then, for $t \in \mathbf{dom}\, g$:

$$\begin{aligned} g(t) &= \mathbf{Tr}\left((I_n + tQDQ^T)^{-1}X^{-1}\right) \\ &= \mathbf{Tr}\left(\left(Q\left(I_n + tD\right)Q^T\right)^{-1}X^{-1}\right) \\ &= \mathbf{Tr}\left(Q\left(I_n + tD\right)^{-1}Q^TX^{-1}\right) \\ &= \mathbf{Tr}\left(\left(I_n + tD\right)^{-1}Q^TX^{-1}Q\right) \\ &= \sum_{i=1}^{n}(1 + tD_{ii})^{-1}\left(Q^TX^{-1}Q\right)_{ii}. \end{aligned}$$

We have $\left(Q^TX^{-1}Q\right)_{ii} = e_i^T\left(Q^TX^{-1}Q\right)e_i = (Qe_i)^TX^{-1}(Qe_i) \geq 0$ (with e_i the i-th vector of the canonical basis of $\mathbb{R}^n$) because $X^{-1} \in \mathbf{S}^n_{++}$.

And each function $t \mapsto (1+tD_{ii})^{-1}$ is convex on $\mathbf{dom}\, g$, because its derivative is $t \mapsto \dfrac{-D_{ii}}{(1 + tD_{ii})^2}$ which is non-decreasing on $\mathbf{dom}\, g$.

Hence the conclusion: g is convex, and therefore $\boxed{f \text{ is convex}}$.

2) $f(X, y) = y^TX^{-1}y$ on $\mathbf{dom}\, f = \mathbf{S}^n_{++} \times \mathbb{R}^n$.

We have seen in class that $\forall y \in \mathbb{R}^n, y^TX^{-1}y = \sup\limits_{x \in \mathbb{R}^n}\left(2y^Tx - x^TXx\right)$.

To prove it again, notice that for each y, the function $x \mapsto 2y^Tx - x^TXx$ has gradient $2y - 2Xx$ and hessian $-2X \prec 0$. It is thus concave, and its gradient is 0 in $x = X^{-1}y$, where the function therefore reaches its maximum, which is $2y^TX^{-1}y - y^TX^{-1}y = y^TX^{-1}y$.

And for each $x \in \mathbb{R}^n$, the function $(X, y) \mapsto 2y^Tx - x^TXx$ is linear, thus convex.

f is then a pointwise supremum of convex functions, so $\boxed{f \text{ is convex}}$.

3) $f(X) = \sum\limits_{i=1}^{n}\sigma_i(X)$ on $\mathbf{dom}\, f = \mathbf{S}^n$.

We will show that $\forall X \in \mathbf{dom}\, f, f(X) = \sup\limits_{\sigma_{\max}(A) \leq 1}\langle A, X\rangle$.

- Let $X = U\Sigma V^T$ be the singular value decomposition of X. Let A be a matrix such that

$\sigma_{\max}(A) \leq 1$. We have

$$\begin{aligned}
\langle A, X \rangle &= \left\langle A, U\Sigma V^T \right\rangle \\
&= \left\langle U^T A V, \Sigma \right\rangle \\
&= \sum_{i=1}^{n} \sigma_i(X) \left(U^T A V\right)_{ii}.
\end{aligned}$$

And $\left(U^T AV\right)_{ii} = e_i^T (U^T AV) e_i = (Ue_i)^T A(Ve_i) \leq \sigma_{\max}(A) \leq 1$ (using the fact that $\sigma_{\max}(A) = \sup_{\|x\|_2 = \|y\|_2 = 1} x^T Ay$, and $\|Ue_i\|_2 = \|e_i\|_2 = 1$ because U is orthogonal, same for Ve_i). So $\langle A, X \rangle \leq \sum_{i=1}^{n} \sigma_i(X)$.

Therefore $\sup_{\sigma_{\max}(A) \leq 1} \langle A, X \rangle \leq f(X)$.

- Now, set $A = UV^T$.

  Then $\langle A, X \rangle = \left\langle UV^T, U\Sigma V^T \right\rangle = \left\langle U^T UV^T V, \Sigma \right\rangle = \langle I_n, \Sigma \rangle = \sum_{i=1}^{n} \sigma_i(X) = f(X)$, and $\sigma_{\max}(A) = 1$ because $A^T A = I_n$ since U and V are orthogonal.

  This shows that $\sup_{\sigma_{\max}(A) \leq 1} \langle A, X \rangle \geq f(X)$.

Finally, we have shown that $f(X) = \sup_{\sigma_{\max}(A) \leq 1} \langle A, X \rangle$.

For each A such that $\sigma_{\max}(A) \leq 1$, the function $X \mapsto \langle A, X \rangle$ is linear, thus convex. And f is a pointwise supremum of convex functions, so $\boxed{f \text{ is convex}}$.

Optional exercises

Exercise 4

1.

- $K_{\mathrm{m}+}$ is a convex cone.

  Let $x, y \in K_{\mathrm{m}+}$, and $\alpha, \beta \geq 0$. Set $z = \alpha x + \beta y$. For $i \in [\![1, n-1]\!]$, we easily have $z_i = \underbrace{\alpha}_{\geq 0} \underbrace{x_i}_{\geq x_{i+1}} + \underbrace{\beta}_{\geq 0} \underbrace{y_i}_{\geq y_{i+1}} \geq \alpha x_{i+1} + \beta y_{i+1} = z_{i+1}$, and $z_n = \underbrace{\alpha}_{\geq 0} \underbrace{x_n}_{\geq 0} + \underbrace{\beta}_{\geq 0} \underbrace{y_n}_{\geq 0} \geq 0$.

  So $z \in K_{\mathrm{m}+}$, and $K_{\mathrm{m}+}$ is a convex cone.

- $K_{\mathrm{m}+}$ is closed.

  $K_{\mathrm{m}+} = \left(\bigcap_{i=1}^{n-1} \left\{ x \in \mathbb{R}^n \mid (e_i - e_{i+1})^T x \geq 0 \right\} \right) \cap \left\{ x \in \mathbb{R}^n \mid e_n^T x \geq 0 \right\}$, and each of these sets is a closed halfspace.

  $K_{\mathrm{m}+}$ is then closed as intersection of closed sets.

- $K_{\mathrm{m}+}$ is solid.

  Set $x := (n, n-1, \cdots, 1)$, then for every $y \in \mathbb{R}^n$ such that $\|y - x\|_\infty \leq \frac{1}{2}$, we have $y \in K_{\mathrm{m}+}$.

  Therefore x is in the interior of $K_{\mathrm{m}+}$, which is then non empty.

- $K_{\mathrm{m}+}$ is pointed.

  Assume $x \in K_{\mathrm{m}+}$ and $-x \in K_{\mathrm{m}+}$.

  Since $K_{\mathrm{m}+} \subset \mathbb{R}_+$, we have $\forall i \in [\![1, n]\!], x_i \geq 0$, and $\forall i \in [\![1, n]\!], -x_i \geq 0$. It follows that $\forall i \in [\![1, n]\!], x_i = 0$ and $x = 0$.

We conclude that $\boxed{K_{\mathrm{m}+} \text{ is a proper cone}}$.

2. $K_{\mathrm{m}+}^* = \left\{ y \in \mathbb{R}^n \mid \forall x \in K_{\mathrm{m}+}, y^T x \geq 0 \right\}$.

For each $y \in \mathbb{R}^n$, we have:

$$y^T x = \sum_{i=1}^{n} y_i x_i = \sum_{i=1}^{n} y_i \left(x_n + \sum_{k=i}^{n-1} (x_k - x_{k+1}) \right) = x_n \sum_{i=1}^{n} y_i + \sum_{k=1}^{n-1} \left((x_k - x_{k+1}) \sum_{i=1}^{k} y_i \right) \tag{1}$$

- Let $y \in K_{\mathrm{m}+}^*$. Since for each $k \in [\![1, n]\!]$, $\sum_{i=1}^{k} e_i \in K_{\mathrm{m}+}$, we have $0 \leq y^T \sum_{i=1}^{k} e_i = \sum_{i=1}^{k} y_i$.

- Conversely, let $y \in \mathbb{R}^n$ be such that $\forall k \in [\![1, n]\!], \sum_{i=1}^{k} y_i \geq 0$. Let $x \in K_{\mathrm{m}+}$. Then $x_n \geq 0$, and for each $k \in [\![1, n-1]\!], x_k - x_{k+1} \geq 0$, so using (1) we have $y^T x \geq 0$.

  Since this holds for each $x \in K_{\mathrm{m}+}$, we have $y \in K_{\mathrm{m}+}^*$.

We have proven that $\boxed{K_{\mathrm{m}+}^* = \left\{ y \in \mathbb{R}^n \mid \forall k \in [\![1, n]\!], \sum_{i=1}^{k} y_i \geq 0 \right\}}$.

Exercise 5

$$f^*(y) = \sup_{x \in \mathbf{dom}\, f} \left(y^T x - f(x)\right)$$

1) $f(x) = \max\limits_{i=1,\dots,n} x_i$ on $\mathbb{R}^n$.

Let $y \in \mathbb{R}^n$.
• First case: there exists $i \in [\![1, n]\!]$ such that $y_i < 0$.
Then for $t \geq 0$, take $x = (0, \dots, 0, \underbrace{-t}_{i^{\text{th}}\text{ position}}, 0, \dots, 0)$ so that $y^T x - f(x) = -ty_i \underset{t \to +\infty}{\longrightarrow} +\infty$, so $f^*(y) = +\infty$.
• Second case: $\forall i \in [\![1, n]\!], y_i \geq 0$ and $\sum\limits_{i=1}^{n} y_i \neq 1$.
Take $x = (\varepsilon t, \dots, \varepsilon t)$ where $\varepsilon = \text{sign}\left(\sum\limits_{i=1}^{n} y_i - 1\right)$, then $y^T x - f(x) = \varepsilon t \left(\sum\limits_{i=1}^{n} y_i - 1\right) \underset{t \to +\infty}{\longrightarrow} +\infty$, so $f^*(y) = +\infty$.
• Third case: $\forall i \in [\![1, n]\!], y_i \geq 0$ and $\sum\limits_{i=1}^{n} y_i = 1$.
Then for any $x \in \mathbb{R}^n$, $y^T x - f(x) = \sum\limits_{i=1}^{n} x_i y_i - \max\limits_{i=1,\dots,n} x_i \leq (\sum\limits_{i=1}^{n} y_i - 1) \max\limits_{i=1,\dots,n} x_i = 0$, and the inequality becomes an equality when $x = (0, \dots, 0)$ for instance.
Therefore $f^*(y) = 0$.

So we have shown

$$\boxed{f^*(y) = \begin{cases} 0 & \text{if } y \succeq 0 \text{ and } \sum\limits_{i=1}^{n} y_i = 1 \\ +\infty & \text{otherwise.} \end{cases}}$$

2) $f(x) = \sum\limits_{i=1}^{r} x_{[i]}$ on $\mathbb{R}^n$.

Let $y \in \mathbb{R}^n$.

• First case: there exists $i \in [\![1, n]\!]$ such that $y_i < 0$.
Set $x_i = -t$, $x_k = 0$ for $k \neq i$, where $t \geq 0$. Then $y^T x - f(x) = -ty_i \underset{t \to +\infty}{\longrightarrow} +\infty$ and $f^*(y) = +\infty$.
Note that to affirm $f(x) = 0$, we need $r < n$. But if $r = n$, then $y^T x - f(x) = -ty_i + t = t(1 - y_i) \underset{t \to +\infty}{\longrightarrow} +\infty$.

• Second case: $\forall k \in [\![1, n]\!], y_k \geq 0$, and there exists $i \in [\![1, n]\!]$ such that $y_i > 1$.
Set $x_i = t \geq 0$ and $x_k = 0$ for $k \neq i$.
Then $y^T x - f(x) = ty_i - t = t(y_i - 1) \underset{t \to +\infty}{\longrightarrow} +\infty$ and $f^*(y) = +\infty$.
Note that to affirm $f(x) = t$, we need $r > 0$. But if $r = 0$, then $y^T x - f(x) = ty_i > t \underset{t \to +\infty}{\longrightarrow} +\infty$.

• Third case: $\forall k \in [\![1, n]\!], 0 \leq y_k \leq 1$ and $\sum\limits_{i=1}^{r} y_i \neq r$.
Take $\varepsilon = \text{sign}\left(\sum\limits_{i=1}^{r} y_i - r\right)$, $t \geq 0$, $x = (\varepsilon t, \cdots, \varepsilon t)$.
Then $y^T x - f(x) = \varepsilon t \left(\sum\limits_{i=1}^{n} y_i - r\right) \underset{t \to +\infty}{\longrightarrow} +\infty$, so $f^*(y) = +\infty$.

- Fourth case: $\forall k \in [\![1,n]\!], 0 \leq y_k \leq 1$, and $\sum\limits_{i=1}^{r} y_i = r$.

Let $x \in \mathbb{R}^n$, and $i : [\![1,n]\!] \to [\![1,n]\!]$ bijective such that $x_{i(1)} \geq x_{i(2)} \geq \cdots \geq x_{i(n)}$.
Then $y^T x - f(x) = \sum\limits_{k=1}^{n} x_{i(k)} y_{i(k)} - \sum\limits_{k=1}^{r} x_{i(k)} = \sum\limits_{k=1}^{r} \underbrace{x_{i(k)}}_{\geq x_{i(r)}} \underbrace{(y_{i(k)} - 1)}_{\leq 0} + \sum\limits_{k=r+1}^{n} \underbrace{x_{i(k)}}_{\leq x_{i(r)}} \underbrace{y_{i(k)}}_{\geq 0}$
$\leq x_{i(r)} \left(\sum\limits_{k=1}^{r} (y_{i(k)} - 1) + \sum\limits_{k=r+1}^{n} y_{i(k)} \right) = x_{i(r)} \underbrace{\left(\sum_{k=1}^{n} y_k - r \right)}_{=0} = 0$, and the inequality becomes an equality for $x = 0$ for instance.
So $f^*(y) = 0$.

Therefore

$$\boxed{f^*(y) = \begin{cases} 0 & \text{if } 0 \preceq y \preceq 1 \text{ and } \sum\limits_{i=1}^{n} y_i = r \\ +\infty & \text{otherwise.} \end{cases}}$$

3) $f(x) = \max\limits_{i=1,\ldots,n} (a_i x + b_i)$ on $\mathbb{R}$ (illustrated in Figure 1).

As suggested by the exercise, we assume that the a_i are sorted in increasing order and that none of the functions $x \mapsto a_i x + b_i$ is redundant.
As a consequence, if we define, for $i \in [\![1, m-1]\!]$, x_i to be such that $a_i x_i + b_i = a_{i+1} x_i + b_{i+1}$, that is $x_i = \dfrac{b_i - b_{i+1}}{a_{i+1} - a_i}$, as well as $x_0 = -\infty$ and $x_m = +\infty$, then we have $f(x) = a_i x + b_i$ for $x \in [x_{i-1}, x_i[$.

Then, for $x, y \in \mathbb{R}$, $yx - f(x) = yx - \sum\limits_{i=1}^{m} (a_i x + b_i) \mathbb{1}_{[x_{i-1}, x_i[}(x) = \sum\limits_{i=1}^{m} ((y - a_i)x - b_i) \mathbb{1}_{[x_{i-1}, x_i[}(x)$.

- If $y > a_m$, then for $x \geq x_{m-1}$ we have $yx - f(x) = \underbrace{(y - a_m)}_{>0} x + b_m \underset{x \to +\infty}{\longrightarrow} +\infty$, and then $f^*(y) = +\infty$.
- Similarly, if $y < a_1$ we have $yx - f(x) \underset{x \to -\infty}{\longrightarrow} +\infty$, and then $f^*(y) = +\infty$.
- Now, assume that $a_1 \leq y \leq a_m$. Let $i \in [\![1, m-1]\!]$ be such that $a_i \leq y \leq a_{i+1}$.

  $x \mapsto yx - f(x)$ is a continuous piecewise affine function (illustrated in Figure 2), with slopes $y - a_1 \geq 0, \ldots, y - a_i \geq 0, y - a_{i+1} \leq 0, \ldots, y - a_m \leq 0$. It is thus non-decreasing on $]-\infty, x_i]$, and non-increasing on $[x_i, +\infty[$.

  It then reaches a maximum at $x = x_i$, which is:

  $f^*(y) = (y - a_i)x_i - b_i = (y - a_i)\dfrac{b_i - b_{i+1}}{a_{i+1} - a_i} - b_i$.

We have shown that:

$$\boxed{f^*(y) = \begin{cases} (y - a_i)\dfrac{b_i - b_{i+1}}{a_{i+1} - a_i} - b_i & \text{if } a_i \leq y \leq a_{i+1} \text{ for } 1 \leq i \leq m-1 \\ +\infty & \text{otherwise.} \end{cases}}$$

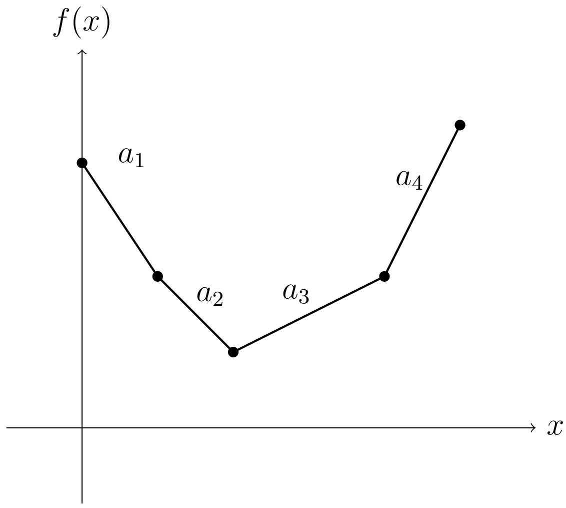

Figure 1: Example of graph for f

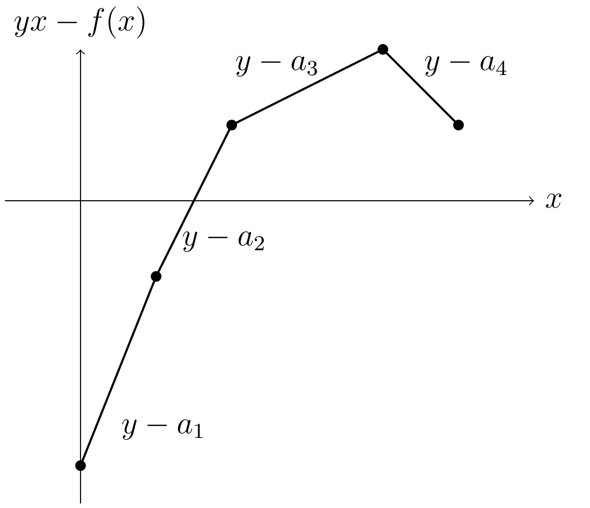

Figure 2: Corresponding graph for $x \mapsto yx - f(x)$ for $y \in [a_3, a_4]$