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Homework n°2 - Exercises
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Image Denoising

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Exercise 5.1

We denote by n the dimension of the initial image u_0, and k the dimension of the low-scale image u_1.

Let us explain how u_1 is obtained from u_0.

- First, we consider the DCT transform of u_0, denoted by $\mathrm{DCT}_n^{iso}(u_0)$.
- Then, we pad to remove high frequencies, and we obtain $\mathrm{ZP}_k(\mathrm{DCT}_n^{iso}(u_0))$.
- Because we changed the size (from n to k), we have to scale the image by a factor $\frac{k}{n} = \sqrt{\frac{k \times k}{n \times n}} = \sqrt{\frac{\mathrm{NUMPIX}(layer)}{\mathrm{NUMPIX}(input)}}$, that is we consider $\sqrt{\frac{\mathrm{NUMPIX}(layer)}{\mathrm{NUMPIX}(input)}}\mathrm{ZP}_k(\mathrm{DCT}_n^{iso}(u_0))$. This is what is done in Algorithm 5, in the function ExtractScale, lines 14 and 16.
- Finally, we convert back the above to an image, which means we get (by linearity of IDCT_k^{iso}) $\sqrt{\frac{\mathrm{NUMPIX}(layer)}{\mathrm{NUMPIX}(input)}}\mathrm{IDCT}_k^{iso}\left(\mathrm{ZP}_k(\mathrm{DCT}_n^{iso}(u_0))\right)$.

So we have $\boxed{u_1 = \sqrt{\frac{\mathrm{NUMPIX}(layer)}{\mathrm{NUMPIX}(input)}}\mathrm{IDCT}_k^{iso}\left(\mathrm{ZP}_k(\mathrm{DCT}_n^{iso}(u_0))\right)}$.

Since $u_0 = \tilde{u}_0 + n_0$ with $n_0 \sim \mathcal{N}(0, \sigma_0^2)$, we have $u_1 = \tilde{u}_1 + n_1$ with $n_1 \sim \mathcal{N}(0, \sigma_1^2)$, where:

$$
\begin{aligned}
\sigma_1^2 &= \mathrm{Var}(n_1) \\
&= \mathrm{Var}(u_1) \\
&= \mathrm{Var}\left(\sqrt{\frac{\mathrm{NUMPIX}(layer)}{\mathrm{NUMPIX}(input)}}\mathrm{IDCT}_k^{iso}\left(\mathrm{ZP}_k(\mathrm{DCT}_n^{iso}(u_0))\right)\right) \\
&= \left(\sqrt{\frac{\mathrm{NUMPIX}(layer)}{\mathrm{NUMPIX}(input)}}\right)^2 \mathrm{Var}\left(\mathrm{IDCT}_k^{iso}\left(\mathrm{ZP}_k(\mathrm{DCT}_n^{iso}(u_0))\right)\right) \\
&= \frac{\mathrm{NUMPIX}(layer)}{\mathrm{NUMPIX}(input)}\sigma_0^2,
\end{aligned}
$$

where, for the last equality, we used the fact that DCT and IDCT preserve the variance (because they are isometries).

We therefore have $\boxed{\sigma_1 = \sigma_0\sqrt{\frac{\mathrm{NUMPIX}(layer)}{\mathrm{NUMPIX}(input)}}}$ i.e., the noise standard deviation at the lower scale has been multiplied by $\sqrt{\frac{\mathrm{NUMPIX}(layer)}{\mathrm{NUMPIX}(input)}}$.