Image Denoising

Homework n°7 - Exercises

Exercise 2.1

$$E_\theta^{\mathrm{F\circ E}}(u,v) = \frac{1}{2\sigma^2}\|u-v\|^2 + \sum_{x\in\Omega}\sum_{i=1}^{N}\phi_i(\mathbf{k}_i * u(x)) + \text{constants}$$

For $x \in \Omega$, we have, if $\mathbf{k}_i$ is of size $n \times n$: $\mathbf{k}_i * u(x) = \sum_{j_1,j_2=-n}^{n} \mathbf{k}_i(x_1 - j_1, x_2 - j_2)u(j_1, j_2)$.
So

$$\frac{\partial\left(\mathbf{k}_i * u(x)\right)}{\partial u_{j_1,j_2}} = \mathbf{k}_i(x_1 - j_1, x_2 - j_2).$$

Then, we have

$$\frac{\partial \phi_i\left(\mathbf{k}_i * u(x)\right)}{\partial u_{j_1,j_2}} = \mathbf{k}_i(x_1 - j_1, x_2 - j_2)\phi_i'\left(\mathbf{k}_i * u(x)\right),$$

and

$$\begin{aligned}\frac{\partial}{\partial u_{j_1,j_2}}\left(\sum_{x\in\Omega}\sum_{i=1}^{N}\phi_i(\mathbf{k}_i * u(x))\right) &= \sum_{x\in\Omega}\sum_{i=1}^{N}\mathbf{k}_i(x_1 - j_1, x_2 - j_2)\phi_i'\left(\mathbf{k}_i * u(x)\right)\\ &= \sum_{i=1}^{N}\left(\overline{\mathbf{k}_i} * \phi_i'\left(\mathbf{k}_i * u\right)\right)_{j_1,j2},\end{aligned}$$

where we used

$$\left(\overline{\mathbf{k}_i} * \phi_i'\left(\mathbf{k}_i * u\right)\right)_{j_1,j_2} = \sum_{x\in\Omega}\mathbf{k}_i(x_1 - j_1, x_2 - j_2)\phi_i'\left(\mathbf{k}_i * u(x)\right).$$

Therefore, we have

$$\nabla E_\theta^{\mathrm{F\circ E}}(u,v) = \frac{1}{\sigma^2}(u-v) + \sum_{i=1}^{N}\overline{\mathbf{k}_i} * \phi_i'\left(\mathbf{k}_i * u\right).$$

Therefore the gradient descent reads $\boxed{u^{t+1} = u^t - \eta\left(\frac{1}{\sigma^2}(u^t - v) + \sum_{i=1}^{N}\overline{\mathbf{k}_i} * \phi_i'\left(\mathbf{k}_i * u^t\right)\right)}$.

Exercise 2.2

$$
\begin{aligned}
\mathcal{L}_\infty(\theta) &= \int \log p_\theta(u) p(u)\, du \\
&= -\int \log\left(\frac{p(u)}{p_\theta(u)}\right) p(u)\, du + \int \log p(u) p(u)\, du \\
&= -\mathrm{KL}\left(p(u) \,||\, p_\theta(u)\right) + \text{constant}.
\end{aligned}
$$

Therefore $\boxed{\text{maximizing } \mathcal{L}_\infty(\theta) \text{ is equivalent to minimizing } \mathrm{KL}\left(p(u) \,||\, p_\theta(u)\right)}$.