Image Denoising

Homework n°8 - Exercises

Exercise 3.1

1. We have:

$$\begin{aligned}
\mathbb{E}_{u,v}\left\{\|\mathcal{F}^\lambda(v)-u\|^2\right\} &= \mathbb{E}_{u,v}\left\{\|\lambda\mathcal{F}(v)+(1-\lambda)v-u\|^2\right\} \text{ by definition of } \mathcal{F}^\lambda \\
&= \mathbb{E}_{u,v}\left\{\|\lambda\left(\mathcal{F}(v)-u\right)+(1-\lambda)(v-u)\|^2\right\} \\
&= \mathbb{E}_{u,v}\left\{\lambda^2\|\mathcal{F}(v)-u\|^2+(1-\lambda)^2\|v-u\|^2\right. \\
&\left.+\ 2\lambda(1-\lambda)\langle\mathcal{F}(v)-u,v-u\rangle\right\} \\
&= \lambda^2\mathbb{E}_{u,v}\left\{\|\mathcal{F}(v)-u\|^2\right\}+(1-\lambda)^2\mathbb{E}_{u,v}\left\{\|v-u\|^2\right\} \\
&+ 2\lambda(1-\lambda)\mathbb{E}_{u,v}\left\{\langle\mathcal{F}(v)-u,v-u\rangle\right\}
\end{aligned}$$

2.

$$\begin{aligned}
\mathbb{E}_{u,v}\left\{\langle\mathcal{F}(v)-u,v-u\rangle\right\} &= \mathbb{E}_{u,v}\left\{\sum_x(\mathcal{F}(v)_x-u_x)(v_x-u_x)\right\} \\
&= \mathbb{E}_u\left\{\mathbb{E}_v\left\{\sum_x(\mathcal{F}(v)_x-u_x)(v_x-u_x)\mid u\right\}\right\} \\
&= \mathbb{E}_u\left\{\mathbb{E}_{-v_x}\mathbb{E}_{v_x}\left\{\sum_x(\mathcal{F}(v)_x-u_x)(v_x-u_x)\mid u\right\}\right\} \\
&= \mathbb{E}_u\left\{\mathbb{E}_{-v_x}\left\{\sum_x(\mathcal{F}(v)_x-u_x)(\mathbb{E}_{v_x}\left\{v_x-u_x\mid u_x\right\})\mid u\right\}\right\} \ (\mathcal{F}(v)_x \perp\!\!\!\perp v_x) \\
&= \mathbb{E}_u\left\{\mathbb{E}_{-v_x}\left\{\sum_x(\mathcal{F}(v)_x-u_x)\times 0\right\}\right\} \text{ because } \mathbb{E}\{v\mid u\}=u \\
&= 0,
\end{aligned}$$

so $\boxed{\mathbb{E}_{u,v}\left\{\langle\mathcal{F}(v)-u,v-u\rangle\right\}=0}$.

3. Using the two previous questions, the MSE is:

$$\mathbb{E}_{u,v}\left\{\|\mathcal{F}^\lambda(v)-u\|^2\right\}=\lambda^2\mathbb{E}_{u,v}\left\{\|\mathcal{F}(v)-u\|^2\right\}+(1-\lambda)^2\mathbb{E}_{u,v}\left\{\|v-u\|^2\right\}.$$

The derivative with respect to λ is

$$2\lambda\mathbb{E}_{u,v}\left\{\|\mathcal{F}(v)-u\|^2\right\}-2(1-\lambda)\mathbb{E}_{u,v}\left\{\|v-u\|^2\right\},$$

which is zero when

$$\begin{aligned}
&\lambda\mathbb{E}_{u,v}\left\{\|\mathcal{F}(v)-u\|^2\right\}-(1-\lambda)\mathbb{E}_{u,v}\left\{\|v-u\|^2\right\}=0 \\
\iff\ &\lambda\left(\mathbb{E}_{u,v}\left\{\|\mathcal{F}(v)-u\|^2\right\}+\mathbb{E}_{u,v}\left\{\|v-u\|^2\right\}\right)=\mathbb{E}_{u,v}\left\{\|v-u\|^2\right\} \\
\iff\ &\lambda=\frac{\mathbb{E}_{u,v}\left\{\|v-u\|^2\right\}}{\mathbb{E}_{u,v}\left\{\|\mathcal{F}(v)-u\|^2\right\}+\mathbb{E}_{u,v}\left\{\|v-u\|^2\right\}}.
\end{aligned}$$

Now, we have:

$$\mathbb{E}_{u,v}\left\{\|v-u\|^2\right\} = \mathbb{E}_u\mathbb{E}_v\left\{\|v - \underbrace{u}_{=\mathbb{E}_v\{v|u\}}\|^2 \mid u\right\} = \mathbb{E}_u\mathbb{E}_v\left\{\left\|v-\mathbb{E}_v\left\{v\mid u\right\}\right\|^2 \mid u\right\} = \mathbb{E}_u\left\{\mathbb{V}\left\{v\mid u\right\}\right\},$$

so in the end the MSE is minimized for

$$\boxed{\lambda^* = \frac{\mathbb{E}_u\left\{\mathbb{V}\left\{v\mid u\right\}\right\}}{\mathbb{E}_{u,v}\left\{\|\mathcal{F}(v)-u\|^2\right\} + \mathbb{E}_u\left\{\mathbb{V}\left\{v\mid u\right\}\right\}}}.$$

4. By applying Proposition 3.4, we have

$$\lambda^* = \frac{\mathbb{E}_u\left\{\mathbb{V}\left\{v\mid u\right\}\right\}}{\mathbb{E}_v\left\{\|\mathcal{F}(v)-v\|^2\right\}}.$$

If we assume that $\mathbb{V}\left\{v\mid u\right\} = d\sigma^2$, then

$$\lambda^* = \frac{d\sigma^2}{\mathbb{E}_v\left\{\|\mathcal{F}(v)-v\|^2\right\}},$$

and the denominator is precisely $R_{\mathrm{N2S}}(\mathcal{F})$, so:

$$\boxed{\lambda^* = \frac{d\sigma^2}{R_{\mathrm{N2S}}(\mathcal{F})}}.$$

Exercise 3.2

$$\begin{aligned}
\mathbb{E}_v\left\{\|\hat{u}(v)-u\|^2 \mid u\right\} &= \mathbb{E}_v\left\{\|\left(\hat{u}(v)-\mathbb{E}_v\left\{\hat{u}(v)\mid u\right\}\right)+\left(\mathbb{E}_v\left\{\hat{u}(v)\mid u\right\}-u\right)\|^2 \mid u\right\} \\
&= \mathbb{E}_v\left\{\|\hat{u}(v)-\mathbb{E}_v\left\{\hat{u}(v)\mid u\right\}\|^2 \mid u\right\} \\
&+ \mathbb{E}_v\left\{\|\mathbb{E}_v\left\{\hat{u}(v)\mid u\right\}-u\|^2 \mid u\right\} \\
&+ 2\mathbb{E}_v\left\{\langle\hat{u}(v)-\mathbb{E}_v\left\{\hat{u}(v)\mid u\right\},\mathbb{E}_v\left\{\hat{u}(v)\mid u\right\}-u\rangle \mid u\right\}.
\end{aligned}$$

Now, notice that $\mathbb{E}_v\left\{\hat{u}(v)\mid u\right\}-u\rangle \mid u$ does not depend on v, so the last term satisfies

$$\begin{aligned}
2\mathbb{E}_v\left\{\langle\hat{u}(v)-\mathbb{E}_v\left\{\hat{u}(v)\mid u\right\},\mathbb{E}_v\left\{\hat{u}(v)\mid u\right\}-u\rangle \mid u\right\} &= 2\langle\mathbb{E}_v\left\{\hat{u}(v)-\mathbb{E}_v\left\{\hat{u}(v)\mid u\right\}\mid u\right\},\mathbb{E}_v\left\{\hat{u}(v)\mid u\right\}-u\rangle \\
&= 2\langle\mathbb{E}_v\left\{\hat{u}(v)\mid u\right\}-\mathbb{E}_v\left\{\hat{u}(v)\mid u\right\},\mathbb{E}_v\left\{\hat{u}(v)\mid u\right\}-u\rangle \\
&= 2\langle 0,\mathbb{E}_v\left\{\hat{u}(v)\mid u\right\}-u\rangle \\
&= 0.
\end{aligned}$$

So we have shown:

$$\mathbb{E}_v\left\{\|\hat{u}(v)-u\|^2 \mid u\right\} = \mathbb{E}_v\left\{\|\hat{u}(v)-\mathbb{E}_v\left\{\hat{u}(v)\mid u\right\}\|^2 \mid u\right\} + \mathbb{E}_v\left\{\|\mathbb{E}_v\left\{\hat{u}(v)\mid u\right\}-u\|^2 \mid u\right\}.$$

But $\|\mathbb{E}_v\left\{\hat{u}(v)\mid u\right\}-u\|^2$ does not depend on v, so the last expected value can be removed, which yields:

$$\boxed{\mathbb{E}_v\left\{\|\hat{u}(v)-u\|^2 \mid u\right\} = \mathbb{E}_v\left\{\|\hat{u}(v)-\mathbb{E}_v\left\{\hat{u}(v)\mid u\right\}\|^2 \mid u\right\} + \|\mathbb{E}_v\left\{\hat{u}(v)\mid u\right\}-u\|^2}.$$