# Discontinous Galerkin methods (intro)

AM274: DG for 1D scalar problems:

## Basic formulation

Instead of strong form

$$q_t + f(q)_x = 0 \tag{1}$$

we consider the weak form

$$\int_\Omega \left( \frac{\partial q}{\partial t} + f(q)_x \right) v \, dx = 0 \tag{2}$$

where v is any function.

- discretize space $\Omega$ into elements $\{T_k\}_{k=1,\ldots,K}$
- represent solution on each element as polynomial of order $N_p$
- solutions are allowed to be discontinous at element boundaries

Numerical solution representation

$$q_h(x,t)|_{T_k} = \sum_{j=1}^{N_p+1} Q_j^k(t)\phi_j^k(x) \tag{3}$$

with $q_h(x,t)|_{T_k} \in V_h^k := span < \{\phi_j^k\}_{j=1,\ldots,N_p+1} >$ and $q_h \in V_h := \bigoplus_{k=1}^K V_h^k$. We now look at the discrete weak form based on $V_h$ and choose the basis functions on $T_k$ as the test functions ($v = \phi_i^k$). After partial integration we get:

$$\int_{T_k} \frac{\partial q_h}{\partial t}\phi_i^k dx - \int_{T_k} f(q_h)\frac{d\phi_i^k}{dx} dx = -\left[\hat{f}\phi_i^k\right]_{x_l^k}^{x_r^k} \tag{4}$$

Left terms are element intern. Right term couples elements together using numerical fluxes just as in FV (e.g upwind, lax friedrich, HLL?). Boundary conditions can be applied by setting

$$\hat{f}(x_l^1)\phi_i^1(x_l^1) \quad \text{and} \quad \hat{f}(x_r^K)\phi_i^K(x_r^K) \tag{5}$$

## Implementation

### Local to global mapping

$$\phi_1^1 \to \phi_1 \tag{6}$$
$$\phi_2^1 \to \phi_2 \tag{7}$$
$$\phi_1^2 \to \phi_3 \tag{8}$$
$$\ldots \tag{9}$$

Define map $M(k,j)$ : local indices $\to$ global indices.

**Reference element**

To calculate more easily we will use an affine map:

$$x = F_k(\hat{x}) = A_k \hat{x} + B_k \tag{10}$$

to map all elements to a reference element. For 1D this is: $\hat{x} \in \hat{T} \equiv [-1, 1]$. We also define the basis functions on $T_k$ in terms of the basis functions on $\hat{T}$.

$$\phi_i^k(x) \equiv \hat{\phi}_i(\hat{x}) \tag{11}$$

$$(\phi_i^k)'(x) = \frac{1}{A_k} \hat{\phi}_i{}'(\hat{x}) \tag{12}$$

**ODE's for solution coefficients**

Instead of evolving Cell averages or similar as in FV we now want to evolve the coefficients $Q_i^k(t)$ to our solution $q_h$ represented in the $\phi$-basis. After substituting eq. (3) into eq. (4) we get the matrix equation

$$\boxed{M_{ij}^k \frac{dQ_j^k(t)}{dt} = \mathrm{RHS}^k(Q(t))_i} \tag{13}$$

with

$$M_{ij}^k = \int_{T_k} \phi_i^k(x) \phi_j^k(x) dx = \int_{\hat{T}} \hat{\phi}_i(\hat{x}) \hat{\phi}_j(\hat{x}) \frac{dx}{d\hat{x}} d\hat{x} \tag{14}$$

$$\mathrm{RHS}^k(Q(t))_i = \int_{T_k} f(q_h) \frac{d\phi_i^k}{dx} dx - \left[\hat{f} \phi_i^k\right]_{x_l^k}^{x_r^k} \tag{15}$$

we now split the RHS into I and II.

$$[\mathrm{RHS}_I^k(Q^k)]_i = \int_{T_k} f(q_h(x,t)) \frac{d\phi_i^k}{dx} dx = (S^k)_{ij} (F_h^k)_j \tag{16}$$

where we represent the flux in the $\phi$-basis using a L2 projection and get coefficients $F_h^k$. (For L2 projection see initial condition) The advection matrix $S_{ij}$ can then also be written in terms of the reference element.

$$S_{ij} \equiv \int_{T_k} \phi_j^k(x) \frac{d\phi_i^k}{dx} dx = \int_{\hat{T}} \hat{\phi}_j^k(\hat{x}) \frac{d\hat{\phi}_i^k}{d\hat{x}} d\hat{x} \tag{17}$$

By integrating $M_{ij}$ and $S_{ij}$ we can now assemble the global $M$ and $S$ as block diagonal matricies.

**Gauss quadrature**

To calculate/approximate those integrals we can use gauss quadrature: weighted sum that approximates an integral.

$$\int_a^b f(x) dx \approx \sum_{k=1}^{n} w_k f(x_k) \tag{18}$$

Those are well tabulated for $\hat{x} \in [-1, 1]$ and for polynomials can even be exact.

**Matrix assembly for boundary terms**

We still need to calculate

$$[\mathrm{RHS}^k_{II}(Q)]_i \equiv \left[\hat{f}\phi^k_i\right]^{x^k_r}_{x^k_l} = \hat{f}(x^k_r)\phi^k_i(x^k_r) - \hat{f}(x^k_l)\phi^k_i(x^k_l) \tag{19}$$

Ideally assemble some kind of interaction matrix and add it to map from Q to RHS(Q) For lax friedrichs like flux this can be done by

$$\mathrm{RHS}(Q) = (S - C_{avg})F_h - C_{jump}Q \tag{20}$$

(have not quite understood this part fully yet) For simpler fluxes like upwind those term in can also be calculated directly from (19) by hand. This will lead to a block tridiagonal matrix.

**Initial condition**

Initalize $q_h(x,0) \in V_h$ such that it approximates $q_0$ as well as possible with respect to L2 projection

$$< Pq_0 - q_0, v >= 0 \quad v \in V_h \tag{21}$$

Leads to system of equations

$$M_{ij}Q_i(0) =< q_0(x), \phi_j(x) > \tag{22}$$

solving this gives us the initial coefficients.

**Boundary conditions**

Can be applied (e.g. for advection) using leftmost and rightmost boundary flux. outflow BC (rightmost):

$$\hat{f}(x^K_r) = f_h(x^K_r) \tag{23}$$

inflow BC (leftmost):

$$\hat{f}(x^1_l) = f(q_{in}) \tag{24}$$

**Temporal discretization**

Discretization of (13) using e.g. forward euler

$$Q^{n+1} = Q^n + \Delta t M^{-1}\mathrm{RHS}(Q^n, t^n) \tag{25}$$

## Example: Advection 1D

using upwind scheme and monomial basis functions for $N_p = 1$. Since $\hat{\phi}_i = \hat{x}^{i-1}$, we can calculate $M_{ij}^k$ and $S_{ij}^k$ by hand. We get:

$$M_{ij}^k = \begin{cases} \frac{2A_k}{i+j-1}, & \text{if } i+j-1 \text{ odd,} \\ 0, & \text{if } i+j-1 \text{ even} \end{cases} = A_k \begin{pmatrix} 2 & 0 \\ 0 & 2/3 \end{pmatrix} \tag{26}$$

$$S_{ij}^k = \begin{cases} \frac{2(i-1)}{i+j-2}, & \text{if } i+j-2 \text{ odd,} \\ 0, & \text{if } i+j-2 \text{ even} \end{cases} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \tag{27}$$

and using that for advection $F_h^k = uQ^k$ we can write $\text{RHS}_I^k(Q_k) = uS^kQ^k$. For the upwind scheme we can calculate the boundary terms directly:

$$[\hat{f}\phi_i^k]_{x_l^k}^{x_r^k} = -f(x_r^k)\phi_i^k(x_r^k) + f(x_r^{k-1})\phi_i^k(x_l^k) \tag{28}$$

$$= u \cdot \sum_j [-\phi_j^k(x_r^k)\phi_i^k(x_r^k)]_{ij} \cdot [Q^k]_j + u \sum_j [\phi_j^{k-1}(x_r^{k-1})\phi_i^k(x_l^k)]_{ij} \cdot [Q^{k-1}]_j \tag{29}$$

those two matricies can be written down as

$$[-\phi_j^k(x_r^k)\phi_i^k(x_r^k)]_{ij} = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \tag{30}$$

$$[\phi_j^{k-1}(x_r^{k-1})\phi_i^k(x_l^k)]_{ij} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \tag{31}$$

and then added to the right position in the global advection matrix.

We get:

$$M_{ij} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2/3 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 2/3 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 2 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 2/3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\text{RHS}(Q_i) = u \cdot \left( \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} - \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 0 & \cdots \\ -1 & -1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & -1 & -1 & 0 & 0 & \cdots \\ -1 & -1 & -1 & -1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 1 & -1 & -1 & \cdots \\ 0 & 0 & -1 & -1 & -1 & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \right) \cdot \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \\ \vdots \end{pmatrix}$$

and can now calculate $q_h(x,t)$ using forward euler as in (25).