Numer Algor (2006) 43:279–288
DOI 10.1007/s11075-006-9059-x

ORIGINAL PAPER

The life and work of André Cholesky

C. Brezinski

Received: 12 December 2006 / Accepted: 19 December 2006 /
Published online: 30 January 2007

Abstract Cholesky's method for solving a system of linear equations with a symmetric positive definite matrix is well known. In this paper, I will give an account of the life of Cholesky, analyze an unknown and unpublished paper of him where he explains his method, and review his other scientific works.

Keywords linear algebra · linear equations · history

1 Prelude

For many years, the life of Cholesky was almost completely unknown, only partial informations were available, and many questions on him were asked on the NA-net. His name was even often misspelled. So, I decided to try to fill the gap, and to find out the missing biographical data.

In France, the personal records concerning a person are open to the public 120 years after his birth. Some days after this required period (Cholesky was born on 15 October 1875), I went to the Fort de Vincennes, near Paris, where the archives of the Army are kept. Using the informations I collected there, I wrote a first biography of Cholesky [4]. Then, a colleague of mine, Yves Dumont, from the Université de la Réunion in Saint-Denis, constructed a website dedicated to Cholesky. In January 2004, I received a letter from Michel Gros, Cholesky's grandson, asking me if I will be interested in helping him to sort his grandfather's personal papers that he recently gave to the École Polytechnique where Cholesky had been a student. My name was given to him by Yves Dumont.

I dedicated this work to John (Jack) Todd with esteem and respect at the occasion of his 95th anniversary.

C. Brezinski (✉)
Laboratoire Paul Painlevé, UMR CNRS 8524, UFR de Mathématiques Pures et Appliquées,
Université des Sciences et Technologies de Lille, 59655–Villeneuve d'Ascq Cedex, Lille, France
e-mail: Claude.Brezinski@univ-lille1.fr

Of course I accepted!

During our first visit to École Polytechnique, we discovered, among Cholesky's papers (six big boxes), an unknown and unpublished manuscript where he explained his method for solving systems of linear equations.

2 The life of Cholesky

André Louis Cholesky (who made himself call René) was born on 15 October 1875 in Montguyon, a village situated 35 km north-east from Bordeaux. He was the son of André Cholesky, head waiter, and of Marie Garnier. He had several brothers and sisters. The family probably came to France with Napoléon's armies, but nothing more is known about its origins. André spent his childhood in his native village, and went then to the "Lycée" (secondary school) in Saint-Jean-D'Angély. He obtained the first part of his "Baccalauréat" in Bordeaux in 1892, and the second part a year after, as usual. On 15 October 1895, he was admitted as a student at the prestigious *École Polytechnique* in Paris, and he had to sign a 3 years contract with the Army. His professors were, in particular, Camille Jordan and Georges Humbert for analysis, Paul Haag for geometry, Octave Callandreau for astronomy and geodesy, and Henri Becquerel for physics. After 2 years at École Polytechnique, he entered at the *École d'Application de l'Artillerie et du Génie* in Fontainebleau, near Paris, where he stayed for two more years. There, he was trained, among other topics, in topography by the Lieutenant-Colonel Charles-Moyse Goulier (I left the military ranks in French), who invented several instruments in use in this domain.

Then Cholesky began his officer's career in October 1899 as a Lieutenant. He was sent to Tunisia from January to June 1902, and again from November 1902 to May 1903. From December 1903 to June 1904, he was in Algeria. On 24 June 1905, he was appointed to the Geographical Service of the Army. He immediately attracted the attention of his chiefs by his sharp intelligence, and his original ideas that he defended with great passion.

At this time, a new triangulation of France had been decided. Cholesky was sent to Dauphiné, a part of the Alps near Grenoble, for measuring the length of the meridian passing through Lyon. The engineers were faced to the problem of finding a simple, fast and precise method for correcting the measuring and the instrumental errors. A system of linear equations had to be solved in the least squares sense, and it certainly was at this occasion that Cholesky had the idea of his method. In his *Carnets*, Cholesky tells us about his every day life during this period, a human testimony; see [10].

On 22 April 1907, Cholesky married his first cousin Anne Henriette Brunet. They will have four children, one posthumous. From November 1907 to June 1908, he was sent to Crete, then occupied by the international troops. His work consisted in the triangulation of the French and British parts of the island. After three months with two other officers, Cholesky remained alone in a difficult situation (the culminating summit of Crete is 2,400 m high and, at the end of May, it was still necessary to melt snow for obtaining water).

In March 1909, Cholesky (Fig. 1) was promoted to Capitaine (Captain), but he had to leave the Geographical Service for a 2 years training period as the head of

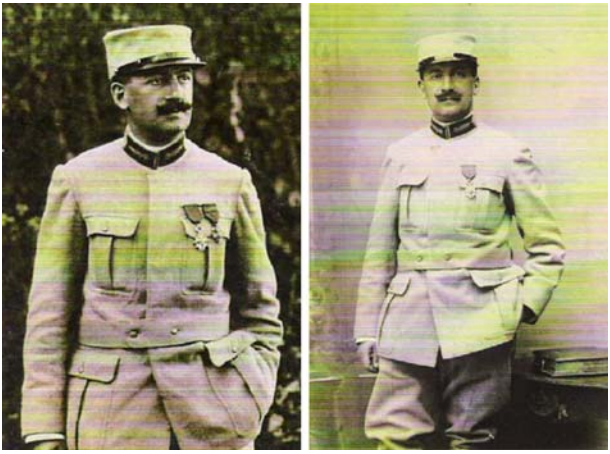


Fig. 1 André Louis Cholesky

an artillery battery. The manuscript on his method is dated from this period. In [1, p. 78], it is mentioned that Cholesky's method was appreciated, and helped to gain time in the levelling operations. However, it is not specified in this document if the remark applies to his method for solving systems of linear equations, or to the double levelling procedure he also invented, and which is still in use today.

In September 1911, Cholesky was appointed to the artillery's headquarters, and went back to the Geographical Service under Général Bourgeois, a well known geodesist and topographer. He received the commandment of the levelling operations in Algeria and Tunisia. He had to map the countries, and to prepare the construction of railways and roads. He also spent some time in Morocco and Sahara. In May 1913, he became the head of the Topographical Service in Tunis. He stayed there until the declaration of war, on 2 August 1914.

Back to France, he was in charge of a battery for some months. At the beginning of the war, the French army was using maps at the scale of $1/80,000$. The artillery had to fire on invisible targets only defined by their positions on a map. More precise maps, indicating the relief, were necessary. But it was not possible to send topographers inside the enemy's lines. It was the beginning of aerial photography. The planes were flying quite low, they were not too stable. The photos were taken obliquely, and they had to be restored horizontally. Cholesky participated to this task, and he had to organize the artillery's shooting. He was one of the officers who understood the best and develop the most the use of geodesy and topography for that purpose. Cholesky was also interested by locating the enemy's batteries by sound, by aerial photography, and even by the sights for shooting from an aeroplane. He wrote many official documents on these problems. He also began to learn English.

From September 1916 to February 1918, Cholesky was sent to Romania as the head of the Romanian Geographical Service. He completely reorganized this Service, showing his great capabilities as a leader. He was promoted Chef d'Escadron, that is Commandant (Commander), on 6 July 1917. Back to France, he participated, with the Army of Général Mangin, at the offensive on the Hindenburg's line passing through the villages of Lassigny, Ribécourt-Dreslincourt, and Tracy-le-Mont. His regiment was involved in the fights on the river Ailette, on 23 August, and in the village of Courson. Mangin was ready to break the enemy front between the river Aisne and the city of Saint-Gobain.

On 31 August 1918, the Commandant Cholesky died at 5:00 AM in a quarry, north of the village of Bagneux (10 km north of Soissons), after the wounds received on the battle field. He was first buried in the military cemetery of Chevillecourt near Autrèches, 15 km west of Soissons. On 24 October 1921, his body was transferred to the military cemetery of Cuts, 10 km south-east of Noyon, tomb 348, square A; see Fig. 2.

3 The manuscript

The method of *triangulation* is used for establishing maps. It goes back to the sixteenth century at least. The region to be mapped is covered by triangles. For refinement, one can use networks of smaller and smaller triangles. One begins to measure the length of a side of the first triangle (called the basis) and then, standing at each corner of the adjacent triangles, only angles are measured. It is easier and less subject to errors to measure angles than distances. Vertical angles have also to be measured for obtaining a map in an horizontal plane (an operation called *levelling*). Then, the usual trigonometric formulae (spherical trigonometry for long distances) give the lengths. Thus, the topographer obtains a network of adjacent triangles. Obviously, the three angles of each triangle (instead of two) could be measured for safety, and a second basis also, thus leading to more equations than unknowns. But one has to take into account the various sources of errors. This is called *compensation*.

The compensation of geodetic networks is due to Carl Friedrich Gauss. Let us consider N measures $l_1, \ldots, l_N$ of n quantities $X_1, \ldots, X_n$ (angles or distances). It is necessary to correct the errors v_i which affect them, and are due to the imperfections

Fig. 2 Cholesky's grave

of the instruments and the experimental flaws. The measures l_i are related to the $n+N$ unknowns $X_1,\ldots,X_n$ and $\nu_1,\ldots,\nu_N$ by means of nonlinear functions f_i. Let $l_i^* = f_i(X_1^*,\ldots,X_n^*)$ be the measures computed from approximated values X_i^*, obtained from only some of the l_is by an adequate procedure. Keeping only the first order term in the Taylor series expansions of the functions f_i leads to a system of linear equations. Let A be the matrix of the partial derivatives of the f_is with respect to the X_j^*s, b the vector with components $b_i = l_i - l_i^*$, x the vector of the *compensations* $x_i = X_i - X_i^*$, and ν the vector of the ν_i. It holds $Ax = b + \nu$. This system is solved in the least squares sense, a method due to Adrien Marie Legendre in 1806. As proved by Gauss around the same time, the errors ν_i are normal random variables, centered and independent, and each l_i is a random variable with standard deviation σ_i. The most probable solution minimizes $\varepsilon^2 = p_1\nu_1^2 + \cdots + p_N\nu_N^2$ where $p_i = 1/\sigma_i^2$. Setting to zero all the partial derivatives of ε^2 with respect to the x_j's results in the system $A^T P\nu = 0$, where P is the diagonal matrix with elements p_i. It is a system of n equations with N unknowns. But $\nu = Ax - b$, and thus $A^T PAx = c$ with $c = A^T Pb$. This system is called the *normal equations*, and it only contains the n unknowns x. So, at the end, we are led to a system with a symmetric and positive definite matrix to solve.

Cholesky presented his method in an unknown and unpublished manuscript entitled *Sur la résolution numérique des systèmes d'équations linéaires* (On the numerical solution of systems of linear equations). It contains eight pages (with only eight lines on the last page), and it is written on a paper of size 21.8 × 32 cm. It is dated 2 December 1910 and since, contrarily to the other manuscripts of Cholesky, very few words are crossed-out, it is almost certain that the method was obtained some time before, and that the paper is a rewriting or a new copy of a preceding text. See Fig. 3 for a print of the first page of the manuscript which is entirely reproduced in [7] and [6].

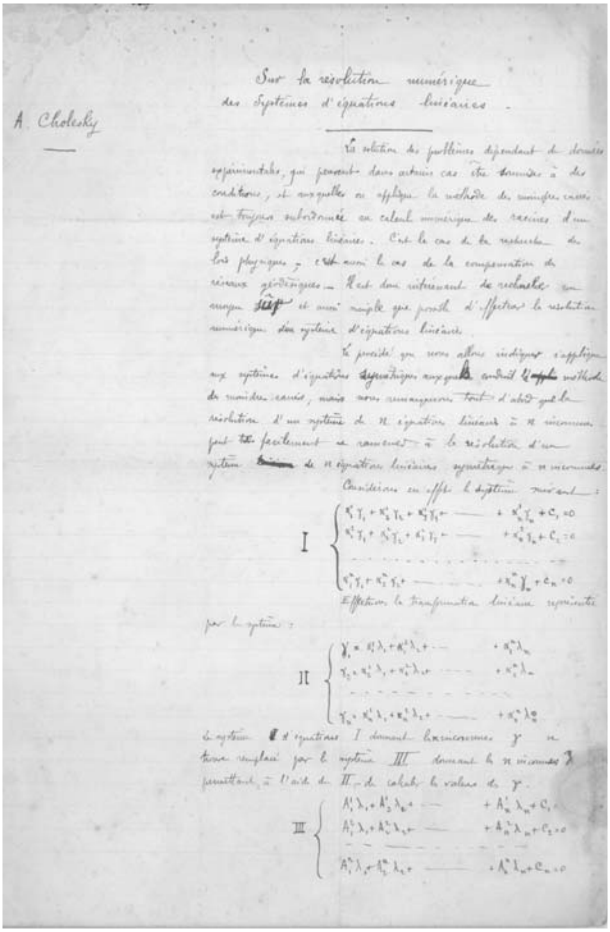
For more details about the other methods for solving systems of linear equations, and the use of least squares in topography, see [6] and [8, Chap. 9]. An account on the history of geodesy, topography and cartography is given in [5].

Let us now present the manuscript. Cholesky first considered the system (I kept his notations)

$$\mathrm{I}\left\{\begin{array}{llll} \alpha_1^1\gamma_1 + \alpha_2^1\gamma_2 + \alpha_3^1\gamma_3 + & \cdots & +\alpha_n^1\gamma_n + C_1 = 0 \\ \alpha_1^2\gamma_1 + \alpha_2^2\gamma_2 + \alpha_3^2\gamma_3 + & \cdots & +\alpha_n^2\gamma_n + C_2 = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots & \cdots & \cdots\cdots\cdots\cdots\cdots \\ \alpha_1^n\gamma_1 + \alpha_2^n\gamma_2 + & \cdots & +\alpha_n^n\gamma_n + C_n = 0. \end{array}\right.$$

Let Υ be the matrix of this system, $\gamma = (\gamma_1,\ldots,\gamma_n)^T$ its solution, and $c = -(C_1,\ldots,C_n)^T$ its right hand side. The system I writes $\Upsilon\gamma = c$. Then, Cholesky carried out the linear transformation represented by the system

$$\mathrm{II}\left\{\begin{array}{l} \gamma_1 = \alpha_1^1\lambda_1 + \alpha_1^2\lambda_2 + \cdots + \alpha_1^n\lambda_n \\ \gamma_2 = \alpha_2^1\lambda_1 + \alpha_2^2\lambda_2 + \cdots + \alpha_2^n\lambda_n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \gamma_n = \alpha_n^1\lambda_1 + \alpha_n^2\lambda_2 + \cdots + \alpha_n^n\lambda_n. \end{array}\right.$$



A. Cholesky

Sur la résolution numérique des systèmes d'équations linéaires

Fig. 3 First page of the unpublished manuscript of Cholesky (by the authorization of the Archives of École Polytechnique)

Setting $\lambda = (\lambda_1, \ldots, \lambda_n)^T$, we thus have $\gamma = \Upsilon^T \lambda$. So, the system I, giving the unknowns γ_i, is replaced by the system III

$$\mathrm{III}\begin{cases} A_1^1\lambda_1 + A_2^1\lambda_2 + \cdots + A_n^1\lambda_n + C_1 = 0 \\ A_1^2\lambda_1 + A_2^2\lambda_2 + \cdots + A_n^2\lambda_n + C_2 = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ A_1^n\lambda_1 + A_2^n\lambda_2 + \cdots + A_n^n\lambda_n + C_n = 0. \end{cases}$$

In matrix terms, this system writes $\Upsilon\Upsilon^T\lambda = c$, and it gives the λ_is, thus allowing the computation of the unknowns γ_i by the system II. The coefficients of the system III are given by the formulae

$$A_p^p = \sum_{k=1}^{k=n} (\alpha_k^p)^2$$

$$A_p^q = \sum_{k=1}^{k=n} \alpha_k^p \alpha_k^q.$$

Let us set $A = \Upsilon\Upsilon^T$. Cholesky noticed that the matrix of this system is symmetric but, maybe, he did not know the notion of positive definiteness. He remarked that the system III could be solved if the γ_is are easily obtained from the system I. This is the case if, in I, the first equation only contains γ_1, if the second equation only contains γ_1 and γ_2, and so on. So, Cholesky proposed to find a system of the form

$$\begin{array}{ll}
\alpha_1^1\gamma_1 & +C_1 = 0 \\
\alpha_1^2\gamma_1 + \alpha_2^2\gamma_2 & +C_2 = 0 \\
\alpha_1^3\gamma_1 + \alpha_2^3\gamma_2 + \alpha_3^3\gamma_3 & +C_3 = 0 \\
\dots\dots\dots\dots\dots\dots\dots & \dots\dots \\
\alpha_1^n\gamma_1 + \alpha_2^n\gamma_2 + \alpha_3^n\gamma_3 + \cdots + \alpha_n^n\gamma_n & +C_n = 0.
\end{array}$$

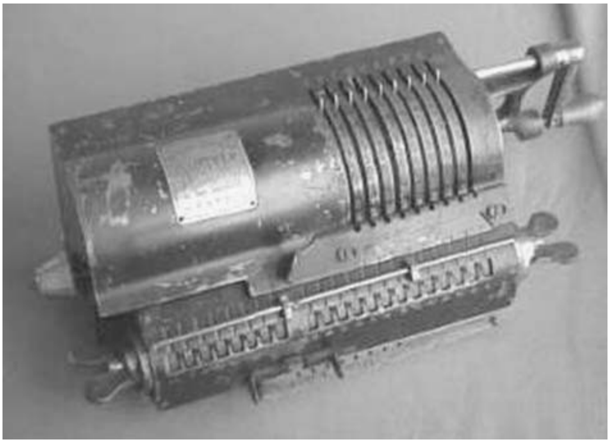
Then, the λ_is can be directly obtained by

$$\begin{array}{rrrrr}
\alpha_1^1\lambda_1 + & \alpha_1^2\lambda_2 + & \cdots & \cdots & +\alpha_1^n\lambda_n - \gamma_1 = 0 \\
 & \alpha_2^2\lambda_2 + & \alpha_2^3\lambda_3 & \cdots & +\alpha_2^n\lambda_n - \gamma_2 = 0 \\
 & & \alpha_3^3\lambda_3 & \cdots & +\alpha_3^n\lambda_n - \gamma_3 = 0 \\
 & & \cdots & \cdots & \dots\dots\dots\dots \\
 & & & & \alpha_n^n\lambda_n - \gamma_n = 0
\end{array}$$

starting from the last equation which gives λ_n. Finally, Cholesky gave the formulae we are all teaching to our students for computing the coefficients α_i^j. Although he used the same notation for the elements of the matrix Υ and for those of these triangular matrices, we perfectly understand that Cholesky factorizes A as $A = LL^T$. Then, he solves $L\gamma = c$, and finally $L^T\lambda = \gamma$. Moreover, Cholesky explained how to implement easily his method on the mechanical calculating machine *Dactyle* (remember that the computations had to be done by soldiers with only a small mathematical background), and how to check that no mistake was made in the application of his method; see Fig. 4.

Then, Cholesky explained that, in the factorization procedure, the matrices of the two triangular systems above could be different, one upper triangular and the other one lower triangular, but that taking the second matrix as the transpose of the first one reduces the propagation of the errors due to the finite precision of

Fig. 4 The calculating machine Dactyle



the computations. After this, Cholesky gave a method for computing the square root of a number, and proved its quadratic convergence rate. It is in fact the standard iterations $x_{n+1} = (x_n + A/x_n)/2$ for $\sqrt{A}$, a procedure attributed to Hero of Alexandria. As in any good numerical analysis paper, Cholesky ends by numerical examples: he was able to solve a system of 10 equations with five decimal digits in 4–5 h. He reported that his method was also successfully used for several systems of dimensions 30, and for one of dimension 56 (but he did not mention how long it took).

Cholesky's method remained unknown outside the circle of French topographers until 1924, when another French officer, the Commandant Benoît, published a paper explaining his fellow's method [3]. Then, a period of 20 years followed without any mentioning of the work. In 1944, the Danish geodesist Henry Jensen published a paper where he compared several methods for solving systems of linear equations [11]. In 1946, John Todd had to teach a numerical analysis course at King's College in London. With his wife, Olga Taussky, they looked into the *Mathematical Reviews* (an easy task at this time!), and they found an analysis of Jensen's paper by Ewald Konrad Bodewig. Jensen was saying that "Cholesky's method seems to possess all advantages." Since the method was clearly explained by Jensen, Taussky and Todd did not try to find the original paper. According to them [13], Otto Toeplitz was aware that a symmetric matrix A could be factored into a product LL^T where L is a lower triangular matrix. Indeed, in a paper published in 1907 (that is before Cholesky, and certainly without knowledge of his work), Toeplitz expressed the elements of L as ratios of determinants, not a method to be used in practice [14]. After Todd's lectures, several colleagues and students of him undertake to study Cholesky's method: Leslie Fox, Harry Douglas Huskey, James Hardy Wilkinson, and Alan Mathison Turing. Then, by 1950, the method was widely known.

Gauss' and Cholesky's methods were later rediscovered many times. Let us mention, in particular, the *square root* method of Tadeusz Banachiewicz in 1938 [2] which is closely related to Cholesky's.

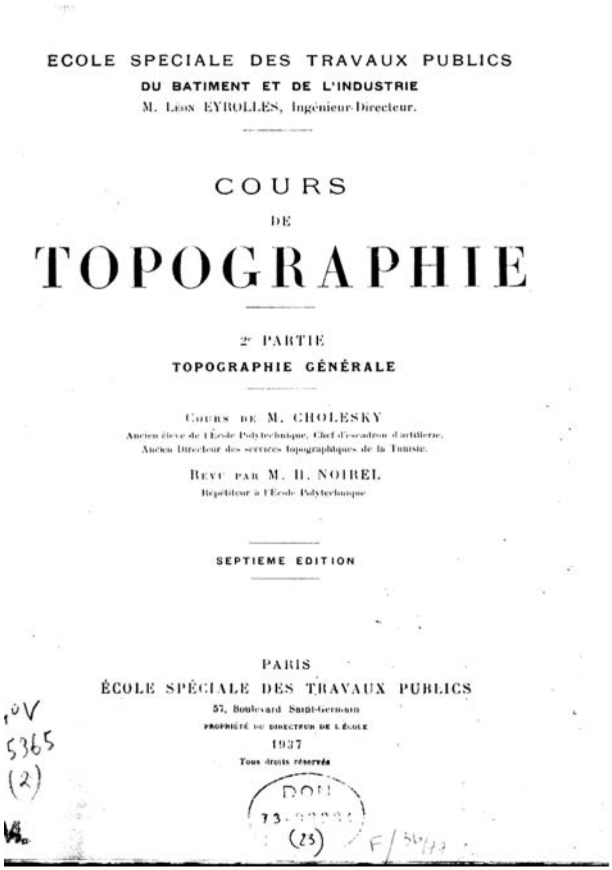
4 Other works

From at least December 1909 to January 1914, Cholesky taught in an engineering school by correspondence, the *École Spéciale des Travaux Publics, du Bâtiment et de l'Industrie*, founded in 1891 by Léon Eyrolles. He wrote many lecture notes, and published a book of 442 pages that had at least seven editions and was still in print in 1937, that is almost 20 years after his death [9]; see Fig. 5.

The archives also contain the manuscripts of two other books: *Complément de Topographie* (239 pages), and *Cours de Calcul Graphique* (83 pages) that will be available soon on the web [15].

Cholesky's archives at École Polytechnique contain many other scientific papers by him

- Three pages with the title *Sur la détermination des fractions de secondes de temps*.
- A 15 pages manuscript entitled *Instructions pour l'exécution des nivellements de précision*.


ECOLE SPECIALE DES TRAVAUX PUBLICS
DU BATIMENT ET DE L'INDUSTRIE
M. Léon EYROLLES, Ingénieur-Directeur.

COURS
DE
TOPOGRAPHIE

2e PARTIE
TOPOGRAPHIE GÉNÉRALE

Cours de M. CHOLESKY
Ancien élève de l'École Polytechnique, Chef d'escadron d'artillerie,
Ancien Directeur des services topographiques de la Tunisie.

Revu par M. H. NOIREL
Répétiteur à l'École Polytechnique

SEPTIEME EDITION

PARIS
ÉCOLE SPÉCIALE DES TRAVAUX PUBLICS
57, Boulevard Saint-Germain
PROPRIÉTÉ DU DIRECTEUR DE L'ÉCOLE
1937
Tous droits réservés

Fig. 5 Cholesky's book on topography

- A eight pages manuscript entitled *Équation de l'ellipsoïde terrestre rapportée à Ox tangente au parallèle vers l'Est, Oy tangente au méridien vers le Nord, Oz verticale vers le zénith.*
- Sixteen pages on *Étude du développement conique conforme de la carte de Roumanie*.
- A three pages manuscript with the title *Instructions sur l'héliotrope-alidade (modèle d'étude 1905)*.
- Three pages and three maps on the construction of railroads.
- Three pages with the title *Remarque au sujet du calcul de correction de mire*.

5 Coda

Cholesky's method is still the choice method for the solution of systems of linear equations when the matrix is symmetric positive definite, such as those coming out from optimization problems or from the discretization of some partial differential equations. It led to the development of incomplete factorization for building preconditioners. It has applications in multigrids, multilevel methods, and domain decomposition. On these topics, see, for example, [12].

Nowadays, with Google, the keyword *Cholesky* gives 713,000 answers, and *Choleski* 77,400.

Acknowledgement I would like to thank Michela Redivo-Zaglia for the improvements she suggested.

References

1. Anonymous, Le Service Géographique de l'Armée. Son Histoire - Son Organisation - Ses Travaux, Imprimerie du Service Géographique de l'Armée, Paris (1938)
2. Banachiewicz, T.: Principes d'une nouvelle technique de la méthode des moindres carrés; Méthode de résolution numérique des équations linéaires, du calcul des déterminants et des inverses et de réduction des formes quadratiques. Bull. Inter. Acad. Polon. Sci., Sér. A, 393–404 (1938)
3. Benoît, C.: Note sur une méthode de résolution des équations normales provenant de l'application de la méthode des moindres carrés à un système d'équations linéaires en nombre inférieur à celui des inconnues, (Procédé du Commandant Cholesky). Bull. Géod. **2**, 67–77 (1924)
4. Brezinski, C.: André Louis Cholesky, in numerical analysis, a numerical analysis conference in honour of Jean Meinguet. Bull. Soc. Math. Belg., 45–50 (1996)
5. Brezinski, C.: Géodésie, topographie et cartographie. Bull. Soc. Amis. Bib. Éc. Polytech. **39**, 33–68 (2005)
6. Brezinski, C.: La méthode de Cholesky. Rev. Hist. Math. **11**, 205–238 (2005)
7. Brezinski, C., Gross-Cholesky, M.: La vie et les travaux d'André-Louis Cholesky. Bull. Soc. Amis. Bib. Éc. Polytech. **39**, 7–32 (2005)
8. Chabert, J.-L., et al.: A History of Algorithms from the Pebble to the Microchip. Springer, Berlin Heidelberg New York (1999)
9. Cholesky, A.: Cours de Topographie. 2è Partie, Topographie Générale, École Spéciale des Travaux Publics, Paris, 7è édition (1937)
10. Cholesky, A.: Restitution du carnet no. 2 et du carnet no. 3. Bull. Soc. Amis. Bib. Éc. Polytech. **39**, 69–79 (2005)
11. Jensen, H.: An attempt at a systematic classification of some methods for the solution of normal equations, Geodaetisk Institut Kobenhan, Meddelelse no. 18, Bianco Lunos Bogtrykkeri A/S, København, 45 (1944)
12. Meurant, G.: Computer Solution of Large Linear Systems. North Holland, Amsterdam, The Netherlands (1999)
13. Taussky, O., Todd, J., Cholesky, A.: Toeplitz and the triangular factorization of symmetric matrices. Numer. Algorithms **41**, 197–202 (2006)
14. Toeplitz, O.: Die Jacobische Transformation der quadratischen Formen von unendlich vielen Veränderlichen. Nachr. Akad. Wiss. Gött. Math.-Phys. Kl. **II**, 101–110 (1907)
15. Tournès, D.: www.rehseis.cnrs.fr/calculsavant/Textes/uncoursineditdec.html