Homework 11 [TI8419 - Multilinear Algebra]

Lucas Abdalah

Professors: André Lima e Henrique Goulart

Table of Contents

Alternating Least Squares (ALS) Algorithm

Problem 1

For the third-order tensor $\mathcal{X} \in \mathbb{C}^{I \times J \times K}$ provided in the file "cpd_tensor.mat", implement the plain-vanilla Alternating Least Squares (ALS) algorithm that estimates the factor matrices $\mathbf{A} \in \mathbb{C}^{I \times R}$, $\mathbf{B} \in \mathbb{C}^{J \times R}$ and $\mathbf{C} \in \mathbb{C}^{K \times R}$ by solving the following problem:

$$(\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}) = \min_{\mathbf{A}, \mathbf{B}, \mathbf{C}} ||\mathcal{X} - \sum_{r=1}^{R} a_r \circ b_r \circ c_r||_F^2$$

where $\mathbf{A} = [a_1, \ldots, a_R]$, $\mathbf{B} = [b_1, \ldots, b_R]$, $\mathbf{C} = [c_1, \ldots, c_R]$.

Considering a successful run, compare the estimated matrices $\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}$ with the original ones (also provided in the same Matlab file). Explain the results.

Hint: An error measure at the ith iteration can be calculated from the following formula:

$$e_{(i)} = \min_{\mathbf{A}, \mathbf{B}, \mathbf{C}} ||[\mathcal{X}]_{(1)} - \hat{\mathbf{A}}_{(i)} \left(\hat{\mathbf{C}}_{(i)} \diamond \hat{\mathbf{B}}_{(i)} \right)^\top ||_F \quad (1)$$

Results

Simulation setup

- The ALS algorithm that estimates the factor matrices of $\hat{\mathcal{X}}$ from a given tensor $\mathcal{X}$, minimizing the distance between them.
- $I, J, K = 8, 4, 5$;
- $R = 3$;
- Initialized from a Normal distribution $\mathcal{N}(0, 1)$.

Discussion

To compare the real data with the estimated factors, we may use the experimental results for NMSE between the given data and obtained as output to ALS.

NMSE for ALS Validation

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X and X_hat: -123.33 dB
A and A_hat: 27.96 dB
B and B_hat: 0.64 dB
C and C_hat: 25.72 dB
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The results are consistent with the proposed scenario, since that for randomly generated $\mathbf{X}$, the algorithm succeeds to obtain factors with small NMSE (dB) values.

Remark: the $\mathrm{NMSE}(\mathbf{B}, \hat{\mathbf{B}})$ presents the smaller error since we choose to fix $\hat{\mathbf{A}}$ and $\hat{\mathbf{C}}$ to estimate $\hat{\mathbf{B}}$ at the first iteration. This choice was arbitrary and its behavior is presented for any factor matrix that is estimated first.

Problem 1 script.
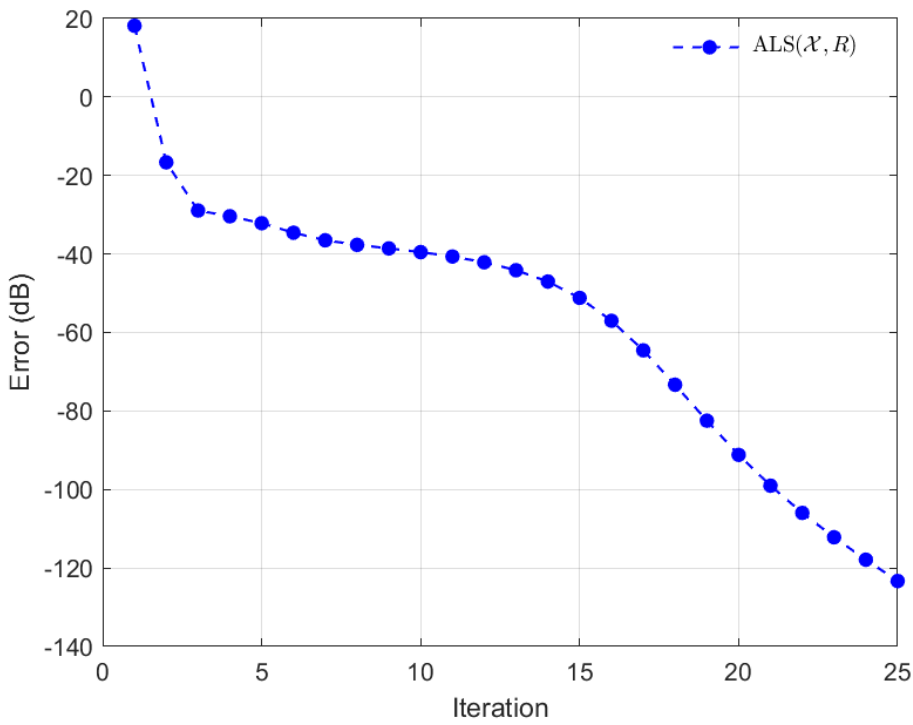
Problem 2

In a Monte Carlo experiment with $M = 1000$ realizations, generate a tensor $\mathcal{X}_{(0)} = \mathrm{CPD}(\mathbf{A}, \mathbf{B}, \mathbf{C})$, where $\mathbf{A} \in \mathbb{C}^{I \times R}$, $\mathbf{B} \in \mathbb{C}^{J \times R}$ and $\mathbf{C} \in \mathbb{C}^{K \times R}$ have unit norm columns with elements randomly drawn from a normal distribution, with $R = 3$.

Let $\mathcal{X} = \mathcal{X}_0 + \alpha \mathcal{V}$ be a noisy version of $\mathcal{X}_0$, where $\mathcal{V}$ is the additive noise term, whose elements are drawn from a normal distribution. The parameter α controls the power (variance) of the noise term, and is defined as a function of the signal to noise ratio (SNR), in dB, as follows

$$\mathrm{SNR}_{\mathrm{dB}} = 10 \log_{10} \left(\frac{||\mathcal{X}_0||_F^2}{||\alpha \mathcal{V}||_F^2} \right) \quad (2)$$

Assuming the SNR range $\{0, 5, 10, 15, 20, 25, 30\}$ dB, find the estimates $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$ and $\hat{\mathbf{C}}$ obtained with the ALS algorithm for $(I, J, K) = (10, 4, 2)$.

Let us define the normalized mean square error (NMSE) measure

$$\mathrm{NMSE}(\mathbf{Q}) = \frac{1}{M} \sum_{m=1}^{M} \frac{||\hat{\mathbf{Q}}(m) - \mathbf{Q}(m)||_F^2}{||\mathbf{Q}(m)||_F^2} \quad (3)$$

where $\mathbf{Q}(m)$ and $\hat{\mathbf{Q}}$ denote the original data matrix and the reconstructed one at the m-th Monte Carlo experiment, respectively. For each SNR value, plot NMSE(A), NMSE(B) and NMSE(C) as a function of the SNR. Discuss the obtained results.

HintHint: Don't forget to take into account the inherent ambiguities of the CP decomposition.

Results

Simulation setup

- 1000 Monte Carlo Runs;
- Each Monte Carlo iteration uses a new matrix initialization from a Normal distribution $\mathcal{N}(0, 1)$;
- The ALS algorithm that estimates the factor matrices of $\hat{\mathcal{X}}$ from a given tensor $\mathcal{X}$, minimizing the distance between them.
- SNR range $\{0, 5, 10, 15, 20, 25, 30\}$;
- $I, J, K = 10, 4, 2$;
- $R = 3$.

Discussion

The Monte Carlo algorithm provides a repeated random sampling to obtain numerical results with the real data, a powerful tool to assess the trend using randomness. The experimental results for NMSE compare the given random data and obtained as output to ALS.

SNR [dB]	NMSE(X) [dB]	NMSE(A) [dB]	NMSE(B) [dB]	NMSE(C) [dB]
0	-7.71	12.32	1.94	13.06
5	-30.53	13.83	1.00	14.69
10	-50.40	15.03	0.82	15.18
15	-69.78	15.17	0.98	15.33
20	-88.74	15.34	0.95	15.97
25	-108.26	15.23	0.89	15.48
30	-127.41	15.31	0.87	15.63

The results are consistent with the second problem scenario, since that for randomly generated data, the algorithm succeeds to obtain factors with small NMSE (dB) values. As well the SNR and NMSE for $\mathbf{X}$ are inversely proportional variables, i.e, as the SNR increases, the NMSE decreases.

The factor matrices remains practically stagnant, with very few variation as the SNR varies. The main notable difference is the remark indicated in the first problem, which indicates that the $\mathrm{NMSE}(\mathbf{B}, \hat{\mathbf{B}})$ presents the smaller error since we choose to fix $\hat{\mathbf{A}}$ and $\hat{\mathbf{C}}$ to estimate $\hat{\mathbf{B}}$ at the first iteration.

Problem 2 script and Figures.