Lista 2 [TI8419 - Multilinear Algebra]

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Problem 1

By using the properties of the outer product, show that

$$\mathcal{X} = \mathbf{a}_{1} \circ \mathbf{b}_{1} \circ \mathbf{c}_{1} + \mathbf{a}_{2} \circ \mathbf{b}_{2} \circ \mathbf{c}_{2}$$

is a rank-one tensor whenever $\mathbf{b}_{1} = \mathbf{b}_{2}$ and $\mathbf{c}_{1} = \mathbf{c}_{2}$. Is this also true in general when $\mathbf{c}_{1} = \mathbf{c}_{2}$ but $\mathbf{b}_{1} \neq \mathbf{b}_{2}$?

1.1) O tensor $\mathcal{X}$ é construido a partir do seguinte modelo: $$\mathcal{X} = \mathbf{a}_{1} \circ \mathbf{b}_{1} \circ \mathbf{c}_{1} + \mathbf{a}_{2} \circ \mathbf{b}_{2} \circ \mathbf{c}_{2}$$

Assumindo $\mathbf{b}_{1} = \mathbf{b}_{2}$ e $\mathbf{c}_{1} = \mathbf{c}_{2}$

$$\begin{aligned} \mathcal{X} &= \mathbf{a}_{1} \circ \mathbf{b}_{1} \circ \mathbf{c}_{1} + \mathbf{a}_{2} \circ \mathbf{b}_{1} \circ \mathbf{c}_{1}\\ &= (\mathbf{a}_{1} + \mathbf{a}_{2}) \circ \mathbf{b}_{1} \circ \mathbf{c}_{1} \end{aligned}$$

1.2) É conveniente observar a soma como um novo vetor: $\mathbf{v}_{1} = \mathbf{a}_{1} + \mathbf{a}_{2}$

1.3) Finalmente, esta representação torna mais evidente que apenas 1 tensor de posto-1 é suficiente para representar $\mathcal{X}$, i.e, tem posto $R = 1$.

$$\boxed{\mathcal{X} = \mathbf{v}_{1} \circ \mathbf{b}_{1} \circ \mathbf{c}_{1}}$$

1.4) Assumindo $\mathbf{b}{1} \neq \mathbf{b}{2}$ e $\mathbf{c}{1} = \mathbf{c}{2}$

$$\begin{align*} \mathcal{X} &= \mathbf{a}{1} \circ \mathbf{b}{1} \circ \mathbf{c}{1} + \mathbf{a}{2} \circ \mathbf{b}{2} \circ \mathbf{c}{1}\ &= (\mathbf{a}{1} \circ \mathbf{b}{1} + \mathbf{a}{2} \circ \mathbf{b}{2}) \circ \mathbf{c}_{1} \end{align*}$$

1.5) Com estas premissas, apenas a reorganização não permite afirmar que $\mathcal{X}$ é representado por um tensor de posto 1.

1.6) Entretanto, ao assumir que existe α tal que $\mathbf{b}{2} = \alpha \mathbf{b}{1}$, a equação fica mais simples.

$$\begin{align*} \mathcal{X} &= \mathbf{a}{1} \circ \mathbf{b}{1} \circ \mathbf{c}{1} + \mathbf{a}{2} \circ \alpha \mathbf{b}{1} \circ \mathbf{c}{1}\ &= \mathbf{a}{1} \circ \mathbf{b}{1} \circ \mathbf{c}{1} + \alpha \mathbf{a}{2} \circ \mathbf{b}{1} \circ \mathbf{c}{1}\ &= (\mathbf{a}{1} + \alpha \mathbf{a}{2}) \circ \mathbf{b}{1} \circ \mathbf{c}{1} \end{align*}$$

É conveniente observar a soma como um novo vetor: $\mathbf{v}{2} = \mathbf{a}{1} + \alpha \mathbf{a}_{2}$

1.7) Finalmente, esta representação torna mais evidente que apenas 1 tensor de posto-1 é suficiente para representar $\mathcal{X}$, i.e, tem posto $R = 1$.

$$\begin{equation*} \mathcal{X} = \mathbf{v}{2} \circ \mathbf{b}{1} \circ \mathbf{c}_{1} \end{equation*}$$

1.8) Entretanto, se não existe α, ou seja $\mathbf{b}{1}$ e $\mathbf{b}{2}$ não são colineares, consequentemente $\mathbf{v}_{2}$ não existe e apenas 1 tensor de posto-1 é insuficiente para representar $\mathcal{X}$. Sendo utilizados 2 tensores de posto-1 para representação, implica em $posto(\mathcal{X}) = 2$.

$$\begin{equation*} \boxed{\mathcal{X} = (\mathbf{a}{1} \circ \mathbf{b}{1} + \mathbf{a}{2} \circ \mathbf{b}{2}) \circ \mathbf{c}_{1}} \end{equation*}$$

Problem 2

Show that the tensor rank is indeed a tensor property: in other words, it is invariant with respect to a multilinear transformation by nonsingular matrices, that is, if

$$\begin{equation*}\mathcal{X} = \mathcal{S} \times_{1} \mathbf{A}^{(1)} \dots \times_{N} \mathbf{A}^{(N)} \end{equation*}$$

where $\mathbf{A}^{(n)} \in \mathbb{C}^{I_{n} \times I_{n}}$ is nonsingular for every n, then

$$\begin{equation*} rank(\mathcal{X}) = rank(\mathcal{S}). \end{equation*}$$

(Hint: write $\mathcal{S}$ as a PD with a minimal number of terms, and then use the properties of the multilinear transformation to bound the rank of $\mathcal{X}$; similarly, use the invertibility of the multilinear transformation to bound the rank of $\mathcal{S}$. More generally, conclude that the same property holds for matrices $\mathbf{A}^{(n)} \in {\mathbb{C}}^{I_{n} \times R_{n}}$ having linearly independent columns (and thus $R_{n} \leq I_{n}$).

2.1) O tensor *core* $\mathcal{S}$ reescrito em função de fatores da decomposição CP resulta em:

$$\begin{equation*} \mathcal{S} = \sum_{r=1}^{R} \mathbf{s}{r}^{(1)} \circ \dots \circ \mathbf{s}{r}^{(N)} \end{equation*}$$

2.2) Também é possível reorganizar a equação em função $\mathcal{S}$ de acordo com as equações (11) e (17) das notas de aula, utilizando as propriedades do operador transposto $\left(^{\top}\right)$, caso $\mathbf{A}^{(n)} \in \mathbb{R}$, e operador hermitiano/autoadjunto $\left(^{H}\right)$

$$\begin{equation*}\mathcal{S} = \mathcal{X} \times_{1} {\mathbf{A}^{(1)}}^{H} \dots \times_{N} {\mathbf{A}^{(N)}}^H \end{equation*}$$

$$\begin{equation*}\mathcal{X} = \mathcal{S} \times_{1} \mathbf{A}^{(1)} \dots \times_{N} \mathbf{A}^{(N)} \end{equation*}$$

Problem 3

Let X P CI1ˆI2ˆI3 be given by X “ a1˝b1˝c1 `a2 ˝b2 ˝c1` a1˝b2˝c2, (1) where the vectors are assumed to satisfy the following: • a1 is not collinear with a2; • b1 is not collinear with b2; • c1 is not collinear with c2. The goal of this exercise is to show that any such tensor has rank three, that is, it cannot be expressed as a sum of fewer terms. We will proceed by steps. 1 (i) First, show that X “ S ˆ1 A ˆ2 B ˆ3 C, where A “ “a1 a2 ‰ , B “ “b1 b2 ‰ , C “ “c1 c2 ‰ , and S¨¨1 “ I2 “ „1 0 0 1 □ , S¨¨2 “ „0 1 0 0 □ . Then, using the result of Exercise 2), conclude that X and S have the same rank. (ii) Hence, it suffices to show that rankpSq “ 3. Suppose, for a contradiction, that rankpSq “ 2. Using the properties of the PARAFAC decomposition, show that this imples the existence of matrices U, V, D1, D2 P C2ˆ2 such that D1, D2 are diagonal and S¨¨1 “ UD1VT, S¨¨2 “ UD2VT. (2) (iii) Now, use the fact that X¨¨1 “ I to show that (2) implies that S¨¨2 can be diagonalized by U, that is, there exists a diagonal matrix D P C2ˆ2 such that S “ UDU ́1. (iv) Conclude that this leads to a contradiction, by taking into account the Jordan form of S¨¨2.

Problem 4

n this last exercise, we will show that, although the tensors of the form considered in the last exercise have rank 3, they are limits of sequences of rank- 2 tensors. Thus, unlike happens for matrices, a sequence of rank-R tensors can converge to a rank-S tensor with S ą R. (i) First, show that the rank-1 tensor Ym “ mpa1 `m 1b2q ˝ pb2` m ́1b1q˝pc1 ` m ́1c2q is equal to X (as given by (1)) plus an Opmq term Zm and an Op1{mq term. (ii) Subtract the Opmq term to get: Xm “ Ym ´Zm. What is the rank of Xm? (iii) Use the expression obtained for Xm to conclude that lim mÑ8 Xm “ X .