TIP8419 - Tensor Algebra
Homework 11

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Alternating Least Squares (ALS) Algorithm

Problem 1 For the third-order tensor $\mathcal{X} \in \mathbb{C}^{I \times J \times K}$ provided in the file "cpd_tensor.mat", implement the plain-vanilla Alternating Least Squares (ALS) algorithm that estimates the factor matrices $\mathbf{A} \in \mathbb{C}^{I \times R}$, $\mathbf{B} \in \mathbb{C}^{J \times R}$ and $\mathbf{C} \in \mathbb{C}^{K \times R}$ by solving the following problem

$$(\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}) \in \underset{\mathbf{A},\mathbf{B},\mathbf{C}}{\arg\min} \left\| \mathcal{X} - \sum_{r=1}^{R} \mathbf{a}_r \circ \mathbf{b}_r \circ \mathbf{c}_r \right\|_F^2,$$

where $\mathbf{A} = [\mathbf{a}_1, \ldots, \mathbf{a}_R]$, $\mathbf{B} = [\mathbf{b}_1, \ldots, \mathbf{b}_R]$, $\mathbf{C} = [\mathbf{c}_1, \ldots, \mathbf{c}_R]$. Considering a successful run, compare the estimated matrices $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$ and $\hat{\mathbf{C}}$ with the original ones (also provided in the same Matlab file). Explain the results.

Hint: An error measure at the ith iteration can be calculated from the following formula:

$$e_{(i)} = \left\| [\mathcal{X}]_{(1)} - \hat{\mathbf{A}}_{(i)} (\hat{\mathbf{C}}_{(i)} \diamond \hat{\mathbf{B}}_{(i)})^T \right\|_F \tag{1}$$

The convergence at the ith iteration can be declared when $(e_{(i-1)} - e_{(i)})/(e_{(i-1)} + \epsilon) < \delta$, where δ is a prescribed threshold value (e.g. $\delta = 10^{-4}$) and ϵ is a small constant (e.g. $\epsilon = 10^{-6}$).

Problem 2 In a Monte Carlo experiment with $M = 1000$ realizations, generate a tensor $\mathcal{X}_0 = \text{CPD}(\mathbf{A}, \mathbf{B}, \mathbf{C})$, where $\mathbf{A} \in \mathbb{C}^{I \times R}$, $\mathbf{B} \in \mathbb{C}^{J \times R}$ and $\mathbf{C} \in \mathbb{C}^{K \times R}$ have unit norm columns with elements randomly drawn from a normal distribution, with $R = 3$. Let $\mathcal{X} = \mathcal{X}_0 + \alpha\mathcal{V}$ be a noisy version of $\mathcal{X}_0$, where $\alpha\mathcal{V}$ is an additive zero-mean white Gaussian noise term. The parameter α controls the power (variance) of the noise term, and is defined as a function of the signal-to-noise ratio (SNR), in dB, as follows:

$$\text{SNR}_{\text{dB}} = 10 \log_{10} \left(\frac{||\mathcal{X}_0||_F^2}{||\alpha\mathcal{V}||_F^2} \right). \tag{2}$$

Considering the SNR levels 0, 5, 10, 15, 20, 25 and 30 (in dB), find the estimates $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$ and $\hat{\mathbf{C}}$ obtained with the ALS algorithm for $(I, J, K) = (10, 4, 2)$.

Let us define the normalized mean square error (NMSE) measure

$$\mathrm{NMSE}(\mathbf{Q}) = \frac{1}{M} \sum_{m=1}^{M} \frac{\|\hat{\mathbf{Q}}(m) - \mathbf{Q}(m)\|_F^2}{\|\mathbf{Q}(m)\|_F^2}, \tag{3}$$

where $\mathbf{Q}(m)$ e $\hat{\mathbf{Q}}(m)$ denote the original data matrix and the reconstructed one at the mth Monte Carlo experiment, respectively. For each SNR value, plot NMSE($\mathbf{A}$), NMSE($\mathbf{B}$) and NMSE($\mathbf{C}$) as a function of the SNR. Discuss the obtained results.

Hint: Don't forget to take into account the inherent ambiguities of the CP decomposition.