Noninvasive Assessment of Atrial Fibrillation Complexity Using Tensor Decomposition Techniques

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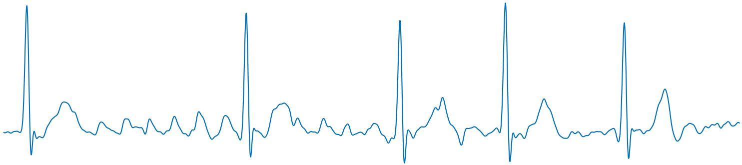
Feb 08th, 2022

Atrial Fibrillation

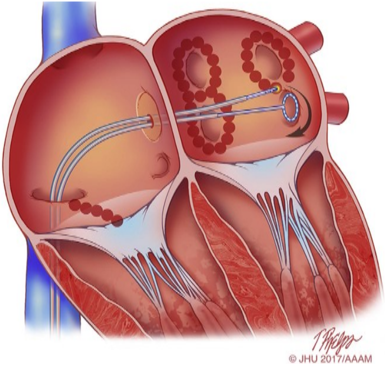
- Atrial Fibrillation (AF) is the most common sustained cardiac arrhythmia encountered in clinical practice.
  - In the EU, the number of adults with AF will double from 2010 to 2060[1].



- The complex electrophysiological mechanisms underlying AF are not completely understood.

[1] Krijthe *et al.*, "Projections on the number of individuals with atrial fibrillation in the European Union, from 2000 to 2060," *Eur Heart J*. 2013.

Step-wise Catheter Ablation (CA)



- Noninvasive techniques to assess AF electrophysiological complexity can help guide step-wise CA in real time.
  - Impact of pulmonary vein isolation (PVI) and other widely used techniques on atrial activity (AA) complexity.

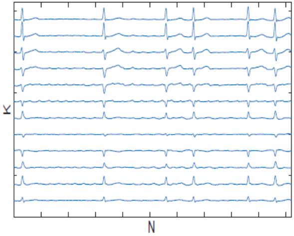
Figure from Tim Helps © 2017 Johns Hopkins University, AAM

Matrix Approach

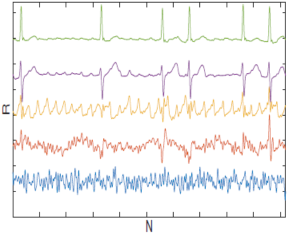
The ECG data matrix can be modeled as:

$$\mathbf{Y} = \mathbf{MS} \in \mathbb{R}^{K \times N}\ , \qquad (1)$$

where $\mathbf{M} \in \mathbb{R}^{K \times R}$ is a mixing matrix and $\mathbf{S} \in \mathbb{R}^{R \times N}$ is the source matrix.



$$= \begin{bmatrix} m_{1,1} & \cdots & m_{1,R} \\ \vdots & \ddots & \vdots \\ m_{K,1} & \cdots & m_{K,R} \end{bmatrix} \times$$



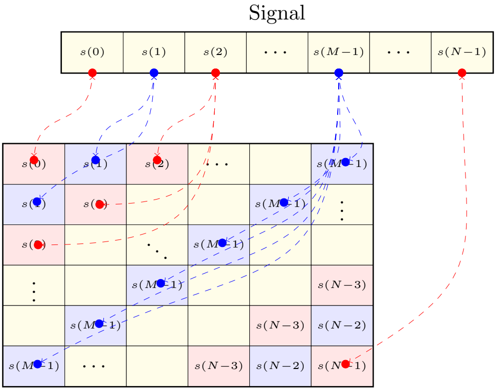
Low-rank Hankel Signal Model

Model Framework

- AA signal during AF can be represented by an all-pole model (2)
- The Hankel matrix has a rank equal to the number of poles (L)

$$s(n) = \sum_{\ell=1}^{L} c_\ell z_\ell^n \quad (2)$$

$$0 \leq n \leq N-1$$



Vandermonde Decomposition

Low-rank Hankel Structure

- The sequence $s(n)$ with N samples is mapped onto an $(M \times M)$ Hankel matrix denoted $\mathbf{H}_s$

- We assume N is odd without loss of generality

$$M = \frac{N+1}{2} \tag{3}$$

Decompose a Low-rank Hankel Structure

- The matrix $\mathbf{H}_s$ accepts Vandermonde decomposition (4)

$$\mathbf{H}_s = \mathbf{V}_s \, \mathbf{D} \, \mathbf{V}_s^{\intercal} \tag{4}$$

Model Framework

Vandermonde Decomposition

- The poles of $s(n)$ are linked to the columns of matrix $\mathbf{V}_s$ (5)

$$\mathbf{V}_s = \begin{bmatrix} 1 & 1 & \dots & 1 \\ z_1 & z_2 & \dots & z_L \\ \vdots & \vdots & & \vdots \\ z_1^{M-1} & z_2^{M-1} & \dots & z_L^{M-1} \end{bmatrix} \in \mathbb{C}^{M \times L}, \tag{5}$$

- As result of the Vandermonde decomposition, the Hankel matrix $\mathbf{H}_s$ has rank at most $\min\{L, M\}$[2]

[2] De Lathauwer, "Blind separation of exponential polynomials and the decomposition of a tensor in rank-(L_r,L_r,1) terms," *SIAM J. Matrix Anal. Appl.*, 2011.

Model Framework

Signal vs. Matrix Complexity

- The more poles a given signal is composed of, the higher the rank of its Hankel matrix
- This observation underlies the use of $rank(\mathbf{H}_s)$ as a measure of signal complexity, where the rank R is equal to L

- To present R equal to L, a Hankel matrix $\mathbf{H}_s$ with L poles needs to map at least N_{min} samples[3]:

$$N_{min} = 2L - 1 \tag{6}$$

[3] Boley, "A General Vandermonde Factorization of a Hankel Matrix," *Int. Lin. Alg. Soc.(ILAS) Symp. on Fast Algorithms for Control, Signals and Image Processing*, 1997.

Vandermonde Matrix

Poles Distance $(\Delta\omega)$

- When the distance between the poles decreases, the columns of $\mathbf{V}_s$ become closer to each other
- It impacts on the Hankel matrix rank

- To illustrate this behavior, we assume two poles $z_1 = e^{j\omega_1}$ and $z_2 = e^{j\omega_2}$, and the corresponding Vandermonde vectors:

$$\mathbf{v}_1 = [1, e^{\jmath\omega_1}, e^{\jmath 2\omega_1}, \ldots, e^{\jmath(M-1)\omega_1}]^\mathsf{T}$$
$$\mathbf{v}_2 = [1, e^{\jmath\omega_2}, e^{\jmath 2\omega_2}, \ldots, e^{\jmath(M-1)\omega_2}]^\mathsf{T}$$

Scalar Product

- Assuming exponential identities and $\Delta\omega = (\omega_2 - \omega_1)$

Vandermonde Matrix and Poles Distance

The scalar product becomes:

$$\cos(\theta) = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{||\mathbf{v}_1||\,||\mathbf{v}_2||} = \frac{\sin(M\frac{\Delta\omega}{2})}{M\sin(\frac{\Delta\omega}{2})} \quad (7)$$

When $\Delta\omega$ tends to zero for a fixed M, we have that:

$$\lim_{\Delta\omega \to 0} \cos(\theta) = 1$$

$\Delta\omega \to 0$

- The columns becomes colinear
- The equivalent Hankel matrix rank (R) is equal to one, built from two damped exponentials, resulting in $R \neq L$

Vandermonde Matrix and Poles Distance

However, we can deduce that increasing the value of M used to build $\mathbf{V}_s$ may compensate for the poles proximity

- Replacing M by its relation with N, we have that:

$$\lim_{N\to\infty} \cos(\theta) = \frac{2\sin\left((N+1)\Delta\omega\right)}{(N+1)\sin\left(\frac{\Delta\omega}{2}\right)}$$

$$\lim_{N\to\infty} \cos(\theta) = 0$$

$N \to \infty$

- The scalar-product gets closer to zero, in such a way that colinearity between $\mathbf{v}_1$ and $\mathbf{v}_2$ is reduced
- Resulting in $R = L$, built from two damped exponentials

Singular Value Decomposition

Singular Value Decomposition

- The Singular Value Decomposition (SVD) of $\mathbf{X} \in \mathbb{C}^{I \times J}$ is given by:

$$\mathbf{X} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^H \tag{8}$$

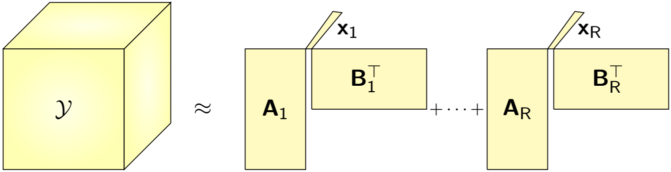
Condition Number

- A way to assess rank-deficiency is by the condition number (σ)
- It takes advantage of the relationship between singular values

$$\sigma = \frac{\lambda_{max}}{\lambda_{min}} \tag{9}$$

Tensor Approach

- The ECG data can be modeled as a 3rd-order tensor $\mathcal{Y}$ via row-Hankelization.
  - Tensor decompositions factorize data as a sum of simpler tensors.

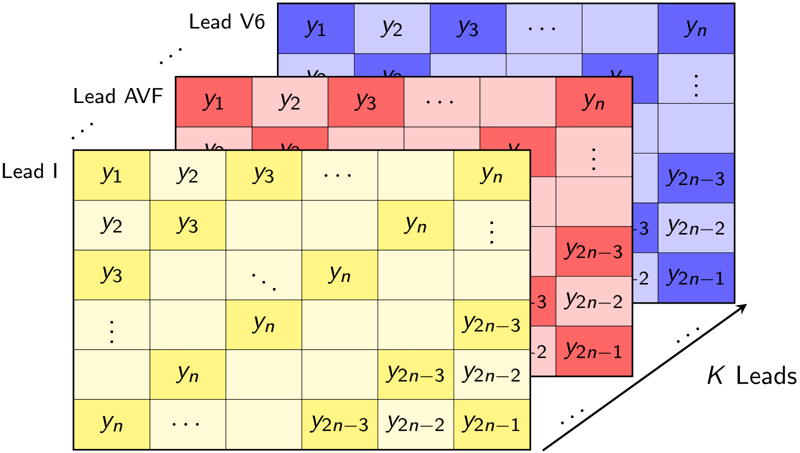


- Block Term Tensor Decomposition (BTD) based on Hankel structure[4].

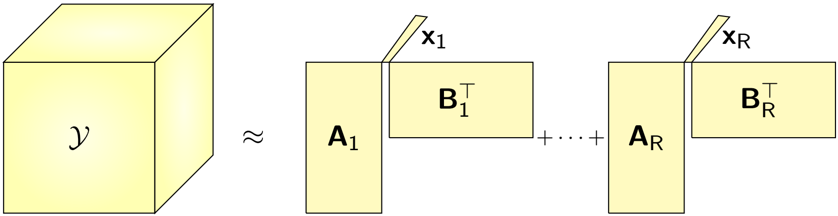
[4] De Lathauwer, "Blind separation of exponential polynomials and the decomposition of a tensor in rank-$(L_r, L_r, 1)$ terms," *SIAM J. Matrix Anal. Appl.*, 2011.

Tensor Approach

- Stack each Hankel matrix in the 3rd-mode of the tensor $\mathcal{Y}$.



BTD-Hankel Model



Challenge

- Parameter estimation
  - R, L
- Factor estimation
  - $\mathbf{A}$, $\mathbf{B}$, $\mathbf{X}$

Constrained Alternating Group Lasso

Classical BTD Approach

- Fixed structure minimizing $f(\mathbf{A}, \mathbf{B}, \mathbf{X})$ with prior knowledge of (R, L)

$$f(\mathbf{A}, \mathbf{B}, \mathbf{X}) \triangleq \left\| \mathcal{Y} - \sum_{r=1}^{R} \left(\mathbf{A}_r \mathbf{B}_r^\top \right) \circ \mathbf{x}_r \right\|_F^2 \tag{10}$$

Constrained Alternating Group Lasso (CAGL) Approach

- Non-fixed structure minimizing $F(\mathbf{A}, \mathbf{B}, \mathbf{X})$ ensuring the Hankel structure
- Penalization term (γ) and $g(\mathbf{A}, \mathbf{B}, \mathbf{X})$ limiting the multilinear ranks and number of blocks
- Allows simultaneous estimation of (R, L) and model factors

$$F(\mathbf{A}, \mathbf{B}, \mathbf{X}) \triangleq f(\mathbf{A}, \mathbf{B}, \mathbf{X}) + \gamma \, g(\mathbf{A}, \mathbf{B}, \mathbf{X}) \tag{11}$$

AF Complexity Index

Signal Complexity

The more poles the signal contains, the more complex it can be considered

- The complexity index proposed in this work is based on the number of poles L contained in a signal.
- The Hankel matrix rank is equal to number of poles L.

Condition Number

Experiment Framework

- We build a Vandermonde matrix from two vectors to illustrate how a small $\Delta\omega$ hamper the construction of a full-rank matrix
- How it impacts the condition number

- Assuming two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ as the columns of the Vandermonde matrix $\mathbf{V}_s$, with $\omega_1 = 0$, and $\omega_2 = \Delta\omega$, *i.e.*:

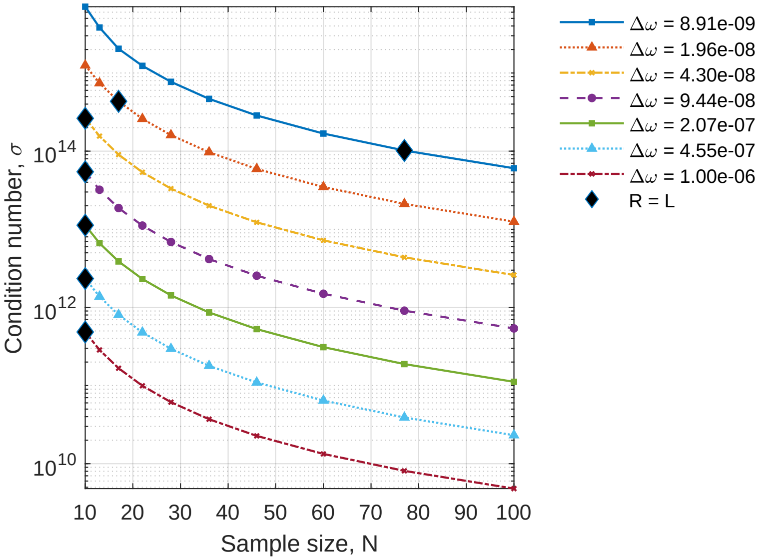
$$\mathbf{v}_1 = [1, 1, 1, \dots, 1]^\mathsf{T}$$

$$\mathbf{v}_2 = [1, e^{\jmath\Delta\omega}, e^{\jmath 2\Delta\omega}, \dots, e^{\jmath(M-1)\Delta\omega}]^\mathsf{T}$$

Numerical Assessment

- Compute the SVD$(\mathbf{H}_s)$ for 10 different values of N, rounding logarithmically spaced values in the interval $[10^1, 10^2]$, and considering 7 different values of $\Delta\omega$.

Condition Number



Condition Number

Numerical Results

- The distances between poles are so small that even increasing the number of samples, σ keeps very high
- This behavior affects rank computation and may prevent us to obtain $R = L$
- The value of σ decreases as we increase the number of samples to the columns, which compensates for the poles proximity as expected from our previous analysis

Sample Size for a Full-Rank Hankel Matrix

Experiment Framework

- We test various distances between poles to assess the minimum amount of samples necessary to obtain the Hankel matrix rank equal to the number of poles contained in the signal

We build $\mathbf{H}_s$ from 100 samples for each $s(n)$, where:

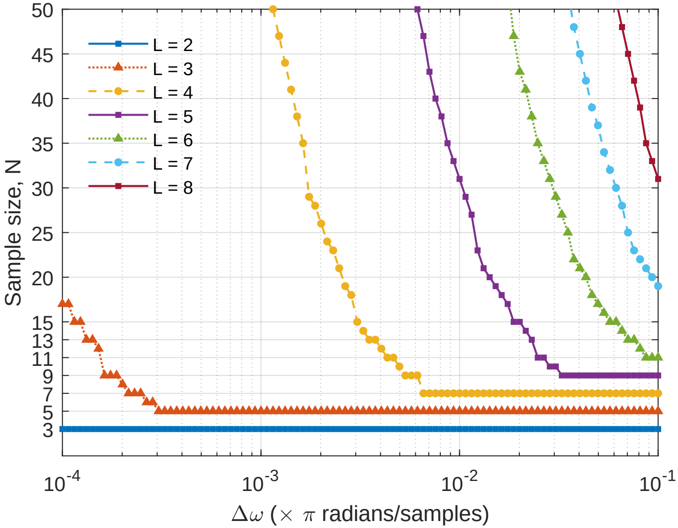
- L is in the [2,8] interval;
- z_l represent equispaced poles;
- $\Delta\omega$ takes 100 logarithmically spaced values in the $[10^{-4}, 10^{-1}]$ interval;
- We assume $c_l = 1$, $l = 1, 2, \ldots, L$ for simplicity, but without loss of generality.

Sample Size for a Full-Rank Hankel Matrix

Numerical Assessment

- To compare the numerical and theoretical results, we use a windowed version of $s(n)$ with $N \leq 100$ samples to build its equivalent Hankel structure
- The number of samples used in window N is a positive integer value between 2 and 100.
- Summarized as follows: for each value of L in the set of positive integers, we vary $\Delta\omega$ from very close poles gradually moving them away from each other at each iteration, noting the value of N for $R = L$.

Sample Size for a Full-Rank Hankel Matrix



Sample Size for a Full-Rank Hankel Matrix

Numerical Results

- The experiment shows that the theoretical values are different from those obtained empirically (One would expect plots with horizontal lines at $2L - 1$ regardless of $\Delta\omega$)
- When the distance gets smaller, the number of samples N necessary in the windowed signal to build $\mathbf{H}_s$ with $R = L$ increases
- This outcome reinforces the hypothesis that the increase of N may compensate for the distance between poles to obtain $R = L$ as anticipated

Noisy Scenario and SNR Threshold

Experiment Framework

- We reproduce the previous setup, but signals are analyzed with various SNR values with a fixed distance between poles.

Considering a scenario with additive white Gaussian noise (AWGN):

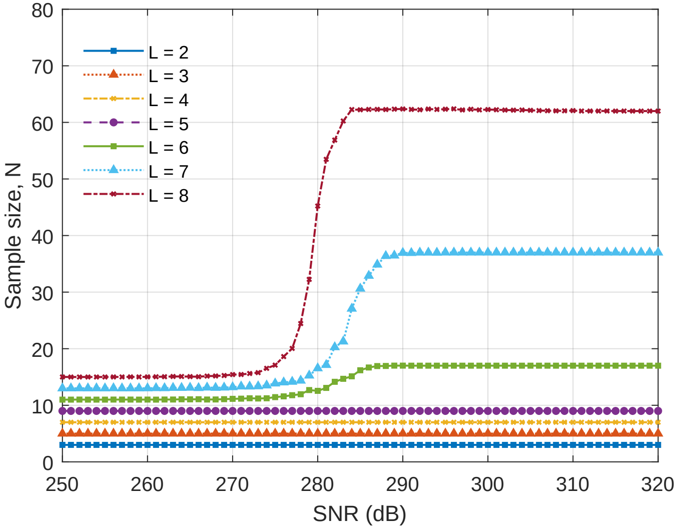
$$y(n) = s(n) + \beta(n)$$

- The mapped signal $y(n)$ is a version of $s(n)$ affected by the AWGN term $\beta(n)$, and with $\Delta\omega = 0.05$.

Numerical Assessment

- We perform 100 Monte Carlo runs, varying the noise for each realization.
- For each value of L in the set of positive integers [2,8], we vary SNR values along the [250,320] interval in 10 dB steps, at each iteration, noting the value of N for $R = L$.

Noisy Scenario and SNR Threshold



Noisy Scenario and SNR Threshold

Numerical Results

- The experiment shows that for values of SNR less than 270 dB, in general the most commonly encountered in real problems, the theoretical result is respected (Expected plots with horizontal lines at $2L - 1$)
- However, for values greater than 270 dB, the theoretical values are once again different from those obtained empirically, but N starts to increase.
- When the SNR gets greater, the number of samples N necessary in the windowed signal to build $\mathbf{H}_s$ with $R = L$ increases

Atrial Source Classification

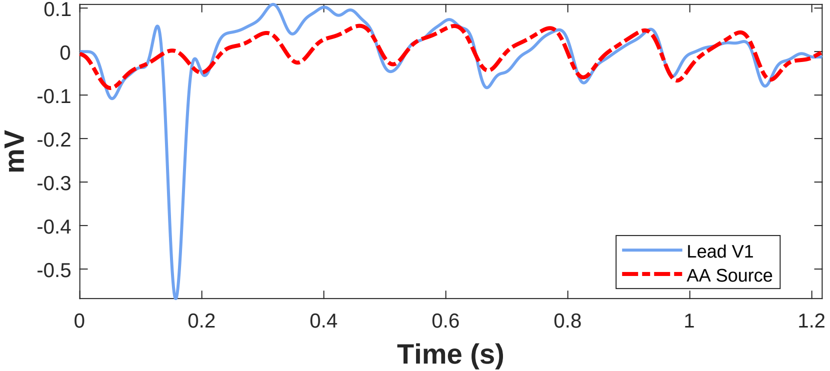
Challenge

After performing CAGL, the automated AA source classification is still a problem

- Spectral concentration (SC), dominant frequency (DF), kurtosis and visual inspection to evaluate AA extraction[5].

[5] De Oliveira and Zarzoso, "Source analysis and selection using block term decomposition in atrial fibrillation", in *Proc. LVA/ICA*, 2018.

AA Source Estimation



- SC = 74.3%
- DF = 6.4 Hz
- Kurtosis = 177.0
- AA Hankel Matrix Rank = 33

Database and Experimental Setup

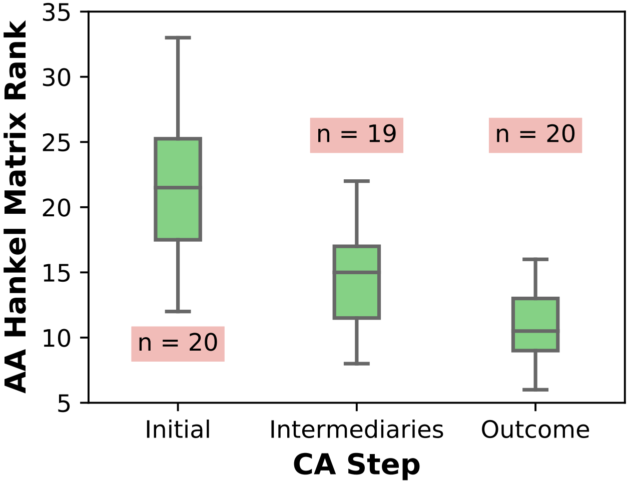
Database

- 20 patients suffering from persistent AF
- 59 ECG segments from 0.72 to 1.42 seconds

Cardiology Department of Princess Grace Hospital Center, Monaco

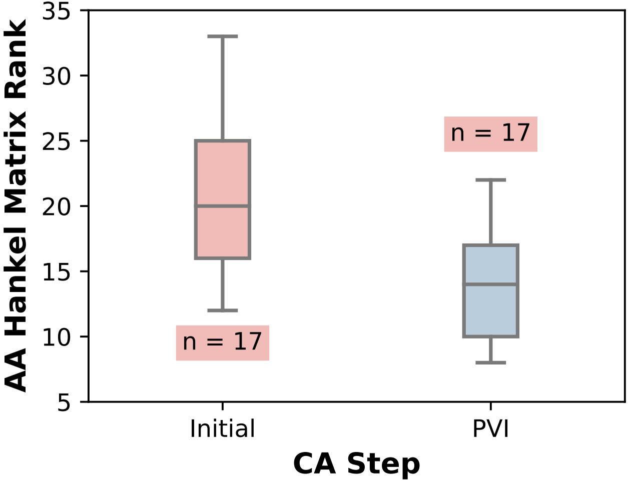
- Hankel-based BTD was implemented using CAGL.

Impact of CA step on AA complexity



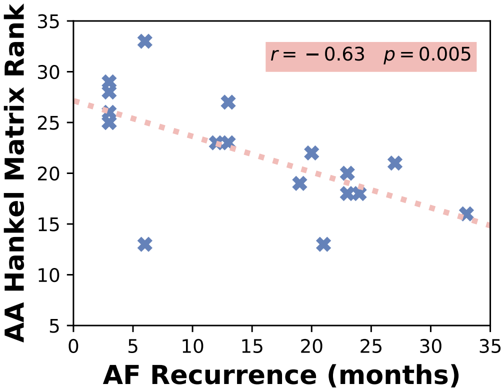
- 20 patients undergoing various CA steps
- 59 ECG segments (1.06 ± 0.2 s)

Impact of PVI on AA complexity



- 17 patients undergoing PVI
- 34 ECG segments

AF Recurrence vs. Complexity Before CA



Relationship

A significant Pearson correlation between AF recurrence and the proposed index

- 18 patients with complete follow-up information

Conclusions

- Numerical features of Hankel matrices associated with complex exponential signal models
- To assure a full rank matrix, a minimum distance between poles is required or the number of mapped samples must increase to compensate for poles proximity
- With a realistic noise level, the Hankel matrix was always full rank, since the noise acts to balance the linear independence between the columns in the associated Vandermonde matrix.
- Jointly extract the AA signal and measure AF complexity via tensor decomposition
- Very short ECG recordings (1.06 ± 0.20 s)
- Validation in 20 patients undergoing CA
  - Expected decreasing AF complexity throughout CA steps
  - Significant correlation with AF recurrence after CA

Further Work

Clinical Impact

- A potential tool to help guide CA in real time

- Performing the experiments for larger matrices, in order to provide more relevant statistical information.
- Explore the BTD context, since it performs the separation of the noise and the low-rank Hankel signal model in different blocks and it could fall within the noiseless scenario.
- Increase number of patients in the database
- Compare the proposed index with other state-of-the-art indices