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Low-Rank Hankel Signal Model: Numerical Results

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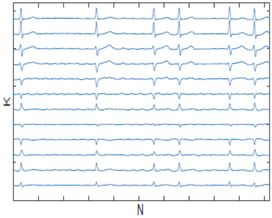
Blind Source Separation

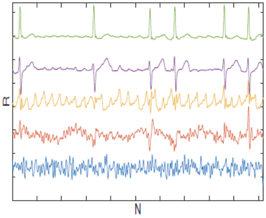
A data matrix can be modeled as:

$$\mathbf{Y} = \mathbf{MS} \in \mathbb{R}^{K \times N} , \tag{1}$$

where $\mathbf{M} \in \mathbb{R}^{K \times R}$ is a mixing matrix and $\mathbf{S} \in \mathbb{R}^{R \times N}$ is the source matrix.



$$= \begin{bmatrix} m_{1,1} & \cdots & m_{1,R} \\ \vdots & \ddots & \vdots \\ m_{K,1} & \cdots & m_{K,R} \end{bmatrix} \times$$



Matrix and Tensor Approach for BSS

Matrix

- Strong mathematical constraints are necessary to assure uniqueness of the decomposition, such as mutual orthogonality between spatial factors and statistical independence
- The best-known methods are based on principal and independent component analysis.

Tensor

- Uniqueness ensured under milder constraints.
- Outperforms matrix-based methods
- A celebrated method is the block term decomposition (BTD)

Low-rank Hankel Structure

BTD Approach for BSS

- Takes advantage of discrete-time signals that can be modeled as linear combinations of exponentials (all-pole models).
- Since the sources can be expressed as low-rank Hankel matrices, the signal separation can be performed via BTD.

Applications

- Speech analysis
- Black-box polynomial analysis
- Biomedical signal analysis (using BTD)

Numerical Experiments

Vandermonde Matrix

- Illustrate how the distance between poles hampers the construction of a full-rank matrix
- How it impacts the singular values and the Hankel matrix rank

Low-rank Hankel Matrices

- Multiple signals with various distances between poles vs. the amount of samples necessary to obtain the Hankel matrix rank equal to the number of poles contained in the signal

Low-rank Hankel Matrices vs. Noisy Signal

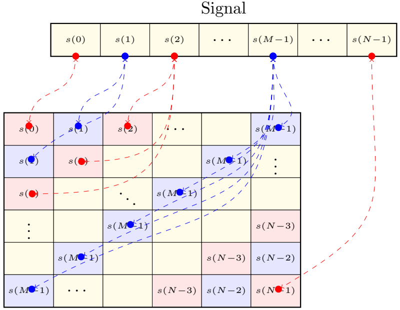
- Previous framework but with noisy signals with different signal-to-noise ratio (SNR).

Model Framework

Low-rank Hankel Structure

- The signal $s(n)$ is represented by an all-pole model (2)
- A Hankel matrix has a rank equal to the number of poles (L)

$$s(n) = \sum_{l=1}^{L} c_l z_l^n, \quad 0 \leq n \leq N-1 \tag{2}$$



Model Framework

Low-rank Hankel Structure

- The sequence $s(n)$ with N samples is mapped onto an $(M \times M)$ Hankel matrix denoted $\mathbf{H}_s$

- We assume N is odd without loss of generality

$$M = \frac{N+1}{2} \tag{3}$$

Decompose a Low-rank Hankel Structure

- The matrix $\mathbf{H}_s$ accepts Vandermonde decomposition (4)

$$\mathbf{H}_s = \mathbf{V}_s \, \mathbf{D} \, \mathbf{V}_s^{\intercal} \tag{4}$$

Model Framework

Vandermonde Decomposition

- The poles of $s(n)$ are linked to the columns of matrix $\mathbf{V}_s$ (5)

$$\mathbf{V}_s = \begin{bmatrix} 1 & 1 & \dots & 1 \\ z_1 & z_2 & \dots & z_L \\ \vdots & \vdots & & \vdots \\ z_1^{M-1} & z_2^{M-1} & \dots & z_L^{M-1} \end{bmatrix} \in \mathbb{C}^{M \times L}, \qquad (5)$$

- As result of the Vandermonde decomposition, the Hankel matrix $\mathbf{H}_s$ has rank at most $\min\{L, M\}$[1]

[1] De Lathauwer, "Blind separation of exponential polynomials and the decomposition of a tensor in rank-(L_r,L_r,1) terms," *SIAM J. Matrix Anal. Appl.*, 2011.

Model Framework

Signal vs. Matrix Complexity

- The more poles a given signal is composed of, the higher the rank of its Hankel matrix
- This observation underlies the use of $rank(\mathbf{H}_s)$ as a measure of signal complexity, where the rank R is equal to L

- To present R equal to L, a Hankel matrix $\mathbf{H}_s$ with L poles needs to map at least N_{min} samples[2]:

$$N_{min} = 2L - 1 \tag{6}$$

[2] Boley, "A General Vandermonde Factorization of a Hankel Matrix," *Int. Lin. Alg. Soc.(ILAS) Symp. on Fast Algorithms for Control, Signals and Image Processing*, 1997.

Vandermonde Matrix

Poles Distance $(\Delta\omega)$

- When the distance between the poles decreases, the columns of $\mathbf{V}_s$ become closer to each other
- It impacts on the Hankel matrix rank

- To illustrate this behavior, we assume two poles $z_1 = e^{j\omega_1}$ and $z_2 = e^{j\omega_2}$, and the corresponding Vandermonde vectors:

$$\mathbf{v}_1 = [1, e^{\jmath\omega_1}, e^{\jmath 2\omega_1}, \ldots, e^{\jmath(M-1)\omega_1}]^\mathsf{T}$$
$$\mathbf{v}_2 = [1, e^{\jmath\omega_2}, e^{\jmath 2\omega_2}, \ldots, e^{\jmath(M-1)\omega_2}]^\mathsf{T}$$

Scalar Product

- Assuming exponential identities and $\Delta\omega = (\omega_2 - \omega_1)$

Vandermonde Matrix and Poles Distance

The scalar product becomes:

$$\cos(\theta) = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{||\mathbf{v}_1||\,||\mathbf{v}_2||} = \frac{\sin(M\frac{\Delta\omega}{2})}{M\sin(\frac{\Delta\omega}{2})} \tag{7}$$

When $\Delta\omega$ tends to zero for a fixed M, we have that:

$$\lim_{\Delta\omega \to 0} \cos(\theta) = 1$$

$\Delta\omega \to 0$

- The columns becomes colinear
- The equivalent Hankel matrix rank (R) is equal to one, built from two damped exponentials, resulting in $R \neq L$

Vandermonde Matrix and Poles Distance

However, we can deduce that increasing the value of M used to build $\mathbf{V}_s$ may compensate for the poles proximity

- Replacing M by its relation with N, we have that:

$$\lim_{N\to\infty} \cos(\theta) = \frac{2\sin\left((N+1)\Delta\omega\right)}{(N+1)\sin\left(\frac{\Delta\omega}{2}\right)}$$

$$\lim_{N\to\infty} \cos(\theta) = 0$$

$N \to \infty$

- The scalar-product gets closer to zero, in such a way that colinearity between $\mathbf{v}_1$ and $\mathbf{v}_2$ is reduced
- Resulting in $R = L$, built from two damped exponentials

Singular Value Decomposition

Singular Value Decomposition

- The Singular Value Decomposition (SVD) of $\mathbf{X} \in \mathbb{C}^{I \times J}$ is given by:

$$\mathbf{X} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H \tag{8}$$

Condition Number

- A way to assess rank-deficiency is by the condition number (σ)
- It takes advantage of the relationship between singular values

$$\sigma = \frac{\lambda_{max}}{\lambda_{min}} \tag{9}$$

Condition Number

Experiment Framework

- We build a Vandermonde matrix from two vectors to illustrate how a small $\Delta\omega$ hamper the construction of a full-rank matrix
- How it impacts the condition number

- Assuming two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ as the columns of the Vandermonde matrix $\mathbf{V}_s$, with $\omega_1 = 0$, and $\omega_2 = \Delta\omega$, *i.e.*:

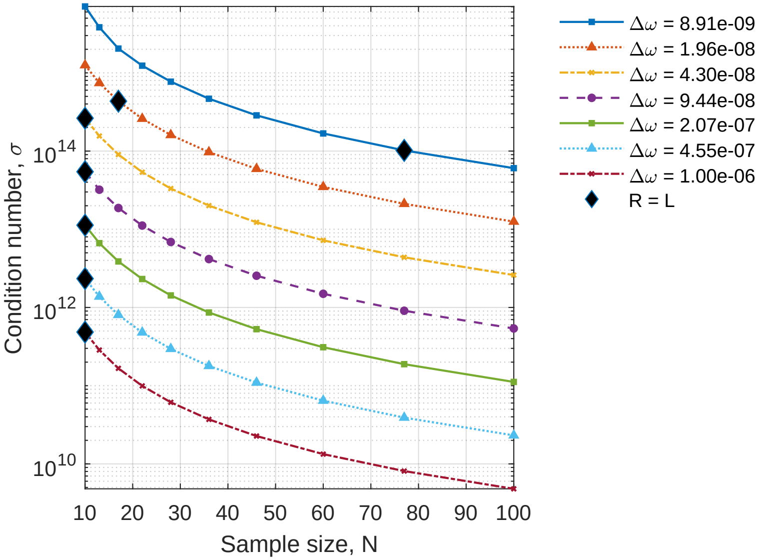
$$\mathbf{v}_1 = [1, 1, 1, \dots, 1]^\intercal$$

$$\mathbf{v}_2 = [1, e^{\jmath\Delta\omega}, e^{\jmath 2\Delta\omega}, \dots, e^{\jmath(M-1)\Delta\omega}]^\intercal$$

Numerical Assessment

- Compute the SVD$(\mathbf{H}_s)$ for 10 different values of N, rounding logarithmically spaced values in the interval $[10^1, 10^2]$, and considering 7 different values of $\Delta\omega$.

Condition Number



Condition Number

Numerical Results

- The distances between poles are so small that even increasing the number of samples, σ keeps very high
- This behavior affects rank computation and may prevent us to obtain $R = L$
- The value of σ decreases as we increase the number of samples to the columns, which compensates for the poles proximity as expected from our previous analysis

Sample Size for a Full-Rank Hankel Matrix

Experiment Framework

- We test various distances between poles to assess the minimum amount of samples necessary to obtain the Hankel matrix rank equal to the number of poles contained in the signal

We build $\mathbf{H}_s$ from 100 samples for each $s(n)$, where:

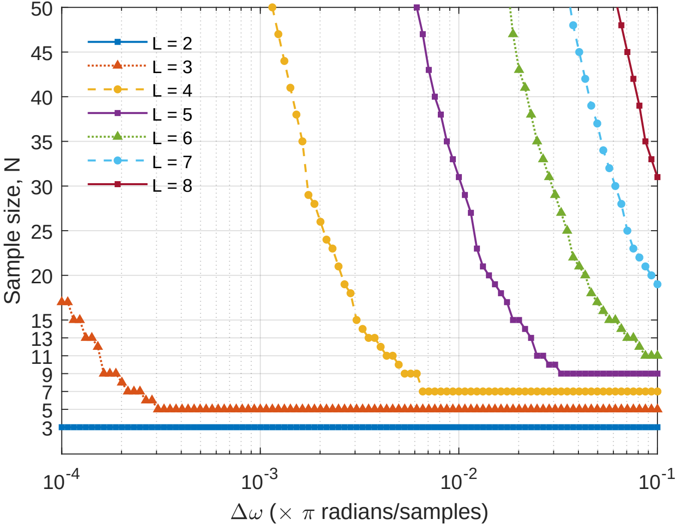
- L is in the [2,8] interval;
- z_l represent equispaced poles;
- $\Delta\omega$ takes 100 logarithmically spaced values in the $[10^{-4}, 10^{-1}]$ interval;
- We assume $c_l = 1$, $l = 1, 2, \ldots, L$ for simplicity, but without loss of generality.

Sample Size for a Full-Rank Hankel Matrix

Numerical Assessment

- To compare the numerical and theoretical results, we use a windowed version of $s(n)$ with $N \leq 100$ samples to build its equivalent Hankel structure
- The number of samples used in window N is a positive integer value between 2 and 100.
- Summarized as follows: for each value of L in the set of positive integers, we vary $\Delta\omega$ from very close poles gradually moving them away from each other at each iteration, noting the value of N for $R = L$.

Sample Size for a Full-Rank Hankel Matrix



Sample Size for a Full-Rank Hankel Matrix

Numerical Results

- The experiment shows that the theoretical values are different from those obtained empirically (One would expect plots with horizontal lines at $2L-1$ regardless of $\Delta\omega$)
- When the distance gets smaller, the number of samples N necessary in the windowed signal to build $\mathbf{H}_s$ with $R = L$ increases
- This outcome reinforces the hypothesis that the increase of N may compensate for the distance between poles to obtain $R = L$ as anticipated

Noisy Scenario and SNR Threshold

Experiment Framework

- We reproduce the previous setup, but signals are analyzed with various SNR values with a fixed distance between poles.

Considering a scenario with additive white Gaussian noise (AWGN):

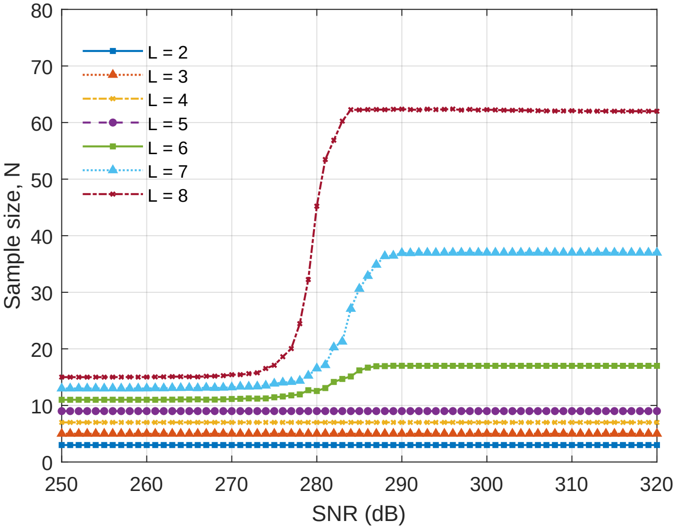
$$y(n) = s(n) + \beta(n)$$

- The mapped signal $y(n)$ is a version of $s(n)$ affected by the AWGN term $\beta(n)$, and with $\Delta\omega = 0.05$.

Numerical Assessment

- We perform 100 Monte Carlo runs, varying the noise for each realization.
- For each value of L in the set of positive integers [2,8], we vary SNR values along the [250,320] interval in 10 dB steps, at each iteration, noting the value of N for $R = L$.

Noisy Scenario and SNR Threshold



Noisy Scenario and SNR Threshold

Numerical Results

- The experiment shows that for values of SNR less than 270 dB, in general the most commonly encountered in real problems, the theoretical result is respected (Expected plots with horizontal lines at $2L - 1$)
- However, for values greater than 270 dB, the theoretical values are once again different from those obtained empirically, but N starts to increase.
- When the SNR gets greater, the number of samples N necessary in the windowed signal to build $\mathbf{H}_s$ with $R = L$ increases

Discussion

Noiseless Scenario

- As the distance between poles decreases, the number required to obtain a full rank matrix increases.
- When the poles get too close to each other, a dependency between the rank, number of poles, amount of samples, and the distance between poles is observed.

Noisy Scenario

- For SNR values commonly encountered in practice, the distance between poles becomes irrelevant.
- The theoretical minimum value is observed, *i.e.*, the rank depends only on the number of poles.

Conclusion

Contributions

- Numerical features of Hankel matrices associated with complex exponential signal models
- To assure a full rank matrix, a minimum distance between poles is required or the number of mapped samples must increase to compensate for poles proximity
- With a realistic noise level, the Hankel matrix was always full rank, since the noise acts to balance the linear independence between the columns in the associated Vandermonde matrix.

Further Work

- Performing the experiments for larger matrices, in order to provide more relevant statistical information.
- Explore the BTD context, since it performs the separation of the noise and the low-rank Hankel signal model in different blocks and it could fall within the noiseless scenario.