Math IA

Mathematics Analysis and Approaches HL

Investigate the relationship between the thickness of an airfoil and the lift it can generate?

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Lucas Salhani

Introduction:

As I learned math in class I always tried to refer to real life applications of things but as we got school went on and the math became more complex it was harder to see real life examples than for example calculating a percentage which we use every day. However the more I get to know about physics the more math is used and it has always interested me to see how physics problems can be resolved using mathematics.

During this investigation I chose to focus on aerodynamics because I want to later on study engineering and thought that it would be a good idea to find out more on these topics and how math is used in those specific areas. More specifically I choose to study the relationship between the thickness because it is a major factor as it will affect the surface area and it will affect the weight the wing can support. In addition this is one of the less regarded factors but it is still important as it directly affects the price of the plane's wing which every company will want to minimize in comparison to looking at the angle of attack of a plane when taking off.

The mathematics that I will be using will first be making a function to match the data modeling a wing for a wing secondly looking at the area under a curve to find the area of the platform of the wing (the bottom side), thirdly I used integrals to find distance between points on a curve and finally as the result where nonlinear I used natural logarithms to make the result non linear.

Theory:

In order to find the lift generated we need to look at the difference in velocity on the top of the wing and on the bottom. This difference in velocity will then generate a lift upward. This lift is generated due to the bernoulli effect that states that under a high velocity there will be a lower pressure and as something has a lower velocity it will create a higher pressure and it is this difference in pressure that causes the lift. Now due to the Bernoulli effect as the air on the upper side of the wing has a longer distance to follow it will then have a higher velocity and therefore create a lower pressure than the lower side which will have a higher pressure creating a lift force.

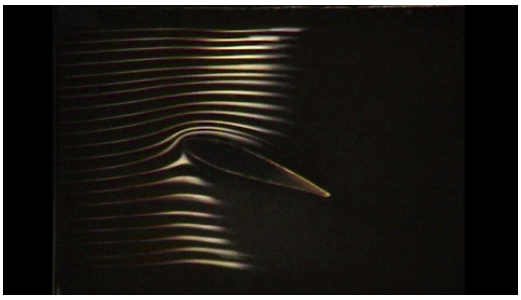
Figure 1. Babinsky Cambridge, Airflow across a wing, 2012)

In figure 1 we can see an experiment conducted by the University of Cambridge and from this experiment we can see that the difference in speed of the air above and under the wing but on the opposite of the equal transit theory which stated that the two air path met at the end of the wing this shows they do not however this theory can work to a certain extend but for this simple calculation it works and will make the physics simpler to focus on the math. This is why the formula I will used to determine the lift is as follow:

$$\Delta P = \frac{1}{2} \times \rho \times \Delta v^2 \ or \ \Delta P = \frac{1}{2} \times \rho \times ({v_1}^2 - {v_2}^2)$$

This equation comes from the Bernoulli theorem. In this equation P = pressure, ρ is the fluid density, and v is the velocity. If we combine it with $pressure = \frac{Force}{area}$ equation we can get the lift generated using this formula: $Lift\,force\,(F) = difference\ in\ pressure\,(\Delta P) \times area\,(A)$ we can then get the final equation of lift that I will use to find the lift:

$$F = \frac{1}{2} \times \rho \times ({v_1}^2 - {v_2}^2) \times \mathrm{A}$$

F = force, $\rho = fluid\ density$, v = velocity and a = area

As we can see from this equation there is no lift coefficient as it will remain a constant as our variable is the height of the wing and it will make the physics simpler in order to focus on the math. Another constant will be the fluid density which is a value of $1.22Kg.m^{-3}$. Then there will be a higher velocity on the upper side if I can calculate the velocity there and compare it to the velocity on the lower side knowing the areas I will then be able to calculate the lift. However in order to better calculate the lift I chose to calculate the lift in different intervals and then add them up in order to have a more precise value. In order to find the velocity on the upper side I will use a ratio of the distance above and under the wing as the same of the speed ratio taking into account the **equal transit** theory saying they are equal to each other. Then by knowing the speed of the wing and the distance between intervals I will then be able to find the velocity on the upper side of the airfoil.

Finding function for wing profile

To find the function for the wing profile I will need to plot points in the shape of the wing then take a certain number of points in order to find the function. The function of this curve is important as it will later be used to find the distance between two points on that curve and also to find the maximum so that we can take the right values in order to stretch the function. I knew that the function would be in the form of $ax^3 + bx^2 + cx + d$ and d = 0 so I could use a simultaneous equation to find the function, this is why I picked four points on that curve in order to find the 4 constants. However we can already deduce for coordinate (0, 0) as if

$f(0) = 0 \Rightarrow a \times 0 + b \times 0 + c \times 0 + d = 0$ *therefore we can say* $d = 0$ and because of this we can use a simultaneous equation using only 3 equations to find 3 unknown.

The function f(x) passed points:

(0, 0) , (0.2, 0.157) , (0.657, 0.1204) , (1, 0)

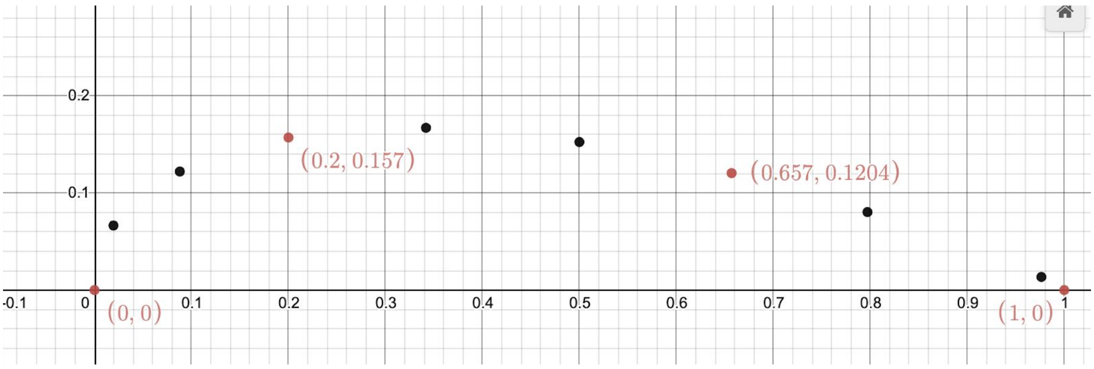

Figure 2

If we consider this and use a simultaneous equation we can find a function. If we put it in a form of $ax^3 + bx^2 + cx + d$ using simultaneous equations we can find the 3 unknown as d will be equal to 0 as said before. First we can put the coordinate as in the form of an equation with the x and y value substituted keeping the constant a, b and c.

$a \times 0 + b \times 0 + c \times 0 = 0$
$r_1 = a \times 0.2^3 + b \times 0.2 + 0.2c + 0 = 0.156$
$r_2 = a \times 0.657^3 + b \times 0.657^2 + 0.657c + 0 = 0.1204$
$r_3 = a \times 1^3 + b \times 1^2 + 1c + 0 = 0$

We can then use technology to solve and get $a = 1.0199$, $b =- 2.202$, $c = 1.18$ and $d = 0$

→ Thus: $f(x) = 1.0199x^3 - 2.202x^2 + 1.18x + 0$

After finding the function mapping the profile airfoil image I plotted other points at intervals of 0.1 in order to make things regular. In addition I chose 10 intervals to make the intervals more precise to give a better value of the lift.

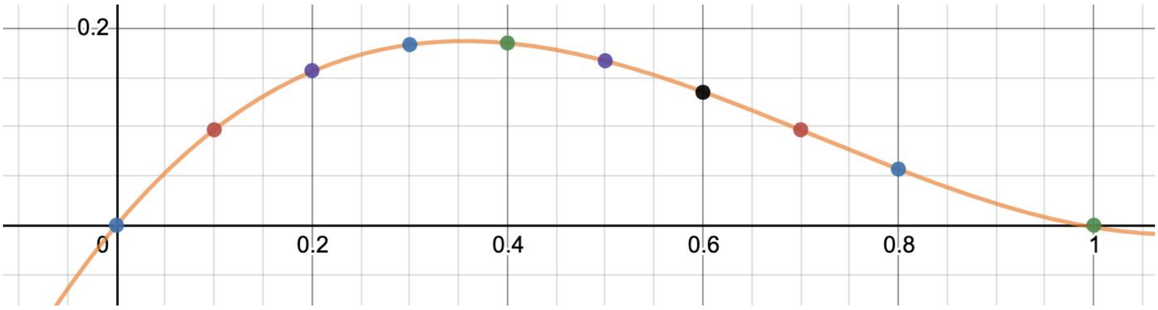

Figure 3

The table below is showing each point at each interval

(0, 0)	(0.5, 0.167)
(0.1, 0.097)	(0.6, 0.135)
(0.2, 0.157)	(0.7, 0.097)
(0.3, 0.183)	(0.8, 0.057)
(0.4, 0.185)	(1, 0)

Finding values for maximum thickness:

I will now need to find how I can stretch the function to find a different maximum height so that I can later use the same function in order to find the distance between points. In addition changing the height will change the surface area which will be needed to calculate and by multiplying by a factor makes the work simpler. To find appropriate values for the maximum thickness I will vertically stretch the function by a number h for the height. This will create a new function which can be represented by $F(x)$ *and will equal* $F(x) = h(f(x))$. Using this we can then find the maximum thickness by doing

$Max\ thickness = h \times (y\ value = putting\ x\ value\ back\ in\ original\ function)$. Now to find this value I will need to differentiate the function to find the turning points see if where is the maximum using the second derivative I will then find if the point is a maximum or minimum or a point of inflection. I will then be able to substitute the value of the maximum point into the original function and get the value of the multiplier.

$$f(x) = 1.02x^3 - 2.2x^2 + 1.18x$$
$$f'(x) = 3.06x^2 - 4.04x + 1.18$$
$$3.06x^2 - 4.04x + 1.18 = 0$$
$$= \frac{4.04 \pm \sqrt{-4.04^2 - 4(3.06)(1.18)}}{2(3.06)}$$
$$= \frac{4.04 + \sqrt{1.8784}}{6.12} \qquad = \frac{4.04 - \sqrt{1.8784}}{6.12}$$
$$x = 0.88 \qquad x = 0.43$$

$$f''(x) = 6.12x - 4.04$$

$f''(0.88) = 1.35$
$f''(0.43) = -1.4$
As f''(0.43) < 0 then *f(x)* has a local maximum at $x = 0.43$ and by substituting the value of x in the original function we find the value of y at $y = 0.18$ making the coordinates of the local maximum at (0.43, 0.18).

I choose to use a multiplier in order to not have to calculate the maximum each time and keep the work concise. By vertically stretching this function by the value of y = 0.18 and I will then obtain different values for the maximum thickness.

Thickness (h)	Multiplier	Maximum thickness in m
0.56	0.18	0.1
0.67	0.18	0.12
0.78	0.18	0.14
0.89	0.18	0.16
1.00	0.18	0.18
1.11	0.18	0.2
1.22	0.18	0.22
1.33	0.18	0.24
1.44	0.18	0.26
1.56	0.18	0.28
1.67	0.18	0.3

Figure 4

When choosing these values the problem was to make it real and to not deviate from the shape of a real wing this is why I choose to keep the value in the range of ≈ 1 in order to not have a shape too different from the wing airfoil shape.

Distance between two points on the curve

In order to find the distance I used an equation to be able to compute the arc length of a function which uses the equation to find points on a line and the mean value theorem to get to this equation. I read about this method on (Paul's Online Notes)

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$\frac{dF(x)}{dx} = h(\,3.06x^2 - 4.04x + 1.18)$$

As we know $(h(3.06x^2 - 4.04x + 1.18))^2 = h^2 \times (3.06x^2 - 4.04x + 1.18)^2$

$$\left(\frac{dF(x)}{dx}\right)^2 = h^2(9.3636x^4 - 12.3624x^3 + 3.6108x^2 - 12.3624x^3 + 16.32x^2 - 4.7672 + 3.6108x^2 - 4.7672x + 1.3924$$

$$\left(\frac{dF(x)}{dx}\right)^2 = h^2(9.3636x^4 - 24.7248x^3 + 23.5416x^2 - 9.5344x + 1.3924$$

$$L = \int_a^b \sqrt{1 + h^2(9.3636x^4 - 24.7248x^3 + 23.5416x^2 - 9.5344x + 1.3924)}$$

Using this final equation I used my GDC to calculate the data for each interval and values of h.

Values of h	The distance between two points on the furve F(x) in metres on the inerval bellow as an element of x									
	[0.0, 0.1]	[0.1, 0.2]	[0.2, 0.3]	[0.3, 0.4]	[0.4, 0.5]	[0.5, 0.6]	[0.6, 0.7]	[0.7, 0.8]	[0.8, 0.9]	[0.9, 1.0]
0.56	0.1144	0.1064	0.1021	0.1004	0.10000	0.1002	0.1003	0.1002	0.1000	0.1002
0.67	0.1264	0.1091	0.1030	0.1005	0.10003	0.1003	0.1005	0.1003	0.1000	0.1003
0.78	0.1264	0.1121	0.1041	0.1007	0.10004	0.1004	0.1007	0.1005	0.1001	0.1004
0.89	0.1333	0.1155	0.1053	0.1009	0.10005	0.1005	0.1009	0.1006	0.1001	0.1005
1	0.1408	0.1192	0.1066	0.1012	0.10006	0.1006	0.1011	0.1008	0.1001	0.1006
1.11	0.1496	0.1233	0.1081	0.1014	0.10008	0.1008	0.1013	0.1009	0.1001	0.1008
1.22	0.1569	0.1276	0.1097	0.1017	0.10009	0.1009	0.1016	0.1011	0.1001	0.1009
1.33	0.1654	0.1321	0.1114	0.1020	0.10010	0.1011	0.1019	0.1014	0.1001	0.1011
1.44	0.1741	0.1368	0.1133	0.1024	0.10013	0.1013	0.1023	0.1016	0.1002	0.1013
1.56	0.1840	0.1422	0.1154	0.1028	0.10016	0.1016	0.1026	0.1018	0.1002	0.1015
1.67	0.1932	0.1473	0.1174	0.1032	0.10018	0.1018	0.1030	0.1021	0.1002	0.1017

Figure 5

<u>Finding velocity above the wing:</u>

As the wing on the upper side will travel a greater distance it means it will travel faster and the ratio between the distance above and under the wing is equal to the velocity ratio above and under the wing. This takes into account the **equal transit** theory in order to make this ratio but considering this ratio we can then find the formula:

$$velocity\ above\ the\ wing\ =\ velocity\ under\ the\ wing\ (m/s)\ \times \frac{Distance\ above\ the\ wing\ in\ m}{distance\ under\ the\ wing\ in\ m}$$

The average speed of the Piper plane I choose has an average speed of 200 km/h according to (Piper PA-38 Tomahawk) and to find my calculation I will convert it into m/s and that makes 55m/s. Below I will show an example for the formula used which I put into an excel spreadsheet.

$$velocity\ above\ the\ wing = \ 55 \times \frac{0.1144}{0.1}$$

$$velocity\ above\ the\ wing\ \ = 62.92\ m/s$$

The intervals are always 0.1 and are the distance under the wing and for the distance over I used the data in the previous table.

Values of h				Velocity above the wing in m/s on each inteval						
	[0.0, 0.1]	[0.1, 0.2]	[0.2, 0.3]	[0.3, 0.4]	[0.4, 0.5]	[0.5, 0.6]	[0.6, 0.7]	[0.7, 0.8]	[0.8, 0.9]	[0.9, 1.0]
0.56	62.92	58.52	56.155	55.22	55	55.11	55.165	55.11	55	55.11
0.67	69.52	60.005	56.65	55.275	55.0165	55.1595	55.275	55.165	55	55.165
0.78	69.52	61.655	57.255	55.385	55.022	55.22	55.385	55.275	55.0275	55.22
0.89	73.315	63.525	57.915	55.495	55.0275	55.275	55.495	55.33	55.033	55.275
1	77.44	65.56	58.63	55.66	55.033	55.33	55.605	55.44	55.033	55.33
1.11	82.28	67.815	59.455	55.77	55.044	55.44	55.715	55.495	55.033	55.44
1.22	86.295	70.18	60.335	55.935	55.0495	55.495	55.88	55.605	55.033	55.495
1.33	90.97	72.655	61.27	56.1	55.055	55.605	56.045	55.77	55.033	55.605
1.44	95.755	75.24	62.315	56.32	55.0715	55.715	56.265	55.88	55.11	55.715
1.56	101.2	78.21	63.47	56.54	55.088	55.88	56.43	55.99	55.11	55.825
1.67	106.26	81.015	64.57	56.76	55.099	55.99	56.65	56.155	55.11	55.935

Figure 6

Platform area

figure 7 (Piper PA-38-112 G-BPPF, wikipedia commons, 2008.)

When drawing my profile wing I tried to focus on one type of plane. As shown in figure 7 I choose a Piper PA-38-112, the reason I choose this plane is that the profile wing shape is an airfoil wing shape with a flat bottom and in addition it has straight leadings and trailing edges meaning the wings are straight and there is a curved end. As a result of this I will be able to calculate the area of this wing using integration by choosing different intervals. The interval I choose are:

$x \in [0.0,\ 0.1],\ x \in [0.1,\ 0.2],\ x \in [0.2,\ 0.3],\ x \in [0.3,\ 0.4],\ x \in [0.4,\ 0.5],\ x \in [0.5,\ 0.6],\ x \in [0.6,\ 0.7],\ x \in$

$x \in [0.8,\ 0.9],\ x \in [0.9,\ 1.0]$

The chosen interval matches the interval on the profile wing as this is the view from the top of the plane where we can imagine figure 2 as a cut of the side of the wing. This is why they need to be the same as the chord on the x axis to match the chord on the x axis of the other image using the same intervals.

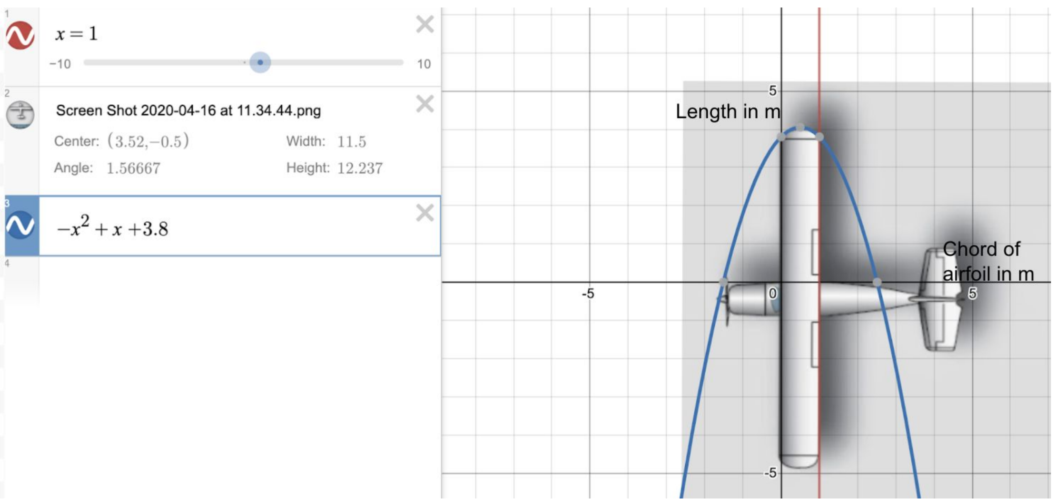

Figure 8

In figure 4 we can see the function that was drawn over an image representing the straight leadings and trailing edges which are the same as on the plane on figure 3. I was then able to determine this quadratic function knowing maximum value at x = 0.5. To find this function I plotted points and knowing the maximum point I was then able to make a good approximation of the function of the tip of the wing on this image.

For $x \in [0.0, 0.1]$ $A_1 = \int_{0}^{0.1} (-x^2 + x + 3.8)dx$

$= \left[\frac{-x^3}{3} + \frac{x^2}{2} + 3.8x\right]_{0}^{0.1}$

$= \left(\frac{-(0.1)^3}{3} + \frac{(0.1)^2}{2} + 3.8(0.1)\right) - \left(\frac{-0^3}{3} + \frac{0^2}{2} + 3.8 \times 0\right)$

$= 0.385\ m^2$

For $x \in [0.1, 0.2]$ $A_2 = \int_{0.1}^{0.2} (-x^2 + x + 3.8)dx = 0.393\ m^2$

For $x \in [0.2, 0.3]$ $A_3 = \int_{0.2}^{0.3} (-x^2 + x + 3.8)dx = 0.399\ m^2$

For $x \in [0.3, 0.4]$ $A_4 = \int_{0.3}^{0.4} (-x^2 + x + 3.8)dx = 0.403\ m^2$

For $x \in [0.4, 0.5]$ $A_5 = \int_{0.5}^{0.4} (-x^2 + x + 3.8)dx = 0.405m^2$

As I said the value of the maximum is at $x = 0.5$ meaning the maximum point lies on the axis of symmetry and therefore the values will be the same so I did not need to calculate them again.

For $x \in [0.5, 0.6]$ $A_6 = \int_{0.5}^{0.6} (-x^2 + x + 3.8)dx = 0.405m^2$

For $x \in [0.6, 0.7]$ $A_7 = \int_{0.6}^{0.7} (-x^2 + x + 3.8)dx = 0.403m^2$

For $x \in [0.7, 0.8]$ $A_8 = \int_{0.7}^{0.8} (-x^2 + x + 3.8)dx = 0.399m^2$

For $x \in [0.8, 0.9]$ $A_9 = \int_{0.8}^{0.9} (-x^2 + x + 3.8)dx = 0.393m^2$

For $x \in [0.9, 1.0]$ $A_{10} = \int_{0.9}^{1.0} (-x^2 + x + 3.8)dx = 0.385m^2$

I then put the calculated result in a table to make the work more neat when I come back to this data.

$x \in [0.0, 0.1]$	$0.385m^2$	$x \in [0.5, 0.6]$	$0.405m^2$
$x \in [0.1, 0.2]$	$0.393m^2$	$x \in [0.6, 0.7]$	$0.403m^2$
$x \in [0.2, 0.3]$	$0.399m^2$	$x \in [0.6, 0.8]$	$0.399m^2$
$x \in [0.3, 0.4]$	$0.403m^2$	$x \in [0.8, 0.9]$	$0.393m^2$
$x \in [0.4, 0.5]$	$0.405m^2$	$x \in [0.9, 1.0]$	$0.385m^2$

Finding the lift force generated by one wing

In order to find the lift I will be using the formula mentioned in the theory part $F = \frac{1}{2} \times \rho \times (v_1^2 - v_2^2) \times A$

This formula will show the lift force in newton however it will only give the value of the lift force for one wing and would need to be doubled in order to find the total lift for the whole plane. As I am not finding the force of the plane but on the airfoil I did not need to multiply by 2 for both wings but these results are used to find the lift force of a wing not a plane. The values for area are the one calculated after figure 8

$$F = \frac{1}{2} \times \rho \times (v_1^2 - v_2^2) \times A$$

- The speed under the wing (v_2) will be = to 55 m/s
- $\rho = fluid\ density\ (air) = 1.22kg.m^{-3}$
- Area in m^2
- Force in newton

value of h					Lift force in Newtons					
	[0.0, 0.1]	[0.1, 0.2]	[0.2, 0.3]	[0.3, 0.4]	[0.4, 0.5]	[0.5, 0.6]	[0.6, 0.7]	[0.7, 0.8]	[0.8, 0.9]	[0.9, 1.0]
0.56	219.333	95.794	31.247	5.961	0.000	2.992	4.469	2.948	0.000	2.845
0.67	424.616	137.989	44.838	7.455	0.448	4.341	7.455	4.424	0.000	4.269
0.78	424.616	186.112	61.611	10.447	0.598	5.991	10.447	7.381	0.725	5.695
0.89	551.918	242.229	80.111	13.446	0.748	7.492	13.446	8.862	0.870	7.122
1	697.963	305.204	100.393	17.954	0.897	8.995	16.450	11.827	0.870	8.551
1.11	879.513	377.305	124.104	20.968	1.196	12.005	19.460	13.312	0.870	11.412
1.22	1038.466	455.543	149.761	25.499	1.346	13.512	23.987	16.287	0.870	12.845
1.33	1233.090	540.291	177.434	30.043	1.495	16.532	28.527	20.759	0.870	15.715
1.44	1442.923	631.942	208.867	36.123	1.944	19.557	34.601	23.749	2.904	18.591
1.56	1694.781	741.198	244.227	42.227	2.393	24.106	39.172	26.744	2.904	21.472
1.67	1941.314	848.268	278.508	48.354	2.693	27.146	45.287	31.247	2.904	24.360

Figure 9

Then I added the values on all the intervals in order to get the total lift force generated for the different values of h. To make it more clear I also added back the values for the maximum thickness as that is what I will be graphing later on.

Maximum thickness	value of h	total lift force
0.1	0.56	365.59
0.12	0.67	635.83
0.14	0.78	713.62
0.16	0.89	926.24
0.18	1	1169.10
0.2	1.11	1460.15
0.22	1.22	1738.12
0.24	1.33	2064.76
0.26	1.44	2421.20
0.28	1.56	2839.22
0.3	1.67	3250.08

Figure 10

Here we can see that the results are a good approximation as an average of 1598.53 newton multiplied by 2 could give a good value for the lift of a plane weighing on average 512Kg empty.

Then I plotted the maximum thickness against the total lift force and the graph is shown below

Maximum thickness	total lift force
0.1	365.59
0.12	635.83
0.14	713.62
0.16	926.24
0.18	1169.10
0.2	1460.15
0.22	1738.12
0.24	2064.76
0.26	2421.20
0.28	2839.22
0.3	3250.08

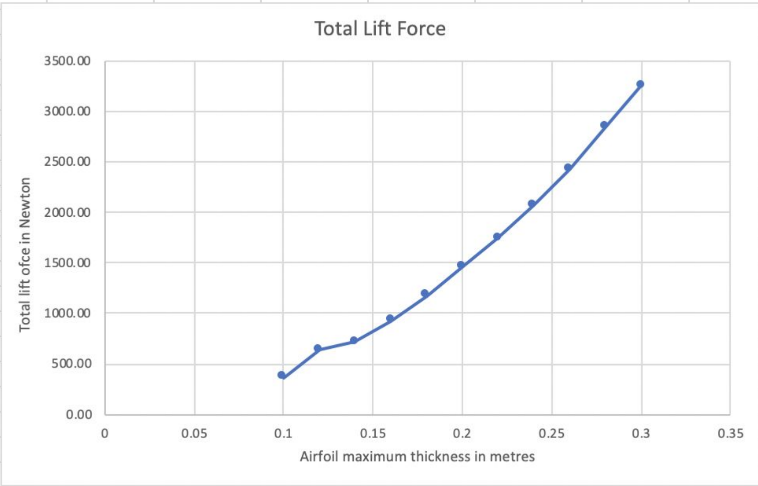

Figure 11

The graph shown has an exponential shape graph from which we can not deduce a mathematical relationship.

This is why I needed to convert the data to natural logarithm so I took the ln(x) value of each of the total lifts.

Maximum thickness in m	total lift force in N	Natural logarithm of total lift force in N
0.1	365.59	5.90
0.12	635.83	6.45
0.14	713.62	6.57
0.16	926.24	6.83
0.18	1169.10	7.06
0.2	1460.15	7.29
0.22	1738.12	7.46
0.24	2064.76	7.63
0.26	2421.20	7.79
0.28	2839.22	7.95
0.3	3250.08	8.09

Figure 12

I then used linear regression to find an equation for the line of best fit for the values

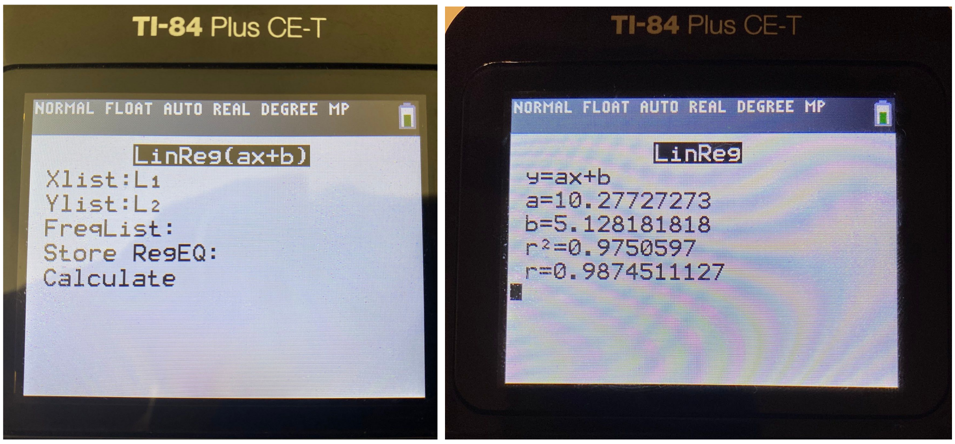

Figure 13

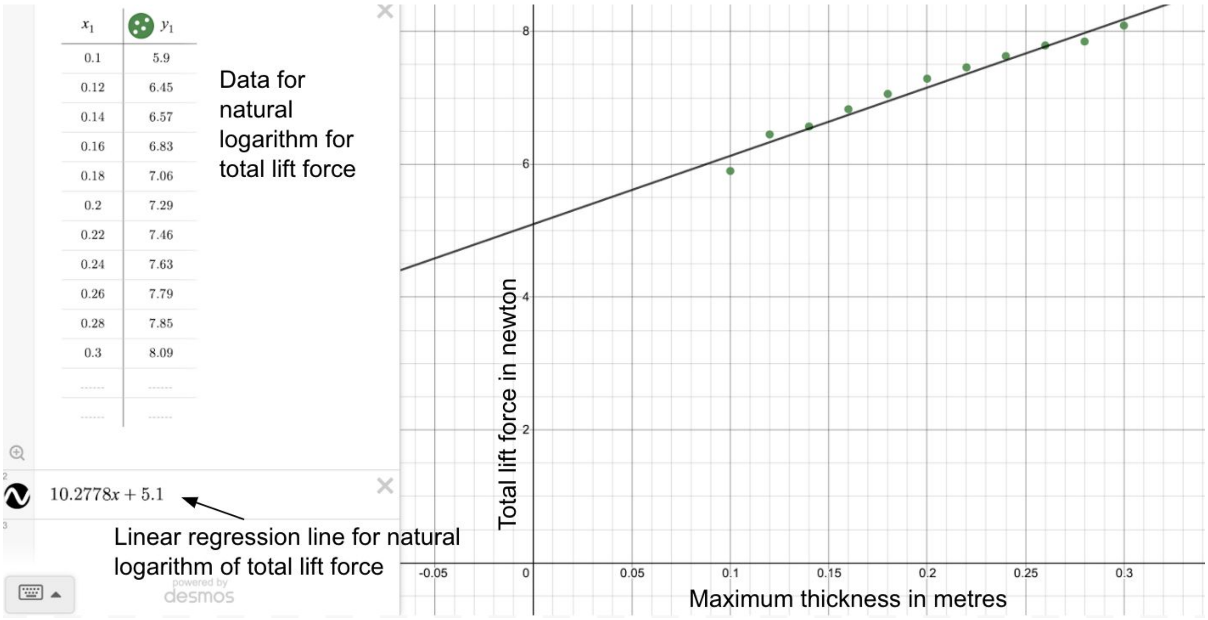

Figure 14

Figure 14 shows the data Natural logarithm data for the values of the total lift force and on the top of the plotted data I drew the function found from the linear regression done using my GDC as shown in figure 13. Because this graph shows a logarithmic scale I need to now transform this equation to find the appropriate function to match the data on graph in figure 11

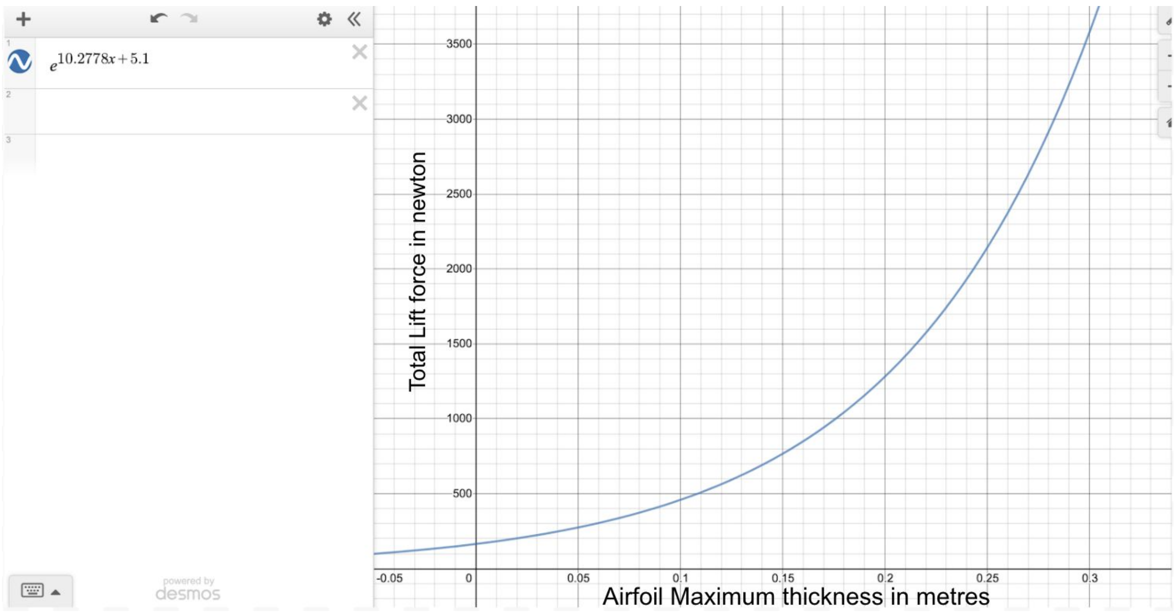

Figure 15

Here in this final graph we can see the equation for the final relationship between the maximum thickness of the airfoil and the total lift force generated with the equation of the relationship being:

$$e^{10.2778x + 5.1}$$

Conclusion and evaluation

The results are shown to be appropriate as the average newton required to lift a piper plane is 3550 newton if we consider that 9.8 newton are required to lift one Kg and the result I got only shows the lift force exerted by one wing. In addition during this experiment I considered the equal transit theory which underestimates the speed of the wind on the upper side of the airfoil. Therefore, the preciseness of this relationship is limited to the use of simple physics principle this is why it will not be the exact relationship but could be considered a good approximation.

As I started the investigation I didn't know how much math could be used in engineering, especially how math could be used to find more and more precise values. During this investigation two things made the calculation precise; the math and the physics. For the purpose of this investigation I chose to focus on the math part and it was very interesting that in order to make it precise not only the physics was important but also the math can demonstrate the level of preciseness. For example at the start I had to choose the number of intervals I wanted to use and the more I used the more precise my work would be. If I would have chosen 2 or 3 intervals I would've gotten very different results. Besides something else that could be interesting to research in the future would be at what point are the number of intervals do not giving a more precise value, is there a sort of limit. In addition it was very exciting to get a hint of how complex math can be used in real life and even more interesting when it is a field I will study in the future.

Throughout this exploration I was able to fulfill my interest in a specific subject and understand how math was used at those in a very specific field of research and how it can be used to improve the precision. I developed a better sense of what values to choose especially for the function of a wing where it took me a long time to find the best point to substitute in my simultaneous equation to get the best curve possible. However a major outcome of this was understanding how to deconstruct all parts of the wing in order to find all the dimensions and putting all this back together to find the lift. The exploration was captivating and made me very excited to learn more about math and physics in specific real life applications.

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