# A brief study of the Lorenz Model 80

Lucas Amaral Taylor

Institute of Mathematics and Statistics

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# Presentation structure

# Notice

This presentation refers to version available at `v1.0`

# General objectives

1. Presentation of the general aspects of the article "*Attractor sets and quasi-geostrophic equilibrium*" by Edward Norton Lorenz
2. Relate the content of the article to concepts presented in class
3. Take an approach in line with the degree course

# Methodology

1. Study the fundamental concepts of geophysical fluids and related areas.
2. Reproduce the construction of the models presented in the article.
3. Perform computer simulations to analyze the behavior of the models.

# *PE Model*: Fluid Characteristics

- **Homogeneous.** The fluid density is uniform throughout the volume;
- **Incompressible.** The volume does not change when subjected to pressure.

# *PE Model*: The Shallow Water Model

Making the necessary notational adjustments in [Sal98], we obtain:

$$\frac{\partial V}{\partial t} + (V \cdot \nabla) V + f\mathbf{k} \times V = -g\nabla z \tag{1}$$

$$\frac{\partial z}{\partial t} + \nabla \cdot (zV) = 0 \tag{2}$$

Where:

- $t$: time;
- $\mathbf{r}$: initial position vector;
- $V(t, r)$: Horizontal velocity;
- $z(t, r)$: Surface height;
- $f$: Coriolis parameter;
- $g$: gravitational acceleration;
- $\mathbf{k}$: vertical unit vector.

# *PE Model*: The Modified Shallow Water Model

The following equations are presented in the paper:

$$\frac{\partial V}{\partial t} = -(V \cdot \nabla)V - f\mathbf{k} \times V - g\nabla z + \nu\nabla^2\mathbf{V} \tag{3}$$

$$\frac{\partial z}{\partial t} = -(V \cdot \nabla)(z - h) - (H + z - H)\nabla \cdot \mathbf{V} + \kappa\nabla^2 z + F \tag{4}$$

Where:

- $t$: time;
- $\mathbf{r}$: initial position vector;
- $H$: mean fluid depth;
- $h(r)$: topographic surface variation;
- $V(t, r)$: horizontal velocity;
- $z(t, r)$: surface height;
- $f$: Coriolis parameter;
- $g$: gravitational acceleration;
- $F$: external forces;
- $\kappa$: viscous diffusion coefficient;
- $\nu$: thermal diffusion coefficient;
- $\mathbf{k}$: vertical unit vector.

# *PE Model*: Diffusion Processes

In equations (3) and (4), we identify two diffusion processes:

1. **Viscous diffusion:** momentum transfer between parts of the fluid due to viscosity *(e.g., honey)*;
2. **Thermal diffusion:** heat transfer by conduction between regions with different temperatures.

*Note: Both processes tend to homogenize their respective properties.*

# *PE Model*: Formation of New Equations

Using the *Helmholtz Decomposition*, which separates the rotational and divergent parts, applied to equation (4), we obtain:

$$V = \nabla\chi + \mathbf{k} \times \nabla\psi \tag{5}$$

Where:

- $\chi$: velocity potential (*divergent part*);
- $\psi$: stream function (*rotational part*);
- **k**: vertical unit vector.

# *PE Model*: Basic Equations

From equations (3) and (5), we derive the following equations:

$$\frac{\partial \nabla^2 \chi}{\partial t} = -\frac{1}{2}\nabla^2(\nabla\chi \cdot \nabla\chi) - \nabla\chi \cdot \nabla(\nabla^2\psi) \times \mathbf{k} + \nabla^2(\nabla\chi \cdot \nabla\psi \times \mathbf{k})$$
$$+ \nabla \cdot (\nabla^2\psi\nabla\psi) - \frac{1}{2}\nabla^2(\nabla\psi \cdot \nabla\psi) + \nu\nabla^4\chi + f\nabla^2\psi - g\nabla^2 z \quad (6)$$

$$\frac{\partial \nabla^2 \psi}{\partial t} = -\nabla \cdot (\nabla^2\psi\nabla\chi) - \nabla\psi \cdot \nabla(\nabla^2\psi) \times \mathbf{k} - f\nabla^2\chi + \nu\nabla^4\psi \quad (7)$$

# *PE Model*: Basic Equations

Following the same process, from equations (4) and (5), we obtain:

$$\frac{\partial z}{\partial t} = -\nabla \cdot (z-h)\nabla\chi - \nabla\psi \cdot \nabla(z-h) \times \mathbf{k} - H\nabla^2\chi + \kappa\nabla^2 z + F \tag{8}$$

The equations (6), (7), and (8) will be the fundamental equations for constructing the low-order model.

# *PE Model:* Objectives of the Simplification Process

1. Convert equations (6), (7), and (8) into a low-order model.
2. Transform an original model composed of primitive atmospheric equations into a system of nine variables.

# *PE Model*: Simplification Process

First, we introduce three dimensionless vectors that satisfy the following condition:

$$\alpha_1 + \alpha_2 + \alpha_3 = 0 \tag{9}$$

Together with the following permutation:

$$(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2) \tag{10}$$

We define the variables $a_i$, $b_i$, and $c_i$

# *PE Model*: Simplification Process

The variables $a_i, b_i$, and $c_i$ are defined as follows:

$$a_i = \alpha_i \cdot \alpha_j$$
$$b_i = \alpha_j \cdot \alpha_i$$
$$c_i = \alpha_j \times \alpha_k \cdot \mathbf{k}$$

Although this relation holds, Lorenz presents an alternative (used in computational applications):

$$b_i = \frac{1}{2}\left(a_i - a_j - a_k\right)$$
$$c_i = c$$

# *PE Model*: Simplification Process

Finally, we define a length $L$ and construct three orthogonal functions:

$$\phi_i = \cos\left(\alpha_i \cdot \frac{r}{L}\right)$$

From these, we obtain:

$$L^2 \nabla^2 \phi_i = -a_i \phi_i$$

$$L^2 \nabla \phi_i \cdot \nabla \phi_k = -\frac{1}{2} b_{ik} \phi_i + \cdots$$

$$L^2 \nabla \cdot (\phi_j \nabla \phi_k) = \frac{1}{2} b_{jk} \phi_i + \cdots$$

$$L^2 \phi_j \cdot \nabla \phi_k \times \mathbf{k} = -\frac{1}{2} c_{jk} \phi_i + \cdots$$

# *PE Model*: Simplification Process

From these, we can introduce the normalized dimensionless variables:

$$t = f^{-1}\tau \tag{11}$$

$$\chi = 2L^2 f^2 \sum x_i \phi_i \tag{12}$$

$$\psi = 2L^2 f^2 \sum y_i \phi_i \tag{13}$$

$$z = 2L^2 f^2 g^{-1} \sum z_i \phi_i \tag{14}$$

$$h = 2L^2 f^2 g^{-1} \sum h_i \phi_i \tag{15}$$

$$F = 2L^2 f^2 g^{-1} \sum F_i \phi_i \tag{16}$$

# *PE Model*: Simplification Process

Next, we apply the variables defined in (11)-(16) to equations (6), (7), and (8), obtaining the following equations:

$$
\begin{aligned}
a_i \frac{dx_i}{d\tau} &= a_i b_i x_i x_k - c(a_i - a_k) x_i y_k c(a_i - a_j) y_i x_k \\
&- 2c^2 y_i y_k - \nu_0 a_i^2 x_i + a_i y_i - a_i z_i && (17) \\
a_i \frac{dy_i}{d\tau} &= -a_i b_k x_i y_k - a_i b_i y_i x_k + c(a_k - a_i) y_i y_k - a_i x_i - \nu_0 a_i^2 y_i && (18) \\
\frac{dz_i}{d\tau} &= -b_k x_i (z_k - h_k) - b_i (z_i - h_i) x_k + c y_i (z_k - h_k) \\
&- c(z_i - h_i) y_k + g_0 a_i x_i - \kappa_0 a_i z_i + F_i && (19)
\end{aligned}
$$

# *PE Model*: Variables

- $x$ – **Velocity potential**: related to flow divergence.
- $y$ – **Stream function**: associated with fluid vorticity.
- $z$ – **Surface elevation**: height of the perturbed surface.

# *PE Model*: Variables

- $\frac{dx}{dt}$ – Gravity waves.
- $\frac{dy}{dt}$ – Associated with fluid vorticity.
- $\frac{dz}{dt}$ – Related to surface height variation and its interaction with vorticity.

# *PE Model:* Brief Observations

- Variables with index 1 correspond to zonally uniform velocity and height fields.
- Variables with indices 2 or 3 represent components associated with waves or large-scale eddies.
- This model will be used in study simulations, following the permutation given in (10).

# *PE Model:* Another Simplification Process

First, we define $U$ and $V$:

$$U_i = -b_i x_i + c y_i \tag{20}$$
$$V_i = -b_k x_i - c y_i \tag{21}$$

Next, $X_i$ and $Y_i$:

$$X_i = -a_i x_i \tag{22}$$
$$Y_i = -a_i y_i \tag{23}$$

# *PE Model*: Another Simplification Process

Applying $U$, $V$, $X_i$, and $Y_i$ in (17), (18), and (19), we obtain:

$$\frac{dX_i}{d\tau} = U_i U_k + V_j V_k - \nu_0 a_i X_i + Y_i + a_i z_i \tag{24}$$

$$\frac{dY_i}{d\tau} = U_i Y_k + Y_j V_k - X_i - \nu_0 a_i Y_i \tag{25}$$

$$\frac{dz_i}{d\tau} = U_i(z_k - h_k) + (z_j - h_j) V_k - g_0 X_i - \kappa_0 a_i z_i + F_i \tag{26}$$

This model also follows the permutations of (10).

# *QG Model*: Model Construction

From equation (17):

- All terms containing $x$ are eliminated, including those with time derivatives.

From equations (18) and (19):

- All nonlinear or topographic terms are eliminated.
- All terms containing $x$ and $z$ are eliminated.

# *QG Model*: Model Construction

Following this process, we obtain:

$$
\begin{aligned}
(a_i g_0 + 1)\frac{dy_i}{d\tau} = {} & g_0 c(a_k - a_j) y_j y_k - a_i(a_i g_0 v_0 + \kappa_0) y_i \\
& - c h_k y_j + c h_j y_k + F_i
\end{aligned}
\tag{27}
$$

# Initial remarks

It is important to note that: for presentation purposes, I have chosen to present the comparison between the models taking **only** as the basis for the criteria for comparing the models.
In the final sections of the article [Lor80], there is a detailed analysis of the structure of the attractor and its relation to the invariant variety, including its qualitative properties.

# Forced dissipative model

First, let's expose a generic forced dissipative model:

$$\frac{dw_i}{dt} = \sum_{j,k}^{N} a_{ijk} w_j w_k - \sum_{j}^{N} b_{ij} w_j + c_i \tag{28}$$

$$A = \sum_{i,j,k}^{N} a_{ijk} w_i w_j w_k \quad \wedge \quad B = \sum_{i,j}^{N} b_{ij} w_i w_j > 0$$

$$C = \sum_{i}^{N} c_i w_i \quad \wedge \quad R^2 = \sum_{i}^{N} w_i^2$$

# Forced dissipative model

- $A$ is a cubic polynomial and represents the nonlinear interactions between the system components;
- $B$ is a quadratic polynomial and represents the dissipation of the system;
- $C$ is a linear polynomial and represents the external forcing;
- $R$ is the squared Euclidean norm and represents the total energy.

And let's define $A_1$ and $C_1$ as the maximum of $A$ and $C$ and $B_1$ as the minimum of $B$

# Special Conditions

The models analyzed in the article [Lor80] must meet the following conditions:

1. The dissipation condition:

$$B_1^2 - 4A_1C_1 > 0 \tag{29}$$

2. Constraint on the coefficients of the system:

$$a_{ijk} = 0, \quad \text{se} \quad j = 1 \quad \vee \quad k = i$$

3. **Volume zero**: The rate of change of the volume satisfies

$$\frac{dV}{dt} = -V \sum_{i=1}^{N} b_{ii}, \tag{30}$$

which implies that the volume decreases exponentially over time due to the dissipation of the system.

# Volume zero

The equation (30) indicates that:

- Since $B$ is positive definite, $V \to 0$ as time progresses.
- This implies that the dynamics of the system is progressively restricted to regions of smaller volume in phase space.
- Any initial surface $S$ generates a sequence of surfaces $S_1, S_2, \ldots,$ where each one has a smaller volume than the previous one.
- In the limit, the succession of surfaces converges to a lower-dimensional subset, characterizing a **dissipative tractor**.

# Relating the generic model to the constructed models

1. **QG model** In the QG model, the total energy is approximately conserved by the quadratic terms. In addition, the dissipation introduced by the diffusion processes acts analogously to the dissipative term of the generic system (28). Thus, the QG model has an attractor of **zero volume**.
2. **PE model**. Although the PE model does not exactly conserve total energy, if $F_1^2 + F_2^2 + F_3^2$ is small enough, the (29) condition is satisfied. In this case, the trajectories of the system remain limited and the attractor has **zero volume**.

# *PE Model*: Simulation Process

Given the model selection, the following process is performed:

1. Selection of constants and initial conditions
2. Discretization method (RK4)
3. Plotting

# *PE Model*: Selection of Constants

```
vetor_a = [1, 1, 3]
vetor_b = [
    0.5 * (vetor_a[0] - vetor_a[1] - vetor_a[2]),
    0.5 * (vetor_a[1] - vetor_a[2] - vetor_a[0]),
    0.5 * (vetor_a[2] - vetor_a[0] - vetor_a[1]),
]
c = math.sqrt(3/4)

f_inv = 10800
vetor_h = [-1, 0, 0]
vetor_f = [0.1, 0, 0]
g_0 = 8
kappa_0 = 1 / 48
nu_0 = kappa_0
```

# *RK4 Discretization Method*: General Characteristics

A method is explicit Runge-Kutta of order 4 if and only if it satisfies three properties:

1. Explicit single-step method;
2. It presents good stability for ordinary differential equations (ODEs).
3. It has a global error of the order of $\mathcal{O}(h^4)$, ensuring high accuracy with a moderate computational cost.

# *RK4 Discretization Method*: Formulation

Taking [A M23] as a reference, we can express the RK44 method as follows:

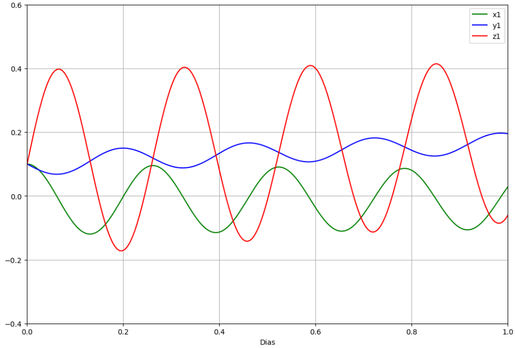
$$\Phi(t, y, h) = \frac{1}{6}\left(\kappa_1 + 2\kappa_2 + 2\kappa_3 + \kappa_4\right) \quad \text{com} \quad \begin{cases} \kappa_1 = f(t, y) \\ \kappa_2 = f\left(t + h/2, y + (h/2)\kappa_1\right) \\ \kappa_3 = f\left(t + h/2, y + (h/2)\kappa_2\right) \\ \kappa_4 = f\left(t + h, y + h\kappa_3\right) \end{cases}$$

# *PE Model*: Initial conditions - default

The first initial condition is the one given by default and is intended to reproduce figure 1 of the article.

```
# Initial conditions
x0 = [0.1, 0, 0]
y0 = [0.1, 0, 0]
z0 = [0.1, 0, 0]

```

# *PE Model*: Results for Standard Initial Condition




(a) Standard condition - 01 day (reproduction)

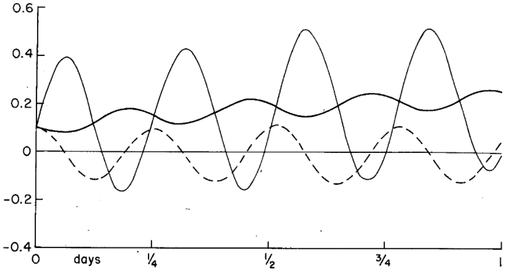
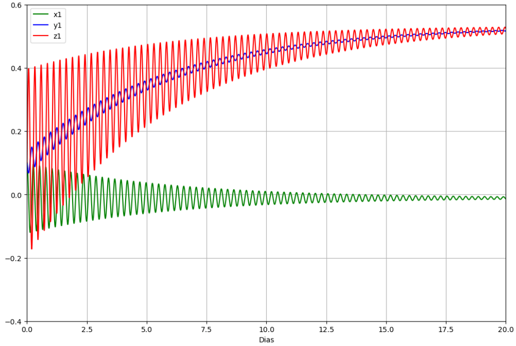


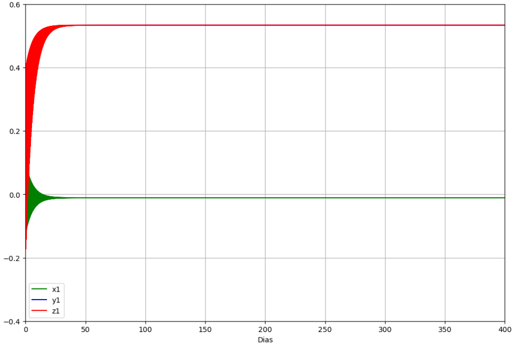

FIG. 1. Variations of $x_1$ (dashed curve), $y_1$ (heavy solid curve) and $z_1$ (thin solid curve) during first day of first numerical solution of PE model.

(b) Standard condition - 01 day (article)

# *PE Model*: Results for Standard Initial Condition




(a) Standard condition - 20 days




(b) Standard condition - 400 days

# Hadley Conditions

The Hadley circulation is a pattern of atmospheric circulation in the tropics, where warm air rises near the equator and descends at higher latitudes, forming a convective cycle.

- In 1735, Hadley incorporated the effect of Earth's rotation, showing that air velocity varies with latitude, influencing the predominant wind direction in the tropics.
- It is based on the conservation of angular momentum, ensuring the balance of atmospheric motion and preventing changes in Earth's rotation.
- Responsible for the predominant winds in the tropics and the redistribution of heat in the atmosphere, influencing global climate patterns.

# *PE Model*: Initial Conditions - Hadley 01

From the article [GM82], we have that the vector values for Hadley conditions are defined as:

$$
\begin{aligned}
x_1 &= -\nu_0 a_1 y_1\,, \\
y_1 &= \frac{F_1}{a_1} v_0 \left(1 + a_1 g_0 + \nu_0^2 a_1^2\right)\,, \\
z_1 &= \left(1 + \nu_0^2 a_1^2\right) y_1 \\
x_2 &= y_2 = z_2 = x_3 = y_3 = z_3 = 0
\end{aligned}
$$

# *PE Model*: Initial Conditions - Hadley 01

Adapting to the code, we have:

```
# Initial conditions
y1 = (
    vector_f[1]
    / vector_a[1]
    * nu_0
    * (1 + vector_a[1] * g_0 + nu_0**2 * vector_a[1] ** 2)
)
z1 = (1 + nu_0**2 * vector_a[1] ** 2) * y1
x1 = -nu_0 * vector_a[1] * y1

hadley01_initial_x = [x1, 0, 0]
hadley01_initial_y = [y1, -(10 ** (-5)), 0]
hadley01_initial_z = [z1, 10 ** (-5), 0]
```

# *PE Model*: Results for Hadley 01 Condition

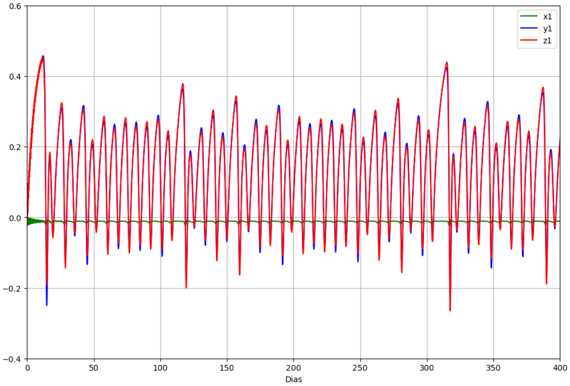



Figure: Hadley 01 - 400 days

# *PE Model*: Results for Hadley 01 Condition

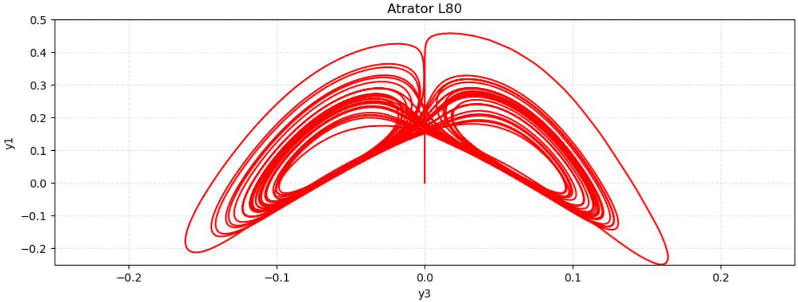



Figure: Hadley 01 - Projection $y_3 \times y_1$

# *PE Model*: Results for Hadley 01 Condition

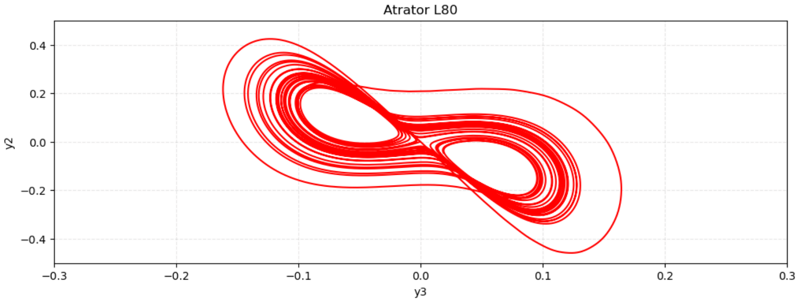



Figure: Hadley 01 - Projection $y_3 \times y_2$

# *PE Model*: Results for Hadley 01 Condition

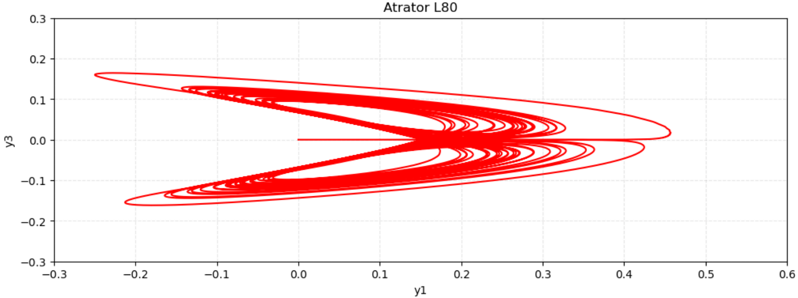



Figure: Hadley 01 - Projection $y_1 \times y_3$

# *PE Model and QG Model*: Initial Conditions - Hadley 02

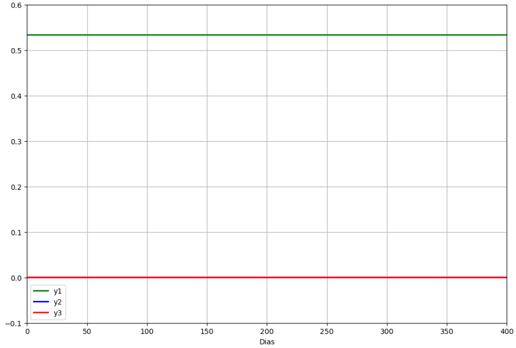
The present condition reproduces the Hadley circulation conditions according to [Lor80]

```
# Initial conditions of the PE model
hadley02_initial_x = [-0.01111, 0, 0]
hadley02_initial_y = [0.53331, 0, 0]
hadley02_initial_z = [0.53354, 0, 0]

```

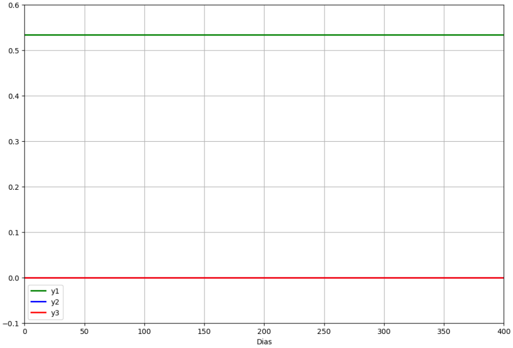
Equivalent to the following condition in the QG Model:

```
# Initial conditions of the QG model
y0 = [0.53333, 0, 0]

```

# *PE Model* and *QG Model*: Results for Hadley 02 Condition




(a) Hadley 02 Condition - 400 days (PE Model)




(b) Hadley 02 Condition - 400 days (QG Model)

# *Some difficulties*: The equations (24)-(26)

1. Although equations (24)-(26) are simplifications of (17)-(19), none of the references used included equations (24)-(26).
2. When attempting to plot (24)-(26), several issues arose, the main ones being: *overflow* and deviation from the nature of equations (17)-(19).

# *Some difficulties*: The equations (24)-(26)

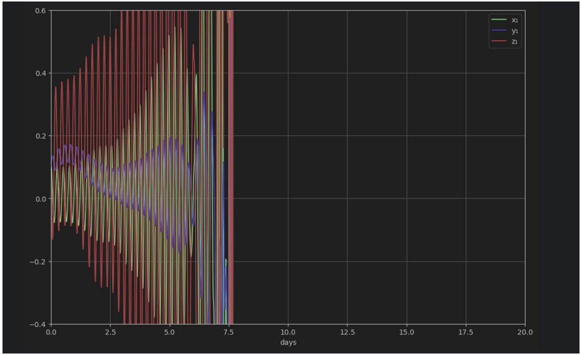
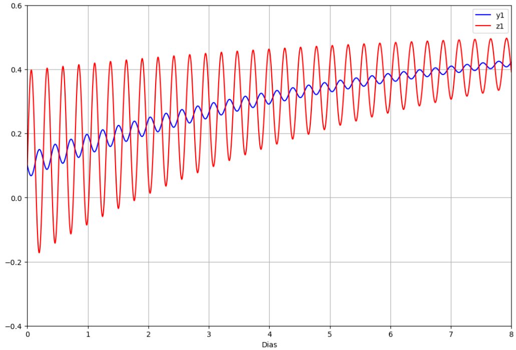



Figure: Attempt to plot equations (24)-(26)

# *Some difficulties*: Equations (17)-(19)




(a) Standard condition - 8 days

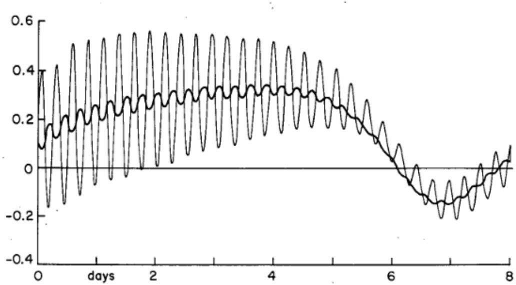



FIG. 2. Variations of $y_1$ (heavy curve) and $z_1$ (thin curve) during first eight days of first numerical solution of PE model.

(b) Standard condition - 8 days (article)

# Github repository

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# Thanks!