MAP2419 – Introduction to the Graduation Project

Institute of Mathematics and Statistics, University of São Paulo (IME-USP)

Project

A Stochastic Approach to the Lorenz 80 Model

Uma abordagem estocástica do Modelo de Lorenz 80

Supervisor:
Prof. Dr. Breno Raphaldini Ferreira da Silva
brenorfs@gmail.com
IME-USP

Student:
Lucas Amaral Taylor
NUSP: 13865062
lucasamtaylor@usp.br
IME-USP

Abstract

This project presents a study of the Lorenz 80 Model, originally proposed by Edward Lorenz (1980), from a stochastic approach inspired by Chekroun et al. (2021). Essential theoretical foundations are covered, such as the Mori-Zwanzig formalism, general properties of stochastic differential equations and the mathematical and physical characteristics of the model itself. The computational development includes the implementation and numerical simulation of the model in *Julia* and *Python*, with an emphasis on scientific libraries for stochastic dynamical systems and data analysis. Finally, an exploratory analysis of different configurations of the noise term is carried out, with the aim of investigating alternative and complementary approaches to the original treatment.

Introduction

In 1980, Edward N. Lorenz published the article *Attractor Sets and Quasi-Geostrophic Equilibrium* (Lorenz, 1980), in which he developed the model that became known as the *Lorenz 80 Model* (L80), with the aim of studying the dynamics of forced-dissipative atmospheric systems.

Starting from shallow water equations with topography, Lorenz constructed a low-order model with nine ordinary differential equations, based on primitive equations. Subsequently, by eliminating the time derivative terms in the divergence equations, yielding a quasi-geostrophic version with only three equations. The model presents two distinct scales of motion: fast oscillations, associated with gravitational waves, and slow oscillations, of a quasi-geostrophic nature. Over time, the fast components dissipate, and the dynamics concentrate on a reduced-dimension invariant manifold, where quasi-geostrophic equilibrium is a good approximation. In addition to the theoretical formulation, Lorenz performed computational simulations of the model, a feat of great importance for the time and for the development of computational mathematics.

In November 2021, the article *Stochastic Rectification of Fast Oscillations on Slow Manifold Closures* (Chekroun et al., 2021) was published, proposing a stochastic approach to slow-fast systems using methods from statistical physics. In this study, the authors used the *L80 Model* as a case study, applying the Mori-Zwanzig (MZ) method.

The MZ method, which originated in statistical physics, separates the dynamics of a system into solved and unresolved parts using projection operators. A central feature of this method is that, when projecting the dynamics onto a subspace of solved variables, the effects of the discarded variables do not disappear. The unresolved variables are incorporated into the effective dynamics in the form of two additional terms: a Markovian term, which represents the influence of the memory of past states, and a noise term, which models the unresolved variability.

In January 2025, student Lucas Amaral Taylor took the course **MAP5007 - Waves in Geophysical Fluids**, offered in the summer program of the Institute of Mathematics and Statistics (IME-USP) and taught by Prof. Dr. Breno Raphaldini Ferreira da Silva. The course aimed to present basic concepts of geophysical fluid dynamics through a mathematical approach (Sistema Janus - USP, 2025). At the end of the course, the student gave a seminar on the topic *A brief study of the Lorenz 80 Model*, whose objective was to present the general aspects of the L80 model.

Finally, this work is an extension of that previous study. Now, instead of exploring the deterministic model of Lorenz (1980), the focus will be on the stochastic approach proposed by Chekroun et al. (2021). Mathematical, statistical, and physical properties involved in the construction and treatment of the model will be analyzed, and computational simulations will be performed.

Objectives

This work is based on three main objectives:

1. **Understanding and manipulation of essential theoretical concepts**

   (a) In-depth study of the MZ method: theory and applications, particularly in slow-fast dynamical systems;
   (b) Analysis of the general properties of stochastic differential equations;
   (c) Study of the particularities of the L80 Model, both in its deterministic and stochastic versions, considering its physical and mathematical implications.

2. **Development of skills in computational tools.**
   (a) Mastery of computational languages aimed at the simulation and analysis of mathematical models, especially Julia and Python, with a focus on the use of scientific libraries;
   (b) Ability to implement, optimize, and interpret computational routines for numerical simulations.

3. **Achievements in L80 model simulations and exploratory analysis.**
   (a) Development of the L80 model in Julia;
   (b) Exploratory analysis of the properties of the noise term.

Methodology

To understand and manipulate the mathematical objects fundamental to the development of the model, we began with an in-depth reading of the basic articles that introduce the main concepts and motivate the study, in particular the works of Chekroun et al. (2017) and Chekroun et al. (2021). Next, to consolidate our understanding of the MZ method, references addressing the formulation and applications of this method will be analyzed, including the texts by Gouasmi et al. (2017), Chorin et al. (2000), Chorin et al. (2002), and Chorin and Hald (2013).

In the development of computational tool skills, the Julia language will be used primarily, with a focus on the SciML library (SciML Team, 2025), designed for simulations with stochastic differential equations, and Python, mainly with the libraries `numpy` and `pandas` for data analysis. For initial familiarization, simulations will be performed based on example 11.7 from (Pavliotis & Stuart, 2008, p. 169).

Finally, in the stage of **simulations of the L80 model and exploratory analysis**, computational simulations of the L80 model will be performed. Initially, we will reproduce the results presented in Chekroun et al. (2021). Then, based on the knowledge and experience acquired throughout the project, we will ideally propose variations in noise modeling, different from those adopted in the original article. The intention is to assess whether alternative or complementary approaches to noise can improve the results obtained in the stochastic formulation.

Work plan

Month	**Activity**
April	Definition of the theme, selection of the supervisor, and survey of the main references
May	Introduction to the MZ method and the *Julia* language.
June	First simulations in Julia using simplified models.
July	Reading about stochastic differential equations and in-depth study of the L80 Model.
August	Initial implementation of the L80 Model.
September	Exploratory analysis of the properties of the noise term.
October	Writing and completion of the monograph.
November	Final review, translation, and preparation for presentation.

Preliminary results

The progress of the project can be followed via the link below:

https://github.com/lucasamtaylor01/Lorenz80_SDE/tree/english-version

References

Chekroun, M. D., Liu, H., & McWilliams, J. C. (2017). The emergence of fast oscillations in a reduced primitive equation model and its implications for closure theories. *Computers & Fluids*, *151*, 3–22. https://doi.org/10.1016/j.compfluid.2016.07.005

Chekroun, M. D., Liu, H., & McWilliams, J. C. (2021). Stochastic rectification of fast oscillations on slow manifold closures. *Proceedings of the National Academy of Sciences*, *118*(48). https://doi.org/10.1073/pnas.2113650118

Chekroun, M. D., Liu, H., & Wang, S. (2015a). *Approximation of stochastic invariant manifolds: Stochastic manifolds for nonlinear spdes i*. Springer International Publishing. https://doi.org/10.1007/978-3-319-12496-4

Chekroun, M. D., Liu, H., & Wang, S. (2015b). *Stochastic parameterizing manifolds and non-markovian reduced equations: Stochastic manifolds for nonlinear spdes ii*. Springer International Publishing. https://doi.org/10.1007/978-3-319-12520-6

Chorin, A. J., & Hald, O. H. (2013). *Stochastic tools in mathematics and science*. Springer New York. https://doi.org/10.1007/978-1-4614-6980-3

Chorin, A. J., Hald, O. H., & Kupferman, R. (2000). Optimal prediction and the mori–zwanzig representation of irreversible processes. *Proceedings of the National Academy of Sciences*, *97*(7), 2968–2973. https://doi.org/10.1073/pnas.97.7.2968

Chorin, A. J., Hald, O. H., & Kupferman, R. (2002). Optimal prediction with memory. *Physica D: Nonlinear Phenomena*, *166*(3), 239–257. https://doi.org/10.1016/S0167-2789(02)00446-3

Evans, L. C. (2014, January). *An introduction to stochastic differential equations*. American Mathematical Society.

Gouasmi, A., Parish, E. J., & Duraisamy, K. (2017). *A priori* estimation of memory effects in reduced-order models of nonlinear systems using the mori–zwanzig formalism. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, *473*(2205), 20170385. https://doi.org/10.1098/rspa.2017.0385

Lorenz, E. N. (1980). Attractor sets and quasi-geostrophic equilibrium. *Journal of Atmospheric Sciences*, *37*(8), 1685–1699. https://doi.org/10.1175/1520-0469(1980)037⟨1685:ASAQGE⟩2.0.CO;2

Pavliotis, G. A., & Stuart, A. (2008). *Multiscale methods: Averaging and homogenization* (Vol. 53). Springer Science & Business Media.

SciML Team. (2025). Stochastic differential equations · differentialequations.jl. https://docs.sciml.ai/DiffEqDocs/stable/tutorials/sde_example/

Sistema Janus - USP. (2025). Ondas em Fluidos Geofísicos [Disponível em: acesso em 08 maio 2025].

Taylor, L. A. (2025). Lorenz80: Estudo do modelo de Lorenz (1980) [Disponível em: acesso em 08 maio 2025]. https://github.com/lucasamtaylor01/Lorenz80