[MATH-15] Indefinite integrals

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Indefinite integrals

We say that $F(x) =$ is a primitive of $f(x)$ when $F'(x) = f(x)$ for all x. This is sometimes written as

$$F(x) = \int f(x)\,dx.$$

Such an expression is called an **indefinite integral**. Mind that an indefinite integral is a function, but a definite integral, i.e. the integral of a function on an interval, as defined in the next lecture, is a number. So, we use the same symbol for two different things.

Primitives are not unique. For instance, both $F_1(x) = x^2/2$ and $F_2(x) = x^2/2 + 1$ are primitives of $f(x) = x$. Since two primitives always differ in a constant, some people write this as

$$\int x\,dx = \frac{x^2}{2} + C,$$

though such expressions are becoming obsolete. I do not use this convention here, taking, implicitly, indefinite integrals as defined up to an additive constant.

Finding a primitive by means of the rules of the symbolic calculus of derivatives is sometimes called **symbolic calculus of integrals**. It is usually one of the main topics of a conventional Calculus course. Although I introduce some simple methods for finding primitives in this lecture, it is not an easy problem. Moreover, in certain cases a simple expression for the primitive does not exist. A famous example is the integral of the Gaussian “bell” curve,

$$\int \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\,dx.$$

Nobody knows how to solve this.

Direct integrals

Some indefinite integrals can be found by merely applying the rules for taking derivatives “backwards”. These are the **direct integrals**. As usual in Calculus, direct integrals become more direct after a bit of training. The following three formulas are very useful, and can be obtained easily by using the derivatives of the exponentials, logarithms and powers, together with the chain rule for composite functions:

- $\displaystyle\int e^u\,u'\,dx = e^u$, where u is a function de x. For instance, taking $u = x^2$, we get

$$\int x\,e^{x^2}\,dx = \frac{e^{x^2}}{2}\,.$$

- $\displaystyle\int u^\alpha\,u'\,dx = \frac{u^{\alpha+1}}{\alpha+1}$. Taking $u = x^2 + 1$ and $\alpha = 1/2$, we get

$$\int x\,\sqrt{x^2+1}\,dx = \frac{\sqrt{(x^2+1)^3}}{3}.$$

- $\displaystyle\int \frac{u'}{u}\,dx = \log u$. Here, taking $u = x^2 - 1$, we get

$$\int \frac{x}{x^2-1}\,dx = \frac{\log\left(x^2-1\right)}{2}\,.$$

Example 1. $\displaystyle\int \frac{2x}{1+x^4}\,dx = \arctan(x^2)$. □

Example 2. $\displaystyle\int \frac{x}{\sqrt{x^2+1}}\,dx = \sqrt{x^2+1}$. □

Example 3. $\displaystyle\int \sin^3 x\,\cos x\,dx = \frac{\sin^4 x}{4}\,.$

Integration by parts

A well known method for solving indefinite integrals is based on the formula of **integration by parts**. It is just the formula of the derivative of a product,

$$d\big(uv\big) = u\,dv + v\,du,$$

but rewrited in a way that helps in the symbolic calculus of integrals:

$$\int u\,dv = u\,v - \int v\,du.$$

This formula allows us to switch from the integral on the left to that on the right, which may be easier (not always). The way in which the formula is applied can be easily understood by looking at the following examples.

Example 4. Taking

$$u = x \Longrightarrow du = dx$$

$$dv = e^x\,dx \Longrightarrow v = e^x,$$

the formula of integration by parts gives

$$\int x\,e^x\,dx = x\,e^x - \int e^x\,dx = x\,e^x - e^x.\ \square$$

Example 5. Taking

$$u = \log x \Longrightarrow du = dx/x$$

$$dv = (x^2+1)dx \Longrightarrow v = (x^3/3) + x,$$

we get

$$\int \big(x^2+1\big)\log x\,dx = x\left(\frac{x^2}{3}+1\right)\,\log x - \int \left(\frac{x^2}{3}+1\right)dx = x\left(\frac{x^2}{3}+1\right)\,\log x - \frac{x^3}{9} - x.\ \square$$

Change of variable

A **change of variable**, also called substitution, is a one-to-one function $x(u)$. In an indefinite integral, writing x and dx in terms of u and du allows us to switch from the indefinite integral of $f(x)$ to the integral

$$\int f\big(x(u)\big)\, x'(u)\, du.$$

If the latter can be solved, this integral gives a function of u. Then, the change is reversed, giving, again, a function of x. This is seen in the examples that follow.

Example 6. Taking the integral

$$\int x\sqrt{x+1}\, dx$$

and the change of variable given by the expression $x = u^2 - 1$. Replacing $\sqrt{x+1} = u$, and $dx = 2u\, du$, we switch to the integral

$$\int \big(u^2-1\big)\, u\,(2u)\, du = 2\int \big(u^4-u^2\big)\, du = 2\left(\frac{u^5}{5}+\frac{u^3}{3}\right).$$

Now, reversing the change, we get

$$\int x\sqrt{x+1}\, dx = 2\sqrt{(x+1)^3}\left(\frac{x+1}{5}+\frac{1}{3}\right). \ \square$$

Example 7. Through the substitution $x = 1-u^2$, the integral $\displaystyle\int \frac{dx}{\sqrt{x(1-x)}}$ becomes

$$\int \frac{-2\, du}{\sqrt{1-u^2}} = -2\, \arcsin u$$

and, therefore,

$$\frac{dx}{\sqrt{x(1-x)}} = -2\, \arcsin\sqrt{1-x}.$$

Homework

A. Use integration by parts to calculate $\displaystyle\int x\,\sin x\, dx$.

B. Calculate $\displaystyle\int \frac{x}{\sqrt{x+1}}\, dx$.