[MATH-18] Local maxima and minima

Miguel-Angel Canela
Associate Professor, IESE Business School

Local maxima and minima

We say that f has a local maximum at X^* when $f(X) \le f(X^*)$ for X in a subdomain $D' \subset D$. Reversing the inequality leads to the definition of local minimum. Since partial derivatives can be interpreted as derivatives of the functions of one variable obtained by keeping constant all the variables except one, we have $\nabla f(X^*) = \mathbf{0}$ for a local extremum point X^* of f. This is a *necessary* condition for a point to be a local extremum point. The points where this condition is satisfied are called **stationary points** of f.

This condition is not sufficient, as shown by the well known counterexample $f(x) = x^3$. There, f is all the time increasing. With two variables, the situation is much more complex, because one can find a point which is local maximum along one direction, and a local minimum along another direction. We call this a **saddle point** of f. The following example is illustrative enough.

Example 1. Let $f(x_1, x_2) = x_1 x_2$. Then

$$\nabla f(x_1, x_2) = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix},$$

so $\nabla f(0,0) = \mathbf{0}$. But f does not have a local maximum nor a local minimum at $(0,0)$, because f is positive on the first and the third quadrants, and negative on the rest of the plane. Moreover, restricting f to the line $x_1 = x_2$, we obtain $f(x_1, x_1) = x_1^2$, with a minimum at $x_1 = 0$. On the other hand, putting $x_1 = -x_2$, we get $f(x_1, -x_1) = -x_1^2$, with a maximum at $x_1 = 0$. $\square$

Linear and quadratic approximations

The extension of Taylor polynomials to functions of several variables is simple, although the notation becomes a bit demanding for polynomials of degree higher than 2. In restrict the discussion to degrees 1 and 2, and I use matrix notation to shorten the expressions.

The **linear approximation** is the Taylor polynomial of degree 1. In matrix notation, the expression is a direct generalization of the formula seen for $n = 1$:

$$f(X^*) + \nabla f(X^*)^{\mathsf{T}}(\mathbf{x} - \mathbf{x}^*).$$

Note that, here, X and $\mathbf{x}$ are the same thing, but taken either as a point or as a vector. This expression can be sued to write the equation of the **tangent hyperplane**, which would be the extension to $n > 1$ of the tangent line of lecture 9.

The quadratic form defined by the Hessian matrix provides an additional term, leading to the **quadratic approximation**. The quadratic term is

$$\frac{1}{2}\big(\mathbf{x} - \mathbf{x}^*\big)^{\mathsf{T}} \nabla^2 f(X^*)\big(\mathbf{x} - \mathbf{x}^*\big).$$

The quadratic approximation can be used as in the case of a function of one variable: for X close to X^*, f and its quadratic approximation share some properties.

Classifying stationary points

Looking at the quadratic approximation, we can decide in almost all cases if a point where the gradient is zero is a local maximum, a local minimum, or none of both. The rule that follows use the Hessian matrix, and is similar to that of functions of one variable, except for the inclusion of saddle points. The (partial) rule is:

- Negative definite: local maximum.
- Positive definite: local minimum.
- Indefinite: saddle point.

In the last case, there are at least two orthogonal directions such that f has a local maximum along one direction, and a local minimum along the other. These directions correspond the eigenvectors of the Hessian matrix.

Leaving aside the fine mathematical details, this rule is easy to justify. In the neighborhood of a stationary point X^*, the quadratic approximation is

$$f(X) \approx f(X^*) + \frac{1}{2}\big(\mathbf{x} - \mathbf{x}^*\big)^{\mathsf{T}} \nabla^2 f(X^*)\big(\mathbf{x} - \mathbf{x}^*\big).$$

If the quadratic term is positive, $f(X) \geq f(X^*)$, and we have a (local) minimum at X^*. If it is negative, $f(X) \leq f(X^*)$, and we have a (local) maximum at X^*. If it is indefinite, we have maxima or minima following the principal directions corresponding to negative or positive eigenvalues. The argument does not work for semidefinite quadratic terms, since we are using an approximation.

Example 1 (continuation). For $f(x_1, x_2) = x_1 x_2$, we have $\nabla f(0,0) = \mathbf{0}$. The Hessian matrix

$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

has eigenvalues 1 and -1. The origin is a saddle point. The eigenvectors,

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

define two directions along which f has a minimum and a maximum, respectively. $\square$

Example 2. For $f(x_1, x_2) = x_1^3 + x_2^3 - 3x_1x_2$, we have

$$\nabla f(x_1, x_2) = \begin{bmatrix} 3x_1^2 - 3x_2 \\ 3x_2^2 - 3x_1 \end{bmatrix}, \qquad \nabla^2 f(x_1, x_2) = \begin{bmatrix} 6x_1 & -3 \\ -3 & 6x_2 \end{bmatrix}.$$

The stationary points arte the solutions of the system of equations given by $\nabla f(X) = \mathbf{0}$, that is,

$$x_2 = x_1^2, \qquad x_1 = x_2^2.$$

The solutions are $(0,0)$ and $(1,1)$. Now,

$$\nabla^2 f(0,0) = \begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix}, \qquad \nabla^2 f(1,1) = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}.$$

The first matrix is indefinite (eigenvalues 3 and -3), so $(0,0)$ is a saddle point. The second one is positive definite (eigenvalues 9 and 3), and, therefore, f has a local minimum at $(1,1)$. Note that it could be seen in advance that f cannot have a local maximum nor a local minimum at $(0,0)$, because, fixing $x_2 = 0$, we obtain the function $f(x_1, 0) = x_1^3$, which is always increasing.

Note that the eigenvalues of the Hessian matrix are the same as in Example 1. The "local" similarity of the two functions will seem natural to you if you note that the quadratic approximation to $f(X)$ at $(0,0)$ is $-6x_1x_2$. $\square$

Homework

A. Is the origin a saddle point of $f(x_1, x_2) = x_1^3 - x_2^3$?

B. Show that $f(x_1, x_2, x_3) = x_1^2 - x_2^2 + x_3^2 - 4x_1 + 6x_2 + 4x_3$ has no local extremum points.

C. Find the local extremum points of $f(x_1, x_2) = x_1^2 - x_1x_2 + x_2^2 + 3x_1 - 2x_2$.