[MATH-07] The product of two matrices

Miguel-Angel Canela
Associate Professor, IESE Business School

The product of two matrices

The definition of the product of two matrices extends the previous definition: in order to multiply two matrices $\mathbf{A}$ and $\mathbf{B}$, we take $\mathbf{B}$ as a pack of column vectors and multiply $\mathbf{A}$ by each column of $\mathbf{B}$, placing the resulting vectors as the columns of the product matrix $\mathbf{A}\,\mathbf{B}$. In mathematical symbols,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1k} \\ b_{21} & b_{22} & \cdots & b_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mk} \end{bmatrix} = \begin{bmatrix} \sum_j a_{1j}b_{j1} & \sum_j a_{1j}b_{j2} & \cdots & \sum_j a_{1j}b_{jk} \\ \sum_j a_{2j}b_{j1} & \sum_j a_{2j}b_{j2} & \cdots & \sum_j a_{2j}b_{jk} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_j a_{nj}b_{j1} & \sum_j a_{nj}b_{j2} & \cdots & \sum_j a_{nj}b_{jk} \end{bmatrix},$$

or, if $\mathbf{a}_1^\mathsf{T}, \ldots, \mathbf{a}_n^\mathsf{T}$ are the rows of $\mathbf{A}$ and $\mathbf{b}_1, \ldots, \mathbf{b}_m$ the columns of $\mathbf{B}$,

$$\begin{bmatrix} \mathbf{a}_1^\mathsf{T} \\ \mathbf{a}_2^\mathsf{T} \\ \vdots \\ \mathbf{a}_n^\mathsf{T} \end{bmatrix} \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_m \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^\mathsf{T}\mathbf{b}_1 & \mathbf{a}_1^\mathsf{T}\mathbf{b}_2 & \cdots & \mathbf{a}_1^\mathsf{T}\mathbf{b}_m \\ \mathbf{a}_2^\mathsf{T}\mathbf{b}_1 & \mathbf{a}_2^\mathsf{T}\mathbf{b}_2 & \cdots & \mathbf{a}_2^\mathsf{T}\mathbf{b}_m \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}_n\mathbf{b}_1 & \mathbf{a}_n^\mathsf{T}\mathbf{b}_2 & \cdots & \mathbf{a}_n^\mathsf{T}\mathbf{b}_m \end{bmatrix}.$$

Note that this definition only makes sense when the number of columns in $\mathbf{A}$ (the length of the rows) equals the number of rows in $\mathbf{B}$ (the length of the columns). The dimensions of the factors and the product satisfy the rule

$$[n \times m]\,[m \times k] = [n \times k].$$

Example 1.

$$\begin{bmatrix} 1 & 2 & 4 \\ -1 & 0 & 1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 17 & 21 \\ 2 & 6 \\ 11 & 6 \end{bmatrix}. \ \square$$

The product of matrices has the following properties (we assume that the dimensions are such that the formulas make sense):

- $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$ (associative property).
- In general, $\mathbf{AB} \neq \mathbf{BA}$ (see the example below).
- $(\mathbf{AB})^\mathsf{T} = \mathbf{B}^\mathsf{T}\mathbf{A}^\mathsf{T}$.
- $\operatorname{tr}(\mathbf{AB}) = \operatorname{tr}(\mathbf{BA})$.
- $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$.

Example 2. For

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},$$

we have

$$\mathbf{AB} = \begin{bmatrix} -1 & 3 \\ -2 & 8 \end{bmatrix}, \qquad \mathbf{BA} = \begin{bmatrix} 4 & 7 \\ 2 & 3 \end{bmatrix},$$

so $\mathbf{AB} \neq \mathbf{BA}$. Noevertheless $\det(\mathbf{AB}) = \det\mathbf{A}\,\det\mathbf{B} = -2$. $\square$

Example 3. A interesting example of non-commutativity of the product of matrices is given by

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} n \end{bmatrix}, \qquad \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Note that the trace of both products is equal to n. □

Inverse matrices

The **identity matrix** of dimension n is defined as

$$\mathbf{I}_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

The subscript n in $\mathbf{I}_n$ is omitted if there is no danger of confusion. For an (n, m)–matrix $\mathbf{A}$,

$$\mathbf{A}\mathbf{I}_m = \mathbf{I}_n\mathbf{A} = \mathbf{A}.$$

So, multiplying a matrix by the identity is like multiplying a number by 1. It is natural, then to define the inverse of a square matrix $\mathbf{A}$ as a matrix $\mathbf{A}^{-1}$ such that

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}.$$

A matrix is said to be **nonsingular** when it has inverse. It can be proved that, for an (n, n)–matrix $\mathbf{A}$, the following assertions are equivalent:

- $\mathbf{A}$ is nonsingular.
- $\det \mathbf{A} \neq 0$.
- The rank of $\mathbf{A}$ is n.
- The only vector in the null space of $\mathbf{A}$ is the zero vector.

If $\mathbf{A}$ and $\mathbf{B}$ are nonsingular and have the same dimension, $\mathbf{AB}$ is also nonsingular, and

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}.$$

Example 4.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}. \square$$

Can you give a rule for the inverses of diagonal matrices?

Example 5. The matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

has no inverse. Indeed, $\mathbf{AB} = \mathbf{I}$ would lead to an impossible equation,

$$\begin{bmatrix} b_{11} - b_{21} & b_{12} - b_{22} \\ -b_{11} + b_{21} & -b_{12} + b_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \square$$

Homework

A. Calculate

$$\begin{bmatrix} 3 & -1 & 2 \\ 1 & 0 & 3 \\ 4 & 0 & 2 \end{bmatrix}^{-1}.$$

B. Solve the equation

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \mathbf{X} = \begin{bmatrix} 3 & 5 \\ 5 & 9 \end{bmatrix},$$

both directly and through the formula

$$\mathbf{X} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 5 \\ 5 & 9 \end{bmatrix}.$$