[MATH-08] Quadratic forms

Miguel-Angel Canela

Associate Professor, IESE Business School

Quadratic forms

Let $\mathbf{A}$ be a symmetric matrix of dimension n. The **quadratic form** associated to $\mathbf{A}$ is the function $Q : E_n \to \mathbb{R}$ defined by

$$Q\mathbf{x} = \mathbf{x}^{\mathsf{T}}\mathbf{A}\mathbf{x} = \sum_{i,j=1}^{n} a_{ij}\, x_i\, x_j.$$

Then $Q\mathbf{x}$ is a **homogeneous** polynomial of degree 2 (meaning that all the terms have the same degree). It can easily be seen that any polynomial with this property can be obtained from a symmetric matrix in this way.

Example 1. The quadratic form $Q\mathbf{x} = x_1^2 + 2x_2^2$ can be obtained from $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. $\square$

Classification of quadratic forms

In Example 1, $Q\mathbf{x} > 0$ for every $\mathbf{x} \neq \mathbf{0}$. A quadratic form Q is said to be **positive definite** when this is true. Note that the eigenvalues of $\mathbf{A}$ are 1 and 2 (positive). Sometimes, one must allow $Q\mathbf{x} = 0$ for some $\mathbf{x}$, so that we will also consider **positive semidefinite** quadratic forms (ie $Q\mathbf{x} \geq 0$ for every $\mathbf{x}$, but $Q\mathbf{x} = 0$ for some $\mathbf{x} = 0$).

¶ Not everybody agrees here. Some call positive definite quadratic forms that satisfy $Q\mathbf{x} \geq 0$ for every $\mathbf{x}$, so that positive semidefinite includes positive definite, such as $x \geq 0$ includes $x = 0$.

Negative definite and semidefinite quadratic forms are defined in the same way. Finally, there are quadratic forms which do not fall in any of these situations. We say that Q is **indefinite** when there is some $\mathbf{x}$ such that $Q\mathbf{x} > 0$, and some $\mathbf{y}$ such that $Q\mathbf{y} < 0$.

Example 2. $Q\mathbf{x} = -2x_1^2 - x_2^2$ can be associated to

$$\mathbf{A} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}.$$

It is easy to see that Q is negative definite. Note that the eigenvalues are -2 and -1. $\square$

Example 3. $Q\mathbf{x} = x_1 x_2$ can be associated to

$$\mathbf{A} = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}.$$

It is easy to see, by playing with the signs of x_1 and x_2, that Q is indefinite. The eigenvalues are $1/2$ and $-1/2$ (different signs). $\square$

Example 4. The polynomial $Qx = 2x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3$ is a quadratic form. The corresponding matrix is

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Q is positive definite, because it can be written as $Q\mathbf{x} = (x_1 + x_2 + x_3)^2 + x_1^2 + x_2^2 + x_3^2$. The eigenvalues of $\mathbf{A}$ are the roots of $\det\big(\mathbf{A} - \lambda\mathbf{I}\big) = -\lambda^3 + 6\lambda^2 - 9\lambda + 4$, ie $\lambda_1 = 1$ (double) and $\lambda_2 = 4$, both positive. □

As the preceding examples suggest, a quadratic form Q defined by a symmetric matrix $\mathbf{A}$ can be classified by examining the signs of the eigenvalues of $\mathbf{A}$. The rules are:

- Positive definite: all the eigenvalues positive (as in Example 1).
- Positive semidefinite: some positive and some zero.
- Negative definite: all negative.
- Negative semidefinite: some negative and some zero.
- Indefinite: some positive and some negative.

Alternative approach

An alternative approach to the classification of a quadratic form is based on determinants. Let $\Delta_n = \det \mathbf{A}$, Δ_{n-1} the determinant obtained by deleting the last row and the last column, Δ_{n-2} that obtained by deleting the two last rows and the two last columns, etc. Finally, let Δ_1 be the top-left term a_{11}. The (partial) rule is:

- Q is positive definite when $\Delta_1 > 0$, $\Delta_2 > 0$, $\Delta_3 > 0$, etc.
- Q is negative definite when $\Delta_1 < 0$, $\Delta_2 > 0$, $\Delta_3 < 0$, etc.

Note that not all the quadratic forms can be classified using determinants. For instance, if $\Delta_n = 0$, the method does not say anything, and we can have a positive semidefinite, a negative semidefinite or an indefinite quadratic form.

Example 4 (continuation). In Example 4, $\Delta_1 = 2$, $\Delta_2 = 3$ and $\Delta_3 = 4$, all positive. □

Homework

A. Classify $Q\mathbf{x} = 3x_1^2 + 8x_2^2 + 6x_1x_3 - 4x_2x_3$.

B. Classify $Q\mathbf{x} = 3x_1^2 - x_2^2 - 8x_3^2 + 2x_1x_2 + 4x_2x_3$.