[MATH-08] Eigenvalues and eigenvectors

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Eigenvalues and eigenvectors

Let $\mathbf{A}$ be an $(n \times n)$–matrix, $\mathbf{v} \neq \mathbf{0}$ an n-vector, and λ a scalar, such that $\mathbf{A}\mathbf{v} = \lambda\,\mathbf{v}$. We say then that $\mathbf{v}$ is an **eigenvector** of $\mathbf{A}$, and that λ is the associated **eigenvalue**. In particular, any non-zero vector of the null space (if it exists) is an eigenvector, with eigenvalue zero.

Just a few numbers can be eigenvalues for a particular matrix. To see why, note that

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \Longleftrightarrow \big(\mathbf{A} - \lambda\mathbf{I}\big)\mathbf{v} = \mathbf{0}.$$

Therefore, λ is an eigenvalue when the null space of $\mathbf{A} - \lambda\,\mathbf{I}$ contains nonzero vectors, i.e. when λ is a solution of the **characteristic equation**

$$\det\big(\mathbf{A} - \lambda\,\mathbf{I}\big) = 0.$$

Example 1. Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$. Here,

$$\det\big(\mathbf{A} - \lambda\,\mathbf{I}\big) = \begin{vmatrix} 1-\lambda & 2 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 2 = 0$$

has two solutions, $\lambda_1 = -1$ and $\lambda_2 = 2$. These are the eigenvalues of $\mathbf{A}$. The eigenvectors associated to $\lambda_1 = -1$ are those vectors satisfying

$$\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longleftrightarrow x_1 + x_2 = 0.$$

The set of these eigenvectors is a linear subspace of dimension 1 (the null space of $\mathbf{A} + \mathbf{I}$). A basis of this subspace is given by the vector

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The eigenvectors associated to $\lambda_2 = 2$ are the solutions of

$$\begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longleftrightarrow x_1 - 2x_2 = 0.$$

These vectors form a linear subspace of dimension 1 (the null space of $\mathbf{A} - 2\,\mathbf{I}$), with basis

$$\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \ \square$$

Example 2. Let me consider now

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

$\mathbf{A}$ has two eigenvalues, $\lambda_1 = 0$ and $\lambda_2 = 3$, which are the solutions of the characteristic equation

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = \lambda^2(3-\lambda) = 0.$$

The eigenvectors associated to $\lambda_1 = 0$ satisfy

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longleftrightarrow x_1 + x_2 + x_3 = 0,$$

which defines a linear subspace of dimension 2 (the null space of $\mathbf{A}$). A basis is given by

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

The eigenvectors associated to $\lambda_2 = 3$ satisfy

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longleftrightarrow x_1 = x_2 = x_3,$$

which defines a linear subspace of dimension 1 (the null space of $\mathbf{A} - 3\,\mathbf{I}$). A basis is given by

$$\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \square$$

Diagonalization

Diagonalization is the key of important multivariate statistical methods, such as principal component analysis. A square matrix $\mathbf{A}$ is said to be diagonalizable if it admits a factorization $\mathbf{A} = \mathbf{B}\boldsymbol{\Lambda}\mathbf{B}^{-1}$, with $\mathbf{B}$ nonsingular and $\boldsymbol{\Lambda}$ diagonal. This is equivalent to the existence of a non-singular matrix $\mathbf{B}$ such that $\boldsymbol{\Lambda} = \mathbf{B}^{-1}\mathbf{A}\mathbf{B}$ is diagonal. This is the origin of the name.

It can be proved that $\mathbf{A}$ is diagonalizable if and only if there exists a basis of eigenvectors of $\mathbf{A}$. Then, the vectors of that basis coincide with the columns of $\mathbf{B}$ and the eigenvalues with the diagonal terms in $\boldsymbol{\Lambda}$. When is this possible? Suppose that the dimension of the matrix is n.

- If the number of eigenvalues coincides with n (the maximum possible), we pick an eigenvector for each eigenvalue, forming a basis, so that the matrix is diagonalizable.
- When the number of eigenvalues is less than n, it can fail to be diagonalizable. If the characteristic equation has complex non-real solutions, it is not diagonalizable. If all the solutions are real, but some are multiple, we associate to each multiple solution a subspace of eigenvectors. If the dimension equals the multiplicity, diagonalization is possible, $\boldsymbol{\Lambda}$ having the multiple eigenvalue repeated according to its multiplicity.

Example 3. The matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

is not diagonalizable, since the characteristic equation $\lambda^2 + 1 = 0$ has no real solution. $\square$

For a diagonalizable matrix, there are two formulas which are sometimes useful, when searching the eigenvalues (or their signs). They relate the sum and the product of the eigenvalues to the trace and the determinant:

- $\operatorname{tr}(\mathbf{A}) = \lambda_1 + \cdots + \lambda_n$.
- $\det \mathbf{A} = \lambda_1 \cdots \lambda_n$.

In any of these formulas, a multiple eigenvalue must be repeated according to its multiplicity.

Example 1 (continuation). In Example 1,

$$\mathbf{B} = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}, \qquad \mathbf{\Lambda} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}.$$

The trace is 1 and the determinant -2. Note that we can find the eigenvalues solving the system

$$\lambda_1 + \lambda_2 = 1, \qquad \lambda_1 \lambda_2 = -2.$$

Of course, this is not possible if the dimension is higher than two. $\square$

Example 2 (continuation). In Example 2,

$$\mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}, \qquad \mathbf{\Lambda} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

The trace equals 3 and the determinant equals 0. $\square$

Homework

A. Find the eigenvalues and eigenvectors of $\mathbf{A} = \begin{bmatrix} -1 & -2 \\ -2 & 2 \end{bmatrix}$.

B. Diagonalize, if possible, $\mathbf{A} = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$.