# [MATH-12] Differential calculus

**Miguel-Angel Canela**
**Associate Professor, IESE Business School**

### Derivatives and monotonicity

Let $f : D \to \mathbb{R}$ be a function of one variable. By definition, the derivative $f'(x)$ is the limit of a quotient. When $f$ is increasing, the numerator and the denominator in this quotient have the same sign. Therefore, it must be $f'(x) \geq 0$ for any $x \in D$. On the other hand, when $f$ is decreasing, they have opposite signs, and hence $f'(x) \leq 0$.

It can be shown that the reciprocal statements are partly true:

- If $f'(x) > 0$ (resp. $f'(x) < 0$) in some interval, then $f$ is increasing (resp. decreasing) in this interval.
- If $f'(x) \geq 0$ (resp. $f'(x) \leq 0$), then $f$ is non-decreasing (resp. non-increasing).

At those points where $f'(x) = 0$, one expects to see a change in the monotonicity. In most cases, this is what happens, but not always, as the following examples show.

**Example 1.** Let $f(x) = x^2 + 2x - 1$. The graph is a parabola, whose vertex is the point $(-1, -2)$. $f$ is decreasing for $x < 0$, and increasing for $x > 0$, so that there is a change at $x_0 = -1$. On the other hand, $f'(x) = 2x + 2$. Now, $f'(x) < 0$ for $x < -1$, and $f(x) > 0$ for $x > -1$. The derivative vanishes at $x_0 = -1$. □

**Example 2.** Let $f(x) = x^3$. Then $f$ is increasing everywhere, but $f'(0) = 0$. There is no change at $x_0 = 0$. □

One would say that the parabola $y = x^2 + 2x - 1$ (Example 1) has a minimum at $x_0 = 0$. We may expect the equation $f'(x) = 0$ to provide a method for finding maxima and minima, although Example 2 shows that not all solutions of this equation would be extremum points. Nevertheless, one must be careful here, because $f'(x) = 0$ is a local condition, that does not inform us about the behaviour of the function far from the point where it is satisfied. We see next that this condition yields local extremum points.

### Local maxima and minima

The definitions of local maximum and minimum points only make sense for the interior of an interval, but not at the extremes. Let $f$ be a function defined in an interval $D$, and suppose that $x_0$ is an interior point of $D$. We say that $f$ has a **local maximum** at $x_0$ when $f(x) \leq f(x_0)$ in some interval $I = [x_0 - r, x_0 + r]$, contained in $D$. The word "local" in this definition reminds us that $x_0$ need not to be the maximum on all $D$, but only on some interval contained in $D$. Reversing the inequality we obtain the definition of **local minimum**.

Suppose that $f$ has a local maximum at $x_0$. Then

$$\frac{f(x_0 + h) - f(x_0)}{h} \leq 0$$

for $h > 0$ and small enough, and

$$\frac{f(x_0 + h) - f(x_0)}{h} \geq 0$$

for $h < 0$. Therefore, if the derivative $f'(x_0)$ exists, it must be zero. The same is true for local minima.

This property provides a method for finding local maxima and minima. For instance, we found in Example 1 a local minimum at $x_0 = -1$. Nevertheless, in Example 2 we have $f'(0) = 0$, but $x_0 = 0$ is neither a local maximum nor a local minimum. This shows that the condition $f'(x_0) = 0$ is not sufficient for $f$ to have a local maximum or minimum at $x_0$.

## Derivatives of higher order

When the derivative $f'$ has a derivative at a point $x_0 \in D$, we call it the **second derivative** of $f$ at $x_0$, and we denote it by $f''(x_0)$. If $f''(x)$ exists at every $x$ in $D$, we can consider the second derivative as a new function $f''$, as we did with the (first) derivative. Taking the derivative of $f''$, we obtain the third derivative $f'''$, and so on.

Second derivatives help to identify local maxima and minima. The rule, whose foundation lies on the Taylor polynomials of the next section, is as follows:

- When $f'(x_0) = 0$ and $f''(x_0) < 0$, $f$ has a local maximum at $x_0$.
- When $f'(x_0) = 0$ and $f''(x_0) > 0$, $f$ has a local minimum at $x_0$.
- Nothing can be said, in general, when $f''(x_0) = 0$, as shown by the following example.

**Example 3.** Consider the functions $f_1(x) = -x^4$, $f_2(x) = x^4$ and $f_3(x) = x^3$. These three functions satisfy $f'(0) = f''(0) = 0$. The first one has a maximum at $x = 0$, the second one has a minimum, and the third one, none of both. □

**Example 4.** Let $f(x) = x\,e^{-x}$. Then

$$f'(x) = e^{-x} - x\,e^{-x} = e^{-x}(1 - x),$$

So, we have:

- $f(x) > 0$ for $x < 1$.
- $f(x) < 0$ for $x > 1$.
- $f'(x) = 0$ at $x = 1$.

Therefore, $f$ is increasing for $x < 1$, decreasing for $x > 1$, and has a local maximum at $x_0 = 1$. This can also be inferred from the sign of the second derivative, since

$$f''(x) = e^{-x}(x - 2) \Longrightarrow f''(1) = -e^{-1} < 0. \ \square$$

## Taylor polynomials

Let $D$ be an interval, $x_0$ an interior point of $D$, and $f : D \to \mathbb{R}$ a function. The **Taylor polynomial** of degree $n$ of $f$, centered at $x_0$, is

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

It is always true that $P_n(x_0) = f(x_0)$. Taylor polynomials are used as approximations of $f(x)$ near $x_0$. Such an approximation is supported by a theorem which is too complex for these notes. Anyway, one can easily manage Taylor polynomials, once the following issues are clearly understood:

- For "reasonable" functions, the approximation improves when $n$ grows, so that we can write

$$f(x) = \lim_{n\to\infty} P_n(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} \left(x - x_0\right)^n.$$

- The approximation improves when $x$ comes close to $x_0$. For $x$ far from $x_0$, it can be very bad, even with a polynomial of high degree.
- The error of the approximation, i.e. the difference between $f(x)$ and the polynomial, cannot be expressed in a simple way.

**Example 5.** Let $f(x) = e^x$. Then $f^{(k)}(x) = e^x$ for any $k$, and the Taylor polynomials at $x_0 = 0$ are

$$P_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}\,.$$

These polynomials approximate the exponential near zero. Unless one uses a Taylor polynomial of high degree, the approximation is poor when $x$ is not close to zero. Table 1 illustrates this. □

**TABLE 1. Approximating the exponential**

| $x$ | $\exp(x)$ | $P_1(x)$ | $P_2(x)$ |
|---|---|---|---|
| 0.001 | 1.0010005 | 1.001 | 1.0010005 |
| 0.01 | 1.01005017 | 1.01 | 1.01005 |
| 0.1 | 1.10517092 | 1.1 | 1.105 |
| 0.25 | 1.28402542 | 1.25 | 1.28125 |
| 0.5 | 1.64872127 | 1.5 | 1.625 |
| 1 | 2.71828183 | 2 | 2.5 |
| 2 | 7.3890561 | 3 | 5 |

**Example 6.** Let $f(x) = \sqrt{x}$ and $x_0 = 1$, so that $f(1) = 1$. By means of the formula for the derivative of a power, one can easily check that

$$f'(1) = \frac{1}{2}, \qquad f''(1) = -\frac{1}{4}, \qquad f'''(1) = \frac{3}{8}.$$

Thus, the Taylor polynomials of degree 1, 2 and 3 are, respectively,

$$P_1(x) = 1 + \frac{x-1}{2}\,,$$

$$P_2(x) = 1 + \frac{x-1}{2} - \frac{(x-1)^2}{8}\,,$$

$$P_3(x) = 1 + \frac{x-1}{2} - \frac{(x-1)^2}{8} + \frac{(x-1)^3}{16}\,. \ \square$$

## Linear and quadratic approximations

The Taylor polynomial of first degree,

$$P_1(x) = f(x_0) + f'(x_0)(x - x_0),$$

is called the **linear approximation** of $f$ at $x_0$. The graph is the tangent line at $x_0$. Some local properties of $f$ at $x_0$ ("local" means near $x_0$) can be easily understood by looking at the tangent.

For instance, when $f'(x_0) > 0$, the tangent has positive slope, and $f$ is increasing. Similarly, when $f'(x_0) < 0$, the tangent has negative slope, and $f$ is decreasing. When $f'(x_0) = 0$, the tangent is horizontal, and $f$ can have a local maximum, a local minimum, or none of both things.

The Taylor polynomial of second degree,

$$P_2(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2,$$

is called the **quadratic approximation**. Other local properties of $f$ at $x_0$ can be seen in the graph of the quadratic approximation, which is a parabola. For instance, when $f'(x_0) = 0$ and $f''(x_0) > 0$ (resp. $f''(x_0) < 0$), the parabola has a local minimum (resp. maximum) at $x_0$, and the same is true for $f$.

## L'Hôpital's rule

The famous **L'Hôpital's rule** is another consequence of the Taylor formula. Let $f$ and $g$ be derivable in an open interval containing $x_0$, with $f(x_0) = g(x_0) = 0$. Then

$$\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = \alpha \implies \lim_{x \to x_0} \frac{f(x)}{g(x)} = \alpha.$$

L'Hôpital's rule is the preferred method to calculate limits of functions. It can be extended to the case $\infty/\infty$, and also $x_0 = \pm\infty$ y $\alpha = \pm\infty$. Nevertheless, mind that it is valid only when there is one of these indeterminations. For instance,

$$\lim_{x \to 1} \frac{x - 1}{x} = 0 \quad \not\Longrightarrow \quad \lim_{x \to 1} \frac{1}{1} = 0.$$

## Homework

**A.** In which interval is the function $f(x) = \sqrt[x]{x}$ increasing?

**B.** Calculate $\lim\limits_{x \to 0} x \log x$.