[MATH-02] Trigonometric functions and complex numbers

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Sine and cosine

The **sine** and **cosine** of an angle, which are involved in the trigonometric formulas used to solve certain problems of elementary geometry, can be used to define functions on the real line. If x is the measure of an angle in **radians**, so that $\pi/2$ corresponds to a right angle (90°), we denote by $\sin x$ and $\cos x$ the sine and the cosine of x, respectively. It is assumed, in this short reminder, that you know the elementary properties of the sine and cosine, and that π is an irrational number, which happens to be the ratio of the length of the circle to the diameter.

Although angles in trigonometry problems are always lower than 2π radians (360°), we wish the sine and cosine to be defined over the whole real line. The extension can be easily done, since summing 360° to an angle does not change the sine and cosine. Thus, sine and cosine functions "repeat" themselves in each interval of length 2π. This gives

$$\sin(x + 2\pi) = \sin x, \qquad \cos(x + 2\pi) = \cos x.$$

Based on this, the sine and cosine functions are said to be **periodic functions**, with period 2π. The graph of the sine function is called **sinusoidal**. The graph of the cosine is the same but displaced.

The variation of the sine and cosine functions in the interval $[0, 2\pi]$ can be easily explained as follows. We take a point (x, y) of the first quadrant of the unit circle ($x^2 + y^2 = 1$), drawing a right-angled triangle whose hypotenuse is the segment joining $(0, 0)$ and (x, y), with horizontal and vertical catheti. Calling θ the angle formed by the positive half-axis OX and the hypotenuse (in this order), we have $\cos\theta = x$ and $\sin\theta = y$ (the hypotenuse has length 1). This gives the fundamental formula (Pythagoras theorem)

$$\sin^2\theta + \cos^2\theta = 1.$$

If θ runs from 0 to 2π, so that (x, y) covers all the circle, this relation is preserved. It follows that sine and cosine values range from -1 to 1. Moreover, if $\sin x = 0$, then $\cos x = \pm 1$, and conversely. More specifically,

$$\sin 0 = \sin\pi = 0, \qquad \sin(\pi/2) = 1, \qquad \sin(3\pi/2) = -1.$$
$$\cos 0 = 1, \qquad \cos\pi = -1, \qquad \cos(\pi/2) = \cos(3\pi/2) = 0.$$

This representation allows for the extension of these functions to negative angles (clockwise). It is easy to check the formulas for **opposite angles**,

$$\sin(-x) = -\sin x, \qquad \cos(-x) = \cos x,$$

for **complementary angles** (summing 90°),

$$\sin(\pi/2 - \theta) = \cos\theta, \qquad \cos(\pi/2 - \theta) = \sin\theta,$$

and for **supplementary angles** (summing 180°),

$$\sin(\pi - \theta) = \sin\theta, \qquad \cos(\pi - \theta) = -\cos\theta.$$

Other trigonometric functions

In trigonometry you have probably met the **tangent** and the **cotangent**, which can be derived from the sine and cosine through the formulas

$$\tan x = \frac{\sin x}{\cos x}\,, \qquad \cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}\,.$$

The tangent and the cotangent are also used to define functions, whose domain is not the whole line, since the points where the denominator is zero must be discounted. Thus, the tangent is not defined at $x = \pm\pi/2,\ \pm 3\pi/2, \ldots$, and the cotangent at $x = \pm\pi,\ \pm 2\pi, \ldots$.

Restricting the sine to the interval $[-\pi/2, \pi/2]$ leaves us with an increasing function, whose inverse is also increasing, with domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$. This inverse is called the **arcsin** function. Analogously, the tangent is increasing in $(-\pi/2, \pi/2)$, with an inverse called the **arctan** function, defined in the whole line, with range $(-\pi/2, \pi/2)$. It may seem to you that these functions are irrelevant for non-mathematicians, but the fact that arctan appears in many integrals makes it specially useful.

Complex numbers

A complex number z is a pair of real numbers a, b, written as $a + bi$. Then a is the **real part**, and b the **imaginary part**, of z. When the imaginary part is zero, we have a real number and, when the real part is zero, a **pure imaginary** number. i is the complex number with zero real part zero and unit imaginary part, called the **imaginary unit**.

We can operate with complex numbers as with real numbers, but keeping in mind that $i^2 = -1$ (so that we could write $i = \sqrt{-1}$). Thus,

$$\begin{aligned}(a_1 + b_1 i) + (a_2 + b_2 i) &= a_1 + a_2 + (b_1 + b_2)i,\\ (a_1 + b_1 i)(a_2 + b_2 i) &= a_1 a_2 - b_1 b_2 + (a_1 b_2 + a_2 b_1)i.\end{aligned}$$

The benefit of introducing complex numbers is that any polynomial equation $P(x) = 0$ has solutions, which are then called the **roots** of the polynomial. For instance, $x^2 + 1$ has no real root, but two roots in the complex field, $x = \pm i$. One of the most important theorems in Mathematics, the **fundamental theorem of algebra**, states that the number of roots of a polynomial is equal to its degree, provided that certain roots, called the **multiple** roots, are counted several times. To see what I mean by that, let me review the case of a second degree polynomial.

The roots of the polynomial $ax^2 + bx^2 + c$ can be found by means of the classical formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\,,$$

which leads to:

- If $b^2 > 4ac$, we get two real solutions.
- If $b^2 = 4ac$, one real solution (double).
- If $b^2 < 4ac$, two complex solutions, which are not real.

Example 1. The polynomial $x^2 + x + 1$ has two (non-real) complex roots,

$$x = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\, i.$$

More specifically, the fundamental theorem states that any polynomial can be expressed as a product of elementary factors,

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = (x - x_1)(x - x_2)\cdots(x - x_n).\ \Box$$

Then $x_1, x_2, \ldots, x_n$ (real or complex) are the roots of the polynomial. In the formula of the second degree equation, we can see that when $\alpha + \beta i$ is a root of a polynomial, the **conjugate** $\alpha - \beta i$ is also a root. Therefore, non-real roots always come in pairs.

Example 2. $x^2 + 1 = (x - i)(x + i)$. $\square$

Example 3. $x^2 + x + 1 = \left(x + \frac{1}{2} - \frac{\sqrt{3}}{2}\, i\right)\left(x + \frac{1}{2} + \frac{\sqrt{3}}{2}\, i\right)$. $\square$

If a root is repeated in the above factorization, we call it a **multiple root**. $\square$

Example 4. $x = 1$ is a double root of $x^3 - x^2 - x + 1$, because of the decomposition $x^3 - x^2 - x + 1 = (x - 1)^2(x + 1)$. $\square$

Modulus and argument

Any complex number can be represented as a point of the plane. To do this, we take the real part as the abscissa, and the imaginary part as the ordinate. Real numbers are represented by the points in the horizontal axis. The distance between the planar representation of a complex number and the origin is called the **modulus** of the complex number, and the angle from the positive half-axis to the segment joining both points is called the **argument**.

In formulas, the **modulus** of $x = a + bi$ is

$$|x| = \sqrt{a^2 + b^2}.$$

For a real number, the modulus coincides with the absolute value. Since $x/|x|$ has unit modulus, it lies in the unit circle (centre $(0, 0)$ and radius 1) in the planar representation of complex numbers. For any point in this circle, the abscissa (i.e. the real part) must be $\cos\theta$ and the ordinate (the imaginary part) must be $\sin\theta$, where θ is the argument. We have thus the **polar representation**

$$x = |x|\big(\cos\theta + i\sin\theta\big).$$

Since the cosine and sine functions are periodic, the argument is not unique. If θ is an argument of x, then $\theta + 2\pi$ is another argument.

Complex exponentials

The following notation is useful. We write, for a real number θ,

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

We call the expression $e^{\theta i}$ a **complex exponential**. Note that, since $\cos^2\theta + \sin^2\theta = 1$, always $|e^{\theta i}| = 1$, so any nonzero complex number x can be written as $x = |x|\, e^{\theta i}$. For instance, it is easy to check that

$$e^{(\pi/2)i} = i, \qquad e^{\pi i} = -1, \qquad e^{-(\pi/2)i} = -i.$$

The advantage of complex exponentials is that, due to some properties of trigonometric functions, we can multiply them as if they were real exponentials, ie summing exponents,

$$e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1 + \theta_2)}.$$

By means of the complex exponential notation, we can easily write, for $n = 1, 2, \ldots$, the n-th **roots of unity**. These are the complex numbers z for which $x^n = 1$ or, equivalently, the roots of the polynomial $x^n - 1$. There are n such numbers, and one of them is 1. Since we have

$$x^n = \big(|x|\, e^{i\theta}\big)^n = |x|^n\, e^{in\theta},$$

the n-th roots of unity can be expressed as

$$x = e^{(2k\pi/n)i}, \quad k = 0, \ldots, n-1.$$

Example 5. For $n = 3$, we see that, besides $e^{0i} = 1$, we have two additional cubic roots of 1:

$$e^{(2\pi/3)i} = -\frac{1}{2} + \frac{\sqrt{3}}{2}\,i, \qquad e^{(4\pi/3)i} = -\frac{1}{2} - \frac{\sqrt{3}}{2}\,i.$$

Note that the three roots can be represented as the vertexes of a regular triangle.

Homework

A. Find the roots of $x^4 - 1$ and $x^4 + 1$.

B. Find the eighth roots of 1.