[MATH-03] Vectors

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Vectors

A **vector** is a one-dimensional array of mathematical objects, called components or terms. In these lectures, I only use vectors whose components are real numbers. Lower case boldface always indicates that the object denoted is a vector. The components of a vector are usually identified by a subscript. In documents, vectors are displayed as columns, with components within square brackets, as in

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

The number of terms is the **length** (also dimension) of the vector. E_n denotes the set of all vectors of length n, usually referred to as the n-dimensional space. An n–vector can be seen as an arrow in $\mathbb{R}^n$. If a and b are two points in $\mathbb{R}^n$, the components of the arrow pointing from a to b are $x_i = b_i - a_i$, for $i = 1, \ldots, n$. For example, the vector

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

can be represented as an arrow from $(1,1)$ to $(2,3)$, or as an arrow from $(-1,-2)$ to $(0,0)$. Note that vectors are free, i.e. the starting point is not fixed. Nevertheless, if we fix it at the origin, the components of a vector coincide with the coordinates of the end point. With this convention, n–vectors are sometimes identified with points in $\mathbb{R}^n$.

The sum of two vectors is done termwise, summing terms at the same position. So, the sum can only be defined for vectors with the same length,

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}.$$

Note that the sum of vectors can be done graphically. If we place the starting point of $\mathbf{y}$ at the end point of $\mathbf{x}$, $\mathbf{x}+\mathbf{y}$ is the arrow from the starting point of $\mathbf{x}$ to the end point of $\mathbf{y}$. An equivalent approach is to place both vectors at the origin and use them to draw a parallelogram. Then, $\mathbf{x}+\mathbf{y}$ is given by the diagonal of the parallelogram that joins the origin and the opposite vertex.

Numbers are sometimes called **scalars** in Mathematics and Physics. The name comes from the distinction made in Physics between scalar magnitudes, like temperature, and vector magnitudes, like force, is classical. The product of a scalar by a vector is defined as

$$\lambda\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{bmatrix}.$$

Linear dependence

We say that a vector $\mathbf{x}$ is a **linear combination** of the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$, when there exist numbers $\alpha_1, \ldots, \alpha_k$ such that

$$\mathbf{x} = \alpha_1 \mathbf{u}_1 + \cdots + \alpha_k \mathbf{u}_k.$$

Alternatively, one can say that $\mathbf{x}$ depends linearly of $\mathbf{u}_1, \ldots, \mathbf{u}_k$. The vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are **linearly independent** when none of them can be expressed as a linear combination of the rest. It is easy to see that $\mathbf{u}_1, \ldots, \mathbf{u}_n$ are linearly independent when the only way to write the zero vector as a linear combination of them is to take all the coefficients equal to zero, i.e. when

$$\alpha_1 \mathbf{u}_1 + \cdots + \alpha_k \mathbf{u}_k = \mathbf{0} \implies \alpha_1 = \cdots = \alpha_k = 0.$$

It follows from this that, if $\mathbf{u}_1, \ldots, \mathbf{u}_n$ are linearly independent and $\mathbf{x}$ is dependent of the $\mathbf{u}_i$, then the representation of $\mathbf{x}$ as a linear combination $\mathbf{x}$ of the $\mathbf{u}_i$ (i.e. the coefficients α_i) are unique.

¶ The zero vector $\mathbf{0}$ is linearly dependent on any set of nonzero vectors.

Example 1. If

$$\mathbf{x} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \qquad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

$\mathbf{x}$ is a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$, because $\mathbf{x} = -\mathbf{u}_1 + 4\mathbf{u}_2 - \mathbf{u}_3$. It is easy to check, because of the positions of zeros and ones, that $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ are linearly independent, and that this representation is unique.

Example 2. The representation of a vector as a linear combination of other vectors is not unique if these are not independent. For instance, take

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \qquad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Then $\mathbf{x} = \mathbf{u}_1 + 2\mathbf{u}_2$, but also $\mathbf{x} = \mathbf{u}_2 + \mathbf{u}_3$.

Subspaces, bases and dimension

A **linear subspace** of E_n is a subset S such that any linear combination of vectors of S gives a vector of S. Equivalently, S is a subspace when:

- $u_1,\ u_2 \in S \Longrightarrow u_1 + u_2 \in S$.
- $u \in S \Longrightarrow \alpha u \in S$ for any number α.

The **dimension** of a linear subspace S is the maximum number of linearly independent vectors of S. Any other vector of S will be then a linear combination of such set, which is said to be a **basis** of the subspace. so, if $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is a basis of S, any vector of S must be a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_k$.

Example 3. The set $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, with

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

is a basis of E_3. Indeed, they are linearly independent, and any vector $\mathbf{x}$ in S can be expressed as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

It follows from this example that the whole space E_n has dimension n, since a basis $e_1, \dots, e_n$ can be found using the same idea. This is the **canonical basis**.

¶ Do not confound bases and dimension. For any subspace, the dimension is a fixed number, whereas a basis is a set of as many vectors as indicated by the dimension. A subspace has infinitely many bases, but only one dimensionality.

Example 4. The set S defined by $S = \{\mathbf{x} \in E_3 : x_1 + x_2 = x_1 - x_3 = 0\}$ is a linear subspace of E_3 of dimension 1. The vector

$$\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

alone is a basis of S, because any vector in S can be expressed as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 \\ x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Example 5. The set $S = \{\mathbf{x} \in E_3 : x_1 - x_2 + 2x_3 = 0\}$ is a linear subspace of E_3. The dimension is 2. This follows from the decomposition

$$\mathbf{x} = x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}.$$

Example 6. The set $S = \{\mathbf{x} \in E_2 : x_1^2 - x_2 = 0\}$ is not a linear subspace of E_2. Indeed, if $x_1 = x_2 = 1$, $\mathbf{x} \in S$ but $2\mathbf{x} \notin S$.

Homework

A. For

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix},$$

check that $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ are linearly independent, and write $\mathbf{x}$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$.

B. Prove that the following set is a basis of E_3:

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

C. Show that, if $\mathbf{u}_1$ and $\mathbf{u}_2$ form a basis of a 2-dimensional subspace S, then $\mathbf{v}_1 = \mathbf{u}_1 - \mathbf{u}_2$ and $\mathbf{v}_2 = \mathbf{u}_1 + \mathbf{u}_2$ form another basis of S. Draw the corresponding arrows. The axes have been rotated 45° clockwise!

D. Given

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{e}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \qquad \mathbf{e}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \qquad \mathbf{e}_4 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 1 \end{bmatrix},$$

prove that $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{e}_4$ are linearly independent, and write $\mathbf{x}$ as a linear combination of them.

E. Find a basis of the subspace $S = \{\mathbf{x} \in E_4 : x_1 + x_2 = x_3 - x_4\}$.