[MATH-16] Definite integrals

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Definite integrals

Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function, and consider a partition

$$x_0 = a < x_1 < \cdots < x_n = b$$

of the interval $[a, b]$ into n subintervals of equal length. Let me denote $y_k = f(x_k)$ and $h = (b-a)/n$. For each sub-interval $[x_k, x_{k+1}]$, the rectangle whose vertexes are

$$(x_k, y_k),\ (x_{k+1}, y_k),\ (x_k, 0),\ (x_{k+1}, 0)$$

has height y_k and width h. The sum of the areas of the n rectangles is then

$$h(y_0 + y_1 + \cdots + y_{n-1}).$$

The limit of this sum, as $n \to \infty$, is the **integral** of f on the interval $[a, b]$. Then, a and b are the **integration limits** and f is the **integrand**. When $f(x) > 0$, the integral coincides with the area of the region delimited by the curve of equation $y = f(x)$, the vertical lines $x = a$ and $x = b$, and the x axis.

There is an elegant formula, called **Barrow's formula**, for the calculation of integrals. If $F(x)$ is a primitive of $f(x)$,

$$\int_a^b f(x)\,dx = \big[F(x)\big]_{x=a}^{x=b} = F(b) - F(a).$$

This formula justifies the use of the integral sign in the calculation of primitives. Integrals, as defined in this lecture, are called **definite integrals**, as opposed to indefinite integrals. Of course, Barrow's formula can only be applied when one knows how to find the primitive.

Example 1. Since $F(x) = (3/2)x^{2/3}$ is a primitive of $f(x) = x^{-1/3}$, Barrow's formula gives

$$\int_1^2 \frac{dx}{\sqrt[3]{x}} = \int_1^2 x^{-1/3}\,dx = \left[\frac{x^{2/3}}{2/3}\right]_{x=1}^{x=2} = \frac{3}{2}\left(2^{2/3} - 1\right) = 0.881102. \ \square$$

Example 2. $\displaystyle\int_0^1 \frac{dx}{1+x} = \Big[\log(1+x)\Big]_{x=0}^{x=1} = \log 2 = 0.693147. \ \square$

Change of variable in a definite integral

When using a change of variable in an definite integral, there is no need of reversing the change. We replace the integration limits by $u_1 = u(x_1)$ and $u_2 = u(x_2)$, obtaining the value of the transformed integral without need of going back to the original integral. The substitution formula for definite integrals is

$$\int_{x_1}^{x_2} f(x)\,dx = \int_{u_1}^{u_2} f\big(x(u)\big)\,x'(u)\,du.$$

Example 3. With the substitution $x = u^2$, we get

$$\int_0^4 \frac{dx}{1+\sqrt{x}} = \int_0^2 \frac{2u\,dt}{1+u} = \int_0^2 2\left(1 - \frac{1}{1+u}\right) du = \Big[2\big(u - \log(1+u)\big)\Big]_{u=0}^{u=2} = 4 - 2\,\log 3.\ \square$$

Improper integrals

Let me consider now $f : [a, +\infty) \longrightarrow \mathbb{R}$, and suppose that the limit

$$\lim_{b\to+\infty} \int_a^b f(x)\,dx$$

exists. Dealing with these limits may be difficult in the general case but, for positive functions, the integral on $[a, b]$ is an increasing function of b, so that there is always a limit, either finite or infinite. We denote this limit by

$$\int_a^{+\infty} f(x)\,dx,$$

and say that this improper integral (what is improper is that one of the integration limits is infinite) converges. Improper integrals with an integration limit equal to $-\infty$ are defined in a similar way.

Example 4. The integral of the exponential $f(x) = e^{-x}$ in $[0, +\infty]$ converges, since

$$\int_0^{+\infty} e^{-x}\,dx = \lim_{b\to+\infty} \int_0^b e^{-x}\,dx = \lim_{b\to+\infty} \Big[-e^{-x}\Big]_{x=0}^{x=b} = \lim_{b\to+\infty} \big(1 - e^{-b}\big) = 1.\ \square$$

Example 5. An example of a divergent integral is

$$\int_0^{+\infty} \frac{dx}{1+x} = \Big[\log(1+x)\Big]_{x=0}^{x=+\infty} = +\infty.\ \square$$

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Numerical integration

A definite integral can be approximated by a sum of n rectangles, with n high enough:

$$\int_a^b f(x)\,dx \approx h(y_0 + y_1 + \cdots + y_{n-1}).$$

Let me show how this works in the integral of Example 1.

Example 1 (continuation). By applying this formula, with $n = 8$ ($h = 0.125$), to the integral of Example 1, we get the (not so bad) approximate value

$$\begin{aligned}\int_1^2 \frac{dx}{\sqrt[3]{x}} &\approx 0.125\left(\frac{1}{\sqrt[3]{1}} + \frac{1}{\sqrt[3]{1.125}} + \frac{1}{\sqrt[3]{1.25}} + \frac{1}{\sqrt[3]{1.375}} + \frac{1}{\sqrt[3]{1.5}} + \frac{1}{\sqrt[3]{1.625}} + \frac{1}{\sqrt[3]{1.75}} + \frac{1}{\sqrt[3]{1.875}}\right)\\ &= 0.894257.\end{aligned}$$

What happens if, instead of taking rectangles of height y_k in the formula of the rectangles, we take y_{k+1}? In this example, we get

$$\begin{aligned}\int_1^2 \frac{dx}{\sqrt[3]{x}} &\approx 0.125\left(\frac{1}{\sqrt[3]{1.125}} + \frac{1}{\sqrt[3]{1.25}} + \frac{1}{\sqrt[3]{1.375}} + \frac{1}{\sqrt[3]{1.5}} + \frac{1}{\sqrt[3]{1.625}} + \frac{1}{\sqrt[3]{1.75}} + \frac{1}{\sqrt[3]{1.875}} + \frac{1}{\sqrt[3]{2}}\right)\\ &= 0.868469.\end{aligned}$$

We see that the exact solution is somewhere in the middle of these two approximations, so that we get a better solution by averaging them:

$$\int_1^2 \frac{dx}{\sqrt[3]{x}} \approx 0.881363. \ \square$$

Algorithms for numerical integration

There are many alternative formulas that improve the approximations based on rectangles. The most popular is the **trapeze formula**, in which the integral is approximated by a sum of areas of trapezes (instead of rectangles),

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\,(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n).$$

Since the area of the trapeze is the average of two rectangles, this approach is equivalent to the average of the preceding example. It can also be seen as approximating the curve $y = f(x)$ by that the polygonal obtained joining the points (x_k, y_k) by segments (this is called a piecewise linear approximation.

For n even, the results of the trapeze formula can be improved with a formula that results from replacing the linear segments by parabolic arcs,

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\big(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 4y_{n-1} + y_n\big).$$

This is the **Simpson formula**.

Example 1 (continuation). By taking $n = 8$ in the Simpson formula, we get

$$\begin{aligned}\int_1^2 \frac{dx}{\sqrt[3]{x}} &\approx \frac{0.125}{3}\bigg(\frac{1}{\sqrt[3]{1}} + \frac{4}{\sqrt[3]{1.125}} + \frac{2}{\sqrt[3]{1.25}} + \frac{4}{\sqrt[3]{1.375}} \\ &\qquad + \frac{2}{\sqrt[3]{1.5}} + \frac{4}{\sqrt[3]{1.625}} + \frac{2}{\sqrt[3]{1.75}} + \frac{4}{\sqrt[3]{1.875}} + \frac{1}{\sqrt[3]{2}}\bigg) \\ &= 0.881103,\end{aligned}$$

improving the preceding approximations. $\square$

Homework

A. Calculate $\displaystyle\int_1^{+\infty} \frac{dx}{1+x^2}$.