

[MATH-02] Vectors

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Vectors

A **vector** is a one-dimensional array of mathematical objects, called components or terms. In these lectures, I only use vectors whose components are real numbers. Lower case boldface always indicates that the object denoted is a vector. The components of a vector are usually identified by a subscript. In documents, vectors are displayed as columns, with components within square brackets, as in

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

The number of terms is the **length** (also dimension) of the vector. E_n denotes the set of all vectors of length n , usually referred to as the n -dimensional space. An n -vector can be seen as an arrow in \mathbb{R}^n . If a and b are two points in \mathbb{R}^n , the components of the arrow pointing from a to b are $x_i = b_i - a_i$, for $i = 1, \dots, n$. For example, the vector

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

can be represented as an arrow from $(1, 1)$ to $(2, 3)$, or as an arrow from $(-1, -2)$ to $(0, 0)$. Note that vectors are free, i.e. the starting point is not fixed. Nevertheless, if we fix it at the origin, the components of a vector coincide with the coordinates of the end point. With this convention, n -vectors are sometimes identified with points in \mathbb{R}^n .

The sum of two vectors is done termwise, summing terms at the same position. So, the sum can only be defined for vectors with the same length,

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}.$$

Note that the sum of vectors can be done graphically. If we place the starting point of \mathbf{y} at the end point of \mathbf{x} , $\mathbf{x} + \mathbf{y}$ is the arrow from the starting point of \mathbf{x} to the end point of \mathbf{y} . An equivalent approach is to place both vectors at the origin and use them to draw a parallelogram. Then, $\mathbf{x} + \mathbf{y}$ is given by the diagonal of the parallelogram that joins the origin and the opposite vertex.

Numbers are sometimes called **scalars** in Mathematics and Physics. The name comes from the distinction made in Physics between scalar magnitudes, like temperature, and vector magnitudes, like force, is classical. The product of a scalar by a vector is defined as

$$\lambda \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{bmatrix}.$$

Linear dependence

We say that a vector \mathbf{x} is a **linear combination** of the vectors $\mathbf{u}_1, \dots, \mathbf{u}_k$, when there exist numbers $\alpha_1, \dots, \alpha_k$ such that

$$\mathbf{x} = \alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k.$$

Alternatively, one can say that \mathbf{x} depends linearly of $\mathbf{u}_1, \dots, \mathbf{u}_k$. The vectors $\mathbf{u}_1, \dots, \mathbf{u}_k$ are **linearly independent** when none of them can be expressed as a linear combination of the rest. It is easy to see that $\mathbf{u}_1, \dots, \mathbf{u}_n$ are linearly independent when the only way to write the zero vector as a linear combination of them is to take all the coefficients equal to zero, i.e. when

$$\alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k = \mathbf{0} \implies \alpha_1 = \dots = \alpha_k = 0.$$

It follows from this that, if $\mathbf{u}_1, \dots, \mathbf{u}_n$ are linearly independent and \mathbf{x} is dependent of the \mathbf{u}_i , then the representation of \mathbf{x} as a linear combination \mathbf{x} of the \mathbf{u}_i (i.e. the coefficients α_i) are unique.

¶ The zero vector $\mathbf{0}$ is linearly dependent on any set of nonzero vectors.

Example 1. If

$$\mathbf{x} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

\mathbf{x} is a linear combination of $\mathbf{u}_1, \mathbf{u}_2$ and \mathbf{u}_3 , because $\mathbf{x} = -\mathbf{u}_1 + 4\mathbf{u}_2 - \mathbf{u}_3$. It is easy to check, because of the positions of zeros and ones, that $\mathbf{u}_1, \mathbf{u}_2$ and \mathbf{u}_3 are linearly independent, and that this representation is unique.

Example 2. The representation of a vector as a linear combination of other vectors is not unique if these are not independent. For instance, take

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Then $\mathbf{x} = \mathbf{u}_1 + 2\mathbf{u}_2$, but also $\mathbf{x} = \mathbf{u}_2 + \mathbf{u}_3$.

Subspaces, bases and dimension

A **linear subspace** of E_n is a subset S such that any linear combination of vectors of S gives a vector of S . Equivalently, S is a subspace when:

- $u_1, u_2 \in S \implies u_1 + u_2 \in S$.
- $u \in S \implies \alpha u \in S$ for any number α .

The **dimension** of a linear subspace S is the maximum number of linearly independent vectors of S . Any other vector of S will be then a linear combination of such set, which is said to be a **basis** of the subspace. so, if $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a basis of S , any vector of S must be a linear combination of $\mathbf{u}_1, \dots, \mathbf{u}_k$.

Example 3. The set $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, with

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

is a basis of E_3 . Indeed, they are linearly independent, and any vector \mathbf{x} in S can be expressed as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

It follows from this example that the whole space E_n has dimension n , since a basis e_1, \dots, e_n can be found using the same idea. This is the **canonical basis**.

¶ Do not confound bases and dimension. For any subspace, the dimension is a fixed number, whereas a basis is a set of as many vectors as indicated by the dimension. A subspace has infinitely many bases, but only one dimensionality.

Example 4. The set S defined by $S = \{\mathbf{x} \in E_3 : x_1 + x_2 = x_1 - x_3 = 0\}$ is a linear subspace of E_3 of dimension 1. The vector

$$\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

alone is a basis of S , because any vector in S can be expressed as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 \\ x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Example 5. The set $S = \{\mathbf{x} \in E_3 : x_1 - x_2 + 2x_3 = 0\}$ is a linear subspace of E_3 . The dimension is 2. This follows from the decomposition

$$\mathbf{x} = x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}.$$

Example 6. The set $S = \{\mathbf{x} \in E_2 : x_1^2 - x_2 = 0\}$ is not a linear subspace of E_2 . Indeed, if $x_1 = x_2 = 1$, $\mathbf{x} \in S$ but $2\mathbf{x} \notin S$.

Homework

A. For

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix},$$

check that \mathbf{u}_1 , \mathbf{u}_2 , and \mathbf{u}_3 are linearly independent, and write \mathbf{x} as a linear combination of \mathbf{u}_1 , \mathbf{u}_2 , and \mathbf{u}_3 .

B. Prove that the following set is a basis of E_3 :

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

C. Find a basis of the subspace $S = \{\mathbf{x} \in E_4 : x_1 + x_2 = x_3 - x_4\}$.