# [MATH-06] Product of matrices

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### The product of a matrix and a vector

The product of a matrix and a vector is defined in such a way that the product of two vectors is just a particular case, that in which the first factor is a 1-row matrix. If we write

$$\begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + \cdots + x_n y_n,$$

the scalar product becomes

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^{\mathsf{T}}\, \mathbf{y}.$$

Thus, the product of a row by a column is a number, that can be considered as a $(1,1)$–matrix. Next, we extend this definition to the product of an $(n,m)$–matrix $\mathbf{A}$ and an $m$–vector $\mathbf{x}$. To do this, we take $\mathbf{A}$ as a pack of row vectors, multiplying each row by $\mathbf{x}$ and placing the numbers that result form these products as the coordinates of a vector, which is the product $\mathbf{Ax}$. In mathematical symbols,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} \sum_j a_{1j} x_j \\ \sum_j a_{2j} x_j \\ \vdots \\ \sum_j a_{nj} x_j \end{bmatrix}.$$

If $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are vectors corresponding to the rows of $\mathbf{A}$, this can be written as

$$\begin{bmatrix} \mathbf{a}_1^{\mathsf{T}} \\ \mathbf{a}_2^{\mathsf{T}} \\ \vdots \\ \mathbf{a}_n^{\mathsf{T}} \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{a}_1^{\mathsf{T}}\mathbf{x} \\ \mathbf{a}_2^{\mathsf{T}}\mathbf{x} \\ \vdots \\ \mathbf{a}_n^{\mathsf{T}}\mathbf{x} \end{bmatrix}.$$

### Matrices as linear operators

The definition of the product of a matrix and a vector allows an interesting interpretation of a matria as a **linear operator**. Let us fix $\mathbf{A}$ and define an operator (functions are called operators in this context)

$$\begin{aligned} T : E_m &\longrightarrow E_n \\ \mathbf{x} &\longmapsto \mathbf{Ax}. \end{aligned}$$

Such operator would be linear, i.e. it would transform a linear combination into a linear combination,

$$T\big(\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2\big) = \alpha_1\, T\mathbf{u}_1 + \alpha_2\, T\mathbf{u}_2.$$

Conversely, any operator with this property can be associated to a matrix. It is easy to check that the columns of the matrix associated to a linear operator coincide with the images of the vectors of the canonical basis under the operator. A one-to-one linear operator $T : E_n \to E_n$ is called a **linear transformation**.

**Example 1.** The matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

defines the operator $T : E_3 \to E_4$ as

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ -x_3 \\ x_1 + x_3 \\ x_2 - 2x_3 \end{bmatrix}. \ \square$$

### The null space

An interesting definition related to linear operators is the **null space** (also called kernel). The null space of $\mathbf{A}$ (equivalently, of $T$), that we denote by $\mathcal{N}(\mathbf{A})$, is the subspace of all vectors $\mathbf{x}$ of $E_m$ such that $\mathbf{A}\mathbf{x} = \mathbf{0}$. The vector $\mathbf{0}$ always belongs to the null space. We may also say that $\mathbf{x} \in \mathcal{N}(\mathbf{A})$ when $\mathbf{x}$ is orthogonal to the rows of $\mathbf{A}$.

The subspace of all vectors $\mathbf{A}\mathbf{x}$, with $\mathbf{x} \in E_n$, is called the **range** of $\mathbf{A}$, denoted by $\mathcal{R}(\mathbf{A})$. Since the vectors of $E_n$ are the linear combinations of the canonical basis, those of $\mathcal{R}(\mathbf{A})$ are the linear combinations of the columns of $\mathbf{A}$. There are as many linearly independent column vectors as indicated by the rank, which thus coincides with the dimension of $\mathcal{R}(\mathbf{A})$. An interesting formula is (mind that, here, $m$ is the number of columns)

$$\dim \mathcal{N}(\mathbf{A}) + \operatorname{rank} \mathbf{A} = m.$$

In particular, when $\mathbf{A}$ is a square matrix, the existence of vectors $\mathbf{x} \neq \mathbf{0}$ such that $\mathbf{A}\mathbf{x} = \mathbf{0}$ is equivalent to $\det(\mathbf{A}) = 0$.

**Example 2** The linear subspace $S = \{\mathbf{x} \in E_3 : x_1 + x_2 = x_1 - x_3 = 0\}$ is the null space of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}.$$

As the rank of this matrix is 2, the null space has dimension 1. This has been previously seen (lecture 3) by finding a basis of $S$. $\square$

### The product of two matrices

The definition of the product of two matrices extends the previous definition: in order to multiply two matrices $\mathbf{A}$ and $\mathbf{B}$, we take $\mathbf{B}$ as a pack of column vectors and multiply $\mathbf{A}$ by each column of $\mathbf{B}$, placing the resulting vectors as the columns of the product matrix $\mathbf{A}\,\mathbf{B}$. In mathematical symbols,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1k} \\ b_{21} & b_{22} & \cdots & b_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mk} \end{bmatrix} = \begin{bmatrix} \sum_j a_{1j}b_{j1} & \sum_j a_{1j}b_{j2} & \cdots & \sum_j a_{1j}b_{jk} \\ \sum_j a_{2j}b_{j1} & \sum_j a_{2j}b_{j2} & \cdots & \sum_j a_{2j}b_{jk} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_j a_{nj}b_{j1} & \sum_j a_{nj}b_{j2} & \cdots & \sum_j a_{nj}b_{jk} \end{bmatrix},$$

or, if $\mathbf{a}_1^\intercal, \ldots, \mathbf{a}_n^\intercal$ are the rows of $\mathbf{A}$ and $\mathbf{b}_1, \ldots, \mathbf{b}_m$ the columns of $\mathbf{B}$,

$$\begin{bmatrix} \mathbf{a}_1^\intercal \\ \mathbf{a}_2^\intercal \\ \vdots \\ \mathbf{a}_n^\intercal \end{bmatrix} \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_m \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^\intercal\mathbf{b}_1 & \mathbf{a}_1^\intercal\mathbf{b}_2 & \cdots & \mathbf{a}_1^\intercal\mathbf{b}_m \\ \mathbf{a}_2^\intercal\mathbf{b}_1 & \mathbf{a}_2^\intercal\mathbf{b}_2 & \cdots & \mathbf{a}_2^\intercal\mathbf{b}_m \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}_n\mathbf{b}_1 & \mathbf{a}_n^\intercal\mathbf{b}_2 & \cdots & \mathbf{a}_n^\intercal\mathbf{b}_m \end{bmatrix}.$$

Note that this definition only makes sense when the number of columns in $\mathbf{A}$ (the length of the rows) equals the number of rows in $\mathbf{B}$ (the length of the columns). The dimensions of the factors and the product satisfy the rule

$$\big[n \times m\big]\big[m \times k\big] = \big[n \times k\big].$$

**Example 3.**

$$\begin{bmatrix} 1 & 2 & 4 \\ -1 & 0 & 1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 17 & 21 \\ 2 & 6 \\ 11 & 6 \end{bmatrix}. \square$$

The product of matrices has the following properties (we assume that the dimensions are such that the formulas make sense):

- $\mathbf{A}\big(\mathbf{BC}\big) = \big(\mathbf{AB}\big)\mathbf{C}$ (associative property).
- In general, $\mathbf{AB} \neq \mathbf{BA}$ (see the example below).
- $(\mathbf{AB})^\mathsf{T} = \mathbf{B}^\mathsf{T}\mathbf{A}^\mathsf{T}$.
- $\operatorname{tr}\big(\mathbf{AB}\big) = \operatorname{tr}\big(\mathbf{BA}\big)$.
- $\det\big(\mathbf{AB}\big) = \det(\mathbf{A})\det(\mathbf{B})$.

**Example 4.** For

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},$$

we have

$$\mathbf{AB} = \begin{bmatrix} -1 & 3 \\ -2 & 8 \end{bmatrix}, \qquad \mathbf{BA} = \begin{bmatrix} 4 & 7 \\ 2 & 3 \end{bmatrix},$$

so $\mathbf{AB} \neq \mathbf{BA}$. Noevertheless $\det\big(\mathbf{AB}\big) = \det\mathbf{A}\,\det\mathbf{B} = -2$. $\square$

**Example 5.** A interesting example of non-commutativity of the product of matrices is given by

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = [n], \qquad \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Note that the trace of both products is equal to $n$. $\square$

## Inverse matrices

The **identity matrix** of dimension $n$ is defined as

$$\mathbf{I}_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

The subscript $n$ in $\mathbf{I}_n$ is omitted if there is no danger of confusion. For an $(n, m)$–matrix $\mathbf{A}$,

$$\mathbf{AI}_m = \mathbf{I}_n\mathbf{A} = \mathbf{A}.$$

So, multiplying a matrix by the identity is like multiplying a number by 1. It is natural, then to define the inverse of a square matrix $\mathbf{A}$ as a matrix $\mathbf{A}^{-1}$ such that

$$\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}.$$

A matrix is said to be **nonsingular** when it has inverse. It can be proved that, for an $(n,n)$–matrix $\mathbf{A}$, the following assertions are equivalent:

- $\mathbf{A}$ is nonsingular.
- $\det \mathbf{A} \neq 0$.
- The rank of $\mathbf{A}$ is $n$.
- The only vector in the null space of $\mathbf{A}$ is the zero vector.

If $\mathbf{A}$ and $\mathbf{B}$ are nonsingular and have the same dimension, $\mathbf{AB}$ is also nonsingular, and

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}.$$

**Example 6.**

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}. \square$$

Can you give a rule for the inverses of diagonal matrices?

**Example 7.** The matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

has no inverse. Indeed, $\mathbf{AB} = \mathbf{I}$ would lead to an impossible equation,

$$\begin{bmatrix} b_{11} - b_{21} & b_{12} - b_{22} \\ -b_{11} + b_{21} & -b_{12} + b_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \square$$

## Homework

**A.** Find the rank and a basis of the null space of

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}.$$

**B.** Check the identity $\mathbf{AB} = \mathbf{BA}$ for

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 0 \\ -1 & -1 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -2 & -1 & -6 \\ 3 & 2 & 9 \\ -1 & -1 & -4 \end{bmatrix}.$$

**C.** Check, for dimension 2, that the trace of the product does not depend on the order of the factors for the matrices:

$$\operatorname{tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & y \\ z & t \end{bmatrix}\right) = \operatorname{tr}\left(\begin{bmatrix} x & y \\ z & t \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right).$$

**D.** Only square matrices can have inverses, because it is impossible that both $\mathbf{AB} = \mathbf{I}$ and $\mathbf{BA} = \mathbf{I}$ if $\mathbf{A}$ and $\mathbf{B}$ are not square. See why this is so in the following case: for a $(2,3)$–matrix $\mathbf{A}$ and a $(3,2)$–matrix $\mathbf{B}$, $\mathbf{AB} = \mathbf{I}$ is possible, but $\mathbf{BA} = \mathbf{I}$ is impossible.

**E.** Find the rank and a basis of the null space of

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 2 & -1 & 1 \\ 2 & 0 & 3 & 5 & 1 \\ 3 & 1 & -1 & 1 & -4 \\ 5 & 2 & 0 & 3 & -5 \end{bmatrix}.$$

**F.** Calculate

$$\begin{bmatrix} 3 & -1 & 2 \\ 1 & 0 & 3 \\ 4 & 0 & 2 \end{bmatrix}^{-1}.$$

**G.** Solve the equation

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \mathbf{X} = \begin{bmatrix} 3 & 5 \\ 5 & 9 \end{bmatrix},$$

both directly and through the formula

$$\mathbf{X} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 5 \\ 5 & 9 \end{bmatrix}.$$