[MATH-01] Functions and limits

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Functions of one variable

Mathematicians denote by $\mathbb{R}$ the set of real numbers, also called the **real line**. Then $\mathbb{R}^n$ stands for the n**-dimensional space**, whose elements are called **points**. In particular, $\mathbb{R}^2$ is the **plane**, identified to the set of all pairs (x, y) of real numbers. x is the **abscissa** and y is the **ordinate**.

A function is a rule that assigns, to each element of some set, a real number (only real-valued functions appear in this course). That set is called the **domain** of the function. Domains are usually specified by **constraints**, such as $0 \le x \le 1$. We use expressions like $f : D \longrightarrow \mathbb{R}$ and $D \xrightarrow{f} \mathbb{R}$, in which D is the domain and f is the function, to denote functions. When $D \subset \mathbb{R}^n$, we have a function of n variables. If x is an element of D (in short $x \in D$), the number assigned to x by f is called the **image** of x, denoted by $f(x)$.

For the moment being, I only consider functions of one variable. When f is one-to-one, i.e. when

$$x_1 \ne x_2 \Longrightarrow f(x_1) \ne f(x_2),$$

it is possible to define the **inverse** function f^{-1}. The domain of f^{-1} is the **range** of f, that is, the set of all $y \in \mathbb{R}$ for which there is a (unique) $x \in D$ such that $y = f(x)$. When the inverse f^{-1} exists, $y = f(x)$ and $x = f^{-1}(y)$ are equivalent formulas.

Not every function has an inverse. An example of a function without inverse is the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. Nevertheless, if we restrict f to the domain of the positive numbers, it has an inverse $f^{-1}(y) = \sqrt{y}$, also defined in the set of positive numbers.

In this lecture, the domain will be an **interval** of the real line. If $a < b$, the open interval defined by a and b is the set

$$(a, b) = \big\{x \in \mathbb{R} : a < x < b\big\}.$$

a and b are then called the **extreme points** of the interval. Similarly, the closed interval $[a, b]$ is given by the constraints $a \le x \le b$. Sometimes, $+\infty$ and $-\infty$ are used as extreme points. Thus,

$$\mathbb{R} = (-\infty, +\infty), \qquad \big\{x \in \mathbb{R} : x \ge 0\big\} = [0, +\infty).$$

The curve of equation $y = f(x)$, i.e. the set of all points (x, y) of the plane which satisfy this formula, is called the **graph** of f. Graphs are easily drawn with mathematical software, even with a spreadsheet, as far as one is able to write a mathematical expression for the function in this software.

The simplest functions are the **linear functions**, given by expressions like $f(x) = ax + b$, in which a and b are constants. The graph of a linear function is a straight line. a is called the **slope** and b the **intercept**. **Quadratic functions** are defined by polynomials of second degree, $f(x) = ax^2 + bx + c$. The graph of a quadratic function is a **parabola**.

A **power** function is defined by an expression $f(x) = x^\alpha$, in which the exponent α can be any (constant) real number. Except for some special values of α (such as for $\alpha = 1/3$), a power function is defined only for $x > 0$. Other elementary functions, such as exponentials, logarithms and trigonometric functions, are introduced in the lectures that follow.

Elementary functions are combined, both algebraically and by forming **composite** functions. The mathematical notation for a composite function is $f_2 \circ f_1$,

$$(f_2 \circ f_1)(x) = f_2\big(f_1(x)\big).$$

For instance, $f(x) = \sqrt{x^2+1}$ can be considered as a composite, $f = f_2 \circ f_1$, with $f_1(x) = x^2 + 1$ and $f_2(u) = \sqrt{u}$.

Monotonic functions

We say that a function f is **increasing** if

$$x_1 < x_2 \Longrightarrow f(x_1) < f(x_2),$$

i.e. when $f(x)$ increases as x increases. The graph of an increasing function is a curve that goes up as x runs from left to right. In the same way, we say that f is **decreasing** when

$$x_1 < x_2 \Longrightarrow f(x_1) > f(x_2).$$

Now, the graph goes down when x runs from the left to the right. The term **monotonic** usually refers to both increasing and decreasing functions, but in some contexts (e.g. in Microeconomics), means just increasing. It is sometimes useful to distinguish between increasing (resp. decreasing) and **non-decreasing** (resp. non-increasing) functions, in the same way as we distinguish between positive ($x > 0$) and non-negative numbers ($x \ge 0$). Thus, a function f is non-decreasing when

$$x_1 < x_2 \Longrightarrow f(x_1) \le f(x_2).$$

Limits

Assigning to each positive integer n a real number x_n, we have a **sequence**. We say that x_0 is the limit of this sequence, in short

$$\lim_{n\to\infty} x_n = x_0,$$

when, for every $\epsilon > 0$, there is some term of the sequence after which we have $|x_n - x_0| < \epsilon$.

This definition can be extended to infinite limits. We say that the sequence has limit $+\infty$ when, for every $M > 0$, there is a term of the sequence after which $|x_n| > M$. The limit $-\infty$ is defined in a similar way.

The following properties are straightforward:

(i) $\lim \big(x_n + y_n\big) = \lim x_n + \lim y_n$.

(ii) $\lim \big(x_n\, y_n\big) = \lim x_n \cdot \lim y_n$.

(iii) $\lim 1/x_n = \dfrac{1}{\lim x_n}$.

Some of these rules work also with infinite limits. They are usually condensed into formulas such as

$$\infty + \infty = \infty, \qquad \infty \cdot \infty = \infty, \qquad 1/0 = \infty, \qquad 1/\infty = 0.$$

These rules do not cover all the cases, so that certain situations require a special treatment. We call them **indeterminations**. There are six indeterminations:

$$\infty - \infty, \qquad 0 \cdot \infty, \qquad 0/0, \qquad \infty/\infty, \quad 0^0, \qquad \infty^0, \qquad 1^\infty.$$

For instance, taking $x_n = n^2$ and $y_n = 1/n$, we get $0 \cdot \infty = \infty$. Nevertheless, taking $x_n = n$ and $y_n = 1/n^2$, we get $0 \cdot \infty = 0$.

Limits of functions

The definition of the limit can be extended to functions. For a function f, we say that y_0 is the limit of f at x_0, in short

$$\lim_{x \to x_0} f(x) = y_0,$$

when for every sequence x_n with limit $x_>0$, the sequence $f(x_n)$ has limit y_0. This is easily understood as: $f(x)$ approaches y_0 when x approaches x_0. You are probably familiarized with the expression "$f(x) \to y_0$ as $x \to x_0$". The extension of this definition to the cases in which x_0 or y_0 (or both) are infinite are easy.

We say that f is continuous at a point x_0 of its domain when $\lim_{x \to x_0} f(x) = f(x_0)$. In practice, this means that the graph of f is not "broken" at any point.

Exponentials

An exponential function is defined by an expression $f(x) = a^x$. The **basis** a can be any positive constant. An exponential function is defined for any $x \in \mathbb{R}$. The key property of exponentials is that they transform sums into products,

$$a^{x_1 + x_2} = a^{x_1}\, a^{x_2}.$$

From this formula, it can be derived that $\left(a^x\right)^\alpha = a^{\alpha x}$, for any α, and also that $a^0 = 1$ and $a^{-x} = 1/a^x$. You are probably familiarized with these formulas. When $a > 1$, we have:

- $x \longmapsto a^x$ is increasing.
- $x > 0 \Longrightarrow a^x > 1$.
- $x < 0 \Longrightarrow 0 < a^x < 1$.

When $0 < a < 1$, the exponential is decreasing, so $0 < a^x < 1$ for $x > 0$, and $a^x > 1$ for $x < 0$. A very special case is given by $a = e$. The exponential function is then denoted by exp, so that $\exp(x) = e^x$. We will see later why this case is so important, but, for the moment being, note that, in most fields, exp is the only exponential being used: when nothing else is said, "exponential" means exp.

Let me also recall that e, approximately 2.718282, is an irrational number. This means that it cannot be written as a ratio between two integers and, therefore, that the sequence of decimal digits is infinite, with no period. It can be defined in two ways, as the limit of

$$x_n = \left(1 + \frac{1}{n}\right)^n,$$

as $n \to \infty$, or as the limit of

$$x_n = 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{3!} + \cdots + \frac{1}{n!} = \sum_{k=0}^{n} \frac{1}{k!}\,.$$

The second definition is more appealing to non-mathematicians, because the convergence is very fast: $n = 10$ gives the first six decimal digits. On the other hand, the convergence in the first definition is very slow, and a very high n is needed for a good approximation.

Logarithms

Reversing the exponential functions we obtain their inverses, called **logarithm** functions. For $a > 0$, the logarithm of basis a is defined by the formula

$$x = \log_a y \iff y = a^x.$$

Since an exponential is always positive, only positive numbers have logarithms. From the properties of the exponentials, those of the logarithms are easily derived:

- $\log_a(x_1 x_2) = \log_a x_1 + \log_a x_2$.
- $\log_a 1 = 0$.
- $\log_a(1/x) = -\log_a x$.
- $\log_a(x^\alpha) = \alpha \log_a x$.

As with the exponentials, we also pay special attention here to the case $a = e$. In this case, we have the **natural logarithm**, sometimes denoted by ln (e.g. in Excel). Therefore, $\ln x = \log_e x$. Nevertheless, the natural logarithm is the only logarithm used in most fields (an exception is the pH in Chemistry). Nowadays, in most places, log with no subscript means natural logarithm. This will be the rule in what follows in these notes.

Conversion formulas

Any exponential can be expressed in terms of exp, and any logarithm in terms of the natural log. For the exponentials, the conversion formula

$$a^x = \exp\big[(\log a)x\big]$$

is easily justified, by taking logs in both sides. Because of the conversion formula, any exponential function $f(x) = a^x$ can be expressed as $f(x) = e^{kx}$, with $k = \log a$. Now, for $k > 0$ we have an increasing exponential, whereas for $k < 0$ (corresponding to $a < 1$), we have a decreasing one. The exponentials are usually found in this form in the scientific literature. The expressions exponential **growth** (for $k > 0$) and **decay** (for $k < 0$) are quite common.

For the logarithms, there is also a conversion formula,

$$\log_a x = \frac{\ln x}{\log a}\,.$$

From this formula, we conclude that any logarithm function is proportional to the natural log. The usefulness of the logarithm comes from the fact that it provides a rescaling that leads to simpler formulas. For instance, in Statistics, the distribution of $Y = \log X$ can be closer than that of X to a normal distribution. To this end, there is no difference between $\log_a$ and log.