[MATH-04] Product of vectors

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Dot product and norm

Let $\mathbf{x}$ and $\mathbf{y}$ be vectors in E_n. The **dot product** of $\mathbf{x}$ and $\mathbf{y}$ is defined as

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \cdots + x_n y_n.$$

Note that the product of two vectors only makes sense when they have the same length. Immediate from the definition are the formulas:

- $\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}$.
- $(\alpha \mathbf{x}) \cdot \mathbf{y} = \alpha \mathbf{x} \cdot \mathbf{y}$.
- $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$.
- $\mathbf{x} \cdot \mathbf{x} \geq 0$, and $\mathbf{x} \cdot \mathbf{x} = 0$ only when $\mathbf{x} = 0$.

From the last one we see that we can take the square root of $\mathbf{x} \cdot \mathbf{x}$. The **norm** or modulus of $\mathbf{x}$ is

$$\|\mathbf{x}\| = (\mathbf{x} \cdot \mathbf{x})^{1/2} = (x_1^2 + \cdots + x_n^2)^{1/2}.$$

If we represent a vector $\mathbf{x}$ as an arrow starting at the origin, the length of the arrow, i.e. the distance between the extremes, coincides with $\|\mathbf{x}\|$ (this is the Pythagoras theorem). Also, the distance between the end points of the arrows associated to $\mathbf{x}$ and $\mathbf{y}$ is

$$\|\mathbf{x} - \mathbf{y}\| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.$$

Orthogonality

The formula

$$\big\|\mathbf{x} + \mathbf{y}\big\|^2 = (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2\,\mathbf{x} \cdot \mathbf{y},$$

looks like the old formula of the square of a sum of two numbers. Suppose that $\|\mathbf{x}\|$ and $\|\mathbf{y}\|$ are the two sides of a parallelogram whose diagonal is $\big\|\mathbf{x} + \mathbf{y}\big\|$. Then, the square of the diagonal coincides with the sum of the squares of the two sides when $\mathbf{x} \cdot \mathbf{y} = 0$. Since Pithagoras theorem says that this is true when the two sides are orthogonal (or perpendicular), a pair of vectors with zero product are called **orthogonal vectors**.

Moreover, it can be shown (an exercise of trigonometry) that the product of two vectors coincides with the product of their norms and the cosine of the angle θ determined by the corresponding arrows, i.e. that

$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\|\,\|\mathbf{y}\|\,\cos\theta.$$

Since the cosine of a right angle (90°) is zero, this formula shows that our definition of orthogonal vectors is consistent with the notion of perpendicularity of the school. A vector with modulus 1 is said to be a **unit vector**. We can transform any non-zero vector $\mathbf{x}$ into a unit vector $\mathbf{u}$ by putting

$\mathbf{u} = (1/\|\mathbf{x}\|)\mathbf{x}$. Note that for two unit vectors the product coincides with the cosine of their angle, so that the formula above can be rewritten as

$$\frac{\mathbf{x}}{\|\mathbf{x}\|} \cdot \frac{\mathbf{y}}{\|\mathbf{y}\|} = \cos\theta.$$

Note that the maximum value for the product of two unit vectors is 1, attained for $\theta = 0$, and the minimum is -1, for $\theta = \pi$. In the preceding formula, this happens when $\mathbf{x}$ and $\mathbf{y}$ are parallel, so that the normalized vectors are equal or opposite.

It can be shown that orthogonal vectors are linearly independent. Nevertheless, orthogonality is stronger than independence, as we see in the next example.

Example. The vectors

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

are linearly independent, but not orthogonal. On the other hand, the vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are not only linearly independent, but also orthogonal. □

Homework

A. Find an orthogonal unit basis for the subspace of E_3 defined by the equation $x_1 + x_2 + x_3 = 0$.