[MATH-09] Derivatives

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The derivative

Let f be a function defined on some interval D of the real line, and x_0 a point in D. The **derivative** of f at x_0 is the limit

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}\,.$$

Alternative expressions for the derivative are

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}\,.$$

In applications, it is frequent to consider a function as defining a relationship between two variables x and y. Then y is identified with $f(x)$, and expressions like y' or dy/dx are used. In the latter expression, it is understood that dx stands for an infinitesimal increment of x and dy for the corresponding increment of y, that is,

$$dy = f(x + dx) - f(x).$$

If there is a derivative of f at x_0, then f is continuous at this point. This is easy to see in the definition above: as $x \to x_0$, the denominator tends to zero, and the only way to make the quotient converge is to have a $0/0$ situation. The reciprocal is not true, the typical example being $f(x) = |x|$ which is continuous at the origin but does not have a derivative (see why plotting this function).

It is easy to check, graphically, that the quotient in the definition of the derivative coincides with the **slope** (the tangent of the angle with the horizontal) of the line through the points $(x_0, f(x_0))$ and $(x, f(x))$. When $x \to x_0$, this line (a secant) converges to the **tangent** to the curve $y = f(x)$ at the point $(x_0, f(x_0))$. Therefore, the equation of the tangent line is

$$y - y_0 = f'(x_0)(x - x_0).$$

Example 1. We find the equation of the tangent to the curve $y = x^2$ at the point $(1, 1)$. Using the rules given later in this lecture, you can see that the derivative of $f(x) = x^2$ at $x_0 = 1$ is $f'(1) = 2$, so that the equation of the tangent line is $y - 1 = 2(x - 1)$. Similarly, the tangent at $(-1, 1)$ is $y - 1 = -2(x + 1)$. □

Numerical derivatives

Since the derivative is a limit, an approximate value of $f'(x_0)$ can be obtained by using a small increment h in the quotient

$$\frac{f(x_0 + h) - f(x_0)}{h}\,.$$

Such an approximation is called a **numerical derivative**. The approximation improves as h gets closer to zero. In practice, as we use a particular electronic device in the calculations, the approximation cannot be improved beyond a certain h, which depends on this device.

¶ Derivatives are called velocities in Physics and Chemistry, marginals in Economics and rates in various fields.

Example 2. Let $f(x) = 2^x$ and $x_0 = 0$, so $f(x_0) = f(0) = 1$. Taking $h = 0.1$, we get the numerical derivative

$$f'(1) \approx \frac{f(0.1) - 1}{0.1} = 0.71773,$$

whereas, taking $h = 0.01$,

$$f'(1) \approx \frac{f(0.01) - 1}{0.01} = 0.69555.$$

By using the rules for symbolic calculus of derivatives, given below, you can easily check that the exact value of the derivative is $f'(1) = \log 2 = 0.69315$. □

Example 3. Sometimes, a numerical derivative is the only thing we can have, because a mathematical expression of the function is not available. Suppose that $x(t)$ is a function of the time t (e.g. a macroeconomic indicator). If we know the values of $x(t)$ for $t_1, \ldots, t_n$, the derivative at $t = t_i$ can be approximated by

$$x'(t_i) \approx \frac{x(t_i) - x(t_{i-1})}{t_i - t_{i-1}}.$$

If the times are equally spaced (as in monthly data), the scale of t can be arranged so, that $t_i - t_{i-1} = 1$. Then, the denominator in this fraction disappears, and the derivative becomes the variation of $x(t)$ from t_{i-1} to t_i. In this case, the derivative can be normalized, by dividing by the previous value $x(t_{i-1})$. A classical example is the price consumer index (CPI), whose normalized derivative is the inflation. Another example is the return of a financial index, which will appear later. □

Derivatives of well known functions

If the derivative of $f : D \to \mathbb{R}$ converges at every point of D, we can define a new function $f' : D \to \mathbb{R}$, by assigning to each $x \in D$ the derivative $f'(x)$. The function f' is a **derivative function**, and f is said to be a **primitive** of f. I give first the derivatives of some usual functions.

- *Constants.* If $f(x)$ is constant, $f'(x) = 0$ everywhere, and conversely.
- *Exponential.* If $f(x) = e^x$, then $f'(x) = e^x$. Hence, $f = f'$ in this case. This is only true for the exponential of basis e, providing an argument for the preeminence of the number e. We will give later the derivative of a general exponential.
- *Natural logarithm.* Let $f(x) = \log x$. Then $f'(x) = 1/x$. I will give later the formula for a general logarithm.
- *Powers.* If $f(x) = x^\alpha$, then $f'(x) = \alpha\, x^{\alpha-1}$.
- *Trigonometric functions.* If $f(x) = \sin x$, then $f'(x) = \cos x$. If $f(x) = \cos x$, then $f'(x) = -\sin x$.
- *Inverse trigonometric functions.* If $f(x) = \arcsin x$, then $f'(x) = 1/\sqrt{1 - x^2}$. If $f(x) = \arctan x$, then $f'(x) = 1/(1 + x^2)$.

Symbolic calculus of derivatives

The process of finding a mathematical expression for the derivative of a function, given a mathematical expression, is the **symbolic calculus** of derivatives. In the symbolic calculus, one uses the rules for taking derivatives in sums, products, quotients and composite functions, together with the derivatives of the elementary functions given above.

The derivative of a sum is given by the formula

$$(f+g)'(x) = f'(x) + g'(x),$$

and the derivative of a product by

$$\big(fg\big)'(x) = f'(x)g(x) + f(x)g'(x).$$

Since the derivative of a constant is null, for α constant we have

$$\big(\alpha f\big)'(x) = \alpha\, f'(x).$$

Example 4. As a particular case, we can apply this rule to the conversion formula for logarithms, so that we get the derivative of a general logarithm,

$$\big(\log_a\big)'(x) = \frac{1}{x\,\log a}\,. \;\square$$

The derivative of a quotient is given by

$$\big(f/g\big)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}\,.$$

In particular,

$$\big(1/f\big)'(x) = -\frac{f'(x)}{f(x)}\,. \;\square$$

Example 5. As an application, we obtain the derivative of the tangent function,

$$\big(\tan)'(x) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}\,. \;\square$$

The derivative of a composite function is given by the **chain rule**,

$$(f_1 \circ f_2)'(x) = f_1'\big(f_2(x)\big)\,f_2'(x). \;\square$$

Example 6. For $f(x) = e^{x^2}$, putting $f_1(x) = e^x$ and $f_2(x) = x^2$, we get $f'(x) = 2xe^{x^2}$. $\square$

Example 7. We can apply the chain rule to the conversion formula given for exponentials, obtaining the derivative of an general exponential $f(x) = a^x$. Indeed, we consider f as a composite function

$$f = f_1 \circ f_2, \quad f_1(x) = \exp(x), \qquad f_2(x) = (\log a)x,$$

so that

$$f'(x) = f_1'(f_2(x))\,f_2'(x) = (\log a)\,\exp(f_2(x)) = (\log a)\exp\big(x\,\log a\big) = (\log a)\,a^x. \;\square$$

As a final application, we apply the chain rule to $g(x) = \log f(x)$, for a positive function f, getting

$$g'(x) = \frac{f'(x)}{f(x)}\,,$$

which is called the **logarithmic derivative** of f. Sometimes the derivative is obtained from the logarithmic derivative.

Example 8. Applying these ideas to

$$f(x) = x^x, \qquad g(x) = x\,\log x,$$

we get

$$f'(x) = g'(x)\,f(x) = \big(\log x + 1\big)\,x^x. \;\square$$

Homework

A. Check: $f(x) = 2^{1/x} \implies f'(x) = -\dfrac{(\log 2)2^{1/x}}{x^2}\,.$

B. Check: $f(x) = \log\left(\dfrac{x-1}{x^2}\right) \implies f'(x) = \dfrac{2-x}{x(x-1)}\,.$