[MATH-06] The product of a matrix and a vector

Miguel-Angel Canela
Associate Professor, IESE Business School

The product of a matrix and a vector

The product of a matrix and a vector is defined in such a way that the product of two vectors is just a particular case, that in which the first factor is a 1-row matrix. If we write

$$\begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + \cdots + x_n y_n,$$

the scalar product becomes

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^\mathsf{T}\,\mathbf{y}.$$

Thus, the product of a row by a column is a number, that can be considered as a $(1,1)$–matrix. Next, we extend this definition to the product of an (n,m)–matrix $\mathbf{A}$ and an m–vector $\mathbf{x}$. To do this, we take $\mathbf{A}$ as a pack of row vectors, multiplying each row by $\mathbf{x}$ and placing the numbers that result form these products as the coordinates of a vector, which is the product $\mathbf{Ax}$. In mathematical symbols,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} \sum_j a_{1j} x_j \\ \sum_j a_{2j} x_j \\ \vdots \\ \sum_j a_{nj} x_j \end{bmatrix}.$$

If $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are vectors corresponding to the rows of $\mathbf{A}$, this can be written as

$$\begin{bmatrix} \mathbf{a}_1^\mathsf{T} \\ \mathbf{a}_2^\mathsf{T} \\ \vdots \\ \mathbf{a}_n^\mathsf{T} \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{a}_1^\mathsf{T}\mathbf{x} \\ \mathbf{a}_2^\mathsf{T}\mathbf{x} \\ \vdots \\ \mathbf{a}_n^\mathsf{T}\mathbf{x} \end{bmatrix}.$$

Matrices as linear operators

The definition of the product of a matrix and a vector allows an interesting interpretation of a matria as a **linear operator**. Let us fix $\mathbf{A}$ and define an operator (functions are called operators in this context)

$$\begin{aligned} T : E_m &\longrightarrow E_n \\ \mathbf{x} &\longmapsto \mathbf{Ax}. \end{aligned}$$

Such operator would be linear, i.e. it would transform a linear combination into a linear combination,

$$T\big(\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2\big) = \alpha_1\, T\mathbf{u}_1 + \alpha_2\, T\mathbf{u}_2.$$

Conversely, any operator with this property can be associated to a matrix. It is easy to check that the columns of the matrix associated to a linear operator coincide with the images of the vectors of the canonical basis under the operator. A one-to-one linear operator $T : E_n \to E_n$ is called a **linear transformation**.

Example 1. The matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

defines the operator $T : E_3 \to E_4$ as

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ -x_3 \\ x_1 + x_3 \\ x_2 - 2x_3 \end{bmatrix}. \square$$

The null space

An interesting definition related to linear operators is the **null space** (also called kernel). The null space of $\mathbf{A}$ (equivalently, of T), that we denote by $\mathcal{N}(\mathbf{A})$, is the subspace of all vectors $\mathbf{x}$ of E_m such that $\mathbf{Ax} = \mathbf{0}$. The vector $\mathbf{0}$ always belongs to the null space. We may also say that $\mathbf{x} \in \mathcal{N}(\mathbf{A})$ when $\mathbf{x}$ is orthogonal to the rows of $\mathbf{A}$.

The subspace of all vectors $\mathbf{Ax}$, with $\mathbf{x} \in E_n$, is called the **range** of $\mathbf{A}$, denoted by $\mathcal{R}(\mathbf{A})$. Since the vectors of E_n are the linear combinations of the canonical basis, those of $\mathcal{R}(\mathbf{A})$ are the linear combinations of the columns of $\mathbf{A}$. There are as many linearly independent column vectors as indicated by the rank, which thus coincides with the dimension of $\mathcal{R}(\mathbf{A})$. An interesting formula is (mind that, here, m is the number of columns)

$$\dim \mathcal{N}(\mathbf{A}) + \operatorname{rank} \mathbf{A} = m.$$

In particular, when $\mathbf{A}$ is a square matrix, the existence of vectors $\mathbf{x} \neq \mathbf{0}$ such that $\mathbf{Ax} = \mathbf{0}$ is equivalent to $\det(\mathbf{A}) = 0$.

Example 2 The linear subspace $S = \{\mathbf{x} \in E_3 : x_1 + x_2 = x_1 - x_3 = 0\}$ is the null space of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}.$$

As the rank of this matrix is 2, the null space has dimension 1. This has been previously seen (lecture 3) by finding a basis of S. $\square$

Homework

A. Find the rank and a basis of the null space of

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}.$$