[MATH-09] Orthogonal matrices

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Orthogonal matrices

Orthogonal matrices are special nonsingular matrices that appear in many applications, in particular in multivariate statistics. $\mathbf{A}$ is said to be orthogonal when its inverse coincides with its transpose, i.e. when $\mathbf{A}^{-1} = \mathbf{A}^{\mathsf{T}}$. This is equivalent to $\mathbf{A}^{\mathsf{T}}\,\mathbf{A} = \mathbf{I}$. It is can be easily proved that a matrix is orthogonal when its columns (or its rows) are pairwise orthogonal unit vectors. This is the easy way of checking the orthogonality of a matrix.

Example 1. The matrix $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is symmetric and orthogonal. Hence, $\mathbf{A}^{-1} = \mathbf{A}$. $\square$

Linear operators defined by orthogonal matrices are sometimes (improperly) called **rotations**. Characterizing orthogonal two-dimensional matrices helps to understand how orthogonal matrices are related to rotations, as we seen next.

A unit vector $\mathbf{u}$ in E_2 can be placed as an arrow in the plane, starting at the origin and ending at a point of the unit circle. This allows for a representation of this vector as

$$\mathbf{u} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix},$$

where θ denotes the angle from the positive x half-axis to the arrow $\mathbf{u}$ (counterclockwise). Now, I call $\mathbf{u}_1$ and $\mathbf{u}_2$ the columns of an orthogonal 2–dimensional matrix $\mathbf{A}$. Since these vectors are orthogonal, $\theta_2 = \theta_1 \pm \pi/2$ (use a picture if you do not see this immediately). If $\theta_2 = \theta_1 + \pi/2$,

$$\cos\theta_2 = -\sin\theta_1, \qquad \sin\theta_2 = \cos\theta_1,$$

so $\mathbf{A}$ can be written as

$$\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Here, the columns correspond to the images of the points $(1,0)$ and $(0,1)$ under the transformation defined by $\mathbf{A}$. It is not hard to see, in a picture, that we have a rotation of angle θ around the origin. A similar argument proves that, when $\theta_2 = \theta_1 - \pi/2$, $\mathbf{A}$ is

$$\mathbf{A} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

What is this? It follows from the identity

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

that $\mathbf{A}$ defines a rotation on top of a change of sign in the second coordinate. Such change-of-sign transformation is called an **inversion**. Therefore, orthogonal matrices of dimension 2 define either a rotation or an inversion plus a rotation.

Diagonalization of symmetric matrices

Not every matrix is diagonalizable, as shown by Example 3. But an important theorem of matrix algebra ensures that every symmetric matrix is diagonalizable. Since, in most applications, the matrices are symmetric (e.g. covariance matrices in statistics, Hessian matrices in differential calculus), this theorem will probably cover your needs.

Theorem. If $\mathbf{A}$ is a symmetric (n,n)–matrix, there is a basis of E_n formed by orthogonal eigenvectors.

Then, for a symmetric matrix, an orthogonal matrix of (unit) eigenvectors can be found. The diagonalization formula becomes $\mathbf{A} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^\intercal$, called the **spectral decomposition**.

Example 2. In Example 2 of the preceding lecture, $\mathbf{B}$ is not orthogonal, but replacing $\mathbf{v}_2$ by

$$\mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix},$$

we get a basis of orthogonal eigenvectors. Now I divide these vectors by the respective norms, obtaining a basis of orthogonal unit eigenvectors,

$$\mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ -2/\sqrt{2} \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}. \square$$

The spectral decomposition is then

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{3} \\ 0 & -2/\sqrt{2} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 0 & & \\ & 0 & \\ & & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & -2/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}. \square$$

Homework

A. Check that the operator defined by $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is a rotation of the plane.

B. Find the spectral decomposition of

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}.$$