[MATH-05] Determinants

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Determinants

Let $\mathbf{A}$ be an (n,n)–matrix:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

The **determinant** of $\mathbf{A}$, denoted by $\det \mathbf{A}$, is a number that results from summing products of terms of $\mathbf{A}$. It is also denoted by $\big|\mathbf{A}\big|$, but I avoid here this notation, leaving the bars for the absolute value of a real number and the modulus of a complex number. Nevertheless, when replacing in a matrix the brackets by bars, I mean the determinant of this matrix, i.e.

$$\det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

The determinant of $\mathbf{A}$ can be defined as the sum of all the possible products of n terms of $\mathbf{A}$, taking one term from each row and one term from each column (there are $n!$ such products), with the same or the opposite sign, according to the following rule: the sign of a product $a_{1m_1} \cdots a_{nm_n}$ is preserved when the number of inversions in the sequence $\big(m_1, m_2, \ldots, m_n\big)$ is even, and changed when the number of inversions is odd.

For n small, the determinant can be calculated easily. For $n = 2$, the rule is

$$\det \mathbf{A} = a_{11}a_{22} - a_{12}a_{21},$$

whereas, for $n = 3$,

$$\det \mathbf{A} = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

¶ You probably learned to pick the factors of these terms using a starlike path. Schoolteachers called this trick the rule of Sarrus.

Example 1. The following calculation illustrates these rules.

$$\begin{vmatrix} 1 & 3 & 0 \\ -1 & 1 & 1 \\ 0 & 5 & 2 \end{vmatrix} = 1\times1\times2+(-1)\times5\times0+3\times1\times0-0\times1\times0-(-1)\times3\times2-1\times5\times1 = 2+6-5 = 3.$$

Properties of the determinant

For $n > 3$, the following properties of the determinant are useful:

- The determinant of a triangular matrix is the product of the diagonal terms.
- A matrix and its transpose have the same determinant: $\det \mathbf{A} = \det(\mathbf{A}^\intercal)$.
- Interchanging two rows (columns) changes the sign of the determinant.
- If two rows (columns) are equal, the determinant is null.
- If one sums to a row (column) a linear combination of the others, the determinant does not change.
- If a row (column) is a linear combination of the others, the determinant is null, and conversely.
- For an (n, n)–matrix $\mathbf{A}$, $\operatorname{rank} \mathbf{A} = n$ if and only if $\det \mathbf{A} \neq 0$.
- The rank of a matrix coincides with the dimension of the biggest squared sub-matrix whose determinant is different of zero.

Example 2. Due to these properties (see Example 1),

$$\begin{vmatrix} 1 & 3 & -1 & 4 \\ 2 & 4 & 3 & 9 \\ -1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -1 & 4 \\ 0 & -2 & 5 & 1 \\ 0 & 0 & 10 & 7 \\ 0 & 0 & 0 & 2/5 \end{vmatrix} = -8.$$

Homework

A. Find the rank and the determinant of

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 2 \\ 3 & 0 & 1 & 2 \\ 0 & 4 & 4 & 3 \end{bmatrix}.$$