f-25-jupyter-hessenberg

May 4, 2021

[1]:

```python
import numpy as np
```

[2]:

```python
def house(x):
    norm_x = np.linalg.norm(x)
    if norm_x == 0:
        v = np.zeros_like(x)
        v[0] = 1
        s = 0
    else:
        u = x / norm_x
        eps = -1 if u[0] >= 0 else +1
        s = 1 + np.abs(u[0])
        v = - eps * u
        v[0] += 1
        v /= s
    return v, s
```

[3]:

```python
rng = np.random.default_rng()
```

[4]:

```python
a = rng.standard_normal((4,4))
a += a.T
a
```

[4]:

```
array([[ 0.66919178,  1.57173286,  3.15378093, -0.98771555],
       [ 1.57173286,  0.00887976,  0.14354289, -0.38916541],
       [ 3.15378093,  0.14354289, -3.99997082, -1.78039984],
       [-0.98771555, -0.38916541, -1.78039984, -0.08153391]])
```

[5]:

```python
v, s = house(a[:, [0]])
h0 = np.eye(4) - s * v @ v.T
h0
```

[5]:

```
array([[-0.17987936, -0.42248322, -0.84773918,  0.26549884],
       [-0.42248322,  0.84872007, -0.30355271,  0.09506803],
       [-0.84773918, -0.30355271,  0.39090238,  0.19075998],
       [ 0.26549884,  0.09506803,  0.19075998,  0.94025692]])
```

[6]:

```python
h0 @ a
```

[6]:

```
array([[-3.72022552e+00, -5.11483739e-01,  2.29029334e+00,
         1.82975303e+00],
       [ 3.19189120e-16, -7.37064347e-01, -1.65648909e-01,
         6.19694680e-01],
       [ 6.38378239e-16, -1.35324093e+00, -4.62038362e+00,
         2.43941444e-01],
       [-2.22044605e-16,  7.96041954e-02, -1.58609612e+00,
        -7.15526377e-01]])
```

[7]:

```
h0 @ a @ h0
```

[7]:

```
array([[-0.57048901,  0.6163526 ,  4.55336797,  1.12099296],
       [ 0.6163526 , -0.51636498,  0.27719827,  0.48100177],
       [ 4.55336797,  0.27719827, -1.34880472, -0.78066661],
       [ 1.12099296,  0.48100177, -0.78066661, -0.96777448]])
```

[8]:

```
def hessenberg_data(a):
    data = np.copy(a)
    n, _ = a.shape
    s = np.empty(n-2)
    for j in range(n-2):
        v, s[j] = house(data[j+1:, [j]])
        data[j+1:, j:] -= (s[j] * v) @ (v.T @ data[j+1:, j:])
        data[:, j+1:] -= (s[j] * (data[:, j+1:] @ v)) @ v.T
        data[j+2:, [j]] = v[1:]
    return data, s
```

[9]:

```
def hessenberg(a):
    data, s = hessenberg_data(a)
    return np.triu(data, -1)
```

[10]:

```
def hessenberg_qh(a):
    data, s = hessenberg_data(a)
    n, _ = a.shape
    h = np.triu(data, -1)
    q = np.eye(n)
    for j in reversed(range(n-2)):
        x = data[j+2:, [j]]
        v = np.vstack([[1], x])
        q[j+1:, j+1:] -= s[j] * v @ (v.T @ q[j+1:, j+1:])
    return q, h
```

[11]:

```
a = np.array(np.arange(25),dtype=float).reshape(5,5)
print(a)
```

```
[[ 0.  1.  2.  3.  4.]
 [ 5.  6.  7.  8.  9.]
 [10. 11. 12. 13. 14.]
 [15. 16. 17. 18. 19.]
 [20. 21. 22. 23. 24.]]
```

```
[12]: q, h = hessenberg_qh(a)
```

```
[13]: np.allclose(q @ q.T, np.eye(5), atol = np.finfo(float).eps)
```

```
[13]: True
```

```
[14]: q @ h @ q.T
```

```
[14]: array([[ 0.,  1.,  2.,  3.,  4.],
             [ 5.,  6.,  7.,  8.,  9.],
             [10., 11., 12., 13., 14.],
             [15., 16., 17., 18., 19.],
             [20., 21., 22., 23., 24.]])
```

```
[15]: np.allclose(q @ h @ q.T, a, atol = np.finfo(float).eps)
```

```
[15]: True
```

```
[16]: h
```

```
[16]: array([[ 0.00000000e+00, -5.47722558e+00,  1.07698418e-15,
              -7.82567726e-16, -5.12822540e-17],
             [-2.73861279e+01,  6.00000000e+01,  2.23606798e+01,
              -1.37628386e-14,  7.93849888e-15],
             [ 0.00000000e+00,  4.47213595e+00, -4.73619917e-16,
               1.76311711e-15,  3.49592482e-15],
             [ 0.00000000e+00,  0.00000000e+00, -2.44814982e-17,
              -9.83087960e-16, -2.44879131e-15],
             [ 0.00000000e+00,  0.00000000e+00,  0.00000000e+00,
              -8.52299065e-17, -2.08626660e-16]])
```

```
[17]: np.tril(h,-2)
```

```
[17]: array([[0., 0., 0., 0., 0.],
             [0., 0., 0., 0., 0.],
             [0., 0., 0., 0., 0.],
             [0., 0., 0., 0., 0.],
             [0., 0., 0., 0., 0.]])
```

```
[18]: b = rng.standard_normal((5, 5))
```

```
[19]: b += b.T
```

```
[20]: b
```

```
[20]: array([[ 2.89794798, -0.69400563,  1.07019376, -1.22072449,  0.28482653],
             [-0.69400563, -2.04521416,  0.12417728,  0.80382975,  1.92577469],
             [ 1.07019376,  0.12417728, -2.02096579,  0.43323169,  0.74397677],
             [-1.22072449,  0.80382975,  0.43323169,  4.04406977, -1.35146984],
             [ 0.28482653,  1.92577469,  0.74397677, -1.35146984,  0.39503089]])
```

```
[24]: qb, hb = hessenberg_qh(b)
```

```
[25]: hb
```

```
[25]: array([[ 2.89794798e+00,  1.78836599e+00,  0.00000000e+00,
               0.00000000e+00,  0.00000000e+00],
             [ 1.78836599e+00,  1.07442304e+00,  3.30668270e+00,
               5.14805792e-16,  7.32230738e-16],
             [ 0.00000000e+00,  3.30668270e+00,  1.26673969e+00,
              -4.43713355e-01,  2.77555756e-16],
             [ 0.00000000e+00,  0.00000000e+00, -4.43713355e-01,
              -1.25266369e+00, -2.40323956e+00],
             [ 0.00000000e+00,  0.00000000e+00,  0.00000000e+00,
              -2.40323956e+00, -7.15578340e-01]])
```

```
[26]: hb[:, 0]
```

```
[26]: array([2.89794798, 1.78836599, 0.        , 0.        , 0.        ])
```

```
[27]: hb[:, 1]
```

```
[27]: array([1.78836599, 1.07442304, 3.3066827 , 0.        , 0.        ])
```

```
[28]: qb @ hb @ qb.T - b
```

```
[28]: array([[ 0.00000000e+00, -2.22044605e-16,  0.00000000e+00,
               0.00000000e+00,  0.00000000e+00],
             [-2.22044605e-16,  1.33226763e-15,  2.49800181e-16,
               0.00000000e+00, -4.44089210e-16],
             [ 0.00000000e+00,  8.04911693e-16,  1.33226763e-15,
               1.66533454e-16, -8.88178420e-16],
             [ 0.00000000e+00, -2.22044605e-16, -1.16573418e-15,
              -8.88178420e-16,  2.22044605e-16],
             [ 0.00000000e+00, -8.88178420e-16, -5.55111512e-16,
               0.00000000e+00,  3.88578059e-16]])
```

```
[ ]:
```