[SIMAX] Block Crout Factorizations: An example:

$$\mathbf{B} \quad = \quad \begin{bmatrix} 4 & -1 & 0 & 1 & 4 & 7 \\ 0 & 4 & -1 & 2 & 5 & 8 \\ 0 & 0 & 4 & 3 & 6 & 9 \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

$\downarrow$ Assumption: B upper trapezoidal:

$$\mathbf{X} \quad := \quad \begin{bmatrix} \mathbf{A} & \mathbf{B}^{\mathrm{T}} \\ \mathbf{B} & \mathbf{0} \end{bmatrix} = \left[\begin{array}{rrrrrr|rrr} 2 & -1 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & -1 & 4 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 & 4 \\ 0 & 0 & -1 & 2 & -1 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & -1 & 2 & -1 & 4 & 5 & 6 \\ 0 & 0 & 0 & 0 & -1 & 2 & 7 & 8 & 9 \\ \hline 4 & -1 & 0 & 1 & 4 & 7 & 0 & 0 & 0 \\ 0 & 4 & -1 & 2 & 5 & 8 & 0 & 0 & 0 \\ 0 & 0 & 4 & 3 & 6 & 9 & 0 & 0 & 0 \end{array}\right]$$

$\downarrow$ Step 1. Permute: $\mathbf{P} = [\mathbf{e}_1, \mathbf{e}_7, \mathbf{e}_2, \mathbf{e}_8, \mathbf{e}_3, \mathbf{e}_9, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6] \in \mathbb{R}^{9\times 9}$

$$\mathbf{Y} \quad = \quad \mathbf{P}^{\mathrm{T}}\mathbf{X}\mathbf{P} \qquad \text{(Permuted Form)}$$

$$= \left[\begin{array}{rr|rr|rr|r|r|r} 2 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & -1 & 0 & 0 & 0 & 1 & 4 & 7 \\ \hline -1 & -1 & 2 & 4 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & -1 & 0 & 2 & 5 & 8 \\ \hline 0 & 0 & -1 & -1 & 2 & 4 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 3 & 6 & 9 \\ \hline 0 & 1 & 0 & 2 & -1 & 3 & 2 & -1 & 0 \\ \hline 0 & 4 & 0 & 5 & 0 & 6 & -1 & 2 & -1 \\ \hline 0 & 7 & 0 & 8 & 0 & 9 & 0 & -1 & 2 \end{array}\right]$$

$\downarrow$ Step 2. Block Crout Factor:

$$\mathbf{Y} \quad = \quad (\mathbf{L_Y} + \mathbf{D_Y})\mathbf{D_Y}^{-1}(\mathbf{L_Y} + \mathbf{D_Y})^{\mathrm{T}} \qquad \text{(Block Crout Factored Form)}$$

$\downarrow$ with $\mathbf{L_Y}$ resp. $\mathbf{D_Y}$

$$\left[\begin{array}{rr|rr|rr|r|r|r} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0.1 & 2 & -0.7 & 3 & 0 & 0 & 0 \\ \hline 0 & 4 & 0.5 & 5 & 0.9 & 6 & 2.3 & 0 & 0 \\ \hline 0 & 7 & 0.9 & 8 & 1.4 & 9 & 5.0 & 3.2 & 0 \end{array}\right], \left[\begin{array}{rr|rr|rr|r|r|r} 2 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1.6 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1.6 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 4.3 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5.1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5.2 \end{array}\right]$$

$\downarrow$ Step 3. Back permute:

$\downarrow$

$$
\begin{aligned}
& \Big\downarrow \quad \text{Step 3. Back permute:} \\
\mathbf{X} \quad = \quad & \mathbf{P}\mathbf{Y}\mathbf{P}^{\mathrm{T}} = (\underbrace{\mathbf{P}\mathbf{L}_{\mathbf{Y}}\mathbf{P}^{\mathrm{T}}}_{=:\mathbf{L}_{\mathbf{X}}} + \underbrace{\mathbf{P}\mathbf{D}_{\mathbf{Y}}\mathbf{P}^{\mathrm{T}}}_{=:\mathbf{D}_{\mathbf{X}}})(\mathbf{P}\mathbf{D}_{\mathbf{Y}}\mathbf{P}^{\mathrm{T}})^{-1}(\mathbf{P}\mathbf{L}_{\mathbf{Y}}\mathbf{P}^{\mathrm{T}} + \mathbf{P}\mathbf{D}_{\mathbf{Y}}\mathbf{P}^{\mathrm{T}})^{\mathrm{T}} \\
=: \quad & (\mathbf{L}_{\mathbf{X}} + \mathbf{D}_{\mathbf{X}})\mathbf{D}_{\mathbf{X}}^{-1}(\mathbf{L}_{\mathbf{X}} + \mathbf{D}_{\mathbf{X}})^{\mathrm{T}} \qquad \text{(Saddle point Factored Form)} \\
& \Big\downarrow \quad \text{with } \mathbf{L}_{\mathbf{X}} \text{ resp. } \mathbf{D}_{\mathbf{X}}
\end{aligned}
$$

$$
\left[\begin{array}{rrrrrr|rrr}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0.1 & -0.7 & 0 & 0 & 0 & 1 & 2 & 3 \\
0 & 0.5 & 0.9 & 2.3 & 0 & 0 & 4 & 5 & 6 \\
0 & 0.9 & 1.4 & 5.0 & 3.2 & 0 & 7 & 8 & 9 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right],
\left[\begin{array}{rrrrrr|rrr}
2 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\
0 & 1.6 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\
0 & 0 & 1.6 & 0 & 0 & 0 & 0 & 0 & 4 \\
0 & 0 & 0 & 4.3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 5.1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 5.2 & 0 & 0 & 0 \\
\hline
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

$$
\begin{aligned}
& \Big\downarrow \quad \text{To determine whether our factors are members of a certain class/family:} \\
\mathbf{X} \quad = \quad & (\mathbf{L}_{\mathbf{X}} + \mathbf{D}_{\mathbf{X}})\mathbf{D}_{\mathbf{X}}^{-1}(\mathbf{L}_{\mathbf{X}} + \mathbf{D}_{\mathbf{X}})^{\mathrm{T}} \\
=: \quad & \mathbf{L}\mathbf{D}\mathbf{L}^{\mathrm{T}} \qquad \text{(``Dollar, Gould, Schilders, Wathen'' Form)} \\
& \Big\downarrow \quad \text{with } \mathbf{L} := \mathbf{L}_{\mathbf{X}} + \mathbf{D}_{\mathbf{X}} \text{ resp. } \mathbf{D} := (\mathbf{D}_{\mathbf{X}})^{-1}
\end{aligned}
$$

$$
\left[\begin{array}{rrr|rrr|rrr}
2 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\
-1 & 1.6 & 0 & 0 & 0 & 0 & -1 & 4 & 0 \\
0 & -1 & 1.6 & 0 & 0 & 0 & 0 & -1 & 4 \\
\hline
0 & 0.1 & -0.7 & 4.3 & 0 & 0 & 1 & 2 & 3 \\
0 & 0.5 & 0.9 & 2.3 & 5.1 & 0 & 4 & 5 & 6 \\
0 & 0.9 & 1.4 & 5.0 & 3.2 & 5.2 & 7 & 8 & 9 \\
\hline
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right],
\left[\begin{array}{rrr|rrr|rrr}
0 & 0 & 0 & 0 & 0 & 0 & 0.2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2 \\
\hline
0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0 \\
\hline
0.2 & 0 & 0 & 0 & 0 & 0 & -0.1 & 0 & 0 \\
0 & 0.2 & 0 & 0 & 0 & 0 & 0 & -0.1 & 0 \\
0 & 0 & 0.2 & 0 & 0 & 0 & 0 & 0 & -0.1
\end{array}\right]
$$

where no entry has been rounded to 0.

Alternative: An extra step after 2:

$$
\begin{array}{rcl}
 & \Big\downarrow & \text{Step 2. Block Crout Factor:} \\
\mathbf{Y} & = & (\mathbf{L}_\mathbf{Y} + \mathbf{D}_\mathbf{Y})\mathbf{D}_\mathbf{Y}^{-1}(\mathbf{L}_\mathbf{Y} + \mathbf{D}_\mathbf{Y})^\mathrm{T} \qquad \text{(Block Crout Factored Form)} \\
 & \Big\downarrow & \text{with } \mathbf{L}_\mathbf{Y} \text{ resp. } \mathbf{D}_\mathbf{Y}
\end{array}
$$

$$
\left[\begin{array}{rr|rr|rr|r|r|r}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 1 & 0.1 & 2 & -0.7 & 3 & 0 & 0 & 0 \\
\hline
0 & 4 & 0.5 & 5 & 0.9 & 6 & 2.3 & 0 & 0 \\
\hline
0 & 7 & 0.9 & 8 & 1.4 & 9 & 5.0 & 3.2 & 0
\end{array}\right],
\left[\begin{array}{rr|rr|rr|r|r|r}
2 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 1.6 & 4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1.6 & 4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 4.3 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 5.1 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5.2
\end{array}\right]
$$

$$
\Big\downarrow \quad \text{Step 2b. Block diagonal} \to \text{Diagonal:}
$$

Note that each step $\mathbf{S}_k := \mathbf{S}_k - (\mathbf{L}_k + \mathbf{D}_k)\mathbf{D}_k^{-1}(\mathbf{D}_k + \mathbf{U}_k)$ where

$$
(\mathbf{L}_k + \mathbf{D}_k)\mathbf{D}_k^{-1}(\mathbf{D}_k + \mathbf{U}_k) = [\mathbf{D}_k; \mathbf{L}_k]\mathbf{D}_k^{-1}[\mathbf{D}_k, \mathbf{U}_k].
$$

If each diagonal block $\mathbf{D}_k$ can be factored:

$$
\mathbf{D}_k = (d_k + u_k)d_k^{-1}(l_k + d_k) \implies \mathbf{D}_k^{-1} = (d_k + u_k)^{-1}d_k(l_k + d_k)^{-1}
$$

then

$$
\begin{array}{rcl}
(\mathbf{L}_k + \mathbf{D}_k)\mathbf{D}_k^{-1}(\mathbf{D}_k + \mathbf{U}_k) & = & [\mathbf{D}_k; \mathbf{L}_k]\mathbf{D}_k^{-1}[\mathbf{D}_k, \mathbf{U}_k] \\
 & = & [(l_k + d_k)d_k^{-1}(d_k + u_k); L](d_k + u_k)^{-1}d_k(l_k + d_k)^{-1}[(l_k + d_k)d_k^{-1}(d_k + u_k), U] \\
 & = & [(l_k + d_k); L(d_k + u_k)^{-1}d_k]d_k^{-1}d_k(l_k + d_k)^{-1}[(l_k + d_k)d_k^{-1}(d_k + u_k), U] \\
 & = & [(l_k + d_k); L(d_k + u_k)^{-1}d_k]d_k^{-1}[(d_k + u_k), d_k(l_k + d_k)^{-1}U].
\end{array}
$$

leads to a **new** LDU type factorization

$$
\mathbf{Y} = (\hat{\mathbf{L}} + \hat{\mathbf{D}})\hat{\mathbf{D}}^{-1}(\hat{\mathbf{D}} + \hat{\mathbf{U}})
$$

where $\mathbf{L}, \mathbf{U}$ are strictly lower respectively upper triangular and where $\mathbf{D}$ is diagonal (not anylonger block-diagonal).

$$
\begin{array}{rcl}
 & \Big\downarrow & \text{Step 2b. Block diagonal} \leftrightarrow \text{Diagonal:} \\
\mathbf{Y} & = & (\hat{\mathbf{L}}_\mathbf{Y} + \hat{\mathbf{D}}_\mathbf{Y})\hat{\mathbf{D}}_\mathbf{Y}^{-1}(\hat{\mathbf{L}}_\mathbf{Y} + \hat{\mathbf{D}}_\mathbf{Y})^\mathrm{T} \qquad \text{(Modified Block Crout Factored Form)} \\
 & \Big\downarrow & \text{with } \hat{\mathbf{L}}_\mathbf{Y} \text{ resp. } \hat{\mathbf{D}}_\mathbf{Y}
\end{array}
$$

$$
\left[\begin{array}{rrr|rrr|rrr}
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & -8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
-1 & 2 & 4 & -8 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & 4 & -8 & 0 & 0 & 0 \\
\hline
0 & 1 & 0.2 & 1.8 & -0.4 & 4.4 & 4.8 & 0 & 0 \\
0 & 4 & 1 & 4 & 1.5 & 4.5 & 3.1 & 5.0 & 0 \\
0 & 7 & 1.8 & 6.2 & 2.4 & 6.6 & 6.2 & 2.7 & 4.5
\end{array}\right],
\left[\begin{array}{rrr|rrr|rrr}
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & -8 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 4.8 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 5.0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4.5
\end{array}\right]
$$

Our Constrained Factored Form is an element of family 10 (from [DGSW] Table4.2):

1. $\mathbf{C} = \mathbf{0}$
2. $[\mathbf{L}_{13}; \mathbf{L}_{23}] = \mathbf{B}^{\mathrm{T}}$
3. All non-zero blocks of $\mathbf{L}$ correctly positioned
4. $\mathbf{L}_{31} = \mathbf{D}_{31}^{-\mathrm{T}}$.

The Block Crout Factorization is a straight forward simple recursive process. However, proper termination (existence of factors $\mathbf{L_Y}$ and $\mathbf{D_Y}$) depends on properties of $\mathbf{A}$ and $\mathbf{B}$. Paper [LAA] only does:

1. Step 2b. An existence proof for the Crout block Factorization Step 2 for
   - $\mathbf{A}$ symmetric positive definite
   - $\mathbf{B}$ *lower* trapezoidal and
   - $\mathbf{B}$ of maximal rank

The obtained Crout block Factorization *can not be used for the construction of a (inexact) preconditioner* because related $\mathbf{L}$ does not contain blocks together form $\mathbf{B}$:

$$\mathbf{B} = \begin{bmatrix} 4 & 0 & 0 & 1 & 4 & 7 \\ -1 & 4 & 0 & 2 & 5 & 8 \\ 0 & -1 & 4 & 3 & 6 & 9 \end{bmatrix} \rightarrow \mathbf{L_X}{+}\mathbf{D_X} = \left[\begin{array}{ccc|ccc|ccc} 2 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 4 \\ \hline 0 & 0.2 & -0.4 & 4.8 & 0 & 0 & 1 & 2.2 & 3.6 \\ 0 & 1 & 1.5 & 3.1 & 5.0 & 0 & 4 & 6 & 7.5 \\ 0 & 1.8 & 2.4 & 6.2 & 2.7 & 4.5 & 7 & 9.8 & 11.4 \\ \hline 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

But it is an efficient factorization because $\texttt{tril}(\mathbf{A}_{11})$ is an unaltered part of $\mathbf{L}$.

Paper [SIMAX] introduces several novelties:

1. Step 2b. A (required!) different existence proof for the Crout block Factorization Step 2 for
   - $\mathbf{A}$ symmetric positive definite
   - $\mathbf{B}$ *upper trapezoidal* and
   - $\mathbf{B}$ of maximal rank
2. Step 3. Modified Incomplete Block Factorization
3. Step 3a. Such that existence can be proven

4. Step 3b. And such that the factorization $(\mathbf{L_X} + \mathbf{D_X})\mathbf{D_X}^{-1}(\mathbf{L_X} + \mathbf{D_X})^{\mathrm{T}}$ is a constrained preconditioner.

5. Step 4. Comparison against an existing method

Suddenly we have an algorithm which generates an exact factorization member of family 10 – and in addition an inexact preconditioner.

[SIMAX] and [LAA] papers both set up the Block Crout factorization. Apart from that [SIMAX] is not related to [LAA] but as mentioned based on ideas by [SHILDERS] who presents an algorithm to calculate a constrained factored form preconditioner.

[SIMAX] opens the road for the construction of algorithms for various incomplete constrained factored form preconditioners. And: Solution of systems $\mathbf{Xu} = \mathbf{b}$ can be done very efficiently with the use of $(\mathbf{L_Y} + \mathbf{D_Y})\mathbf{D_Y}^{-1}(\mathbf{L_Y} + \mathbf{D_Y})^{\mathrm{T}}(\mathbf{P}^{\mathrm{T}}\mathbf{u}) = \mathbf{P}^{\mathrm{T}}\mathbf{b}$.