Dimensional Analysis

Notes for group meeting

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In the 2011 Boulder Summer School on Hydrodynamics, I was introduced to dimensional analysis as a tool to obtain insight into a physical problem. Obviously, in the past I checked units on the left-hand side and right-hand side of an equation to make sure they are equal and that the equation made physical sense. Also, I have used dimensional analysis to convert between various types of units e.g. feet to miles. But this was the extend of it. However, since dimensional analysis was presented as a powerful tool when used with caution during the summer school, I decided to learn more about it which resulted in the notes I present below. I hope after reading these notes you (as I) will not be surprised that power laws appear in physical problems as well as get a better understanding why we encounter rational as well as irrational exponents in scaling laws. Rational exponents are connected to dimensional analysis whereas irrational exponents known as anomalous dimensions to renormalization group theory.

1 Introduction

The first step in modeling any physical phenomena is the identification of the relevant variables, and then relating these variables via known physical laws. For sufficiently simple phenomena we can usually construct a quantitative relationship among these variables from first principles; however, for many complex phenomena such an *ab initio* theory is often difficult, if not impossible. In these situations modeling methods are indispensable, and one of the most powerful modeling methods is dimensional analysis. Dimensional analysis is rapid; we don't need to write down the equations of motion of a mechanical system, but on the other hand this analysis does not give as complete information as might be obtained by carrying through a detailed analysis. We have encountered dimensional when we checked our units to ensure that the left- and right- hand sides of an equation had the same units. In a sense, this is all there is to dimensional analysis, although *checking units* is certainly the most trivial example of dimensional analysis. Here we will use dimensional analysis to actually solve problems, or at least infer some information about the solution. The majority of the discussion is taken out of [1] and [2].

The basic principle of dimensional analysis was known to Isaac Newton (1686) who referred to it as the "Great Principle of Similitude" [3]. James Clerk Maxwell played a major role in establishing modern use of dimensional analysis by distinguishing mass, length, and time as fundamental units, while referring to other units as derived [4]. The 19th-century French mathematician Joseph Fourier made important contributions [5] based on the idea that physical laws like $F = ma$ should be independent of the units employed to measure the physical variables. That is nature does not care if we measure lengths in centimeters or inches or light-years. This led to the conclusion that meaningful laws must be homogeneous equations in their various units of measurement, a result which was eventually formalized in the Buckingham-Π theorem.

Through dimensional analysis, researches including Newton, Fourier, Maxwell, Rayleigh and Kolomogorov have been able to obtain remarkable deep results that have sometimes changed entire branches of science. The mathematical techniques required to derive these results turn out to be simple and accessible to all. Hence many people tried to attack similar problems through dimensional analysis. But they almost always failed. Dimensional analysis was cursed and reproached for being untrusworthy and unfounded, even mystical. The reason for this lack of success was that only a few people understood the content and real abilities of dimensional analysis.

2 Warmup

Application of the methods of dimensional analysis to simple problems, particular in mechanics are made by every student of physics. Let us analyze two such problem in order to refresh our minds.

2.1 Simple Pendulum

First consider the illustrative problem of the simple pendulum. Our goal is to find, without going through a detailed solution of the problem, certain relations which must be satisfied by the various measurable quantities in which we are interested. The usual procedure is a follows:

1. make a list of all the quantities on which the answer may depend on
2. write down the dimensions of these quantities
3. combine these quantities in such a way that the relation remains true no matter what size of the units in terms of which the quantities are measured

Now we apply this method to find how the time of swing of a simple pendulum depends on the variables which determine the behavior. Our list of quantities is as follows: Our goal is to find

quantity	*symbol*	*dimensions*
time of swing	t	T
length of pendulum	l	L
mass of pendulum	m	M
acceleration of gravity	g	LT^{-2}
angular amplitude of swing	θ	no dimensions

t as a function of l, m, g, and θ such that the functional relation still holds when the size of the fundamental units (M, T, L) is changed. Suppose that we have found the relation and write

$$t = f(l, m, g, \theta).$$

The magnitude of the time of the swing only depends on the size of the unit of time, and is not changed if the units of mass or length are changed. Hence the quantities on the right-hand-side of the equation must be combined in such a way that together they are also unchanged when the units of mass and length are changed. First we notice that the unit of mass only affects the magnitude of m and no other quantity depends on M which means if the unit of mass changes no other quantity can compensate this change and hence the time of the swing cannot depend on mass:

$$t = f(l, g, \theta).$$

Next l and g must enter together in such a way that the magnitude of the argument is unchanged when the size of the unit of length is changed and the unit of time is kept constant. To accomplish this l must be divided by g

$$t = f(l/g, \theta).$$

Moreover, l/g must enter the unknown function in such a way that the combination has dimensions of T. Also since θ is dimensionless the magnitude of the angular momentum can enter the unknown function in any way and we see by inspection that the final result is to be written as

$$t = \sqrt{l/g}\Psi(\theta),$$

which we obtained from dimensional considerations alone. $\Psi(\theta)$ is subject to no restrictions and we know from mechanics that $\Psi(\theta)$ is very near a constant independent of θ and approximately equal to 2π for a pendulum starting at zero angular velocity from initial angle θ_0 for small θ_0. But dimensional analysis does not reveal this constant. Indeed, to fix the numerical constant we need a real theory of the phenomena in question or perform a simple experiment where we measure the period of oscillation of a weight hung on a thread.

2.2 Atomic bomb explosion

Now let us consider a second, very famous, problem. In 1947, a sequence of photographs of the first atomic bomb explosion in New Mexico in 1945 were published in *Life* magazine. The photographs show the expansion of the shock wave caused by the blast at successive times in ms. G.I. Taylor used this information to give a very accurate estimate of the strength from dimensional analysis [6]. His goal was to find the dependence of the shock wave front described by a radius R on the energy released, the time elapsed, and the air density. To use dimensional analysis we fill in our table: leading to the following relation

quantity	*symbol*	*dimensions*
energy deposited in explosion	E_0	ML^2T^{-2}
time elapsed since explosion	t	T
air density	ρ	ML^{-3}
radius of shock wave	R	L

$$R = f(E_0, \rho, t).$$

Now this functional equation is not quite so easy to solve by inspection as the previous problem, and hence we have to use a little algebra on it by solving the so called dimensional formula.

3 Dimensional Formula

Next we derive the dimensional formula. Suppose the value of a certain variable x_1 depends only on the variables $x_2, x_3, x_4, \ldots$. So

$$x_1 = f(x_2, x_3, x_4, \ldots).$$

Now we show that the function f is a function, such that

$$x_1 = C x_2^{\alpha} x_3^{\beta} x_3^{\gamma} ...,$$

where α, β, γ and C are constant coefficients. x_1, x_2, etc stand for numbers which are the measures of particular kinds of physical quantities. Thus x_1 might stand for the number which is a measure of velocity. The arguments in the functional relation above fall into two groups, depending on the way in which the numbers are obtained physically. The first group is called *primary quantities* e.g. mass, length, time and the second group is called *secondary quantities* which are measured in terms of the primary quantities e.g. velocity. For both quantities it must be true that the ratio of the numbers measuring any two concrete examples of a primary/secondary quantity must be independent of the size of the fundamental units. This requirement that the ratio be constant is a requirement for dimensional analysis to work. Hence dimensional analysis cannot be applied to systems which do not meet this requirement.

Next we formulate this restriction analytically. Let's call the primary quantities $a, b, c, ...$ which are combined in a certain way to measure the secondary quantity. The combination is represented by the functional symbol f

$$secondary\ qunatity = f(a, b, c, ...).$$

Now if there are two concrete examples of the secondary quantity, the associated primary quantities have different numerical values we have two sets $f(a_1, b_1, c_1, ...)$ and $f(a_2, b_2, c_2, ...)$ and the ratio of these two sets must be a constant

$$\frac{f(a_1, b_1, c_1, ...)}{f(a_2, b_2, c_2, ...)} = constant.$$

We now change the size of the fundamental units e.g. a is measured $1/x$th as large and the number measuring a will be x times as large, or xa. In the same way the unit measuring b is $1/y$th as large and so on. Now for our two concrete examples we have $f(xa_1, yb_1, zc_1, ...)$ and $f(xa_2, yb_2, zc_2, ...)$ and their ratio must be the same constant leading to the following relation

$$\frac{f(a_1, b_1, c_1, ...)}{f(a_2, b_2, c_2, ...)} = \frac{f(xa_1, yb_1, zc_1, ...)}{f(xa_2, yb_2, zc_2, ...)}.$$

Next we solve the this equation for the unknown function f rewriting the above expression

$$f(xa_1, yb_1, zc_1, ...) = f(xa_2, yb_2, zc_2, ...) \frac{f(a_1, b_1, c_1, ...)}{f(a_2, b_2, c_2, ...)}.$$

Differentiating partially with respect to x where f_x denotes the partial derivative in x

$$a_1 f_x(xa_1, yb_1, zc_1, ...) = a_2 f_x(xa_2, yb_2, zc_2, ...) \frac{f(a_1, b_1, c_1, ...)}{f(a_2, b_2, c_2, ...)}$$

and setting $x, y, z = 1$ we have

$$a_1 \frac{f_x(a_1, b_1, c_1, ...)}{f(a_1, b_1, c_1, ...)} = a_2 \frac{f_x(a_2, b_2, zc_2, ...)}{f(a_2, b_2, c_2, ...)}.$$

This holds for all $a_1, b_1, c_1, ...$ and $a_2, b_2, c_2, ...$ Hence we can hold $a_2, b_2, c_2, ...$ constant and let $a_1, b_1, c_1, ...$ vary. Equivalent to the product rule we first let a_1 vary and hold $a_2, b_2, c_2, ...$ constant since they don't depend on a_1. Hence after dropping the subscript $(a_1 \to a)$ we write

$$\frac{a}{f}\frac{\partial f}{\partial a} = constant$$

or

$$\frac{1}{f}\frac{\partial f}{\partial a} = \frac{constant}{a},$$

which integrates to

$$f = C_1(b, c, ...)a^{constant}.$$

Here the factor C_1 is a function of the other variables $b, c, ...$ To obtain C_1 we can repeat the process integrating with respect to $y, z, ..$ to obtain the dimensional formula

$$f = Ca^{\alpha}b^{\beta}c^{\gamma}...$$

where α, β, γ and C are constants. Hence we have shown that in dimensional formulas the fundamental units always enter as products of powers e.g. the dimensions of any physical quantity can be expressed in terms of a power-law monomial.

3.1 Return to atomic bomb explosion

In section *2.2* we started the atomic bomb explosion problem. We were not able to solve it by inspection and hence apply now the dimensional formula. We already decided that the radius of the shock wave R is given as

$$R = f(E_0, \rho, t).$$

Using the dimensional formula we have

$$R = CE_0^{\alpha}\rho^{\beta}t^{\gamma}.$$

The dimensions on the left-hand-side and right-hand-side must coincide:

$$L = \left(\frac{ML^2}{T^2}\right)^{\alpha}\left(\frac{M}{L^3}\right)^{\beta}T^{\gamma}.$$

Equating the exponents of like dimensions gives:

$$\begin{aligned} L: 1 &= 2\alpha - 3\beta \\ M: 0 &= \alpha + \beta \\ T: 0 &= -2\alpha + \gamma \end{aligned}$$

from which we get

$$\alpha = 1/5,\ \beta = -1/5,\ \gamma = 2/5$$

and therefore

$$R = C\left(\frac{E_0}{\rho}\right)^{1/5}t^{2/5}.$$

When is this solution going to work? First, the size of the bomb, R_0, did not enter (that allowed to form the scaling combination for R), which means that we have effectively treated the bomb as a point source. But this is fine since every source looks like a point source if we are very far away. Therefore, our solution is valid for $R >> R_0$. Second, I wonder now if we forgot another quantity - namely the air pressure. Looking at our approach adding the air pressure ($ML^{-1}T^{-2}$) we will have three equations and four unknowns. Hence the constant C must depend on the pressure p, $C(p)$ and we need to use more information! Taylor actually included p in his calculation and plotted measured values of R as a function of t in a double logarithmic plot. He came to the conclusion that $C(p) \approx 1$.Hence every shock wave is very well described by

$$log(R) = \frac{1}{5} log\left(\frac{E_0}{\rho}\right) + \frac{2}{5} log(t).$$

Since ρ is usually known, it is easy to estimate the energy E_0 from fitting this equation into measured $R(t)$ data. Hence Taylor was able to estimate the energy of the explosion to be around 10^{20} erg (30 000 tons of T.N.T). He published the energy of the explosion in 1950 [8] which caused in his words "much embarrassment" in the American government circles because this figure was considered top secret and not declassified till the 1960s.

4 Buckingham-Π Theorem

Next to applying the dimensional formula one can use another method called Buckingham-Π theorem [7] which is often a bit more work but a safer method. This method simply states that the dimensional formula can be rescaled into an equivalent *dimensionless* statement. The Buckingham-Π theorem states how every physically meaningful equation involving n variables can be equivalently rewritten as an equation of $n - m$ dimensionless parameters of the form $\Pi_1, \Pi_2, ..., \Pi_{n-m}$, where m is the number of fundamental dimensions used e.g. mass, time, length. Then the solution to the physical problem is of the form

$$\Pi_1 = f(\Pi_2, \Pi_3, ..., \Pi_{n-m}).$$

Or equivalently, in dimensional analysis one needs to form all possible independent dimensionless quantities. The proof is of this theorem is omitted here but one can find a formal proof in *chapter 0* of Barenblatt's book [2].

4.1 Dimensional analysis of Pythagoras' theorem

In this section, we use the Buckingham-Π theorem to derive Pythagoras' theorem. The area of a right triangle, A_c, is completely determined by the hypotenuse c and the smaller of its acute angles ϕ. In this problem we have only one fundamental unit, length L, since ϕ is dimensionless and 3 variables leading to $3 - 1 = 2$ dimensional groups. The two independent dimensionless groups are

$$\Pi_1 = A_c/c^2 = f(\Pi_2),$$

where $\Pi_2 = \phi$ giving

$$A_c = c^2 f(\phi).$$

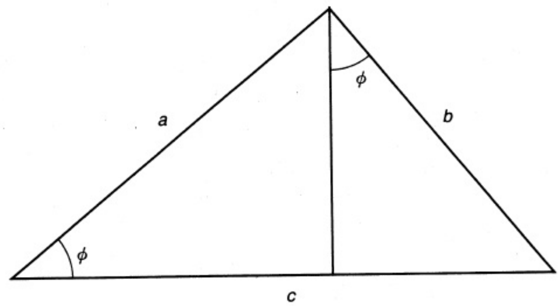

Figure 1: A proof of Pythagoras' theorem using dimensional analysis [2]

The altitude perpendicular to the hypotenuse of this triangle divides it into two similar right triangles with hypotenuses equal to the sides a and b of the larger triangle. We can use the above equation to write

$$A_a = a^2 f(\phi),\ A_b = b^2 f(\phi).$$

Since the sum of these two equals the total area $A_c = A_a + A_b$, $f(\phi)$ cancels and we have proven Pythagoras' theorem

$$c^2 = a^2 + b^2$$

without knowing any trigonometry.

4.2 Drag on sphere

Next we wan to compute the drag F_d on a sphere of radius R in a fluid with density ρ and kinematic viscosity ν. First we need to identify the number of independent dimensions which is 3 corresponding to mass M, time T, and length L. Next we make a list of, to us, important physical quantities The problem can be described by 5 quantities. Hence we need $5-3=2$ dimensionless Π

quantity	*symbol*	*dimensions*
drag force	F_d	MLT^{-2}
fluid density	ρ	ML^{-3}
sphere radius	R	L
fluid velocity	v	L/T
kinematic viscosity	ν	L^2/T

groups. One of them must involve the drag force F_d. The first dimensionless group is the Reynolds number, Re

$$\Pi_1 = \frac{vR}{\nu} \equiv Re.$$

The Re number is the ratio of inertial force to viscous force. The other dimensionless group must have drag force F_d in it

$$\Pi_2 = \frac{F_d}{\rho v^2 R^2}$$

and the Buckingham-Π theorem $\Pi_2 = f(\Pi_1)$ gives

$$\Pi_2 = \frac{F_d}{\rho v^2 R^2} = f(Re)$$

so that

$$F_d = \rho v^2 R^2 f(Re).$$

Looking at the result is is physically plausible that the drag depends on Re number. Now we are left to explore the asymptotics of $f(Re)$ which is a key part of the Π theorem. Hence we need to answer what happens if $Re >> 1$ which describes turbulent flow or $Re << 1$ which describes laminar flow also called Stokes flow. To find the dimensionless factor $f(Re)$ we can either perform measurements, use physical insight, or need to analyze the Navier Stokes equation. For low Re numbers it turns out that $f(Re) = Re^{-1}$ and

$$F_d = \rho \nu v R = \eta R v$$

where $\eta = \rho\nu$ is the viscosity and F_d is the well known Stokes drag.

4.3 Thermodynamic equations

At first sight the application to dimensional analysis in thermodynamics is puzzling due to the appearance of so-called logarithmic constants. Some equations in thermodynamics do not appear to be complete equations or to be dimensionally homogeneous. These equations often involve constants which cannot change numerical magnitude by some factor when the size of the fundamental units is changed, but must change by the addition of some term. But a rearrangement of terms is possible which throw the formula in a conventional form. Rearrangement is always possible if the formula has had a theoretical derivation and the logarithmic constant only appears as a formal exception. This constant is due to the under-determined constant of integration arising from the fact that energy, or work, or entropy, or thermodynamic potential has no absolute significance, but is only the difference between two values and the coordinates of the initial point which fix the origin of entropy (may be chosen at pleasure). For an example see [1] pages 74-75.

In thermodynamics you will commonly find expressions in which the logarithm is taken of a quantity with dimensions which are particularly common in thermodynamics like $log(P)$. This is fine since on expanding expressions like

$$\frac{dlog(P)}{dT} = \frac{1}{P}\frac{dP}{dT}$$

we obtain zero dimension in P. The occurrence of such expressions is not contrary to the Π-theorem since in such a case the slope of the curve dP/dT would be one of the variables in which dimensionless products are to be expressed, and there is evidently no exception.

5 Anomalous Dimensions

In the last section I want to talk about the anomalous dimensions and the connection to renormalization group theory. For an extensive discussion I refer to Nigel Goldenfeld's book [9]. So far we have talked about scaling which means in its simplest form that measurable quantities depend upon each other in a power law fashion. So far in any scaling laws, the power law or exponent is a rational fraction which can be deduced from dimensional considerations. But is this always th case? NO! For example, other phenomena such as continuous phase transitions have a critical point and can be characterized by a critical exponent which is not a simple fraction. For instance, let us examine the behavior of the heat capacity C near a phase transition. We vary the temperature T of the system while keeping all the other thermodynamic variables fixed, and find that the transition occurs at some critical temperature T_c. When T is near T_c, the heat capacity C typically has a power law behavior:

$$C\|T_c - T\|^{-\alpha}$$

where from experimental measurements the critical exponent $0 > \alpha$ is not an obviously simple fraction and hence we see the appearance of a so called an anomalous scaling dimension. Since until now we have thought we should only expect rational exponents, anomalous scaling dimensions seem to be in odds with dimensional analysis.

5.1 Renormalization

It is understood that an anomalous dimension reflects the presence of a microscopic length scale, which affects the behavior of thermodynamic and correlation functions asymptotically close to the critical point. However, the value of the anomalous dimension does not generally depend on the microscopic length scale, although in principle it could. Mathematically, the situation can be summarized as follows. Suppose that we are interested in some quantity F that depends in principle on a and ζ. Then, it is only legitimate to replace $x = a/\zeta$ by 0 in he function $F(a/\zeta)$ if $F(x)$ is not singular in the limit $x \to 0$. There are three possibilities for this limit:

1. intermediate asymptotics of the first kind: $F(x) \to 0$ as $x \to 0$.
2. intermediate asymptotics of the second kind: $F(x) \sim x^{-\alpha}\Phi(x)$ as $x \to 0$. with $\alpha > 0$ and $\Phi(x)$ regular in the limit that $x \to 0$.
3. None of the above.

It turns out that the Π-theorem falls in case (1) and that critical phenomena in case (2). To solve the problem of anomalous dimensions both in critical phenomena and in other problems discussed by Barenblatt [2] such as in fluid dynamics, one applies a renormalization procedure known as renormalization group theory. This procedure introduces a new length scale into the problem from which anomalous dimension appear. Hence starting from dimensional analysis specifically the Π-theorem, we can express a problem in dimensionless variables $\Pi, \Pi_0, \Pi_1, ..$, with solution

$$\Pi = f(\Pi, \Pi_0, \Pi_1, ..).$$

The asymptotics is said to be of second kind if, with an appropriate choice of exponents $\alpha, \alpha_1, \alpha_2, ...$ the asymptotic behavior is of the form

$$\Pi \sim \Pi_0^{\alpha} g\left(\frac{\Pi_1}{\Pi_0^{\alpha_1}}, ...\right), \quad as\ \Pi \to 0,$$

where g is a scaling function. The exponents $\{\alpha\}$ i.e. the anomalous dimensions cannot be predicted by dimensional analysis, but must be determined by solving the problem directly by applying for example renormalization group theory. In this theory the first step is the renomalization process i.e. extension of dimensional analysis, taking into account renormalization effects. The second step is to combine the renormailzation group with an approximation scheme, such as perturbation theory, in order to estimate the values of the anomalous dimensions. As mentioned for a detailed discussion refer to *chapter 10* in [9].

6 Summary

We have learned that the dimensions of any physical quantity can be expressed in terms of a power-law monomial. Second, the Buckingham-Π theorem can be a powerful tool in deriving physical equations to within dimensionless factors (usually negligible in order of magnitude estimates). We have demonstrated that the seemingly trivial concept of dimensional analysis are capable of producing results with a great deal of content, especially when the difference between the number of governing parameters is not too large. Thus correctly choosing the right set of governing parameters is crucial. The set of governing parameters may be determined relatively easily if a mathematical formulation of the problem is available. This must include the governing variables and constant parameters of the problem, which appear in the equations, boundary condition, initial condition, and so forth. On the other hand correctly choosing the set of governing parameters for problems that do not have an explicit mathematical formulation depends primarily on the intuition of the researcher. In such problems, success in applying dimensional analysis involves a correct understanding of which governing parameters are essential and which may be neglected. Finally, dimensional analysis is deep and non-trivial and ultimately rooted in the renormalization group theory which gives physical insight into anomalous dimensions.

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