Similarity, Self-Similarity, and Intermediate Asymptotics

G. I. Barenblatt
Academy of Sciences of the USSR
Moscow, USSR

Translated from Russian by
Norman Stein

Translation editor
Milton VanDyke
Stanford University
Stanford, California

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PODOBIE, AVTOMODEL'NOST', PROMEZHUTOCHNAYA ASIMPTOTIKA
G. I. Barenblatt
ПОДОБИЕ, АВТОМОДЕЛЬНОСТЬ, ПРОМЕЖУТОЧНАЯ АСИМПТОТИКА
Г. И. Баренблатт

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To the glowing memory of
his beloved mother, Dr. Nadezhda Veniaminovna Kagan,
physician-virologist, heroically lost for the sake of
the healthy future of humanity, the author dedicates his work.

Foreword

Professor Grigorii Isaakovich Barenblatt has written an outstanding book that contains an attempt to answer the very important questions of how to *understand* complex physical processes and how to *interpret* results obtained by numerical computations.

Progress in numerical calculation brings not only great good but also notoriously awkward questions about the role of the human mind. The human partner in the interaction of a man and a computer often turns out to be the weak spot in the relationship. The problem of formulating rules and extracting ideas from vast masses of computational or experimental results remains a matter for our brains, our minds.

This problem is closely connected with the recognition of patterns. It is not just a coincidence that in both the Russian and English languages the word "obvious" has two meanings–not only something easily and clearly understood, but also something immediately evident to our eyes. The identification of forms and the search for invariant relations constitute the foundation of pattern recognition; thus, we identify the similarity of large and small triangles, etc.

Let us assume now that we are studying a certain process, for example, a chemical reaction in which heat is released, and whose rate depends on temperature. For a wide range of parameters and initial conditions, a completely definite type of solution is obtained–flame propagation. The chemical reaction occurs in a relatively narrow region separating the cold combustible substance from the hot combustion products; this region moves relative to the combustible substance with a velocity that is independent of the initial conditions. (Of course, the very occurrence of combustion depends on the initial conditions.)

This result can be obtained by direct numerical integration of the partial differential equations that describe the heat transfer, diffusion, chemical reaction, and (in some cases) hydrodynamics. Such a computational approach is difficult; the result is obtained in the form of a listing of quantities such as temperature, concentration, etc., as functions of temporal and spatial coordinates. To make manifest the flame propagation, i.e., to extract from the mass of numerical material the regime of uniform temperature propagation, $T(x - ut)$, is a difficult problem! It is necessary to know the type of the solution in advance in order to find it; anyone who has made a practical attempt to apply mathematics to the study of nature knows this truth.

The term "self-similarity" was coined and is by now widespread: a solution $T(x, t_1)$ at a certain moment t_1 is similar to the solution $T(x, t_0)$ at a certain earlier moment. In the case of uniform propagation considered above, similarity is replaced by simple translation. Similarity is connected with a change of scales:

$$T = \left(\frac{t_1}{t_0}\right)^n T\left(x\left(\frac{t_1}{t_0}\right)^m, \ t_0\right)$$

or

$$T = \varphi(t)\Psi(x/\xi(t)).$$

In geometry, this type of transformation is called an affine transformation. The existence of a function ψ that does not change with time allows us to find a similarity of the distributions at different moments.

Barenblatt's book contains many examples of analytic solutions of various problems. The list includes heat propagation from a source in the linear case (for constant thermal conductivity) and in the nonlinear case, and also in the presence of heat loss. The problem of hydrodynamic propagation of energy from a localized explosion is also considered. In both cases, the problem in its ordinary formulation—without loss—was solved many years ago; in these problems the dimensions of the constants that characterize the medium (its density, equations of state, and thermal conductivity) and the dimensions of energy uniquely dictate the exponents of self-similar solutions.

However, with properly introduced losses the problems turn out to be essentially different. If $dE/dt = -\alpha E^{3/2}/R^{5/2}$, $dR/dt = \beta E^{1/2}/R^{3/2}$ (E being the total energy referred to the initial density of the gas, R the radius of the perturbed domain, and t the time) so that $dE/dR = -\gamma E/R$, then conservation of energy does not hold:

$$E \sim R^{-\gamma}, \quad E = E_0 R_0^{\gamma} R^{-\gamma} \neq \text{const};$$

however, self-similarity remains.

The dimensionless numbers α, β, and γ depend on the functions describing the solution, but the equations that determine these solutions contain indeterminate exponents. Mathematically we have to deal with the determination, from nonlinear ordinary differential equations and their boundary conditions, of certain numbers that can be called eigenvalues.

The new exponents in the problem are not necessarily integers or rational fractions; as a rule they are transcendental numbers that depend continuously on the parameters of the problem, including the parameters of energy loss. Thus arises a new type of self-similar solution, which we shall call the second type, reserving the title of first type for the case when naive dimensional analysis succeeds.

An important point arises here. The solution does not describe the point source asymptotically: if R_0 (the value of R at $t = 0$) is taken to be equal to zero, then it must necessarily be that $E_0 = \infty$ for $t = 0$, which is physically inconsistent. Hence the new solution is considered as an intermediate asymptotics. We assume that up to a certain finite time t_0 there is no loss. At this moment, when the radius of the perturbed domain reaches the finite value R_0, we switch on the loss. Or, more generally, we can start with a finite energy E created by some other means and which has already spread out to the finite radius R_0. It is assumed that asymptotically for sufficiently large time the solution assumes a self-similar form corresponding to the given loss.

We want to emphasize the asymptotic character of the self-similar solution for $t \gg t_0$. In nonlinear problems, exact special solutions sometimes appear to be useless: since there is no principle of superposition, one cannot immediately find a solution of the problem with arbitrary initial conditions.

Here asymptotic behavior is the key that partially plays the role of the lost principle of superposition. However, for arbitrarily given initial conditions this asymptotic behavior must be proved. The problem is difficult, and in many cases numerical computations give only a substitute for rigorous analytic proof.

The preceding arguments may seem unusual in a Foreword; but I want, using the simplest examples, to introduce the reader as quickly as possible to the advantages and difficulties of the new world of solutions of the second kind.

There are also other types of solutions, among which convergent spherical shock waves are the most important. In this case there is no external loss, but the region in which self-similarity holds is contracting; it is therefore impossible to assume that the entire energy is always concentrated in the shrinking region, and this energy in fact decreases according to a power law, since part of the energy remains in the exterior regions of the gas. Again it is necessary to find the exponents as eigenvalues of a nonlinear operator.

The specific character of this class of equations is connected with the finiteness of the speed of sound; a point where the phase velocity of propagation of a self-similar variable is equal to the velocity of sound plays a decisive role in the construction of the solution.

Barenblatt also discusses in his book another problem of analogous type: the problem of a strong impulsive load in a half-space filled with gas. This problem abounds in paradoxes. In particular: why do the laws of conservation of energy and momentum not make it possible to determine the exponents? The answer to this question is contained in Chapter 4, and it would be against the rules to give it here in the Foreword.

Problems of nonlinear propagation of waves on the surface of a heavy fluid, described by the Korteweg–deVries equation, give a remarkable example. Here there are long-established and well-known solutions describing solitary waves (called "solitons"), propagating with a velocity dependent on the amplitude. This example is remarkable in that there exist theorems proving the stability of solitons even after their collisions, and theorems determining the asymptotic behavior of initial distributions of general type, which are transformed into a sequence of solitons. At first suggested by numerical computations, these properties are now rigorously proved by analytic methods of extraordinary beauty. In these solutions all the properties of ideal self-similar solutions of the second kind appear.

In some sense the problems of turbulence, considered at the end of the book, differ from those mentioned above. These are farther from my interests and I will not dwell on them here. A complete outline of all that is contained in the book can be found in the table of contents and should not be sought in the Foreword.

We shall now return to the general nature of the book as a whole; we shall not hesitate to repeat for the general situation some considerations that have already been presented above in connection with simple examples.

The problems are chosen carefully. Each of them taken separately is a pearl, important and cleverly presented. In the solution of many of the problems the role of the author was essential, and this gives to the presentation the flavor of something lived. But I must emphasize that the importance of this book far exceeds its value as a collection of interesting special examples; from the special problems considered, very general ideas develop.

Most of the problems are nonlinear. What is the use of special solutions if there is no principle of superposition? The fact is that as a rule these special solutions represent the asymptotics of a wide class of other more general solutions that correspond to various initial conditions. Under these circumstances the value of exact special solutions increases immensely. This aspect of the question is reflected in the title of the book in the words "intermediate asymptotics." The value of solutions as asymptotics depends on their stability. The questions of the stability of a solution and of its behavior under small perturbations are also considered in this book; in particular, there is presented a rather general approach to the stability of invariant solutions developed in a paper by Barenblatt and myself.

The very idea of self-similarity is connected with the group of transformations of solutions. As a rule, these groups are already represented in the differential (or integrodifferential) equations of the process. The groups of transformations of equations are determined by the dimensions of the variables appearing in them; the transformations of the units of time, length, mass, etc. are the simplest examples. This type of self-similarity is characterized by power laws with exponents that are simple fractions defined in an elementary way from dimensional considerations.

Such a course of argument has led to results of immense and permanent importance. It is sufficient to recall the theory of turbulence and the Reynolds number, linear and nonlinear heat propagation from a point source, and a point explosion. Nevertheless, we shall see that dimensional analysis determines only a part of the problem, the tip of the iceberg; we shall call the corresponding solutions *solutions of the first kind*. We shall reserve the name *solutions of the second kind* for the large and ever growing class of solutions for which the exponents are found in the process of solving the problem, analogous to the determination of eigenvalues for linear equations. For this case, conservation laws and dimensional considerations prove to be insufficient.

The establishment of an intrinsic connection between nonlinear propagation problems with solutions of the type $f(x-ut)$ and self-similar problems with solutions of the form $t^n f(x/t^m)$ has turned out to be a very important step. The general procedures for determining the speed parameter u and the powers n, m have, as it turned out, many points of contact. By the same token, self-similarity touches on a new stream of problems arising from the theory of combustion and from applications to chemical technology. Barenblatt's book contains concrete, detailed consideration of certain problems, giving a wealth of information. It also contains brilliant generalizations, foresights touching on developments of the future, and hints about discoveries not yet made.

You can read this book and study it, but you can also use it as a source of inspiration. Possibly this is the best compliment for a book with a title that sounds so special.

Ya. B. Zel'dovich
Member, Academy of Sciences of the USSR

Preface

It is most useful for researchers, both theoreticians and experimentalists, to know what the methods of similarity are, what self-similarity is, how to recognize it, and how to use it. To give a view of this range of problems is the object of the present book.

Together with some well-known material included for completeness of exposition, the book presents a good deal of material that is comparatively new and not yet the subject of a traditional presentation. Naturally, the choice of material, and especially the mode of exposition, are affected by the subjective approach and personal scientific interests of the author. I hope, nevertheless, that the book reflects in some measure the state of affairs in the range of problems considered, which are again attracting the immediate attention of mechanicians and theoretical physicists.

Over the course of many years I have had the privilege of continuous scientific and personal contact with Yakov Borisovich Zel'dovich. My interest in generalized self-similar solutions, to which this book is largely devoted, was first aroused by his work. His comments on the manuscript of the book were important and useful. I am pleased to thank Yakov Borisovich here.

I thank Andrei Sergeevich Monin, whose friendly attention has been a continuing stimulus for me. I also thank my colleagues Dr. V. A. Gorodtsov, Prof. V. M. Entov, Dr. V. D. Larichev, Dr. G. M. Reznik, and Dr. E. V. Teodorovich for attentively reading the manuscript of the book and making useful comments. To the charming Mrs. I. A. Viktorova and her assistants I am very grateful for help in preparing the manuscript.

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G. I. Barenblatt

Contents

Introduction

1. A phenomenon is called *self-similar* if the spatial distributions of its properties at various moments of time can be obtained from one another by a similarity transformation.† Establishing self-similarity has always represented progress for a researcher: self-similarity has simplified computations and the representation of the characteristics of phenomena under investigation. In handling experimental data, self-similarity has reduced what would seem to be a random cloud of empirical points so as to lie on a single curve or surface, constructed using self-similar variables chosen in some special way. Self-similarity of the solutions of partial differential equations has allowed their reduction to ordinary differential equations, which often simplifies the investigation. Therefore with the help of self-similar solutions researchers have attempted to envision the characteristic properties of new phenomena. Self-similar solutions have also served as standards in evaluating approximate methods for solving more complicated problems.

The appearance of computers changed the general attitude toward self-similar solutions but did not decrease the interest in them. Previously it was considered that the reduction of partial to ordinary differential equations simplified matters, and hence self-similar solutions attracted attention, first of all, because of the simplicity of obtaining and analyzing them. Gradually the situation grew more complicated, and in many cases it turned out that the simplest method of numerically solving the boundary-value problems for systems of ordinary equations that resulted from the construction of self-similar solutions was a computation by the method of stabilization of the solutions of the partial differential equations. Nevertheless, self-similarity continued as before to attract attention as a profound physical fact indicating the presence of a certain type of stabilization of the processes under investigation, valid for a rather wide range of conditions. Moreover, self-similar solutions were used as a first step in starting numerical calculations on computers. For all these reasons the search for self-similarity was undertaken at once, as soon as a new domain of investigation was opened up.

†The fact that we identify one of the independent variables with time is of no significance.

Instructive examples of self-similarity are given by several highly idealized problems in the mathematical theory of heat conduction. Suppose that at the initial instant $t = 0$, and at a certain point in an infinite space filled with a heat-conducting medium, a finite amount of heat E is supplied instantaneously. Then at time t the increment in temperature u at an arbitrary point of the space is given by the relation

$$u = \frac{E}{c\,(2\sqrt{\pi \varkappa t}\,)^3} \exp\left(-\frac{r^2}{4\varkappa t}\right). \tag{0.1}$$

Here c is the specific heat of the medium and $\varkappa$ its thermal diffusivity (thermometric conductivity), which are properties of the medium that are assumed to be constant, and r is the distance of the point at which the observation is made from the central point at which heat was supplied initially.

The structure of (0.1) is instructive: there exist a temperature scale $u_0(t)$ and a linear scale $r_0(t)$, both depending on time,

$$u_0(t) = \frac{E}{c\,(\varkappa t)^{3/2}}, \quad r_0(t) = \sqrt{\varkappa t}, \tag{0.2}$$

such that the spatial distribution of temperature, when expressed in these scales, ceases to depend on time:

$$\frac{u}{u_0} = f\left(\frac{r}{r_0}\right),$$
$$f(\xi) = \frac{1}{8\pi^{3/2}} \exp\left(-\frac{\xi^2}{4}\right), \quad \xi = \frac{r}{r_0}. \tag{0.3}$$

The example just considered is typical. Suppose that we are faced with a problem of mathematical physics in two independent variables r and t, requiring the solution of a system of partial differential equations. In this problem self-similarity means that we can choose variable scales $u_0(t)$ and $r_0(t)$ such that in the new scales the properties of the phenomenon can be expressed by functions of one variable:

$$u = u_0(t)\, U(\xi), \quad \xi = r/r_0(t). \tag{0.4}$$

The solution of the problem thus reduces to the solution of a system of ordinary differential equations for the function $U(\xi)$.

It was natural to attempt to clarify the nature of self-similarities. Here at a certain stage the attraction of dimensional analysis played an essential role. By dimensional analysis we mean a very simple system of concepts and rules which nevertheless deserves to be examined.

Suppose we have some relationship defining a quantity a as a function of n

parameters $a_1, a_2, \ldots, a_n$:

$$a = f(a_1, a_2, \ldots, a_n). \tag{0.5}$$

If this relationship has some physical meaning, (0.5) must reflect the indisputable fact that although the numbers $a, a_1, \ldots, a_n$ express the values of corresponding quantities in a definite system of units of measurement, the physical law represented by this relation does not depend on the arbitrariness in the choice of units. In order to explain what follows from this, we shall divide the quantities $a, a_1, a_2, \ldots, a_n$ into two groups. The first group, $a_1, \ldots, a_k$, includes the governing quantities with independent dimensions (for example, length, velocity, and density). The second group, $a, a_{k+1}, \ldots, a_n$, contains quantities whose dimensions can be expressed in terms of the dimensions of the quantities in the first group. Thus, for example, the quantity a has the dimensions of the product $a_1^p a_2^q \ldots a_k^r$, the quantity a_{k+1} has the dimensions of the product $a_1^{p_{k+1}} a_2^{q_{k+1}} \ldots a_k^{r_{k+1}}$, etc. The exponents $p, q, \ldots$ are obtained by a simple calculation. Thus the quantities

$$\Pi = \frac{a}{a_1^p a_2^q \ldots a_k^r}, \quad \Pi_1 = \frac{a_{k+1}}{a_1^{p_{k+1}} a_2^{q_{k+1}} \ldots a_k^{r_{k+1}}}, \ldots \tag{0.6}$$

turn out to be dimensionless, so that their values will be one and the same no matter how we choose the units of measurement. Hence the fact that the relationship (0.5) has a physical significance independent of the choice of units of measurement actually means that it can be expressed in the form of an equation,

$$\Pi = \Phi(\Pi_1, \ldots, \Pi_{n-k}), \tag{0.7}$$

connecting the dimensionless quantity Π with the dimensionless quantities $\Pi_1, \ldots, \Pi_{n-k}$, which are, however, k fewer in number than the original dimensional governing parameters $a_1, \ldots, a_n$. If in (0.7) we now return to the original dimensional variables $a, a_1, \ldots, a_n$, then we find that the function of n arguments $f(a_1, \ldots, a_n)$ can in fact be expressed in terms of a function of fewer arguments, fewer by as many of the quantities $a_1, \ldots, a_n$ as have independent dimensions.

Let us now apply dimensional analysis to the heat-conduction problem considered above. The increment in temperature u that we seek satisfies the equation of heat conduction,

$$\partial_t u = \varkappa r^{-2} \partial_r r^2 \partial_r u, \tag{0.8}$$

under conditions expressing the fact that the temperature is constant at infinity and at the initial instant everywhere except at the center,

$$u(r, 0) \equiv 0, \quad r \neq 0; \quad u(\infty, t) = 0, \tag{0.9}$$

and the fact that at the initial instant an amount of heat E is concentrated at the center:

$$4\pi \int_0^{r_*} r^2 u(r, 0)\, dr = E/c = Q. \tag{0.10}$$

Here r_* is an arbitrarily small radius whose size is immaterial, since at the initial instant all the heat is concentrated at a point. Of course such an initial temperature distribution $u(r, 0)$ is actually a so-called generalized function. It is evident that the increment in temperature u depends on the time t, the thermal diffusivity $\varkappa$, the quantity $Q = E/c$, and also on the distance r of the observation point from the center. All these quantities are dimensional and their numerical values depend on the choice of units of length, time, and temperature. Here the thermal diffusivity $\varkappa$ has the dimensions of length squared divided by time, and the quantity Q has the dimensions of temperature multiplied by the length cubed; this is evident from the fact that the left and right sides of (0.8) and (0.10) must have the same dimensions. Hence the quantity $(\varkappa t)^{1/2}$ has the dimensions of length, the quantity $Q/(\varkappa t)^{3/2}$ has the dimensions of temperature, and the quantities

$$\Pi = \frac{u}{Q(\varkappa t)^{-3/2}}, \quad \Pi_1 = \frac{r}{\sqrt{\varkappa t}}$$

are dimensionless. The dimensionless parameter Π_1 is the only independent quantity that can be formed from the quantities $t, \varkappa, Q$, and r, the first three of which have independent dimensions. The relation

$$u = f(t, \varkappa, Q, r)$$

for the desired solution must, according to the above, be representable in the form of a relation between the dimensionless quantities:

$$\Pi = \Phi(\Pi_1). \tag{0.11}$$

From this we get

$$u = \frac{Q}{(\varkappa t)^{3/2}} f(\xi), \quad \xi = \Pi_1 = \frac{r}{\sqrt{\varkappa t}}, \tag{0.12}$$

i.e., relation (0.3). Thus, we have in this case succeeded in establishing the self-similarity of the solution and in determining the scales $u_0(t)$ and $r_0(t)$ using only dimensional analysis.

Equation (0.12) gives

$$\partial_t u = \frac{Q}{(\varkappa t)^{3/2}}\left\{-\frac{3}{2t} f - \frac{1}{2t}\xi f'(\xi)\right\},$$
$$\partial_r u = \frac{Q}{(\varkappa t)^{3/2}} \frac{1}{\sqrt{\varkappa t}} f'(\xi), \quad \partial^2_{rr} u = \frac{Q}{(\varkappa t)^{3/2}} \frac{1}{\varkappa t} f''(\xi). \tag{0.13}$$

Substituting these expressions into the partial differential equation (0.8), we obtain for the function f the ordinary differential equation

$$\frac{d^2 f}{d\xi^2} + \frac{2}{\xi}\frac{df}{d\xi} + \frac{1}{2}\xi\frac{df}{d\xi} + \frac{3}{2} f = 0. \tag{0.14}$$

Further, condition (0.9) at infinity gives $u(\infty, t) = Q(\varkappa t)^{-3/2} f(\infty) = 0$, whence $f(\infty) = 0$. It is easy to find a bounded solution of (0.14) under the condition $f(\infty) = 0$ to within a constant:

$$f = A \exp(-\xi^2/4). \tag{0.15}$$

The constant A is determined as follows. In the problem considered one has the law of conservation in time of the total amount of heat, which is equal to the amount of heat concentrated at the center at the initial instant:

$$4\pi c \int_0^\infty r^2 u(r, t)\, dr = \text{const} = E = cQ. \tag{0.16}$$

Relation (0.12) gives

$$\int_0^\infty r^2 u(r, t)\, dr = QA \int_0^\infty \xi^2 e^{-\xi^2/4}\, d\xi = 2QA\sqrt{\pi}.$$

From this and (0.16) we find the value of the constant $A = 1/(8\pi^{3/2})$, which completes the solution of the problem.

In many other cases too, considerations of dimensional analysis turn out to be quite sufficient for proving the self-similarity of the solution starting from the formulation of the mathematical problem, and for obtaining expressions for the scales and the self-similar variables. A well-known book by Sedov (1959; first Russian edition 1944) contains a large collection of examples illustrating the application of dimensional analysis for establishing self-similarity and determining the self-similar variables, and also presents a general approach applicable in such cases. We shall see later, however, that solutions for which dimensional analysis is sufficient to establish self-similarity are relatively rare among all self-similar solutions; as a rule the situation is more complicated.

2. Self-similarities, even as understood from the point of view of dimensional analysis, have been regarded by the majority of researchers merely as isolated exact solutions of special problems—elegant, sometimes rather useful, but all the same limited in their significance as properties of physical theories. It was only gradually realized that the significance of these solutions is much broader. In fact they do not describe merely the behavior of physical systems under certain special conditions. They also describe the "intermediate-asymptotic" behavior of solutions of wider classes of problems in the range where these solutions no longer depend on the details of the initial and/or boundary conditions, yet the system is still far from being in a limiting state.

For the example considered, this means that the solution (0.1) does not describe merely the temperature distribution in infinite space under the action of an instantaneous point source. It also describes the temperature distribution within a finite region of length scale Λ provided that at the initial instant the same amount of heat is concentrated not at a point but in a finite region Ω of length scale $\lambda \lll \Lambda$ (not necessarily even symmetric).† The temperature is measured at distances from the center of Ω much larger than λ, and at the same time much less than Λ, at times for which the size of the heated region is considerably larger than λ and at the same time much less than either Λ or the distance from the boundaries. This "intermediate-asymptotic" property of solutions of the type with an instantaneous point source is proved exactly in the mathematical theory of heat conduction.

It is precisely the consideration of self-similarities as intermediate asymptotics that allows one to understand properly the role of dimensional analysis in establishing self-similarity and determining the self-similar variables. As it turns out, dimensional considerations are far from being always sufficient to establish self-similarity. What is more, one can even assert that as a rule they are not.

Zel'dovich (1956) first explicitly distinguished a particular class of self-similar solutions for which dimensional analysis is insufficient for establishing self-similarity and determining the self-similar variables. He called these *self-similar solutions of the second kind*.‡ It should be noted in fact that analogous solutions were also considered earlier by Guderley (1942), L. D. Landau and K. P. Staniukovich (cf. Staniukovich, 1960; von Weizsäcker, 1954). In the well-known book of Zel'dovich and Raizer (1967, translation of first Russian edition

†The symbol $a \lll b$ means that there exists a range of values of a quantity x such that $x \gg a$, but $x \ll b$.

‡The term "self-similarity of the second kind" was used by Ya. B. Zel'dovich in his earlier papers in a narrower sense than we use it here.

of 1963), and also in the paper of Brushlinskii and Kazhdan (1963) a detailed analysis was given of the solutions of this type known at that time.

3. To understand what constitutes the intrinsic nature of the classification of self-similarities in the simplest case, one can modify somewhat the problem of heat conduction considered above: let the specific heat of the medium now be equal to one constant c if the medium is heated and to another constant c_1 if the medium is cooled. For this modified problem the equation of heat conduction (0.8) is replaced by the equation

$$\begin{aligned} \partial_t u &= \varkappa r^{-2}\partial_r r^2 \partial_r u \quad (\partial_t u \geqslant 0), \\ \partial_t u &= \varkappa_1 r^{-2}\partial_r r^2 \partial_r u \quad (\partial_t u \leqslant 0). \end{aligned} \tag{0.17}$$

Here $\varkappa$, $\varkappa_1$ are constants,† in general distinct. It appears that on applying dimensional analysis one can construct a solution of equation (0.17) of instantaneous source type for $\varkappa_1 \neq \varkappa$ in virtually the same way as was just done for the classical equation of heat conduction with $\varkappa_1 = \varkappa$. In fact, to the parameters t, $\varkappa$, Q, r governing the solution there is added in this case only an additional parameter $\varkappa_1$ of the same dimensions as $\varkappa$. Hence it seems at a first glance that the solution can be represented in the same form as (0.12), and the additional constant dimensionless parameter $\varkappa_1/\varkappa$ does not change the situation. One can show, however, that for $\varkappa_1 \neq \varkappa$ such a solution simply does not exist!

The resolution of this apparent paradox is the following. We are not actually interested in the solution of the idealized problem of an instantaneous point source of heat, but rather in the behavior for large t of the solution corresponding to the discharge of a quantity of heat at the initial instant in a region of finite diameter λ. In the classical case $\varkappa_1 = \varkappa$, these are one and the same, so we automatically first sought a solution of point-source type for $\varkappa_1 \neq \varkappa$ also. If, however, the heat at the initial instant is concentrated in a region of finite size λ, then the problem involves a new parameter λ having dimensions of length and a new dimensionless parameter

$$\Pi_2 = \frac{\lambda}{\sqrt{\varkappa t}} = \eta. \tag{0.18}$$

The parameter η immediately spoils the self-similarity, since the solution can no longer be expressed as a function of one variable, but has the form

$$u = \frac{Q}{\sqrt{(\varkappa t)^3}} f(\xi, \eta, \varkappa_1/\varkappa). \tag{0.19}$$

†Equation (0.17) describes heat conduction in materials in which pores appear in the material being cooled. It occurs also in the theory of filtration of fluids in porous media.

Just as for $\varkappa_1 = \varkappa$, we are interested in the behavior of the solution for small η. However for $\varkappa_1 \neq \varkappa$ it is impossible simply to pass to the limiting form of the solution corresponding to $\eta = 0$ in (0.19). The reason for this is trivial: it turns out that for small η the function f behaves like

$$f = \eta^{\alpha}\varphi(\xi, \varkappa_1/\varkappa), \tag{0.20}$$

where φ is a finite quantity and α is a constant that depends on $\varkappa_1/\varkappa$—it is non-zero for $\varkappa_1 \neq \varkappa$, but equal to zero for $\varkappa_1 = \varkappa$. If one tries to pass to the limit $\eta \to 0$ with $\varkappa_1 \neq \varkappa$, then on the right of (0.20) one gets either zero or infinity depending on the sign of α, i.e., one gets an empty relation. Hence without passing to the limit we substitute (0.20) into the general representation (0.19) of the solution while regarding η as small, i.e., the time as large or λ as small. Thus we obtain a self-similar asymptotic representation of the solution of the original non-self-similar problem, valid for large t,

$$u = \frac{A}{(\varkappa t)^{(3+\alpha)/2}}\varphi\left(\xi, \frac{\varkappa_1}{\varkappa}\right), \quad A = \text{const}\, Q\lambda^{\alpha}. \tag{0.21}$$

Equation (0.21) shows that the representation for large times is given not by a solution of point-source type but by another self-similar solution. Let us now decrease the size λ of the region of the initial heat discharge. It is evident that the solution is such that in order to leave the asymptotic representation at large time invariant we must vary the output Q of the source in such a way that the product $Q\lambda^{\alpha}$ preserves its magnitude.

If we now substitute the expression (0.21) for the solution into (0.17) we obtain for the function φ an ordinary differential equation containing the quantity α as parameter. It turns out that for arbitrary α this equation has no solution with the necessary properties. However, for each value of the parameter $\varkappa_1/\varkappa$ there exists one value of α for which the required solution of the ordinary differential equation exists. Thus, to determine φ and the parameter α one obtains a nonlinear eigenvalue problem. Under such a direct construction of self-similar intermediate asymptotics the constant A remains undetermined. It is impossible to find it from an integral conservation law of the type of (0.16) when $\varkappa_1 \neq \varkappa$ since in this case the equation of bulk heat balance assumes the nonintegrable form

$$\frac{d}{dt}\int_0^{\infty} r^2 u(r, t)\, dr = -(\varkappa_1 - \varkappa)\left[r^2 \partial_r u\right]_{r=r_0} \tag{0.22}$$

[$r_0(t)$ being the coordinate of the point at which, at the given moment, the derivative $\partial_t u$ vanishes].

Thus, the behavior of the solution for large t turns out to be self-similar also for $\varkappa_1 \neq \varkappa$, but the self-similarity here is not the same as for $\varkappa_1 = \varkappa$. First of all, the parameter λ, having the dimension of length, which spoils the self-similarity of the original problem, does not disappear from the limiting solution. Further, dimensional analysis does not in this case allow us to find the self-similar variables or to establish the self-similarity of the limiting solution, starting from the mathematical formulation of the problem; the fact is that the dimensions of the constant A are unknown in advance and must be found in the course of the solution. Finally, the constant A turns out to be undefined. To find it one must "match" the constructed solution with the solution of the original non-self-similar problem, for example by means of numerical computation. (Numerical computation has confirmed the transition at large times from the solution of the non-self-similar problem to the constructed self-similar asymptotics.)

A peculiar situation arises also with similarity laws. Dimensional analysis allows one to obtain to within a constant the law of attenuation of temperature at the center in the classical case $\varkappa_1 = \varkappa$. In fact, the increment of temperature at the center depends only on the quantities Q, $\varkappa$, and t, from which one can construct only one quantity with the dimensions of temperature, $Q(\varkappa t)^{-3/2}$; and it is impossible to form a dimensionless combination, since their dimensions are independent. It is therefore clear that

$$u_{\max} = \text{const}\,\frac{Q}{(\varkappa t)^{3/2}} \tag{0.23}$$

($u_{\max}$ being the temperature increment at the center). According to the solution given above, const $= 1/(8\pi^{3/2})$. A naive application of dimensional analysis, i.e., the assumption of heat discharge at a point, would lead to the same similarity law also for $\varkappa_1 \neq \varkappa$, although in this latter case, as (0.21) shows, such a similarity law does not hold. In fact,

$$u_{\max} = \text{const}\,\frac{Q\lambda^{\alpha}}{(\varkappa t)^{(3+\alpha)/2}}, \tag{0.24}$$

so that although the law of attenuation is also a power law, one cannot now obtain the exponent by dimensional analysis. The peculiar situation is that it is impossible to say in advance, before the entire non-self-similar problem has been solved, whether or not one can use dimensional analysis to analyze the similarity laws.

It is now easy to understand what happens in the general case (Barenblatt and Sivashinskii, 1969, 1970). Self-similar solutions are always obtained for degenerate problems in which the parameters of the problem that have the dimensions of the independent variables (characteristic length, characteristic time, etc.) are equal to zero or infinity. In the contrary case, the arguments would include ratios of independent variables in these parameters, and there would be no self-similarity. This means that upon passage from nondegenerate formulations of the problem, corresponding to finite values of the parameters, to the degenerate one, some dimensionless parameters, which we denote by Π_1, Π_2, ..., tend to zero or infinity. Here the function Φ in (0.7),

$$\Pi = \Phi(\Pi_1, \Pi_2, \ldots),$$

can

(1) tend to a finite limit different from zero;
(2) tend to zero, or infinity, or in general tend to no limit but have for small (or large) Π_1, Π_2, ... a power-law asymptotics:

$$\Phi = \Pi_1^{\alpha}\Phi_1\left(\frac{\Pi_2}{\Pi_1^{\beta}}, \ldots\right); \tag{0.25}$$

(3) tend to no finite limit and have no power-law asymptotics for small (or large) Π_1, Π_2, ...

In case (1), for sufficiently large (or small) Π_1, Π_2, ... one can simply replace the function Φ by its limiting expression, corresponding to Π_1, Π_2, ... equal to zero (or infinity). Here the number of its arguments is correspondingly diminished, and the corresponding dimensional parameters (for example, the size of the domain of initial discharge of heat for $\varkappa_1 = \varkappa$) turn out to be immaterial and drop out of consideration. This is called *complete* self-similarity in the parameters Π_1, Π_2, ...

In case (2), the relation (0.7) can, for small (or large) Π_1, Π_2, ..., be rewritten, using (0.25), in the form

$$\Pi^* = \Phi_1(\Pi^{**}, \ldots), \quad \Pi^* = \Pi\Pi_1^{-\alpha}, \quad \Pi^{**} = \Pi_2\Pi_1^{-\beta}. \tag{0.26}$$

Thus, in this case too the number of arguments in (0.7) is reduced. The parameters Π^* and Π^{**} are completely analogous in their structure to ordinary parameters of similarity, being combinations of powers of governed and governing parameters. The difference, however, consists in the facts that, first, the quantities Π^*, Π^{**} contain dimensional parameters a_{k+1}, a_{k+2} that spoil the self-similarity of the original problem, and second, they cannot be obtained by dimensional analysis. An example of such parameters is, in the modified heat-conduction problem the quantity

$$\frac{u\,(\varkappa t)^{(3+\alpha)/2}}{Q\lambda^{\alpha}};$$

it contains the size λ of the region of initial heat discharge raised to a power α that is not defined by dimensional considerations. In such cases one speaks of *incomplete* self-similarity in the parameters $\Pi_1, \Pi_2, \ldots$.

Finally, in case (3) the parameters $\Pi_1, \Pi_2, \ldots$ remain essential no matter how large or small they are and no self-similarity in them ensues. The nature of the classification of self-similar solutions becomes transparent now. If the passage to the limit from the solution of the non-self-similar original problem to a self-similar intermediate asymptotics corresponds to complete self-similarity in a dimensionless parameter that spoils self-similarity in the original problem, the self-similar solution is a solution of the first kind. If the passage to the limit corresponds to incomplete self-similarity, the self-similar solution is a solution of the second kind. The real difficulty is that similarity methods usually apply when the solution of the complete non-self-similar problem is unknown. Hence, *a priori*, it is impossible to say with which type of self-similarity we are dealing.

4. Clarification of the nature of self-similarity of the second kind was aided by the establishment of a close connection between the classification of self-similarities and of nonlinear progressive waves (Barenblatt and Zel'dovich, 1972). Progressive waves are solutions of the form

$$u = f(\zeta - \lambda\tau + c) \tag{0.27}$$

(ζ being the spatial and τ the temporal variable, λ the constant speed of propagation of the wave, and c a constant), for which the distributions of properties at different moments of time are obtained from one another by means of a simple translation. It is well known that progressive waves are subdivided into two types. For waves of the first type the speed of propagation λ is found from the conservation laws alone and is independent of the internal structure of the wave. Examples of such waves are shock waves in gas dynamics and detonation waves. For waves of the other type the speed of propagation λ is found from the condition for the existence of a global solution describing the internal structure of the wave, and is completely determined by that structure. An example of a wave of this type is a flame wave or the propagation wave of a gene having an advantage in the struggle for existence. It should be noted that consideration of the latter problem in the classical paper of Kolmogorov, Petrovskii, and Piskunov (1937) was generally the first example of the exact analysis of the intermediate asymptotics of nonlinear problems.

We now set $\zeta = \ln x$, $\tau = \ln t$ and $c = -\ln A$ in (0.27). Then this expression is transformed into the self-similar form

$$f\left(\ln \frac{x}{At^{\lambda}}\right) = F\left(\frac{x}{At^{\lambda}}\right). \tag{0.28}$$

By this transformation the classification of self-similar solutions into solutions of the first and second kinds is put into one-to-one correspondence with the classification of progressive waves mentioned above.

5. Self-similarity is connected (Barenblatt and Zel'dovich, 1971, 1972) with a generally nonlinear eigenvalue problem, the existence of a solution of which guarantees the existence of a self-similar intermediate asymptotics in the large. A nontrivial problem is found for the set of eigenvalues in this problem—the spectrum determined by the possible values of the exponents in the self-similar variables. Everything is simple if the spectrum consists of one point, as in the modified heat-conduction problem considered above. But if the spectrum consists of more than one point, in particular if it is continuous, the exponents in the self-similar variables can depend on the initial conditions of the original non-self-similar problem. A remarkable example here is provided by the self-similar interpretation of the well-known Korteweg–de Vries equation (for details cf. Chapter 8).

Recently, ideas connected with the concept of incomplete self-similarity and self-similar solutions of the second kind have been used to solve many important problems, which are of independent, nonillustrative interest. Some of these problems are considered below. A peculiar place is occupied by the analysis of incomplete self-similarity in the theory of turbulence, where a complete mathematical formulation of the problem is lacking up to this time and the comparison of similarity laws with experimental data is of decisive importance in estimating self-similarity.

The concepts of self-similarity are widely used by physicists in quantum field theory and in the theory of phase transitions (where self-similarity is called "scaling"). In these theories there exist no properly formulated boundary-value problems and hence the recipes proposed and the statements expressed have to a considerable extent an intuitive character. In fact, scaling is precisely what we understand today by self-similarity of the second kind. It would seem to me useful for those interested in scaling to look at how this concept works in other situations, where in fact one has precise formulations of the boundary-value problems, and the origins of the self-similar asymptotics can be traced directly.

What has been said defines the contents of this book; the table of contents gives a more complete and precise representation. The brief account of dimensional analysis and similarity in Chapter 1 will prove necessary because dimensional analysis is used essentially in everything expounded later. Our account of dimensional analysis and similarity as a whole is essentially different from that found in the literature although in its general ideas it follows the excellent book of Bridgman (1931), which is lately becoming undeservedly forgotten.

1

Dimensional Analysis and Similarity

1. Dimensions

Physical quantities are expressed in terms of numbers that are obtained by measurements—direct or indirect comparison with corresponding *units of measurement*. The units of measurement are divided into *fundamental* and *derivative* ones. The fundamental units of measurement are defined arbitrarily in the form of certain standards, artificial or natural, while the derivative ones are obtained from the fundamental units of measurement by virtue of the definitions of physical quantities, which are always the indications of some method, at least conceptual, of measuring them. Thus, speed is by definition the ratio of the distance traversed in a given interval of time to the magnitude of that interval. Hence as unit of speed one can (but need not!) take the ratio of the unit of length to the unit of time in the given system. In exactly the same way density is by definition the ratio of a certain mass to the volume containing it. Hence as unit of density one can take the ratio of the unit of mass to the unit of volume, i.e., to the cube of the unit of length.

A set of fundamental units of measurement sufficient to measure the properties of the class of phenomena under consideration is called a *system of units of measurement*. Thus, in mechanics one often uses the cgs system of units of measurement in which one takes as unit of mass 1 gram (g), which is 1/1000 of the mass of a certain specifically manufactured and carefully preserved standard; as unit of length 1 centimeter (cm), which is 1/100 of the length of another standard†; and as unit of time 1 second (sec), which is 1/86,400 part of the mean solar day.‡ A unit of speed in this system is 1 cm/sec, that of acceleration is 1 cm/sec^2, and that of force is 1 dyne, which is 1 g · cm/sec^2. We emphasize that in the definition of systems of units of measurement there is no requirement of minimality, i.e., of the minimality of the set of fundamental units of

†A more precise definition of this standard: a length equal to 1,650,763.73 times the wavelength in vacuum of the radiation corresponding to the transition between the levels $2p_{10}$ and $5d_5$ of the atom krypton-86.

‡A more precise definition of a second: 1/31,556,925.9747 part of the tropical year at 12 o'clock on January 0th, 1900, ephemeral time.

measurement. One can consider, for example, the system in which the unit of mass is 1 gram, the unit of length 1 centimeter, the unit of time 1 second, and the unit of speed 1 knot ($\sim$0.5 meter per second).

Let us call a set of systems of units of measurement that differ from one another only in the magnitude of the fundamental units of measurement *a class of systems of units of measurement*. The cgs system of units of measurement just mentioned is contained in the class of systems of units of measurement in which the fundamental units of measurement are

$$\mathrm{g}/M, \quad \mathrm{cm}/L, \quad \mathrm{sec}/T, \tag{1.1}$$

where M, L, T are abstract positive numbers showing by how many times the fundamental units of mass, length, and time are decreased in passing from the original cgs system to another system of the given class. This class is denoted by MLT†. In particular, to the MLT class belongs the SI (International System), widely introduced recently, in which the unit of mass is taken to be 1 kilogram = 1000 grams, which is the complete mass of the above mentioned standard of mass; the unit of length is taken to be 1 meter = 100 cm, which is the complete length of the standard of length mentioned above; and the unit of time is taken to be 1 second. Thus, upon passage from the cgs system to the SI system $M = 1:1000$, $L = 1:100$, $T = 1$. Often one also uses systems of the FLT class, in which the fundamental units of measurement have the form

$$\mathrm{kgf}/F, \quad \mathrm{cm}/L, \quad \mathrm{sec}/T. \tag{1.2}$$

(Here kgf is kilogram of force, a unit of force: the force which, applied to a mass equal to the mass of the standard kilogram, gives it an acceleration of $9.80665\ \mathrm{m/sec^2}$.)

By the *dimensions* of a physical quantity is meant the function determining by how many times one alters the numerical value of this quantity upon passage from the original system of units of measurement to another system in the given class. The dimensions of a quantity φ are denoted, following Maxwell's suggestion, by $[\varphi]$. We emphasize specifically the fact that the dimensions depend on the class of systems of units of measurement. Quantities are called *dimensionless* if their numerical values are identical for all systems of units of measurement in a given class; all others are called *dimensional*. The dimensions of a dimensionless quantity are equal to one.‡

†A notation for a class of systems of units of measurement is obtained by writing down one after the other the symbols for the quantities whose units of measurement are taken as fundamental. Simultaneously these symbols denote the number of times by which one reduces the corresponding unit of measurement upon passage from the original system to the other system of the given class.

‡Usually by the dimensions is meant the expression for the unit of measurement of a given quantity in terms of fundamental ones. The definition given above is essentially a refinement of this.

Some examples: If the unit of mass is decreased by M times, the unit of length by L times, and the unit of time by T times, then the numerical value of any measured force increases by MLT^{-2} times. Thus, for the dimensions of force in the class of MLT systems we have

$$[F]=MLT^{-2}. \tag{1.3}$$

Analogously the dimensions of mass in the FLT class have the form $[M] = FT^2L^{-1}$, the dimensions of energy in the MLT class are $[E] = ML^2T^{-2}$, etc.

In all these cases the dimensions are represented by a power monomial, and this is not accidental–the dimensions are always a power monomial. This fact follows from a naturally formulated but actually profound statement (which is a consequence of the fundamental general physical principle of covariance): *within a given class all systems are equivalent*, so that the choice of the original system is of no significance for the characteristics of the given class.

Let us now prove that the dimensions are always a power monomial. We consider some class of systems of units of measurement PQ . . . . By virtue of the equivalence of systems within a given class, the dimensions of any quantity a depend only on the quantities $P, Q, \ldots$:

$$[a]=\varphi(P, Q, \ldots). \tag{1.4}$$

If there were to exist some distinguished system within the given class, then the position of any system of units of measurement within the given class would be defined by its relation to that distinguished one. In this case, among the arguments of the dimension function would appear also the ratio of the magnitude of the fundamental units of the original system to the corresponding units of the distinguished system. By virtue of the assumption of the principle of equivalence of systems of units of measurements within a given class, this is not so. Hence the arguments of the dimension function are just the quantities $P, Q, \ldots$, independent of which system is chosen as the original one.

We choose in the class PQ . . . three systems of units: (0), (1), and (2), where the system (1) is obtained from the system (0) by decreasing the fundamental units of measurement by $P_1, Q_1, \ldots$ times, and the system (2) is obtained from the system (0) by decreasing the fundamental units of measurement by $P_2, Q_2, \ldots$ times. By the above, upon transition from the system (0) to the system (1) the numerical value of a considered quantity a increases by $\varphi(P_1, Q_1, \ldots)$ times, and upon transition from the system (0) to the system (2) by $\varphi(P_2, Q_2, \ldots)$ times. Hence it follows that the numerical values of the quantity a in the systems (2) and (1) differ by $\varphi(P_1, Q_1, \ldots)/\varphi(P_2, Q_2, \ldots)$ times. Furthermore, by virtue of the equivalence of systems within the given class, passage from the system (2) to the system (1) depends only on these systems and does not depend on what system was taken as the zeroth. However, the ratios of the fundamental units of measurement in these systems are respectively P_1/P_2, Q_1/Q_2, ... Hence the numerical value of the quantity a must, upon this transition, increase by $\varphi(P_1/P_2, Q_1/Q_2, \ldots)$ times. Thus, we have calculated the variation in the numerical value of the quantity a upon transition from the system (2) to the system (1) in two ways. Comparing the results, we get a functional equation for the dimension function φ:

$$\frac{\varphi(P_1, Q_1, \ldots)}{\varphi(P_2, Q_2, \ldots)} = \varphi\left(\frac{P_1}{P_2}, \frac{Q_1}{Q_2}, \ldots\right). \tag{1.5}$$

This equation is easily solved: we differentiate both sides with respect to P_1 and set $P_1 = P_2 = P$, $Q_1 = Q_2 = Q$, ... We have

$$\frac{\varphi'_P(P, Q, \ldots)}{\varphi(P, Q, \ldots)} = \frac{1}{P}\varphi'_P(1, 1, \ldots) = \frac{\alpha}{P}, \tag{1.6}$$

where α is a constant quantity, i.e., independent of P, Q, ... Integrating (1.6), we find

$$\varphi(P, Q, \ldots) = P^{\alpha}\varphi_1(Q, \ldots), \tag{1.7}$$

where the function φ_1 is now independent of P. Repeating the argument for the remaining variables we get

$$\varphi = [a] = P^{\alpha}Q^{\beta} \ \ldots . \tag{1.8}$$

(The constant factor finally obtained on the right side is equal to one, since for $P = Q = \ldots = 1$ the system of units of measurement is unchanged and there is also no change in the numerical value of the quantity a.) Thus, the dimensions of a physical quantity are always expressed by a power monomial.

One says that the quantities $a_1, a_2, \ldots, a_k$ have *independent dimensions* if none of the dimensions of these quantities can be represented as a product of powers of the dimensions of the remaining quantities. For example, the dimensions of density (ML^{-3}), speed (LT^{-1}), and force (MLT^{-2}) are independent; on the other hand the dimensions of density, speed, and pressure ($ML^{-1}T^{-2}$) are dependent.

One can always pass to a system of units of measurement inside a given class $PQ \ldots$ such that any quantity, say a_1, from a collection of quantities $a_1, \ldots, a_k$ with independent dimensions will increase its numerical value by an arbitrary factor $A > 0$, and the others remain invariant. In fact, let the dimensions of the quantities $a_1, \ldots, a_k$ in the chosen class of systems of units of measurement $PQ \ldots$ have the form

$$[a_1] = P^{\alpha_1}Q^{\beta_1} \ldots, \ldots, [a_k] = P^{\alpha_k}Q^{\beta_k} \ldots . \tag{1.9}$$

We have by definition to construct a system in which the relations

$$\begin{aligned} P^{\alpha_1}Q^{\beta_1}\ldots &= A, \\ P^{\alpha_2}Q^{\beta_2}\ldots &= 1, \\ \cdots\cdots \\ P^{\alpha_k}Q^{\beta_k}\ldots &= 1 \end{aligned} \tag{1.10}$$

hold. Taking logarithms shows that to find the logarithms of the transition factors $\ln P$, $\ln Q$, ..., we have the system of linear algebraic equations

$$\begin{aligned} \alpha_1 \ln P + \beta_1 \ln Q + \ldots &= \ln A, \\ \alpha_2 \ln P + \beta_2 \ln Q + \ldots &= 0, \\ \cdots\cdots\cdots \\ \alpha_k \ln P + \beta_k \ln Q + \ldots &= 0. \end{aligned} \tag{1.11}$$

This system always has at least one solution. In fact, the number of unknowns $\ln P$, $\ln Q$, ... in it is not less than the number of equations, since otherwise the dimensions of the quantities $a_1, \ldots, a_k$, which are expressible in terms of $P, Q, \ldots$, would obviously be dependent. If the number of unknowns is greater than the number of equations, then the solvability of the system (1.11) is obvious. And if the number of unknowns is equal to the number of equations, then the single-valued solvability of (1.11) follows from the fact that the determinant

$$\begin{vmatrix} \alpha_1 & \beta_1 & \ldots \\ \alpha_2 & \beta_2 & \ldots \\ \cdot & \cdot & \cdot \\ \alpha_k & \beta_k & \ldots \end{vmatrix} \tag{1.12}$$

is different from zero, since otherwise the dimensions of the quantities $a_1, \ldots, a_k$ would again not be independent.

2. Dimensional Analysis

The relationships found in physical theories or experiments can always be represented in the form

$$a = f(a_1, \ldots, a_k, a_{k+1}, \ldots, a_n), \tag{1.13}$$

where the quantities $a_1, \ldots, a_n$ are called governing parameters. Any investigation ultimately comes down to determining one or several dependencies of the form (1.13). Let the arguments $a_1, \ldots, a_k$ have independent dimensions, and the dimensions of the arguments $a_{k+1}, \ldots, a_n$ be expressible in terms of the dimensions of the governing parameters $a_1, \ldots, a_k$ in the following way:

$$
\begin{aligned}
&[a_{k+1}] = [a_1]^{p_{k+1}} \ldots [a_k]^{r_{k+1}}, \\
&\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\
&[a_n] = [a_1]^{p_n} \ldots [a_k]^{r_n}.
\end{aligned}
\tag{1.14}
$$

The dimensions of the governed quantity a must be expressible in terms of the dimensions of the governing parameters $a_1, \ldots, a_k$:

$$[a] = [a_1]^p \ldots [a_k]^r. \tag{1.15}$$

If this were not so then the dimensions of the quantities $a, a_1, \ldots, a_k$ would be independent, and according to the above it would be possible, by altering the system of units of measurement inside the given class, to change the quantity a arbitrarily, leaving the quantities $a_1, \ldots, a_k$ (and, consequently, all the governing parameters $a_1, \ldots, a_n$ also) unchanged. This would mean that the quantity a depends not only on the parameters $a_1, \ldots, a_n$, i.e., that the list of governing parameters in the relation (1.13) must be incomplete. Thus, there exist numbers $p, \ldots, r$, such that (1.15) holds. We therefore set

$$
\begin{aligned}
&\Pi_1 = \frac{a_{k+1}}{a_1^{p_{k+1}} \ldots a_k^{r_{k+1}}}, \\
&\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\
&\Pi_{n-k} = \frac{a_n}{a_1^{p_n} \ldots a_k^{r_n}}, \\
&\Pi = \frac{a}{a_1^p \ldots a_k^r}.
\end{aligned}
\tag{1.16}
$$

One can rewrite (1.13), using the quantities from (1.16), in the form

$$
\begin{aligned}
\Pi = \frac{f(a_1, \ldots, a_n)}{a_1^p \ldots a_k^r} = \frac{1}{a_1^p \ldots a_k^r} f(a_1, \ldots, a_k, \\
\Pi_1 a_1^{p_{k+1}} \ldots a_k^{r_{k+1}}, \ldots, \Pi_{n-k} a_1^{p_n} \ldots a_k^{r_n}) = \\
= F(a_1, \ldots, a_k, \Pi_1, \ldots, \Pi_{n-k}).
\end{aligned}
\tag{1.17}
$$

The quantities $\Pi, \Pi_1, \Pi_2, \ldots, \Pi_{n-k}$ are obviously dimensionless, and upon transition from one system of units to any other inside the given class their numerical values remain unchanged. At the same time, according to the above, one can pass to a system of units of measurement such that any of the parameters $a_1, \ldots, a_k$, for example, a_1, is changed by an arbitrary factor, and the remaining ones are unchanged. Upon such a transition the first argument in

(1.17) is changed arbitrarily, and all the other arguments of the function F remain unchanged as well as its value Π. Hence if follows that $\partial F/\partial a_1 = 0$ and, entirely analogously, $\partial F/\partial a_2 = 0, \ldots, \partial F/\partial a_k = 0$. Therefore the relation (1.17) is in fact represented by a function of $n-k$ arguments,

$$\Pi = \Phi(\Pi_1, \ldots, \Pi_{n-k});$$

or, what is the same, the function f has the special form

$$f(a_1, \ldots, a_k, a_{k+1}, \ldots, a_n) =$$
$$= a_1^p \ldots a_k^r \Phi\left(\frac{a_{k+1}}{a_1^{p_{k+1}} \ldots a_k^{r_{k+1}}}, \ldots, \frac{a_n}{a_1^{p_n} \ldots a_k^{r_n}}\right). \quad (1.18)$$

This fact constitutes the content of the central (and essentially the only nontrivial) statement of *dimensional analysis*, the Π-theorem, explicitly formulated and proved for the first time apparently by E. Buckingham:

Suppose there exists a physical regularity, expressed as the dependence of a certain quantity, in general dimensional, on dimensional governing parameters. This relationship can be expressed as the dependence of a dimensionless quantity on dimensionless combinations of the governing parameters. The number of these dimensionless combinations is equal to the difference of the total number of governing parameters and the number of governing parameters with independent dimensions.

One should note that dimensional analysis is intuitively completely obvious, and its implicit use began long before the Π-theorem was explicitly formulated and formally proved; here, one should mention first of all the names of Fourier, Maxwell, Reynolds, and Rayleigh. The use of dimensional analysis in the construction of special solutions of systems of partial differential equations will be the subject of detailed consideration below. Here we merely note that dimensional analysis applies advantageously to the preliminary analysis of physical phenomena and to the handling of experimental data. In fact, one can assume that to determine the dependence of some quantity or another on some governing parameter it is necessary to measure this quantity for a minimum of ten values of the given argument (it being understood that the number ten is entirely arbitrary). Thus, for the experimental determination of the quantity a as a function of n governing parameters it would be necessary to perform 10^n experiments. According to the Π-theorem, the matter reduces to determining a function of $n-k$ dimensionless arguments $\Pi_1, \ldots, \Pi_{n-k}$, for the determination of which 10^{n-k} experiments suffice, which is less by 10^k times. Thus, the difficulty of determining the desired function is reduced by as many orders as there are quantities with independent dimensions among the governing parameters.

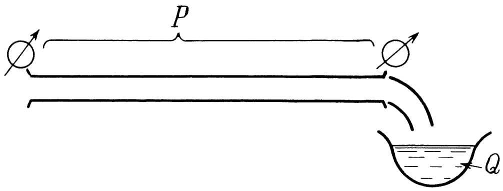


Figure 1.1. Scheme of the experiments of E. Bose, D. Rauert, and M. Bose. They measured the time τ to fill a container of volume Q and pressure drop P at the ends of a pipe for steady flow of various fluids through the pipe.

Here is an instructive example. In 1909–1911 the physical chemists E. Bose, D. Rauert, and M. Bose (Bose and Rauert, 1909; Bose and Bose, 1911) published a series of experimental investigations based on the following scheme (Fig. 1.1). They measured the time τ to fill a container of given volume Q and the pressure drop P at the ends of a pipe for steady flow of various fluids through the pipe: water, chloroform, bromoform, mercury, ethyl alcohol, etc. The results of the experiments were, as usual, represented in the form of a series of curves for the pressure drop as a function of the filling time for various fluids, like those in Fig. 1.2. This work was noted by the subsequently well-known mechanician von Kármán (1911, 1957), who subjected their results to a treatment from the point of view, using contemporary terminology, of dimensional analysis. Von Kármán's arguments can be presented as follows. The pressure drop P at the ends of the tube must depend on the filling time τ of the container, the volume Q of the container, the viscosity coefficient μ of the fluid, and its density ρ:

$$P = f(\tau,\ Q, \mu,\ \rho). \tag{1.19}$$

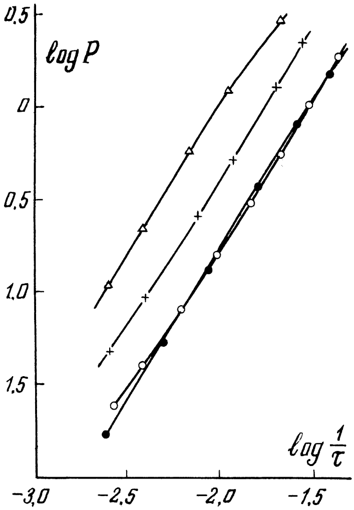


Figure. 1.2. Results of the experiments of E. Bose, D. Rauert, and M. Bose in the original form: ○, water; ●, chloroform; +, bromoform; △, mercury. The curves are different for different fluids.

As is evident $n = 4$ in the present case. The dimensions of the parameters, for definiteness in the class MLT, are expressed by the following relations:

$$[P] = \frac{M}{LT^2}, \quad [\tau] = T, \quad [Q] = L^3, \quad [\mu] = \frac{M}{LT}, \quad [\rho] = \frac{M}{L^3}. \quad (1.20)$$

It is easy to see that the first three governing parameters τ, Q, μ have independent dimensions, while the dimensions of the fourth governing parameter, the density ρ, are expressed in terms of the dimensions of the first three: $[\rho] = [\mu][\tau][Q]^{-2/3}$. Thus $k = 3$, so that $n - k = 1$, and dimensional analysis gives

$$\Pi = \Phi(\Pi_1), \quad \Pi = \frac{P}{\mu\tau^{-1}}, \quad \Pi_1 = \frac{\rho}{\mu\tau Q^{-2/3}}. \quad (1.21)$$

Consequently, according to (1.21), in coordinates Π_1, Π, all the experimental points should lie on a single curve. As Fig. 1.3 shows, this is extremely well confirmed. It is clear that dimensional analysis performed in advance could reduce the bulk of experimental work of physical chemists by many times.

Here is another example.† During an atomic explosion in a domain small enough to be considered a point, a large amount of energy E is released quickly (one can consider it instantaneously). From the center of the explosion a powerful shock wave spreads, the pressure behind it amounting at first to hundreds of thousands of atmospheres. This pressure is much greater than the initial air pressure, whose magnitude can accordingly be neglected in the initial stage of the explosion. Thus, the radius of the front of the shock wave r_f at time t after the explosion depends on E, t, and the initial air density ρ_0:

$$r_f = r_f(E, t, \rho_0).$$

It is evident that $n = 3$. The dimensions of the governing parameters in the class MLT are respectively $[E] = ML^2T^{-2}$, $[t] = T$, and $[\rho_0] = ML^{-3}$. It is easy to see that k is also equal to 3, i.e., $n - k = 0$, so that the function Φ in (1.18) does not depend on any argument, i.e., becomes a constant in this case: $\Phi = C$. Further, it is easy to show that

$$\Pi = r_f (Et^2/\rho_0)^{-1/5},$$

whence

$$r_f = C(Et^2/\rho_0)^{1/5}.$$

†Theoretical investigations of the corresponding problem in gas dynamics have been published in papers by Sedov (1946) and Taylor (1950), and it will be considered in more detail in the next chapter.

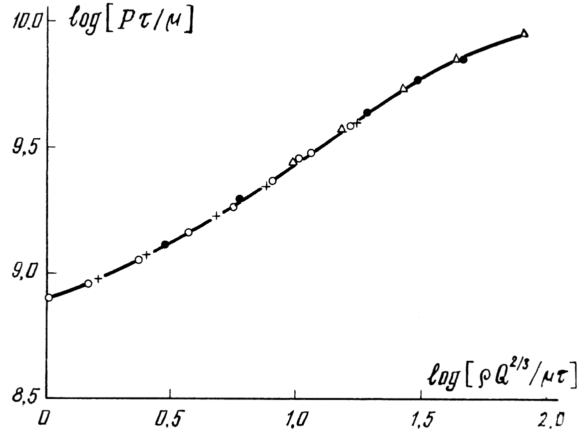


Figure 1.3. Results of the experiments of E. Bose, D. Rauert, and M. Bose in the form in which Th. von Kármán presented them, using dimensional analysis. All the experimental points lie on a single curve.

This formula shows that if by some means one measures the radius of the shock wave at various instants of time, then in logarithmic coordinates $\frac{5}{2}\log r_f$, $\log t$ the experimental points must lie on the straight line

$$\frac{5}{2}\log r_f = \frac{5}{2}\log CE^{1/5}\rho_0^{-1/5} + \log t,$$

having slope equal to one. G. I. Taylor confirmed this extremely well, making use of a movie film of the spread of the fireball, taken by J. Mack at the time of an American nuclear test (Fig. 1.4). As a more detailed calculation shows (cf. the next chapter), the factor C is close to one. Knowing this, from the experimental dependence of the radius of the front on time one can determine the

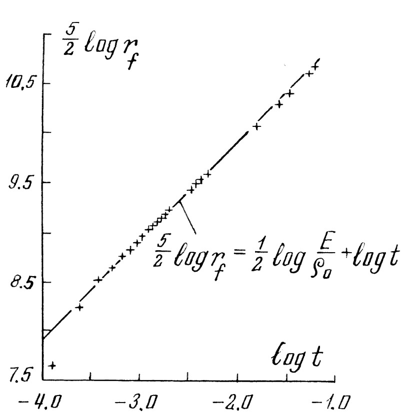


Figure 1.4. Propagation of the shock wave of an atomic explosion. The experimental points determined from J. Mack's motion picture lie, in the coordinates $\frac{5}{2}\log r_f$, $\log t$, on a straight line with slope equal to unity over a large time interval. Analysis of Mack's film enabled G. I. Taylor to determine the energy of the explosion.

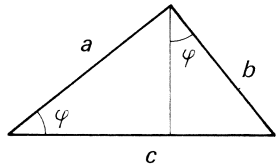


Figure 1.5. Proof of the Pythagorean theorem with the help of dimensional analysis.

energy of the explosion. Taylor's publication of this quantity (which turned out to be equal to $\sim 10^{21}$ ergs) evoked in its time, in his words, considerable confusion in American governmental circles, since this number was considered entirely secret, although Mack's film was not secret.

We give another rather more amusing example of the application of dimensional analysis, using it to prove the Pythagorean theorem [cf. also Migdal (1977)]. The area S of a right triangle is determined by the size of its hypotenuse c and, for definiteness, the lesser of the acute angles φ: $S = f(c, \varphi)$. Obviously, dimensional analysis gives $S = c^2 \Phi(\varphi)$. The altitude perpendicular to the hypotenuse (Fig. 1.5) divides the basic triangle into two right triangles that are similar to it, and whose hypotenuses are the legs a and b of the basic triangle. Thus, their areas are equal to $S_1 = a^2 \Phi(\varphi)$, $S_2 = b^2 \Phi(\varphi)$, where $\Phi(\varphi)$ is just the same as in the case of the basic triangle. But the sum of the areas S_1 and S_2 is equal to the area of the basic triangle S: $S = S_1 + S_2$, whence $c^2 \Phi(\varphi) = a^2 \Phi(\varphi) + b^2 \Phi(\varphi)$, so that $c^2 = a^2 + b^2$, which was to be proved. It is evident that the theorem is based essentially on Euclidean geometry: in Riemannian and Lobachevskian geometry there is an intrinsic parameter λ having the dimensions of length, and the proof does not go through: to the number of arguments of the function Φ one must append the ratio of the hypotenuse to λ.

The examples just considered show that apparently trivial considerations of dimensional analysis can yield quite significant results. The most important element here is a proper definition of the set of governing parameters. Finding the set of governing parameters is simple if one has a mathematical formulation of the problem–it is the set of independent variables and parameters of the problem appearing in the equations, boundary conditions, initial conditions, etc., that determine a solution of the problem in a unique way. The proper choice of governing parameters in a problem that does not have an explicit mathematical formulation is dependent, first of all, on the intuition of the investigator–success here depends on a proper understanding of which parameters are really important and which can be neglected. This question will be considered in detail below.

3. Similarity

In most cases, before some large and expensive structure is built, such as a ship or airplane, in order to obtain its best characteristics under future working conditions, effort is devoted to the testing of models–modeling. In order to model, it is necessary to know how to relate the results of experiments on models to the actual manufactured product; if this is not known, modeling is useless. For the purpose of rational modeling, the concept of *similar phenomena* is basic.

Phenomena are called *similar* if they differ only in the numerical values of the governing parameters, and moreover so that the corresponding dimensionless quantities $\Pi_1, \Pi_2, \ldots, \Pi_{n-k}$ for them coincide. In connection with this definition of similar phenomena, the dimensionless quantities $\Pi_1, \Pi_2, \ldots, \Pi_{n-k}$ are called *similarity parameters.*

Let us consider two similar phenomena, one of which will be called the prototype and the other the model; it should be understood that this terminology is just a convention. For both phenomena there is some relation of the form

$$a=f(a_1, \ldots, a_k, a_{k+1}, \ldots, a_n), \tag{1.22}$$

where the function f is the same in both cases by the definition of similar phenomena, but the numerical values of the governing parameters $a_1, \ldots, a_n$ are different. Thus,

$$a_p=f\left(a_1^{(p)}, a_2^{(p)}, \ldots, a_n^{(p)}\right), \quad a^m=f\left(a_1^{(m)}, a_2^{(m)}, \ldots, a_n^{(m)}\right), \tag{1.23}$$

where the index p denotes quantities related to the prototype and the index m denotes quantities related to the model. Using dimensional analysis we find for both phenomena

$$\begin{aligned} \Pi^{(p)} &= \Phi\left(\Pi_1^{(p)}, \Pi_2^{(p)}, \ldots, \Pi_{n-k}^{(p)}\right), \\ \Pi^{(m)} &= \Phi\left(\Pi_1^{(m)}, \Pi_2^{(m)}, \ldots, \Pi_{n-k}^{(m)}\right), \end{aligned} \tag{1.24}$$

where the function Φ must be the same for the model and the prototype. Since by the definition of similar phenomena,

$$\Pi_1^{(m)}=\Pi_1^{(p)}, \ \Pi_2^{(m)}=\Pi_2^{(p)}, \ \ldots, \ \Pi_{n-k}^{(m)}=\Pi_{n-k}^{(p)},$$

it follows that

$$\Pi^{(m)}=\Pi^{(p)}. \tag{1.25}$$

Returning to dimensional variables, we get from (1.25)

$$a_p=a_m\left(\frac{a_1^{(p)}}{a_1^{(m)}}\right)^p\left(\frac{a_2^{(p)}}{a_2^{(m)}}\right)^q \cdots \left(\frac{a_k^{(p)}}{a_k^{(m)}}\right)^r, \tag{1.26}$$

which is a simple rule for recalculating the results of measurements on the similar model for the prototype, for which direct measurement may be difficult to carry out for one reason or another.

The conditions for similarity of the model to the prototype—equality of the similarity parameters $\Pi_1, \Pi_2, \ldots, \Pi_{n-k}$ for both phenomena—show that it is necessary to choose the governing parameters $a_{k+1}^{(m)}, \ldots, a_n^{(m)}$, of the model so as to guarantee the similarity of the model to the prototype:

$$a_{k+1}^{(m)} = a_{k+1}^{(p)} \left(\frac{a_1^{(m)}}{a_1^{(p)}}\right)^{p_{k+1}} \left(\frac{a_2^{(m)}}{a_2^{(p)}}\right)^{q_{k+1}} \cdots \left(\frac{a_k^{(m)}}{a_k^{(p)}}\right)^{r_{k+1}}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \tag{1.27}$$

$$a_n^{(m)} = a_n^{(p)} \left(\frac{a_1^{(m)}}{a_1^{(p)}}\right)^{p_n} \left(\frac{a_2^{(m)}}{a_2^{(p)}}\right)^{q_n} \cdots \left(\frac{a_k^{(m)}}{a_k^{(p)}}\right)^{r_n},$$

whereas the model parameters $a_1^{(m)}, a_2^{(m)}, \ldots, a_k^{(m)}$ can be chosen arbitrarily.

We consider some simple illustrative examples.

(1) Steady motion of a body in an unbounded viscous incompressible fluid is to be modeled. The model body is geometrically similar to the prototype, differing from it only in size, and both have the same orientation with respect to the motion. The governing parameters are the characteristic length scale of the body, for example the diameter l of a cross section, the speed of motion V, the viscosity μ of the fluid, and its density ρ. The dimensions of these quantities in systems of the class MLT are

$$[l] = L, \quad [V] = LT^{-1}, \quad [\mu] = ML^{-1}T^{-1}, \quad [\rho] = ML^{-3}.$$

Thus $n = 4$, $k = 3$, and $n - k = 1$, so that aside from obvious geometric parameters we have only a single dynamical similarity parameter

$$\Pi_1 = \frac{\rho V l}{\mu}. \tag{1.28}$$

Arnold Sommerfeld called this the Reynolds number (the usual notation being $\Pi_1 = \mathrm{Re}$) in honor of the English investigator Osborne Reynolds, who had exceptional success in applying one of the first ideas of similarity to hydrodynamics. To guarantee similarity, it is necessary that this parameter be the same for the model and the prototype.

An analogous system of governing parameters, and consequently the same criterion for similarity, holds for internal problems, for example the flow of viscous fluid in a pipe. In this case one can take as characteristic length scale the diameter D of the pipe, and as characteristic speed V the average speed of the flow across a section of the pipe. It is natural to define the dimensionless drag force of the body as

$$\Pi = \frac{F}{\frac{1}{2}\rho V^2 S} \tag{1.29}$$

($S \sim l^2$ being the area of a cross section) and the dimensionless pressure drop in the pipe as

$$\Pi = \frac{dp/dx}{\frac{1}{2}\rho V^2/D}\,. \tag{1.30}$$

Introduction of the factor $\frac{1}{2}$ in both (1.29) and (1.30) follows tradition.

The graphs of Figs. 1.6a and 1.6b show the dependence of these quantities on the Reynolds number for flow past a cylinder and for the motion of fluid in a smooth circular-cylindrical pipe. The graphs are constructed using data from numerous experiments, and for each case they confirm very well the existence of a universal relation $\Pi = \Phi(\mathrm{Re})$. It is evident that these curves have a rather complicated character: sections with a smooth variation of Π alternate with sharp decreases or increases, there are parts on which Π is almost independent of Re, etc. All this attests to the variation of the flow conditions with change of the Reynolds number, which is the unique parameter governing the global structure of the flow.

Modeling of the motion of a body is usually carried out in the same fluid as the actual motion. Then the product Vl must coincide for the model and the actual case, i.e., the speed of motion of a scale model grows in inverse proportion to the decrease of the size of the model compared with the prototype. It is easy to show that the drag forces of the model and prototype bodies are identical, so that the recalculation coefficient is equal to one.

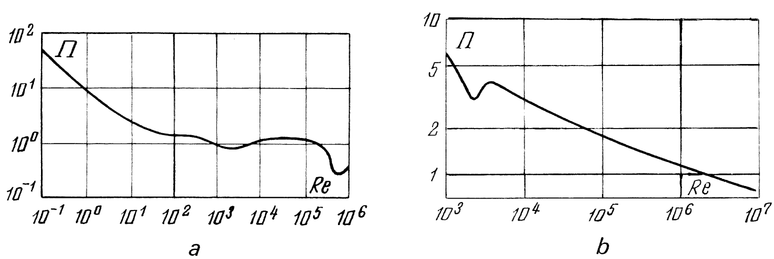


Figure 1.6. Variations with Reynolds number of the dimensionless drag force for flow past a cylinder (Fig. 1.6a) and of the dimensionless pressure drop per unit length for flow through a pipe (Fig. 1.6b). The complicated nature of the curves illustrates the changes in regimes of flow with variation of Reynolds number.

(2) A ship of streamlined shape moves rapidly over the surface of a fluid. The principal contribution to the resistance of a high-speed ship comes from the surface waves created by its motion; the contribution of viscosity to resistance can, to a rough approximation, be considered small for fast ships of streamlined form. The drag force F of the ship is governed by its size l, its speed of motion V, the density of the fluid ρ, and the acceleration of gravity g; the last parameter is essential because the force of gravity turns out to be a decisive influence on waves. In the class MLT the dimensions of the quantities considered are

$$[F]=MLT^{-2},\ [l]=L,\ [g]=LT^{-2},\ [\rho]=ML^{-3},\ [V]=LT^{-1}.$$

Again $n=4$, $k=3$, and $n-k=1$, and the unique dynamic similarity parameter has the form

$$\Pi_1=\frac{V}{\sqrt{lg}}\ . \tag{1.31}$$

It is called the Froude number (the usual notation being $\Pi_1 = \mathrm{Fr}$) in honor of the famous English engineer-shipbuilder William Froude. Thus the quantity V^2/l must be identical for the model and the prototype (because the parameter g can be changed only with great difficulty with the help of refined artifices, usually not applicable), so that the ratio of model and actual speeds must be proportional to the square root of the modeling scale $l^{(m)}/l^{(p)}$. The rule for recalculating the drag force from the model to the prototype in the same fluid has the form

$$F^{(p)}=F^{(m)}\left(\frac{V^{(p)}}{V^{(m)}}\right)^2\left(\frac{l^{(p)}}{l^{(m)}}\right)^2=F^{(m)}\left(\frac{l^{(p)}}{l^{(m)}}\right)^3, \tag{1.32}$$

i.e., the drag force is proportional to the cube of the modeling scale.

If one does not neglect the role of viscosity, then a new similarity parameter appears—the Reynolds number $\mathrm{Re}=\rho Vl/\mu$. Modeling with simultaneous consideration of both similarity parameters in the same fluid turns out to be impossible. In fact, it is then required that the product Vl and the ratio V^2/l be equal for the model and the prototype, and this is only possible if the model and the prototype are identical, which makes modeling absurd. Hence for illustration we have restricted ourselves to the case when the viscous drag is small compared with the wave drag; and as a matter of fact the contribution of viscous drag is modeled separately from the wave drag with the help of practical schemes specially developed for this purpose.

(3) The following example relates to modeling the atmospheres of planets, which is now receiving wide attention (Golitsyn, 1973). We emphasize that the question is not of modeling specific flows in the atmosphere of one planet or

another, but of the similarity parameters that govern the globar dynamic and thermal properties of planetary atmospheres.

The global properties of a planetary atmosphere are governed by (1) the mean surface density of solar energy q entering the atmosphere in unit time; (2) the radius r of the planet; (3) the Stefan-Boltzmann constant σ, governing for given temperature the outgoing flow of radiation; (4) the specific heat at constant pressure c_p of the gas in the planetary atmosphere (which is considered thermally and calorically ideal); (5) the specific heat at constant volume c_v of the gas in the atmosphere; (6) the angular velocity ω of rotation of the planet; (7) the gravitational acceleration g of the planet; and (8) the mass m of a column of gas in the planetary atmosphere having unit base area. The dimensions of these parameters in the class of systems $MLT\theta$ (θ being the dimension of temperature) are

$$[q]=MT^{-3},\quad [r]=L,\quad [\sigma]=MT^{-3}\theta^{-4},$$
$$[c_p]=[c_v]=L^2T^{-2}\theta^{-1},\quad [\omega]=T^{-1},$$
$$[g]=LT^{-2},\quad [m]=ML^{-2}.$$

It is evident that in this case $n=8$ and $k=4$. Thus, in this problem there are four similarity parameters,

$$\Pi_1=\frac{\omega}{r^{-1}c_p^{1/2}q^{1/8}\sigma^{-1/8}};$$
$$\Pi_2=\frac{g}{c_p q^{1/4}\sigma^{-1/4}r^{-1}};$$
$$\Pi_3=\frac{m}{rc_p^{-3/2}q^{5/8}\sigma^{3/8}};$$
$$\Pi_4=\gamma\equiv c_p/c_v.$$

The first parameter is the rotational Mach number, which is to within a factor of order unity equal to the ratio of the speed of rotation of the planet at the equator to the speed of sound. Indeed, the speed of sound c at equilibrium temperature T_e is equal to $(c_pT_e(\gamma-1))^{1/2}$, and the equilibrium temperature itself is equal to $T_e=(q/\sigma)^{1/4}$. Further, by tradition, instead of the parameters Π_2 and Π_3 one considers the quantities inverse to them, $\Pi_g=\Pi_2^{-1}$ and $\Pi_M=\Pi_3^{-1}$. The parameter Π_g is† the ratio of the "height of the homogeneous atmosphere" $(c_p-c_v)T_e/g$ for the given planet to its radius r. The parameter Π_M, the so-called energy similarity parameter, represents some relative measure of the thermal inertia of the atmosphere of the planet.

Thus the basic similarity parameters that govern the global properties of the

†To within a factor $\gamma/(\gamma-1)$.

TABLE 1.1

Planet	Π_1	Π_g	Π_M
Mercury	no atmosphere		–
Venus	7.6×10^{-3}	8.3×10^{-4}	1×10^{-5}
Earth	1.43	1.2×10^{-3}	1.17×10^{-3}
Mars	1.05	3.2×10^{-3}	3.3×10^{-2}
Jupiter	15.6	2.4×10^{-4}	$< 10^{-4}$
Saturn	14.7	5.5×10^{-4}	$< 10^{-4}$
Uranus	7.5	1×10^{-3}	$< 10^{-5}$
Neptune	6	6×10^{-4}	$< 10^{-5}$

planetary atmosphere are Π_1, Π_g, and Π_M. The values of these parameters for the planets of the solar system are given in Table 1.1. The table shows that the values of Π_g and Π_M are small for all the planets. Hence G. S. Golitsyn made the assumption that the values of these parameters are unessential for the global characteristics of planetary atmospheres. On the other hand, the parameter Π_1 varies over a rather wide range: for Venus and Mercury it is small, for Earth and Mars it is near unity, and finally, for the giant planets Jupiter and Saturn it is large. Because of this it is assumed (Golitsyn, 1973) that planetary atmospheres should be classified just by the magnitude of the rotational similarity parameter Π_1.

For slowly rotating planets with dense atmospheres (Venus, and Earth to a certain approximation), all three governing similarity parameters are small: $\Pi_1 \ll 1$, $\Pi_g \ll 1$, $\Pi_M \ll 1$. Hence Golitsyn (1973) assumes that the corresponding dimensional parameters g, ω, m are unessential. This makes it possible for him to propose a simple formula for the total kinetic energy E of atmospheric circulation. In fact, since under the assumptions made $E=f(q, c_p, r, \sigma, \gamma)$, the standard procedure of dimensional analysis gives

$$E = 2\pi \frac{\sigma^{1/8} q^{7/8} r^3}{c_p^{1/2}} \Phi(\gamma), \tag{1.33}$$

where the factor 2π is introduced for convenience. Golitsyn's formula (1.33) allows one to estimate the mean speed of atmospheric motion. In fact, $E = 2\pi r^2 m U^2$ whence, using (1.33), we find

$$U = \left(\frac{E}{2\pi r^2 m}\right)^{1/2} = \left(\frac{\Phi(\gamma)}{\gamma - 1}\right)^{1/2} \Pi_M^{1/2} c, \tag{1.34}$$

where c is again the speed of sound. It is evident that to within a factor depending on γ the parameter Π_M for a slowly rotating planet with dense atmosphere is equal to the square of the Mach number. Golitsyn (1973) cited data showing that (1.33) and (1.34) give the proper order of magnitude of the wind speeds on Venus, Earth, and Mars.

These examples show that dimensional considerations play a decisive role in establishing rules for modeling and criteria for similarity. The crucial step in modeling, as in any application of dimensional analysis to cases where an exact mathematical formulation of the problem is missing, lies in the proper choice of a system of governing parameters. Often (cf. in particular the last example) the procedure is as follows: one takes as governing parameters all quantities that can, in the investigator's opinion, have an influence on the phenomena, no matter how hypothetical. As governing parameters with independent dimensions one takes the governing parameters that are definitely known to be essential, and with respect to the remaining ones, one looks at the numerical values of the corresponding similarity parameters Π_i. If these values are very small or very large, the corresponding dimensional parameter a_{k+i} is considered unessential and is discarded.

In many cases one can actually proceed in this way. It is important, however, that in general this is not so, and one should be very careful about arguments similar to this. One should see in them not a proof of the possibility of disregarding one parameter or another, but a strong conjecture. The last assertion is essentially trivial: it is not necessarily true that a function $\Pi = \Phi(\Pi_1, \ldots, \Pi_i, \ldots, \Pi_{n-k})$ converges to a definite and moreover finite limit for small or large values of the argument Π_i. Only the existence of such a limit (and in fact even a sufficiently rapid convergence to it) can justify neglecting a governing parameter when the corresponding similarity parameter is very large or very small. Subsequent discussion will show us that such crudeness of analysis can lead to serious mistakes.

2

The Application of Dimensional Analysis to the Construction of Exact Special Solutions to Problems of Mathematical Physics. Self-Similar Solutions

1. Strong Thermal Waves

Dimensional analysis allows one, in certain cases, to obtain, by means of completely standard methods, exact special solutions to complicated problems of mathematical physics that reduce to initial, boundary, or mixed problems for partial differential equations or systems of such equations. These solutions are expressed in terms of solutions of boundary-value problems for systems of ordinary differential equations.

An indication of the general method of applying dimensional analysis to obtain exact special solutions is provided by the example of the thermal and gas-dynamic phenomena arising in the initial stages of a nuclear explosion in a gas.[†] We shall discuss the corresponding solutions here, although these solutions are well known and have been explained more than once, and we shall do this for two reasons. First of all, the application of dimensional analysis is well demonstrated by these solutions. Secondly, and this is more important for us, we shall indicate explicitly here some assumptions that are not ordinarily mentioned, and which are as a matter of fact very strong hypotheses made in formulating the corresponding problems. These hypotheses turn out to be valid for the specific problems considered in the present chapter. However, as we shall see later, apparently small complications of the problems that at first glance leave the considerations of dimensional analysis unaltered make these hypotheses

[†]A more detailed consideration of the physics of gas-dynamic phenomena in strong explosions can be found in the monograph of Zel'dovich and Raizer (1967). For similar phenomena arising from the action of a focused laser impulse on matter, see the monograph of Raizer (1977).

inapplicable, and we encounter a paradox whose resolution will lead us to self-similar solutions of a new type.

Thus, at the very initial stage of a nuclear explosion, immediately following the release of energy, the hot gas is still at rest. Strong thermal waves propagate through the motionless gas. The radiation transfer of energy at this stage takes place with a speed many times exceeding the speed of sound; hence the hydrodynamic transfer of matter can be neglected at this stage. The thermal conductivity of the gas is then determined basically by radiation, and the coefficient of thermal conductivity λ can be considered a power function of the temperature u:

$$\lambda = \lambda_0 u^n. \tag{2.1}$$

The value of n is roughly 5. The dependence of the specific heat c on temperature is substantially weaker, and to a first approximation can be neglected. We write the equation for the conservation of energy in the form

$$c\partial_t u + \operatorname{div} \mathbf{q} = 0,$$

where $\mathbf{q} = -\lambda \operatorname{grad} u$ is the heat flux and t the time. We have

$$\begin{aligned} \operatorname{div} \mathbf{q} &= -\operatorname{div} \lambda \operatorname{grad} u = -\lambda_0 \operatorname{div} u^n \operatorname{grad} u = \\ &= -\frac{\lambda_0}{n+1} \operatorname{div} \operatorname{grad} u^{n+1} = -\frac{\lambda_0}{n+1} \Delta (u^{n+1}). \end{aligned}$$

In the case of interest, spherically symmetric wave propagation, we have, by virtue of the symmetry of the problem $\Delta(u^{n+1}) = r^{-2}\partial_r r^2 \partial_r u^{n+1}$ (r being the distance from the center), and the equation of heat conservation finally assumes the form

$$\partial_t u = \varkappa r^{-2} \partial_r r^2 \partial_r u^{n+1}. \tag{2.2}$$

Here $\varkappa = \lambda_0/(n+1)c$ is a constant.

We consider a solution of this equation under the initial conditions and conditions at infinity

$$\begin{gathered} u(r, 0) = 0 \quad (r \neq 0); \quad 4\pi c \int_0^{r_*} u(r, 0) r^2 dr = E, \\ u(\infty, t) = 0 \quad (t > 0), \end{gathered} \tag{2.3}$$

where r_* is an arbitrary positive number. This corresponds to the instantaneous release at the initial moment at the point that is the center of the explosion of a definite finite amount of heat E, with the initial temperature equal to zero everywhere except at the center of the explosion.†

For the solution u the governing parameters will obviously be the independent variables r and t and the constant parameters $\varkappa$ and $Q = E/c$ (because the parameters E and c occur only as this ratio) that appear in the equation and the initial conditions:

$$u = f(t, \varkappa, Q, r). \tag{2.4}$$

The dimensions of the governing parameters are

$$[t] = T, \ [\varkappa] = L^2T^{-1}\theta^{-n}, \ [Q] = \theta L^3, \ [r] = L, \tag{2.5}$$

where θ is a symbol for the temperature dimension.

We now apply dimensional analysis. It is evident that in this case $n = 4$ and $k = 3$. Choosing as governing parameters with independent dimensions $t, \varkappa$, and Q, we obtain by virtue of the Π-theorem

$$\Pi = \Phi(\Pi_1), \tag{2.6}$$

where

$$\Pi = \frac{u}{[Q^2(\varkappa t)^{-3}]^{1/(3n+2)}}, \quad \Pi_1 = \frac{r}{[Q^n \varkappa t]^{1/(3n+2)}} = \xi.$$

Hence we find

$$u = [Q^2(\varkappa t)^{-3}]^{1/(3n+2)}\, \Phi(\xi). \tag{2.7}$$

Calculating the required derivatives of u with respect to t and r with the help of (2.7), and substituting into (2.2) and (2.3), we obtain for the function $\Phi(\xi)$ the ordinary differential equation

$$\frac{d^2\Phi^{n+1}}{d\xi^2} + \frac{2}{\xi}\frac{d\Phi^{n+1}}{d\xi} + \frac{1}{3n+2}\xi\frac{d\Phi}{d\xi} + \frac{3}{3n+2}\Phi = 0. \tag{2.8}$$

Equation (2.7) shows that for any t,

$$4\pi c \int_0^\infty u(r, t)\, r^2\, dr = 4\pi E \int_0^\infty \xi^2 \Phi(\xi)\, d\xi = \text{const}.$$

†The asymptotic meaning of the solution under such initial conditions will be considered in detail below. The parameter r_* is an arbitrary positive number, since $u(r, 0) = 0$ for $r \neq 0$.

From this and (2.3) we get the conditions

$$\int_0^\infty \Phi(\xi)\xi^2 d\xi = 1/4\pi, \quad \Phi(\infty)=0. \tag{2.9}$$

To this we also add the requirement of continuity of the function Φ and of the derivative $d\Phi^{n+1}/d\xi \sim \Phi^n d\Phi/d\xi$. This follows from the hypothesis of continuity at any instant of time $t>0$ of the temperature, which is proportional to Φ, and of the heat flux $\mathbf{q} = -\lambda \operatorname{grad} u = -[\lambda_0/(n+1)] \operatorname{grad} u^{n+1}$, which is proportional to $d\Phi^{n+1}/d\xi$. The last requirement is nontrivial–it shows that for $\Phi \neq 0$ the derivative $d\Phi/d\xi$ must be continuous; at the same time, at points where Φ vanishes, the derivative $d\Phi/d\xi$ can suffer a finite or even infinite discontinuity, provided only that $d\Phi^{n+1}/d\xi$ be continuous.

Integration of (2.8) gives a solution satisfying the second condition of (2.9) in the form

$$\Phi = K(\xi_0^2 - \xi^2)^{1/n} \quad (\xi \leqslant \xi_0), \qquad \Phi \equiv 0 \quad (\xi \geqslant \xi_0), \tag{2.10}$$

where $K = [n/2(n+1)(3n+2)]^{1/n}$. To determine the remaining constant ξ_0 we use the first condition of (2.9). We have

$$K\int_0^{\xi_0} (\xi_0^2 - \xi^2)^{1/n}\xi^2 d\xi = K\xi_0^{\frac{3n+2}{n}} \int_0^1 (1-\zeta^2)^{1/n}\zeta^2 d\zeta = 1/4\pi, \tag{2.11}$$

whence, using the expression of the integral in terms of beta functions (Abramowitz and Stegun, 1964), we find

$$\xi_0 = [2\pi K B(3/2,\ (n+1)/n)]^{-\frac{n}{3n+2}}. \tag{2.12}$$

Here B is the symbol for Euler's beta function.

Thus the temperature distribution finally assumes the form

$$u = \left[\frac{E^2}{c^2\varkappa^3 t^3}\right]^{\frac{1}{3n+2}} K\left(\xi_0^2 - \frac{r^2}{(E^n c^{-n}\varkappa t)^{\frac{2}{3n+2}}}\right)^{1/n} \tag{2.13}$$

for $r \leqslant r_f(t) = \xi_0(n)[(E/c)^n \varkappa t]^{1/(3n+2)}$, and $u \equiv 0$ for $r \geqslant r_f(t)$.

From (2.13) follows the simple relation

$$\frac{u}{u(0,t)} = \left(1 - \frac{r^2}{r_f^2}\right)^{1/n} \quad (r \leqslant r_f),$$

$$u \equiv 0 \quad (r \geqslant r_f).$$

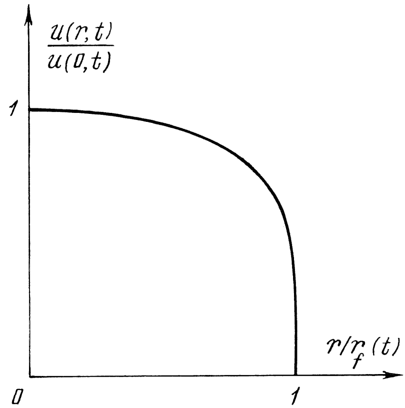


Figure 2.1. Distribution of temperature behind a strong thermal wave in the self-similar variables $u(r, t)/u(0, t)$, $r/r_f(t)$.

This function is shown in Fig. 2.1 for $n = 5$. It is curious that for $n > 0$, in contrast to the linear case, one has "finite speed of heat propagation"; the perturbation zone is bounded: $r_f(t) < \infty$ for any finite t. Passing to the limit $n \to 0$ we recover the known solution (0.1) of the linear equation of heat conduction for an instantaneous point source. In this case $r_f(t) = \infty$ for any $t > 0$. The solution discussed above was obtained by Zel'dovich and Kompaneets (1950) and Barenblatt (1952). The latter paper considered the mathematically equivalent problem of gas filtration.

It is clear that in the case of heat propagation in one dimension all the arguments proceed completely analogously; by exactly the same reasoning one can obtain from dimensional considerations the classical solution of instantaneous source type for the linear equation of heat conduction in one dimension. Since we shall soon need this solution, and more importantly its derivation, we dwell on it briefly. Thus we seek a solution to the equation

$$\partial_t u = \varkappa \partial^2_{xx} u \tag{2.14}$$

under the initial conditions and condition at infinity

$$u(x, 0) = Q\delta(x), \text{ i.e., } \begin{cases} u(x, 0) \equiv 0 \quad (x \neq 0), \\ \int\limits_{-\infty}^{\infty} u(x, 0)\, dx = Q, \end{cases} \tag{2.15}$$

$$u(\infty, t) \equiv 0.$$

The desired solution obviously depends on the governing parameters $t, \varkappa, Q$, and x, whose dimensions are respectively T, L^2T^{-1}, UL, and L (U being the dimension of the temperature u). As governing parameters with independent dimensions we choose the first three. In the present case $n = 4$ and $k = 3$, and dimensional analysis gives

$$\Pi = \Phi(\Pi_1), \quad \Pi = \frac{u}{Q(\varkappa t)^{-1/2}}, \quad \Pi_1 = \frac{x}{\sqrt{\varkappa t}},$$

whence

$$u = \frac{Q}{\sqrt{\varkappa t}} f\left(\frac{x}{\sqrt{\varkappa t}}\right).$$

Substituting this expression into (2.14) and using (2.15), we obtain the well-known solution

$$u = \frac{Q}{2\sqrt{\pi \varkappa t}} \exp\left(-\frac{x^2}{4\varkappa t}\right). \tag{2.16}$$

One can also get from considerations of dimensional analysis the law of temperature decay at a maximum point; the same arguments give, to within a constant,

$$u_{\max} \sim \frac{Q}{\sqrt{\varkappa t}};$$

and from (2.16) we find

$$u_{\max} = \frac{Q}{2\sqrt{\pi \varkappa t}}.$$

We will have to reconsider the above derivation of (2.16). While it is typical and seems entirely transparent, this derivation in fact contains underwater reefs that become completely apparent after a seemingly small and insignificant modification of the problem (see the following chapter).

2. Strong Blast Waves

The solution obtained in Section 1 describes the phenomenon of strong explosion only at the very initial thermal stage. As time passes, the speed of radiative transfer of energy decreases and quickly becomes small compared with the speed of sound. There arises in the heated gas an intense shock wave, which outstrips the thermal wave and initiates transition of the flow to the subsequent gas-dynamic stage. At this stage it is necessary to consider the motion of the gas, which can be considered adiabatic. We recall the well-known equations [cf.

Kochin, Kibel', and Roze 1964; Landau and Lifschitz 1959)] for adiabatic motion of a gas in the case of spherical symmetry in which we are interested. The first equation, Newton's law written for a unit volume of gas, is

$$\frac{dv}{dt} \equiv \partial_t v + v\partial_r v = -\frac{1}{\rho}\partial_r p.$$

Here v is the radial component of velocity, p the pressure, ρ the density of the gas, r the radial coordinate measured from the center of the explosion, and t the time. The only force acting is the pressure drop in the radial direction, and the mass of a unit volume is equal to the density of the gas. Further, the conservation law for the mass of the gas is satisfied:

$$\partial_t \rho + \operatorname{div} \rho \mathbf{v} = 0.$$

In the case of spherical symmetry, when the radial velocity is the only nonzero component, $\operatorname{div} \rho \mathbf{v} = r^{-2}\partial_r r^2 \rho v = (2/r)\rho v + \partial_r \rho v$. Finally, by virtue of the adiabaticity of the motion, one has the equation of conservation of entropy in a fluid particle:

$$\frac{ds}{dt} \equiv \partial_t s + v\partial_r s = 0.$$

Here s is the entropy of a unit mass, which in the case considered of a thermodynamically ideal gas is equal to $s = c_v \ln p/\rho^\gamma$; c_v is the specific heat of the gas at constant volume, and γ is the ratio of the specific heats at constant pressure and constant volume. Thus the basic equations of motion for the gas can be written in the form

$$\begin{aligned} &\partial_t v + v\partial_r v + \partial_r p/\rho = 0, \\ &\partial_t \rho + \partial_r (\rho v) + 2\rho v/r = 0, \\ &\partial_t (p/\rho^\gamma) + v\partial_r (p/\rho^\gamma) = 0. \end{aligned} \tag{2.17}$$

We consider here the exact solution to the problem of the gas motion arising from instantaneous release at the center of the explosion of a finite amount of energy E. The gas is assumed to be initially at rest, its pressure is equal to zero, and the initial density of the gas is equal to ρ_0 everywhere except at the center of the explosion.† A classical, i.e., smooth solution of this problem does not exist, and we shall seek a piecewise-smooth solution: the perturbed domain,

†The asymptotic meaning of this solution will also be considered below.

inside which the solution varies continuously and is described by (2.17), is bounded by a shock wave, which is a sphere of radius $r_f(t)$ on which the properties of the motion–the pressure, density, and velocity–change discontinuously. Outside this sphere the state of rest of the gas is preserved, and the initial pressure of the gas is also equal to zero by assumption. Thus the conditions of conservation (continuity of flux) of mass, momentum, and energy at the front of the shock wave are written in the following form (where index f denotes the value of a quantity immediately behind the shock wave, i.e., for $r = r_f - 0$):

$$\rho_f(v_f - D) = -\rho_0 D,$$

$$\rho_f(v_f - D)^2 + p_f = \rho_0 D^2,$$

$$\rho_f(v_f - D)\left[\frac{\gamma}{\gamma - 1}\frac{p_f}{\rho_f} + \frac{(v_f - D)^2}{2}\right] = -\rho_0 \frac{D^3}{2}.$$

Here $D = dr_f/dt$ is the speed of propagation of the shock wave through the ambient gas. (We recall that the energy flux is equal to the product of the mass flux and the sum of the kinetic energy per unit mass and the enthalpy per unit mass.) The last two relations are conveniently written in the form

$$\rho_f(v_f - D)\,v_f + p_f = 0,\quad \rho_f(v_f - D)\left[\frac{p_f}{(\gamma - 1)\rho_f} + \frac{v_f^2}{2}\right] + p_f v_f = 0.$$

Solving the continuity equations for the flux of mass, momentum, and energy, we find the density, pressure, and speed behind the shock wave:

$$p_f = \frac{2}{\gamma + 1}\rho_0 D^2,\quad \rho_f = \frac{\gamma + 1}{\gamma - 1}\rho_0,\quad v_f = \frac{2}{\gamma + 1}D. \tag{2.18}$$

Further, the energy of a unit volume of gas is equal to $\rho(v^2/2 + c_v T) = \rho[v^2/2 + p/(\gamma - 1)\rho]$ (T being the absolute temperature). Hence the initial conditions for the problem of a point explosion can be written in the form

$$\begin{gathered}\rho(r, 0) = \rho_0,\quad p(r, 0) \equiv 0,\quad v(r, 0) \equiv 0\quad (r \neq 0),\\ 4\pi \int_0^{r_*} \rho\left[\frac{v^2}{2} + \frac{1}{\gamma - 1}\frac{p}{\rho}\right] r^2\,dr = E\quad (t = 0).\end{gathered} \tag{2.19}$$

Here r_* is an arbitrary positive quantity† and E is the energy released at the center at the initial moment. (Obviously these initial conditions are given, as before,

†Since $v(r, 0) = 0$ and $p(r, 0) = 0$ for $r \neq 0$.

in terms of generalized functions.) Finally, we have the obvious condition of absence of influx of matter and energy at the center for $t > 0$,

$$v(0,\ t) \equiv 0 \quad (t > 0). \tag{2.20}$$

Analysis of (2.17) and the conditions (2.18), (2.19), and (2.20) shows that the properties p, ρ, and v of the gas motion depend on the governing parameters

$$t,\ E,\ \rho_0,\ r,\ \gamma, \tag{2.21}$$

whose dimensions in the class of systems of units of measurement MLT are, respectively,

$$T,\ ML^2T^{-2},\ ML^{-3},\ L,\ 1. \tag{2.22}$$

The radius of the shock front depends on the same parameters (2.21) with the exception of r. Thus $n = 5$ and $k = 3$, and taking the first three as the governing parameters with independent dimensions we get

$$\Pi_1 = r\left(Et^2/\rho_0\right)^{-1/5} = \xi, \quad \Pi_2 = \gamma. \tag{2.23}$$

From this and dimensional analysis it follows that†

$$\begin{gathered} p = \rho_0 \frac{r^2}{t^2} P(\xi,\ \gamma), \quad \rho = \rho_0 R(\xi,\ \gamma), \quad v = \frac{r}{t} V(\xi,\ \gamma), \\ r_f = \xi_0(\gamma)\left(Et^2/\rho_0\right)^{1/5}, \quad D = \frac{2}{5}\xi_0\left(Et^{-3}/\rho_0\right)^{1/5}. \end{gathered} \tag{2.24}$$

Substituting (2.24) into (2.17), we obtain for the functions P, V, and R the system of ordinary differential equations

$$\begin{gathered} \left(V - \frac{2}{5}\right) R \frac{dV}{d\ln\xi} + \frac{dP}{d\ln\xi} - RV + RV^2 + P = 0, \\ \frac{dV}{d\ln\xi} + \left(V - \frac{2}{5}\right)\frac{d\ln R}{d\ln\xi} + 3V = 0, \\ \frac{d}{d\ln\xi}\left(\ln\frac{P}{R^\gamma}\right) - \frac{2(1-V)}{V - 2/5} = 0. \end{gathered} \tag{2.25}$$

†Here, following tradition, we have deviated somewhat from the formal recipe for applying dimensional analysis. For example, for the pressure we should write

$$p = E^{2/5} t^{-6/5} \rho_0^{3/5} \Phi\left(\Pi_1,\ \Pi_2\right).$$

The notation of (2.24) is obtained if we write $\Phi = \Pi_1{}^2 P$, and analogously for the velocity.

The boundary conditions (2.18) give

$$P(\xi_0-0)=\frac{8}{25(\gamma+1)}, \quad V(\xi_0-0)=\frac{4}{5(\gamma+1)}, \quad R(\xi_0-0)=\frac{\gamma+1}{\gamma-1}. \tag{2.26}$$

Further, it follows from (2.24) that the bulk energy of the gas in the perturbed region is constant in time, i.e., is an integral of the motion:

$$4\pi\int_0^{r_f}\rho\left(\frac{v^2}{2}+\frac{p}{(\gamma-1)\rho}\right)r^2\,dr=$$

$$=4\pi\rho_0\frac{E}{\rho_0}\int_0^{\xi_0}R(\xi)\left[\frac{V^2(\xi)}{2}+\frac{P(\xi)}{(\gamma-1)R(\xi)}\right]\xi^4\,d\xi=\mathrm{const.} \tag{2.27}$$

By virtue of the initial condition (2.19), the constant on the right side of (2.27) is equal to E, whence

$$\int_0^{\xi_0}R(\xi)\left[\frac{V^2(\xi)}{2}+\frac{P(\xi)}{(\gamma-1)R(\xi)}\right]\xi^4\,d\xi=\frac{1}{4\pi}. \tag{2.28}$$

The solution of (2.25) under the conditions (2.26) is found, as in the problem of strong thermal waves, in the explicit form

$$\left(\frac{\xi_0}{\xi}\right)^5=C_1V^2\left(1-\frac{3\gamma-1}{2}V\right)^{\nu_1}\left(\frac{5}{2}\gamma V-1\right)^{\nu_2},$$

$$R=C_2\left(\frac{5}{2}\gamma V-1\right)^{\nu_3}\left(1-\frac{3\gamma-1}{2}V\right)^{\nu_4}\left(1-\frac{5}{2}V\right)^{\nu_5}, \tag{2.29}$$

$$P=C_3R\left(1-\frac{5}{2}V\right)V^2\left(\frac{5}{2}\gamma V-1\right)^{-1}.$$

Here

$$C_1=\left[\frac{5}{4}(\gamma+1)\right]^2\left[\frac{5(\gamma+1)}{7-\gamma}\right]^{\nu_1}\left(\frac{\gamma+1}{\gamma-1}\right)^{\nu_2},$$

$$C_2=\left(\frac{\gamma+1}{\gamma-1}\right)^{\nu_3+\nu_5+1}\left(\frac{5(\gamma+1)}{7-\gamma}\right)^{\nu_4}, \quad C_3=\frac{\gamma-1}{2},$$

$$\nu_1=\frac{13\gamma^2-7\gamma+12}{(3\gamma-1)(2\gamma+1)}, \quad \nu_2=-\frac{5(\gamma-1)}{2\gamma+1}, \quad \nu_3=\frac{3}{2\gamma+1}, \tag{2.30}$$

$$\nu_4=\frac{13\gamma^2-7\gamma+12}{(2-\gamma)(3\gamma-1)(2\gamma+1)}, \quad \nu_5=-\frac{1}{2-\gamma}.$$

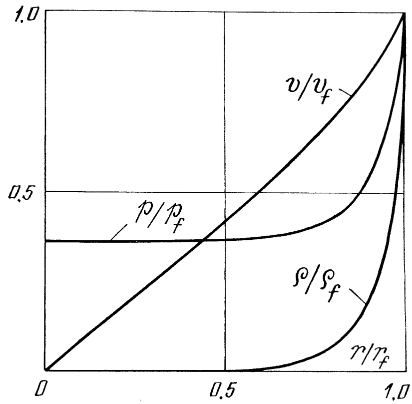


Figure 2.2. Distributions of gas pressure, density, and velocity behind a strong blast wave in the self-similar variables ρ/ρ_f, p/p_f, v/v_f, r/r_f.

It is easy to verify that (2.29) also satisfies (2.20). The dependence of ξ_0 on γ is determined by substituting (2.29) into (2.28), in principle completely analogously to the way this was done in the previous problem. Calculation shows, for example, that for $\gamma = 1.4$, $\xi_0 = 1.033$, i.e., close to unity.

Figure 2.2 shows the distributions of pressure, density, and speed in the form of the dependences on r/r_f for $\gamma = 1.4$ (air) of the universal functions (identical for all instants of time)

$$\frac{p}{p_f}, \quad \frac{\rho}{\rho_f}, \quad \frac{v}{v_f}, \tag{2.31}$$

where p_f, ρ_f, and v_f are the values of the quantities just behind the shock wave.

The solution discussed in this section was obtained by Sedov (1946), and somewhat less completely by Taylor (1950).†

3. Self-Similarity. Intermediate Asymptotics

The solution of the problem of a strong adiabatic explosion, as well as the solutions considered earlier of the problems of strong thermal waves and of an instantaneous heat source, have the very important feature of self-similarity. This consists in the fact that the distributions of all properties in both problems (temperature in the first problem; pressure, density, and speed in the second) are at various instants of time similar, i.e., we obtain one from another by a similarity transformation. Thus, if we choose time-dependent scales $r_0(t)$ for the spatial variable and $u_0(t)$ for any property of the phenomenon, then its distribution at various instants can be expressed in the form $u_0(t)f(r/r_0(t))$. Hence it follows

†In particular, the possibility of obtaining a solution in finite form is connected with the law, pointed out by L. I. Sedov, of conservation of energy in time not only inside the entire perturbed region but also inside any sphere bounded by radius $\xi = \text{const} < \xi_0$, i.e., $r = \text{const}\,(Et^2/\rho_0)^{1/5}$.

that if we describe these distributions in "self-similar coordinates" $u/u_0(t)$, $r/r_0(t)$ (u being the property considered), then at various instants of time all these distributions are represented by a single curve. Thus, in the problem of strong thermal waves the spatial scale is $r_0(t) = [(E/c)^n \varkappa t]^{1/(3n+2)}$ and the temperature scale is $u_0(t) = [(E/c)^2 (\varkappa t)^{-3}]^{1/(3n+2)}$, in the strong explosion problem $r_0(t) = (Et^2/\rho_0)^{1/5}$ and $u_0(t) = \rho_0$ for the density, $u_0(t) = \rho_0^{3/5} E^{2/5} t^{-6/5}$ for the pressure, and $u_0(t) = (Et^{-3}/\rho_0)^{1/5}$ for the speed. We have already encountered the possibility of such a reduction earlier, and have mentioned the advantages that it provides. The situation is just the same here: the establishment of the fact of self-similarity in the present case reduces by an order of magnitude the number of values required for practical determination of the desired distribution.

As was already mentioned in the Introduction, self-similar solutions are encountered in many branches of mathematical physics. Their discovery has attracted considerable attention, because in complicated nonlinear problems, reducing them to boundary-value problems for ordinary differential equations by obtaining such solutions has frequently been the only means of breaking through the analytic difficulties and reaching a qualitative understanding of the phenomena. Moreover, self-similar solutions have been widely used as standards for evaluating all kinds of approximation methods, irrespective of the immediate urgency of the problems. It is now essential for us to emphasize that self-similar solutions are of basic value not only and not mainly as exact solutions of isolated albeit urgent specific problems, but above all as intermediate-asymptotic representations of the solutions of an immeasurably wider class of problems.

For example, we consider this point for the problem of strong thermal waves. Of course the release of energy in a nuclear explosion does not actually take place at a point but in a finite domain of some size d, it is not spherically symmetric, and the initial temperature T_0 is not equal to zero. Hence, strictly speaking, the governing parameters in (2.4) should also include the parameters d and T_0 and the polar angles φ and ψ. Hence it follows immediately that in addition to the parameter Π_1 the function Φ in (2.6) will be determined by the four other dimensionless parameters

$$\Pi_2 = d/[(E/c)^n \varkappa t]^{\frac{1}{3n+2}}, \quad \Pi_3 = T_0/[(E/c)^2 (\varkappa t)^{-3}]^{\frac{1}{3n+2}}, \tag{2.32}$$
$$\Pi_4 = \varphi, \qquad \Pi_5 = \psi.$$

It is clear intuitively, however, and well confirmed by numerical computations, that the asymmetry of the region of initial release is essential only in the very first moments, when the thermal wave has spread to a distance of the order of one or two times the size of the initial region of heat release. At these distances the various details of the initial heat discharge, which are different from case to case, are not recorded and are of no interest. We abandon their consideration, i.e., we shall be interested in the propagation of strong thermal waves only at the stage when the wave has gone a distance $r_f(t)$, large compared with the size of the initial region of heat discharge. This means that $r_f(t) \gg d$, and from this and the fact that $r_f(t)$ is of order $[(E/c)^n \varkappa t]^{1/(3n+2)}$ it follows that we must here have $t \gg d^{3n+2}/\varkappa(E/c)^n$. But for such t the parameter Π_2 is much smaller than unity. One ordinarily assumes that if some similarity parameter has a value much smaller or much larger than unity then the dependence on that parameter, and consequently also on the corresponding dimensional parameter, can be neglected. In this special case this turns out to be correct, so that for $r \gg d$ and $t \gg d^{3n+2}/\varkappa(E/c)^n$ the dependence of the solution on the parameters Π_2, Π_4, Π_5 is unessential.

Further, since the explosion is strong, the temperature in the region traversed by the thermal wave is at first very high, much higher than the initial temperature T_0. But the temperature near the center of the wave is of order $[(E/c)^2(\varkappa t)^{-3}]^{1/3n+2}$, whence it follows that for $t \ll (E/c)^{2/3}/\varkappa T_0^{(3n+2)/3} = T_2$ the parameter $\Pi_3 \ll 1$, and thus the initial temperature is unessential. Keeping in mind that for such t, $r_f \ll (E/cT_0)^{1/3}$, we find that for a sufficiently strong and sufficiently concentrated explosion (large E and small d) the characteristic time scales of the problem,

$$T_1 = d^{3n+2}/\varkappa(E/c)^n, \quad T_2 = (E/c)^{2/3}/\varkappa T_0^{\frac{3n+2}{3}},$$

and the spatial scales of the problem,†

$$L_1 = d, \quad L_2 = (E/cT_0)^{1/3},$$

are strongly separated from each other, i.e., are such that $T_1 \lll T_2$ and $L_1 \lll L_2$. The self-similar solution we have obtained describes the phenomenon of a strong and concentrated explosion well at times and distances from the center large enough to make the influence of the asymmetry of the initial conditions and the size of the domain of original heat release disappear, and at the same time small enough so as to make the difference of the original temperature from zero negligible:

$$d^{3n+2}/\varkappa(E/c)^n \ll t \ll (E/c)^{2/3}/\varkappa T_0^{\frac{3n+2}{3}}, \qquad (2.33)$$
$$d \ll r \ll (E/cT_0)^{1/3}.$$

†As a matter of fact, the scales T_2 and L_2 are bounded also by the beginning of the gas motion.

We therefore say that the self-similar solution is an *intermediate asymptotics* of the phenomenon described. By *intermediate asymptotics* in the general case one means the following. Suppose in the problem there are two constant governing quantities having the dimensions of the independent variable x_i: $X_i^{(1)}$ and $X_i^{(2)}$. An intermediate asymptotics is an asymptotic representation of the solution as $x_i/X_i^{(1)} \to 0$ but $x_i/X_i^{(2)} \to \infty$.

The situation is quite analogous in the problem of describing the gas-dynamic stage of a strong explosion. In this case we must take into consideration that the energy release occurs not at a point but in a sphere of radius R_0 (the radius R_0 corresponding to the time when the strong shock wave outstrips the thermal wave), and that outside this sphere the ambient gas of density ρ_0 is under not zero but some finite pressure p_0. The solution discussed above represents an intermediate asymptotics describing the gas-dynamic stage of the explosion for

$$T_1 = \left(\frac{\rho_0 R_0^5}{E}\right)^{1/2} \ll t \ll \left(\frac{\rho_0 E^{2/3}}{p_0^{5/3}}\right)^{1/2} = T_2,$$
$$L_1 = R_0 \ll r \ll \left(\frac{E}{p_0}\right)^{1/3} = L_2, \tag{2.34}$$

i.e., for times and at distances from the center of the explosion sufficiently large that the influence of the size of the region of initial energy discharge vanishes, and at the same time sufficiently small that the influence of the counter-pressure p_0 is not yet felt. We give some figures. Under the conditions of the first American atomic explosion at Alamogordo, $\rho_0 \sim 10^{-3}$ g/cm^3, $E \sim 10^{21}$ ergs, $p_0 \sim 10^6$ dynes/cm^2, and $R_0 \sim 10^3$ cm $= 10$ m, whence for the temporal and spatial bounds on the domain of applicability of the self-similar intermediate asymptotics we find $T_1 \sim 10^{-4}$ sec, $T_2 \sim 1$ sec, $L_1 \sim 10^3$ cm, and $L_2 \sim 10^5$ cm. One should note that as a matter of fact the upper bound on the applicability of the self-similar intermediate asymptotics lies lower because of the influence of viscous erosion of the front.

The situation is quite the same in the general case. Self-similar solutions are always solutions of degenerate problems in which constant parameters having the dimensions of the independent variables appearing in the problem assume zero or infinite values, so that as a rule self-similar solutions correspond to singular initial or boundary conditions, as in the examples just considered. Hence self-similar solutions are always intermediate asymptotic solutions of nondegenerate problems.†

The idea is widespread that obtaining self-similar solutions is always connected with dimensional analysis, i.e., with similarity, so that by applying dimensional analysis to the formulation of a degenerate problem that has some self-similar

†More precisely, stable self-similar solutions (see Chapter 9).

solutions, one can always obtain the form of the solution, i.e., the expression for the self-similar variables. After obtaining the exact solution it is easy to find the class of nondegenerate problems for which the self-similar solution considered is an intermediate asymptotics. This is actually the situation for some solutions. The examples considered in the present chapter have demonstrated this very well, and indicated a general approach that is applicable in similar cases. It is essential, however, that the cases in which the construction of self-similar solutions is exhausted by dimensional analysis constitute, as is sometimes said, only the visible part of the iceberg. As a rule, the situation is different: there exist extensive classes of problems for which, although a self-similar intermediate asymptotics exists, it cannot be obtained from the original formulation of the problem by applying dimensional considerations. The form of the self-similar variables in these cases is obtained from the solution of nonlinear eigenvalue problems, and sometimes even from some additional considerations. We emphasize again that it is not a question here of exceptions but rather of the rule: the set of self-similar solutions that cannot be obtained from similarity considerations is considerably richer than the set of self-similar solutions whose form is completely determined by similarity considerations. Subsequent examination will show what the situation is here: modifying the problems considered in this chapter apparently slightly, and moreover in such a way that at first glance all similarity considerations used, and hence also everything deduced from them, must remain valid, we will arrive at a contradiction. Resolving the contradiction reveals that the passage to the limit from a certain non-self-similar problem to its self-similar intermediate asymptotics is far from being always uniform. Investigation of nonuniform passages to the limit will lead us to a new class of self-similar solutions.

3

Modified Problem of an Instantaneous Heat Source: Self-Similar Solution of the Second Kind

1. Modified Problem of an Instantaneous Heat Source

Consideration of the problems to which we now turn will show that the elementary situations that we met in the examples of self-similar solutions of the previous chapter are actually rare exceptions, and that as a rule the situation is substantially more complicated.

We begin with a modified problem of an instantaneous heat source. The modification consists in changing the equation for the temperature u at those points where the body is cooling: instead of the classical equation of heat conduction (2.14) this function satisfies an equation with discontinuous coefficient of thermal diffusivity,

$$\begin{aligned} \partial_t u &= \varkappa \partial^2_{xx} u \quad (\partial_t u \geqslant 0), \\ \partial_t u &= \varkappa_1 \partial^2_{xx} u \quad (\partial_t u \leqslant 0), \end{aligned} \tag{3.1}$$

where $\varkappa_1$ is a constant that is in general different from $\varkappa$, so that the coefficient of thermal diffusivity depends upon whether the body is heating or cooling at a given point. Such an equation describes heat conduction in a medium in which pores arise when the body is cooled. It is encountered also in the theory of fluid motion in porous media, and its derivation will be given in the following section. It is essential that the steplike behavior of the coefficient of thermal diffusivity is connected with the difference in specific heat for heating and cooling. However, the thermal conductivity does not depend on the direction of the change of temperature, so that the condition of continuity of heat flux requires the continuity of the derivative $\partial_x u$.

Thus we are interested in a solution of (3.1) that is continuous with continuous derivatives with respect to both independent variables. As was proved by Kamenomostskaya (1957) a solution to the initial-value problem for (3.1) with an arbitrary sufficiently smooth function $u(x, 0)$, which decreases monotonically and sufficiently rapidly with increasing $|x|$, exists, is unique, and has a continuous derivative with respect to t and two continuous derivatives with respect to x.

2. Derivation of the Basic Equation

This equation occurs in the theory of filtration of an elastic fluid in an elasto-plastic porous medium. A short derivation of it is given below; the reader who is not interested in the actual physical meaning of the modified problem can skip this section without damage to his understanding of what follows.

As is known, the equation for the conservation of fluid mass in the filtration of fluids in porous media has the form

$$\partial_t (m\rho) + \operatorname{div} \rho \mathbf{v} = 0.$$

Here m is the porosity of the medium—the relative volume occupied in the medium by the pores; ρ is the density of the fluid; $\mathbf{v}$ is the velocity of filtration, equal in magnitude to the volume of the fluid passing through a section of unit area of the porous medium normal to the flow; and t is the time. The velocity of filtration is proportional to the pressure gradient; this constitutes the content of Darcy's law, which is basic for the theory of filtration and is analogous in its formulation to Fourier's law in the theory of heat conduction:

$$\mathbf{v} = -\frac{k}{\mu} \operatorname{grad} p.$$

Here k is the so-called coefficient of permeability, determining the resistance of the porous medium to the fluid leaking through it, and μ is the coefficient of viscosity of the fluid. The liquid is assumed to be weakly compressible, so that its density grows linearly with increasing pressure:

$$\rho/\rho_0 = 1 + \beta_f (p - p_0),$$

where β_f is the coefficient of compressibility of the fluid, and p_0 and ρ_0 are the reference pressure and density of the fluid. The porous medium is also considered to be weakly compressible. Its porosity m can, as experiments show, be considered to a first approximation to depend only on σ, the first invariant of the stress tensor† (one third of the sum of the principal stresses) acting on the skeleton of the porous medium: $m = m(\sigma)$. If the porous medium is elastic,

†Here it is convenient to assume a compressive stress as positive.

then

$$m/m_0 = 1 - \beta_r(\sigma - \sigma_0),$$

where β_r is the coefficient of compressibility of the porous medium, σ_0 is the reference value of σ (for increasing stress the medium compresses, $\beta_r > 0$), and m_0 is the corresponding value of the porosity. Under conditions of a deep-lying porous stratum the total stress state of the fluid-porous medium system is fixed, since the fluid and the porous skeleton together restrain the burden of higher-lying strata. Hence $\sigma + p = \sigma_0 + p_0$, whence $\sigma - \sigma_0 = -(p - p_0)$. Substituting these relations into the equation for conservation of fluid and discarding terms of order less than the highest in $\beta(p - p_0)$, we find [for details see Shchelkachev (1959); Collins (1961); Barenblatt, Entov, and Ryzhik (1972)] that for filtration of an elastic fluid in an elastic porous medium under the conditions of a deep-lying porous stratum the pressure of the fluid $p(r, t)$ satisfies the classical linear equation of heat conduction,

$$\partial_t p = \varkappa \Delta p. \tag{3.2}$$

Here $\varkappa$ is the so-called coefficient of piezoconductivity, analogous to the coefficient of thermal diffusivity, and equal to $k/\mu(m_0\beta_f + \beta_r)$.

Now, as often happens, let the porous medium be irreversibly deformable. Then [for details see Barenblatt and Krylov (1955); Barenblatt, Entov, and Ryzhik (1972)]

$$\partial_t m = -m_0\beta_r\partial_t\sigma = m_0\beta_r\partial_t p$$

for increasing σ (decreasing fluid pressure, because the total stress state of the fluid-porous medium system remains unchanged, $\sigma + p = \sigma_0 + p_0$, $\partial_t\sigma = -\partial_t p$) and

$$\partial_t m = -m_0\beta_{r1}\partial_t\sigma = m_0\beta_{r1}\partial_t p$$

for decreasing σ (increasing fluid pressure), where β_{r1} is not equal to β_r. Hence, the equation for excess fluid pressure, i.e., the difference of initial and instantaneous pressure $u(r, t) = p_0 - p(r, t)$, assumes the form

$$\partial_t u = \varkappa(\partial_t u)\,\Delta u, \tag{3.3}$$

where $\varkappa(\partial_t u)$ is a step function: $\varkappa(\partial_t u) = \varkappa$ for $\partial_t u > 0$ and $\varkappa(\partial_t u) = \varkappa_1$ for $\partial_t u < 0$. The coefficients $\varkappa$ and $\varkappa_1$ are determined by the properties of the fluid and the deformation properties of the medium, being different for loading of the stratum by the burden of the higher-lying strata (a drop in fluid pressure) and for unloading (a subsequent increase in fluid pressure)†:

$$\varkappa = \frac{k}{\mu(m_0\beta_f + \beta_r)}, \quad \varkappa_1 = \frac{k}{\mu(m_0\beta_f + \beta_{r1})}.$$

Thus the analog of thermal conductivity, the quantity k/μ, is identical for loading and unloading, whereas the analog of specific heat, the quantity $m_0\beta_f + \beta_r$, is different for loading and unloading.

In particular, for the one-dimensional problem of rectilinear parallel fluid motion (filtration to a drainage gallery or from it), (3.3) assumes the form of the basic equation (3.1).

3. Direct Application of Dimensional Analysis to the Modified Problem of an Instantaneous Heat Source

We now try to find the solution of the problem of the instantaneous removal of a finite mass of a fluid from a small region of an elasto-plastic layer. It would seem that obtaining this solution reduces, in view of the linear dependence of the fluid density on pressure, to constructing the solution to (3.1) for a problem of instantaneous point-source type. We try to construct such a solution with the help of dimensional analysis; but later analysis will show that the matter is actually more complicated.

Thus a solution of (3.1) is sought, satisfying the initial condition and condition at infinity

$$u(x, 0) \equiv 0 \quad (x \neq 0); \qquad \int_{-\infty}^{\infty} u(x, 0)\, dx = Q; \quad u(\infty, t) \equiv 0. \qquad (3.4)$$

As is well-known and was already mentioned above, for the case $\varkappa_1 = \varkappa$ (the classical equation of heat conduction or of filtration in an elastic porous medium) such a solution exists, is self-similar, and takes the form (2.16). It would appear that for $\varkappa_1 \neq \varkappa$ the dimensional considerations would proceed exactly the same as in the case $\varkappa_1 = \varkappa$, because the list of governing parameters in the modified problem has been increased, compared with the classical problem of an instantaneous heat source, only by the dimensionless constant parameter $\varepsilon = \varkappa_1/\varkappa$. Hence it would seem at first glance that the desired solution must be expressed

†It is assumed that at each point of the porous medium the processes of loading and unloading take place only once. One can also consider more complicated processes, but we shall not do so here.

in the form

$$u=\frac{Q}{\sqrt{\varkappa t}}\Phi(\xi,\ \varepsilon),\quad \xi=\frac{x}{\sqrt{\varkappa t}},\tag{3.5}$$

where the function Φ is continuous with continuous derivative with respect to ξ and is an even function: $\Phi(-\xi,\varepsilon)=\Phi(\xi,\varepsilon)$. Further, the loading domain ($\partial_t u\geqslant 0$) must correspond by virtue of the self-similarity of the problem to

$$|x|\geqslant x_0(t)=\xi_0\sqrt{\varkappa t},$$

where ξ_0 is a constant depending on ε; and for the unloading domain ($\partial_t u\leqslant 0$),

$$0\leqslant|x|\leqslant x_0(t).$$

However for $\varkappa_1\neq\varkappa$ there does not exist a solution of (3.1) in the form (3.5) that is continuous and has a continuous derivative with respect to x (continuity of the fluid flow) which satisfies the natural conditions of symmetry and vanishes at infinity. In order to see this, we substitute (3.5) into (3.1) and obtain for Φ the ordinary differential equation with discontinuous coefficient of the highest derivative

$$\begin{gathered}\varepsilon\frac{d^2\Phi}{d\xi^2}+\frac{1}{2}\frac{d}{d\xi}\xi\Phi=0\quad(0\leqslant\xi\leqslant\xi_0),\quad \varepsilon=\frac{\varkappa_1}{\varkappa},\\ \frac{d^2\Phi}{d\xi^2}+\frac{1}{2}\frac{d}{d\xi}\xi\Phi=0\quad(\xi_0\leqslant\xi<\infty),\end{gathered}\tag{3.6}$$

where the point $\xi=\xi_0$ corresponds to the vanishing of the quantity $d(\xi\Phi)/d\xi$ which, as is easily seen from (3.5), is proportional to the derivative $\partial_t u$. Integrating, we have

$$\begin{gathered}\varepsilon\frac{d\Phi}{d\xi}+\frac{1}{2}\xi\Phi=c_1\quad(0\leqslant\xi\leqslant\xi_0),\\ \frac{d\Phi}{d\xi}+\frac{1}{2}\xi\Phi=c_2\quad(\xi_0\leqslant\xi<\infty).\end{gathered}\tag{3.7}$$

By virtue of symmetry and the absence of influx at $x=0$ for times $t>0$, $d\Phi/d\xi=0$ for $\xi=0$, and as $\xi\to\infty$ the function $\xi\Phi$ tends to zero (the total amount of removed fluid is finite at each instant and Φ must be integrable). Hence $c_1=c_2=0$. Integrating the preceding equations we obtain

$$\begin{gathered}\Phi=c_3\exp\left(-\frac{\xi^2}{4\varepsilon}\right)\quad(0\leqslant\xi\leqslant\xi_0),\\ \Phi=c_4\exp\left(-\frac{\xi^2}{4}\right)\quad(\xi_0\leqslant\xi<\infty),\end{gathered}\tag{3.8}$$

where c_3 and c_4 are new constants. The condition of continuity of the function $u(x, t)$ and its derivative with respect to x reduces to the requirement of continuity of Φ and $d\Phi/d\xi$ at $\xi = \xi_0$, and from this and the previous equations we get a linear system of homogeneous algebraic equations for determining c_3 and c_4:

$$
\begin{aligned}
c_3 \exp\left(-\frac{\xi_0^2}{4\varepsilon}\right) &= c_4 \exp\left(-\frac{\xi_0^2}{4}\right),\\
c_3 \frac{\xi_0}{\varepsilon} \exp\left(-\frac{\xi_0^2}{4\varepsilon}\right) &= c_4 \xi_0 \exp\left(-\frac{\xi_0^2}{4}\right).
\end{aligned}
\tag{3.9}
$$

For $\varepsilon \neq 1$, i.e., $\varkappa_1 \neq \varkappa$, this system evidently has no nontrivial solution for any finite ξ_0, since its determinant is different from zero. Thus, it is proved that there exists no nontrivial solution in the form of (3.5) of the problem posed.

4. Resolution of the Paradox. Self-Similar Intermediate Asymptotics. Nonlinear Eigenvalue Problem

In order to resolve the paradox that has arisen, we note that the conditions (3.4) have a limiting character and are described by a generalized function. A solution satisfying those conditions, if it existed, would have to be an asymptotics for large times for the class of solutions satisfying the initial conditions (Fig. 3.1) described by ordinary smooth functions of the form

$$u(x, 0) = \frac{Q}{l} u_0\left(\frac{x}{l}\right), \tag{3.10}$$

where l is some length scale characterizing the size of the region from which fluid was removed at the initial instant, and

$$Q = \int_{-\infty}^{\infty} u(x, 0)\, dx, \qquad \int_{-\infty}^{\infty} u_0(\zeta)\, d\zeta = 1,$$

where $u_0(\zeta)$ is an even function that is dimensionless and smooth, and decreases monotonically rapidly (faster than any power) with the growth of the absolute value of its argument. For such initial conditions one can be certain that a solution of the Cauchy problem exists, is unique, and has continuous derivatives with respect to x up to the second order and a continuous derivative with respect to t; this follows from the general theorems proved by Kamenomostskaya (1957). However, a new dimensional governing parameter l has appeared in this problem, and the solution is no longer self-similar. In fact, the standard procedure, based on dimensional analysis and demonstrated above in several cases,

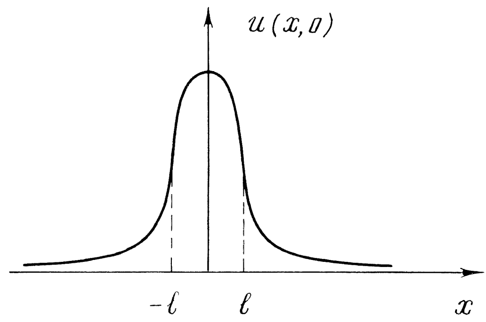


Figure 3.1. To resolve the paradox of the nonexistence of the required solution of (3.1), a non-self-similar problem with smooth initial data is considered, whose solution certainly exists.

gives here

$$u=\frac{Q}{\sqrt{\varkappa t}}F(\xi,\ \eta,\ \varepsilon);\quad \eta=\frac{l}{\sqrt{\varkappa t}}\,. \tag{3.11}$$

The self-similar exact special solution of instantaneous source type for the case $\varkappa_1 = \varkappa$ considered above corresponds to the singular initial condition obtained from (3.10) for $l = 0$. But a solution of instantaneous source type is broader than a simple exact special solution to a single problem. In fact (3.11), being valid also for $\varepsilon = 1$, shows that $\eta \to 0$ as $t \to \infty$. Choosing x appropriately, we can carry out this passage to the limit so that $\xi = x/(\varkappa t)^{1/2}$ remains constant; in the limit we obtain the known self-similar solution indicated above. Thus, as was already remarked, a self-similar solution to the problem with singular initial data is, for $\varkappa = \varkappa_1$, an asymptotics of a wide class of solutions of initial-value problems for large times. The nonexistence of a solution of the problem with singular initial data means that for $\varkappa \neq \varkappa_1$ the function $F(\xi, \eta, \varepsilon)$ has no limit as $\eta \to 0$ that is finite and nonzero. Nevertheless, there still exists a self-similar asymptotics of the solution (3.11). It turns out that there exists a real number α, depending on ε, such that the limit

$$\lim_{\eta\to 0}\eta^{-\alpha}F(\xi,\ \eta,\ \varepsilon)=f(\xi,\ \varepsilon)$$

exists and is finite and nonzero; the justification of this will be given below.

If so, then as $\eta \to 0$ we have for the function $F(\xi, \eta, \varepsilon)$ the asymptotic representation†

$$F(\xi,\ \eta,\ \varepsilon)=\eta^{\alpha}f(\xi,\ \varepsilon)+0(\eta^{\alpha}). \tag{3.12}$$

†The symbol $O(x)$ denotes, as usual, a quantity of order x; the symbol $o(x)$ denotes a quantity small compared with x.

Therefore as $t \to \infty$ the asymptotic form of the solution of the problem considered cannot be expressed in the form (3.5), but has the form

$$u = \frac{Ql^{\alpha}}{(\varkappa t)^{\frac{1+\alpha}{2}}} f(\xi, \varepsilon). \tag{3.13}$$

We now observe that the convergence of η to zero for finite ξ can be realized also by a passage to the limit as $l \to 0$ with fixed x and t. As is well known, in the classical case with $\varkappa_1 = \varkappa$ and $\varepsilon = 1$, such a passage to the limit again yields a solution of instantaneous source type. Equation (3.13) shows that if this passage to the limit is carried out leaving Q fixed, then for $\alpha \neq 0$ the limit of the solution will be equal to zero or infinity, depending on whether α is positive or negative. For $\alpha \neq 0$, in order to obtain, in the limit $l \to 0$ with x and t unchanged, the same limiting expression for the solution of the problem that is obtained for finite l and $t \to \infty$, it is necessary to proceed to the limit $l \to 0$ with Q simultaneously tending to infinity or to zero, depending on the sign of α, and moreover with the product Ql^{α} remaining finite. The self-similar solution obtained by such a passage to the limit does not have the form (3.5) but can be expressed in the form

$$u = \frac{A}{(\varkappa t)^{\frac{1+\alpha}{2}}} \Phi(\xi, \varepsilon), \quad A = \beta \lim_{l \to 0} Ql^{\alpha}; \quad x_0(t) = \xi_0 \sqrt{\varkappa t}\,. \tag{3.14}$$

Here β is a dimensionless constant that depends on the normalization of the function $\Phi(\xi, \varepsilon)$, and the parameter α is the remnant of the parameters Q and l after the limiting process. The parameter α can be determined by carrying out, for example by means of numerical calculation, the limiting passage from a solution of the non-self-similar problem to the self-similar asymptotics (see below). In direct construction of a self-similar solution by substitution of (3.14) into the basic equation and initial conditions, the parameter α is unknown and subject to determination. Thus, the determination of the parameter α—the remnant of the non-self-similar parameters Q and l—appears explicitly in the statement of the problem, constituting a part of the determination of the self-similar solution.

We now note that for the solution (3.14) the "moment"

$$\int_{-\infty}^{\infty} |x|^{\alpha} u(x, t)\, dx \tag{3.15}$$

is finite, different from zero, and invariant in time if the integral

$$\int_{-\infty}^{\infty} |\xi|^{\alpha} \Phi(\xi, \varepsilon)\, d\xi \tag{3.16}$$

is finite and nonzero. The solution (3.14) itself corresponds to singular initial data; however this singularity is no longer the classical delta function, as in the case $\varepsilon = 1$.

The function $\Phi(\xi, \varepsilon)$ is conveniently normalized by the relation

$$\Phi(0, \varepsilon) = 1. \tag{3.17}$$

Substituting (3.14) into (3.1), we obtain for the function $\Phi(\xi, \varepsilon)$ the ordinary equation with discontinuous coefficient of the highest derivative

$$\begin{gathered}\varepsilon \frac{d^2\Phi}{d\xi^2} + \frac{1}{2}\xi \frac{d\Phi}{d\xi} + \frac{1+\alpha}{2}\Phi = 0 \quad (0 \leqslant \xi \leqslant \xi_0),\\ \frac{d^2\Phi}{d\xi^2} + \frac{1}{2}\xi \frac{d\Phi}{d\xi} + \frac{1+\alpha}{2}\Phi = 0 \quad (\xi_0 \leqslant \xi < \infty).\end{gathered} \tag{3.18}$$

Here ξ_0 is the point at which $d^2\Phi/d\xi^2$ vanishes, or, what is by virtue of (3.18) the same, the relation

$$\xi \frac{d\Phi}{d\xi} + (1+\alpha)\Phi = 0 \tag{3.19}$$

is satisfied, where the quantity standing on the left side is proportional to $\partial_t u$. The function $\Phi(\xi, \varepsilon)$ is even by virtue of the natural symmetry of the solution. Because of the absence of influx at $x = 0$ for times $t > 0$, it satisfies the boundary condition

$$\frac{d\Phi(0, \varepsilon)}{d\xi} = 0. \tag{3.20}$$

Moreover, the function $\Phi(\xi, \varepsilon)$ must be continuous along with its first derivative with respect to ξ everywhere, and in particular for $\xi = \xi_0$. (We recall that this follows from the fact that the fluid pressure and flux are continuous.) A solution of the equation (3.18) with discontinuous coefficient of the highest derivative can be expressed simply in terms of well-known special functions: the so-called confluent hypergeometric functions or the parabolic cylinder functions related to them (Abramowitz and Stegun, 1964). For $0 \leqslant \xi \leqslant \xi_0$ a solution of (3.17) satisfying (3.20) has the form

$$\Phi = C \exp\left(-\frac{\xi^2}{8\varepsilon}\right)\left[D_\alpha\left(\frac{\xi}{\sqrt{2\varepsilon}}\right) + D_\alpha\left(-\frac{\xi}{\sqrt{2\varepsilon}}\right)\right], \tag{3.21}$$

where C is a constant and D_α is the symbol for the parabolic cylinder function. From (3.17) we obtain

$$C = \frac{1}{2D_\alpha(0)} = \frac{\Gamma((1-\alpha)/2)}{2^{1+\alpha/2}\sqrt{\pi}}.$$

For $\xi \geqslant \xi_0$ a solution of (3.18) for which the integral of (3.16) converges can be expressed in the form

$$\Phi = F \exp\left(-\frac{\xi^2}{8}\right) D_\alpha\left(\frac{\xi}{\sqrt{2}}\right), \tag{3.22}$$

where F is a constant; a second linearly independent solution decays like $\xi^{-\alpha-1}$ at infinity, and the integral (3.16) diverges.† Requiring that the condition

$$\xi \frac{d\Phi}{d\xi} + (1+\alpha)\Phi = 0 \qquad \text{for} \qquad \xi = \xi_0 \pm 0$$

be satisfied, and using the recursion relations for the derivatives of the parabolic cylinder functions and the expression for parabolic cylinder functions in terms of confluent hypergeometric functions (Abramowitz and Stegun, 1964), we obtain

$$D_{\alpha+2}\left(\frac{\xi_0}{\sqrt{2}}\right) = 0, \quad M\left(-\frac{\alpha}{2} - 1, \frac{1}{2}, \frac{\xi_0^2}{4\varepsilon}\right) = 0, \tag{3.23}$$

where $M(a, b, z)$ is the symbol for the confluent hypergeometric function. These equations must determine the dependence on ε of the parameter α and the quantity ξ_0. Further, the condition of continuity of the function Φ for $\xi = \xi_0$ determines the constant F:

$$\begin{aligned} F = C &\left[D_\alpha\left(\frac{\xi_0}{\sqrt{2\varepsilon}}\right) + D_\alpha\left(-\frac{\xi_0}{\sqrt{2\varepsilon}}\right)\right] \times \\ &\times \exp\left[\frac{\xi_0^2}{8}\left(1 - \frac{1}{\varepsilon}\right)\right]\left[D_\alpha\left(\frac{\xi_0}{\sqrt{2}}\right)\right]^{-1} = \\ = \frac{\Gamma((1-\alpha)/2)}{2^{1+\alpha/2}\sqrt{\pi}} &\left[D_\alpha\left(\frac{\xi_0}{\sqrt{2\varepsilon}}\right) + D_\alpha\left(-\frac{\xi_0}{\sqrt{2\varepsilon}}\right)\right] \times \\ &\times \exp\left[\frac{\xi_0^2}{8}\left(1 - \frac{1}{\varepsilon}\right)\right]\left[D_\alpha\left(\frac{\xi_0}{\sqrt{2}}\right)\right]^{-1}. \end{aligned} \tag{3.24}$$

By virtue of (3.19) the requirement of continuity of the derivative $d\Phi/d\xi$ is satisfied automatically.

Thus, assuming the existence of a self-similar intermediate asymptotics to the solution of the original non-self-similar initial-value problem with the con-

†Like the solution of the initial-value problem for the classical equation of heat conduction, the solution of the present problem must decrease at infinity faster than any power of x so that the integral (3.16) will converge.

dition (3.10), we have arrived at the classical situation of a nonlinear eigenvalue problem [nonlinear because the coordinate ξ_0 of the point of discontinuity of the coefficient of the highest derivative in (3.18) is unknown in advance and must be found in the course of solving the problem]. For arbitrary α the basic equation (3.18) does not have a solution of the required smoothness. However if the system (3.23) is solvable, then for α satisfying (3.23) the solution satisfies all the requirements.

To complete our investigation of the solution, it remains to elucidate the solvability of the system of transcendental equations (3.23) that determine α and ξ_0. Solving the first equation with respect to $\xi_0/2^{1/2}$, we obtain the monotonically increasing function of α represented in Fig. 3.2a by curve 1. Solving the second equation with respect to $\xi_0/(2\varepsilon)^{1/2}$ we get the monotonically decreasing function of α represented in Fig. 3.2a by curve 2. For any given ε the corresponding dependence of $\xi_0/2^{1/2}$ on α is obtained by simple extension or contraction of curve 2 along the vertical axis. For $\varepsilon = 1$, i.e., $\varkappa_1 = \varkappa$, curves 1 and 2 intersect at $\alpha = 0$, in accord with the known results for the classical case, giving the coordinate of the point of inflection of the function $\Phi(\xi, 1) = \exp(-\xi^2/4)$, $\xi_0 = 2^{1/2}$. For $\varepsilon \neq 1$, the point of intersection of curves 1 and 2 is unique, and the corresponding variation of α with ε is shown in Fig. 3.2b. It is evident that the quantity α is positive for $\varepsilon > 1$ and negative for $\varepsilon < 1$. We note that for large ε curves 1 and 2 of Fig. 3.2a have other branches also. The second branch of curve 1 starts at the point $\xi_0 = 0$, $\alpha = 1$ and, increasing

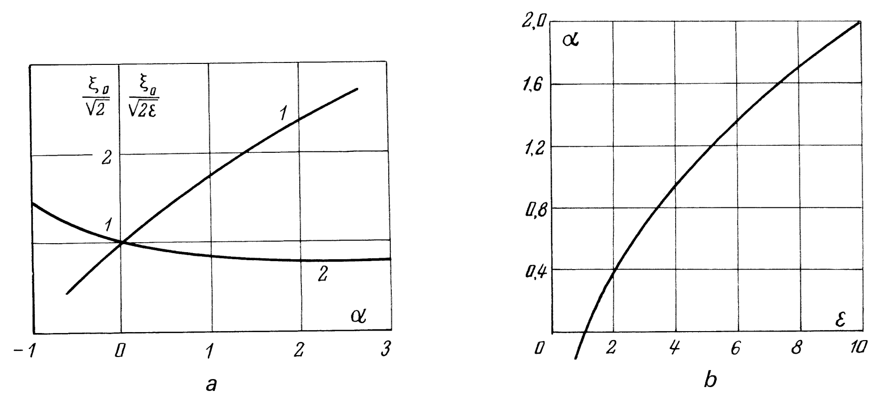


Figure 3.2a. Investigation of the solvability of the system of transcendental equations (3.23). The dependence of $\xi_0/2^{1/2}$ on α, determined from the first equation (curve 1) and the dependence of $\xi_0/(2\varepsilon)^{1/2}$ on α, determined from the second equation (curve 2). Curve 1 is monotonically increasing, curve 2 is monotonically decreasing; the intersection point of the curves exists and is unique.

Figure 3.2b. Dependence of the eigenvalue α on the ratio $\varepsilon = \varkappa_1/\varkappa$. For $\varepsilon < 1$, α is negative; for $\varepsilon > 1$, α is positive. For $\varepsilon = 1$ (the classical linear equation of heat conduction), $\alpha = 0$.

monotonically, goes below the first branch. The second branch of curve 2 is two-valued and is situated above the first branch of this curve. The points corresponding to intersections of curve 1 with these branches, which exist for sufficiently large ϵ, are physically unreal.

Thus, if in fact an asymptotics in our problem is self-similar, then the construction of an asymptotic representation of a solution reduces to the solution of a nonlinear eigenvalue problem. A solution of this problem determines the self-similar asymptotics only to within the constant $A = \beta Q l^{\alpha}$ or, what is the same, to within the dimensionless constant β. In the classical case $\varepsilon = 1$, when $\alpha = 0$ this constant (cf. Introduction) is found from the integral conservation law

$$\int_{-\infty}^{\infty} u(x, t)\, dx = \int_{-\infty}^{\infty} u(x, 0)\, dx = Q, \tag{3.25}$$

which is valid also for non-self-similar motions. This conservation law does not hold for $\varkappa_1 \neq \varkappa$ ($\varepsilon \neq 1$), being replaced by the nonintegrable relation

$$\frac{d}{dt}\int_{-\infty}^{\infty} u(x, t)\, dx = -2(\varkappa_1 - \varkappa)(\partial_x u)_{x = x_0(t)} \neq 0, \tag{3.26}$$

which is easy to obtain if one integrates (3.1) with respect to x from $x = -\infty$ to $x = \infty$ and takes into account the fact that $\varkappa$ suffers a discontinuity at $x = \pm x_0(t)$. Hence one cannot define the constant A from the initial conditions using the integral conservation law; the constant A is a more complicated functional of the initial pressure distribution, i.e., the function $u(x, 0)$. We note that if in place of $u(x, 0)$ one takes as initial distribution the function $u(x, t_1)$ corresponding to any moment of time $t = t_1 > 0$, then the constant A is unchanged, so in this sense A is an "integral" of (3.1).

The self-similar asymptotics is no longer a solution to the problem of an instantaneous point source. In fact, the amount of fluid Q that must be removed at the initial instant from a region with characteristic length l must change as this length decreases if one wants to get one and the same limiting representation of the solution for large t; Q increases for $\varepsilon > 1$ and decreases for $\varepsilon < 1$, in such a way that the product $Q l^{\alpha}$ is constant.

5. Numerical Calculations

It would be very important to have a rigorous proof that the solution of any problem with initial conditions of the form (3.10) for a function $u(x, 0)$ that decreases sufficiently rapidly at infinity (say even vanishing identically for sufficiently large values of the argument) tends for large t to the constructed self-similar asymptotics, i.e., that as $t \to \infty$ $u(x, t)$ actually converges to a self-similar solution of the form (3.14) with the necessary α. Thus the validity would be proved of the fundamental assumption we have made, that there exists a number α such that the limit

$$\lim_{\eta \to 0} \eta^{-\alpha} F(\xi, \eta, \varepsilon) = f(\xi, \varepsilon)$$

exists and is finite and nonzero. So far this has not been done analytically. In order to see that the self-similar solution (3.14) is actually an asymptotic representation of the solutions of a sufficiently wide class of Cauchy problems of the form (3.10), and to determine the quantity A for $\varepsilon \neq 1$, we can only offer at the present time the numerical calculation of a problem starting from non-self-similar initial conditions.

Such a calculation was performed by V. M. Uroev. Namely, he obtained a solution to the non-self-similar initial-value problem for (3.1) with $\varkappa = 1$ and $\varkappa_1 = \varepsilon$ and various initial conditions for various values of ε. As a result of the calculations it turned out that the solution rather rapidly approaches the self-similar asymptotics

$$u = \frac{A}{(\varkappa t)^{\frac{1+\alpha}{2}}} \Phi\left(\frac{x}{\sqrt{\varkappa t}}, \varepsilon\right),$$

where $\alpha = 0.397$ for $\varepsilon = 2$, for example, and $\alpha = 0.948$ for $\varepsilon = 4$, which agrees quite well with the values of α obtained by solving the nonlinear eigenvalue problem.

Figure 3.3 shows values of the function $u(x, t)(\varkappa t)^{(1+\alpha)/2}/A$ for various values of the time t with $\varepsilon = 2$ and the initial conditions $u(x, 0) = 10$ $(0 \leqslant x \leqslant 0.1)$ and $u(x, 0) \equiv 0$ $(0.1 \leqslant x < \infty)$. The quantity A and the exponent α were determined in a preliminary way (Fig. 3.4) by the asymptotics for large t of the law of attenuation of the maximum, which turned out to be a power: $u_{\max} = A(\varkappa t)^{-(1+\alpha)/2}$. It is evident that with increasing t the solution converges rapidly to the self-similar one. In Fig. 3.3 points are also shown that are the values of the function $\Phi(\xi, \varepsilon)$, obtained by solving the nonlinear eigenvalue problem. As was expected, these points fall well on the curve of $u(x, t)(\varkappa t)^{(1+\alpha)/2}/A$ that corresponds to $t \to \infty$. Thus numerical calculation confirms the assumption made about the character of the self-similar asymptotics of the solution of the problem considered, and allows one to compute the constants A and α and to make a comparison of the second of these constants with the eigenvalue determined from the nonlinear eigenvalue problem. It turns out that, in complete accord with the assumption made, varying the initial conditions alters the constant A but leaves the constant α unchanged.

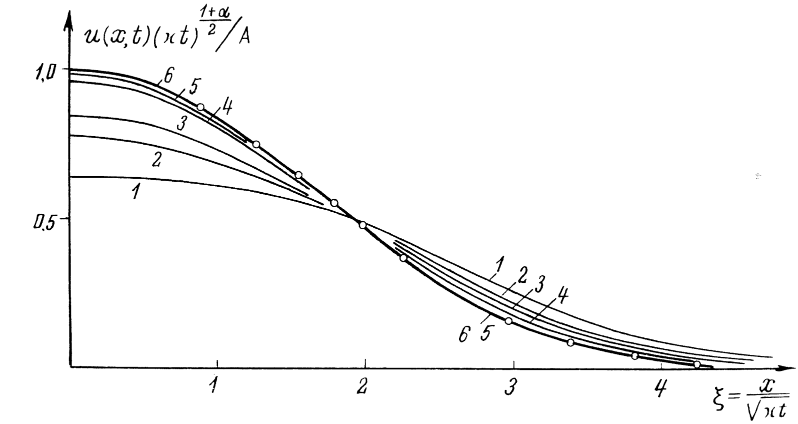


Figure 3.3. Transition to a self-similar intermediate asymptotics of the solution to the non-self-similar problem of (3.1) with $\varepsilon = 2$ *and initial data* $u(x, 0) = 10$ $(0 \leqslant x \leqslant 0.1)$, $u(x, 0) \equiv 0$, $(x > 0.1)$. *Curves 1-6 correspond respectively to* $t = 0.001, 0.002, 0.003, 0.015, 0.040, 0.225$ *and all greater values. The points shown are the values of the function determined by solving the nonlinear eigenvalue problem.*

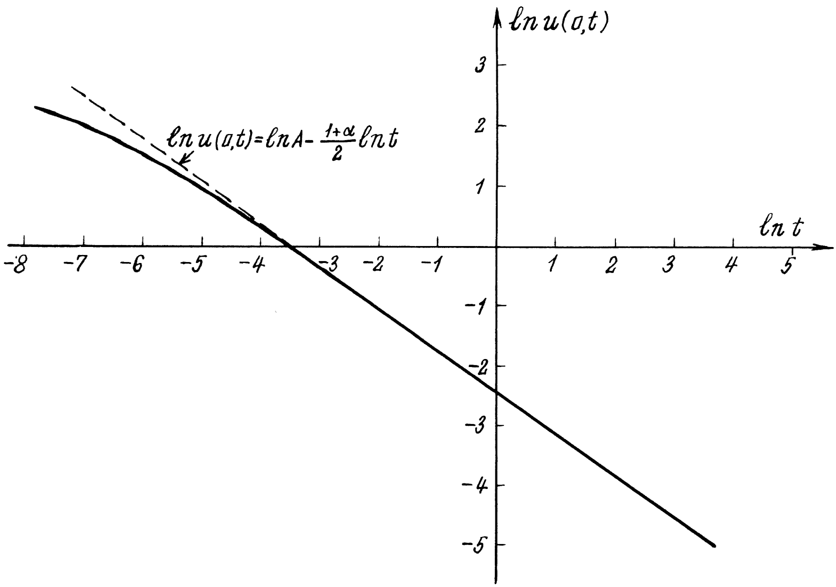


Figure 3.4. Determination of the parameters A and α *of the self-similar intermediate asymptotics from the law of decay of the maximum.*

We make an important comment on similarity laws. The solution obtained gives for the coordinate of the point of discontinuity of the thermal diffusivity

$$x_0(t) = \xi_0 \sqrt{\varkappa t}\,, \tag{3.27}$$

and for the change in the value of u at the maximum point

$$u_{\max} = \frac{A}{(\varkappa t)^{\frac{1+\alpha}{2}}}\,. \tag{3.28}$$

The first of these relations is obtained easily from "naive" considerations of similarity, i.e., by applying dimensional analysis proceeding from the concept of an instantaneous point source. For the second relation, this is impossible to do in principle, despite the fact that the similarity law (3.28) has a power form and is completely determined by the dimensions of the quantity A. The point is that the dimensions of the quantity A are unknown in advance, and to determine them it is necessary to solve the nonlinear eigenvalue problem formulated above.

The problem presented above was considered in the paper of Barenblatt and Sivashinskii (1969).

4

The Problem of a Strong Explosion with Energy Loss or Deposition at the Front of the Shock Wave and the Problem of an Impulsive Load: Self-Similar Solutions of the Second Kind

1. Statement of the Modified Problem of a Strong Explosion

We now make what seems at first an insignificant modification in the problem of a strong explosion that was considered in Chapter 2. We assume that at the front of the strong shock wave there occurs for some reason a loss of energy (for example, due to radiation), or an influx of energy (for example, due to chemical reaction). In this case the flux of energy at the front is not conserved, and the equation of energy balance at the shock front assumes the form

$$\rho_f(v_f - D)\left[\frac{\gamma}{\gamma-1}\frac{p_f}{\rho_f} + \frac{(v_f - D)^2}{2}\right] - \rho_f(v_f - D)\,\varepsilon = -\frac{\rho_0 D^3}{2}.$$

Here ε is the intensity of loss ($\varepsilon < 0$) or deposition ($\varepsilon > 0$) of energy in unit time in a unit mass of gas passing through the front. Here, as before, p is the pressure, ρ the density, v the speed of the gas, and D the speed of propagation of the shock wave, and subscript f denotes quantities just behind the wave front, i.e., for $r = r_f - 0$. We also assume as before that ahead of the wave the gas is at rest at zero pressure and has density ρ_0.

Performing the same transformations as were used in Chapter 2 in considering an ordinary strong explosion, we write the equation of energy balance across the front in the form

$$\rho_f(v_f - D)\left[\frac{p_f}{(\gamma-1)\rho_f} + \frac{v_f^2}{2}\right] + p_f v_f - \rho_f(v_f - D)\,\varepsilon = 0. \quad (4.1)$$

In the model problem considered it is assumed that the intensity of energy loss or deposition in a unit mass is proportional to the temperature:

$$\varepsilon = \tilde{A} T_f = C p_f / \rho_f,$$

where $\tilde{A}$ and C are constants. (This is necessary in order that the asymptotics obtained be self-similar.) We emphasize immediately that our concern here is more with a mathematical model than with a completely adequate analysis of the physical phenomenon. It is convenient to introduce the new notation

$$C = \frac{\gamma_1 - \gamma}{(\gamma_1 - 1)(\gamma - 1)}.$$

For $\gamma_1 = 1$ we have $C = -\infty$, which means that all the thermal energy of the gas particles is absorbed at the front. For γ_1 increasing from unity to γ the constant C grows from $-\infty$ to zero, and the fraction of lost energy decreases. The case $\gamma_1 = \gamma$ corresponds to the absence of energy loss or deposition–an ordinary strong explosion. For $\gamma_1 > \gamma$ there is a deposition of energy at the wave front.

Using the expression assumed for ε and the new notation we can rewrite (4.1) in the form

$$\rho_f (v_f - D) \left(\frac{p_f}{(\gamma_1 - 1)\rho_f} + \frac{v_f^2}{2} \right) + p_f v_f = 0, \tag{4.2}$$

i.e., in the same form as for the ordinary problem of a strong explosion, but with a different adiabatic exponent: in place of the adiabatic exponent γ in (4.2) there appears γ_1, the effective adiabatic exponent at the front, taking into account the loss or deposition of energy. The conditions of continuity of mass and momentum flux at the wave front, just as in the ordinary problem of a strong explosion, assume the form

$$\rho_f (v_f - D) = -\rho_0 D, \quad \rho_f (v_f - D) v_f + p_f = 0, \tag{4.3}$$

and these conditions do not contain the adiabatic exponent γ. The equations of gas motion in the region of continuous motion also remain unchanged:

$$\begin{aligned} &\partial_t v + v\,\partial_r v + \frac{1}{\rho}\,\partial_r p = 0, \\ &\partial_t \rho + \partial_r (\rho v) + \frac{2}{r}\,\rho v = 0, \\ &\partial_t (p/\rho^\gamma) + v\,\partial_r (p/\rho^\gamma) = 0. \end{aligned} \tag{4.4}$$

The condition of absence of influx of matter and energy at the center of the explosion for $t > 0$ also preserves its form:

$$v(0, t) = 0. \tag{4.5}$$

Thus we have obtained almost the same problem as earlier, with the one difference that the adiabatic exponent is different in the conditions at the front of the wave and in the equations describing the motion of the gas in the region of continuous motion.

2. Direct Application of Dimensional Analysis to the Modified Problem of a Strong Point Explosion

We now attempt, just as before, to construct a self-similar solution of this problem, corresponding to the instantaneous release of a finite amount of energy E at a point–the center of the explosion. It would seem that nothing in our arguments has to change. In fact, the only new governing parameter in the problem we are considering compared with the ordinary problem of a strong explosion is the constant dimensionless parameter γ_1, so that the dimensional considerations remain as before, and at first glance it would seem that the desired solution must, for the same reason as in Chapter 2, be representable in the form

$$p = \rho_0 \frac{r^2}{t^2} P(\xi, \gamma, \gamma_1), \quad \rho = \rho_0 R(\xi, \gamma, \gamma_1), \quad v = \frac{r}{t} V(\xi, \gamma, \gamma_1),$$
$$r_f(t) = \xi_0(\gamma, \gamma_1)(Et^2/\rho_0)^{1/5}. \tag{4.6}$$

However, there exists no solution of our new problem for $\gamma_1 \neq \gamma$ having the form (4.6) in a reasonable class of functions. In order to see this, we note, first of all, that the functions P, V, and R must satisfy the same system of ordinary differential equations as in the ordinary problem of a strong explosion, since the equations of gas motion in the region of continuous motion are unchanged, and the form of self-similar solution we seek remains essentially the same.

We now show by a method close to that by which the corresponding consideration in the ordinary problem of a strong explosion was carried out in the books of Sedov (1959) and Landau and Lifschitz (1959), that a solution of this system satisfying the condition (4.5) of absence of energy influx at the center for $t > 0$ satisfies the relation

$$P = \frac{\gamma - 1}{2\gamma} \frac{RV^2(V - 2/5)}{(2/5\gamma - V)} \tag{4.7}$$

for all ξ in the interval $0 \leqslant \xi \leqslant \xi_0$. If $r_* = \xi(Et^2/\rho_0)^{1/5}$ and if p, ρ, and v are given by (4.6), then as can be verified by simple substitution, the energy contained in a sphere of radius $r_*(t)$, equal to

$$4\pi \int_0^{r_*} r^2 \left(\frac{\rho v^2}{2} + \frac{p}{\gamma - 1}\right) dr,$$

is constant in time. We differentiate this expression with respect to time and equate the result to zero, noting beforehand that $dr_*(t)/dt = 2r_*/5t$ and that by virtue of the equation of conservation of energy in the case of spherical symmetry one has the relation

$$\partial_t \left(\rho \frac{v^2}{2} + \frac{p}{\gamma - 1}\right) = -\frac{1}{r^2} \partial_r r^2 \left(\rho \frac{v^2}{2} + \frac{\gamma}{\gamma - 1} p\right) v.$$

We have

$$\begin{aligned}
&\frac{d}{dt} 4\pi \int_0^{r_*(t)} r^2 \left(\rho \frac{v^2}{2} + \frac{p}{\gamma - 1}\right) dr = \\
&= 4\pi r_*^2 \left(\frac{\rho v^2}{2} + \frac{p}{\gamma - 1}\right)_{r = r_*(t)} \frac{dr_*}{dt} + \\
&+ 4\pi \int_0^{r_*(t)} r^2 \partial_t \left(\frac{\rho v^2}{2} + \frac{p}{\gamma - 1}\right) dr = \\
&= 4\pi r_*^2 \left(\frac{\rho v^2}{2} + \frac{p}{\gamma - 1}\right)_{r = r_*(t)} \frac{2r_*}{5t} - \\
&- 4\pi r_*^2 \left(\frac{\gamma p}{\gamma - 1} + \frac{\rho v^2}{2}\right)_{r = r_*(t)} v = 0.
\end{aligned} \tag{4.8}$$

In transforming the last integral it is essential to make use of the condition of absence of energy influx at the origin for $t > 0$,

$$\lim_{r \to 0} r^2 \left(\frac{\gamma p}{\gamma - 1} + \frac{\rho v^2}{2}\right) v = 0.$$

Substituting (4.6) into (4.8), we get (4.7).

Further, the conditions (4.2) and (4.3) at the front are, just as in the ordinary problem of a strong explosion (Chapter 2), reduced to the form

$$p_f = \frac{2}{\gamma_1 + 1} \rho_0 D^2, \quad v_f = \frac{2}{\gamma_1 + 1} D, \quad \rho_f = \frac{\gamma_1 + 1}{\gamma_1 - 1} \rho_0. \tag{4.9}$$

By virtue of (4.6), $D = dr_f/dt = 2r_f/5t$. Substituting the remaining expressions (4.6) into (4.9), we reduce the conditions at the front to the prescription of boundary values of the functions P, V, and R at the front $\xi = \xi_0$:

$$P(\xi_0, \gamma, \gamma_1) = \frac{8}{25(\gamma_1 + 1)}, \quad V(\xi_0, \gamma, \gamma_1) = \frac{4}{5(\gamma_1 + 1)},$$
$$R(\xi_0, \gamma, \gamma_1) = \frac{\gamma_1 + 1}{\gamma_1 - 1}, \tag{4.10}$$

so that the point (4.10) must lie on the integral curve sought. On the other hand, the integral curve must lie on the surface determined by (4.7). It is easy to see, however, that these requirements are incompatible, since for $\gamma_1 \neq \gamma$, the point (4.10) does not lie on the surface (4.7): relations (4.7) and (4.10) are incompatible. The contradiction thus obtained proves the nonexistence for $\gamma_1 \neq \gamma$ of a nontrivial solution having the form (4.6). We note that one can also see this from other and simpler considerations. If the solution had the form (4.6), the total energy of the gas in the perturbed domain $\mathfrak{E}$ would be constant, which is proved in exactly the same way as earlier [cf. relation (2.27)]. However for $\gamma_1 \neq \gamma$ the total energy of the gas in the perturbed domain must vary because of the loss or deposition of energy at the front of the wave:

$$\frac{d\mathfrak{E}}{dt} = -4\pi r_f^2 \rho_f (v_f - D)\,\varepsilon = -4\pi r_f^2 \rho_0 D \frac{\gamma - \gamma_1}{(\gamma - 1)(\gamma_1 - 1)} \frac{p_f}{\rho_f} \neq 0.$$

3. Resolution of the Paradox. Intermediate Asymptotics

Again, as in the analogous situation of the previous chapter, we deviate from an exact formulation of the degenerate self-similar problem. We recall that a solution corresponding to a point explosion makes sense only if it is an intermediate asymptotics for a solution corresponding to the initial release of energy in a small but finite domain. Hence we turn to consideration of the problem in which the energy at time $t = 0$ is released not at a point, but a sphere of radius R_0; otherwise both problems coincide.

In this case, we must add R_0 to the governing parameters of the problem, so that there appears not one but two dimensionless independent variables,

$$\xi = r\left(\frac{Et^2}{\rho_0}\right)^{-1/5}, \quad \eta = R_0\left(\frac{Et^2}{\rho_0}\right)^{-1/5} \tag{4.11}$$

and according to dimensional analysis the speed, density, and pressure of the gas are expressed in the form

$$
\begin{aligned}
v &= \frac{r}{t} V(\xi, \eta, \gamma, \gamma_1), \\
p &= \rho_0 \frac{r^2}{t^2} P(\xi, \eta, \gamma, \gamma_1), \\
\rho &= \rho_0 R(\xi, \eta, \gamma, \gamma_1).
\end{aligned} \tag{4.12}
$$

For $\gamma = \gamma_1$ the solution to the problem of a strong point explosion is on the one hand a solution of a singular limiting problem corresponding to $R_0 = 0$, and on the other hand the limit of the solution (4.12) as $t \to \infty$. For $\gamma_1 \neq \gamma$, as we have just explained, there exists no nontrivial solution of the problem corresponding to $R_0 = 0$. However, we are interested not in a solution of the limiting problem, but in an asymptotic representation for large t of the solution of the non-self-similar problem with $R_0 \neq 0$. For increasing t and fixed r, both ξ and η tend to zero. We shall see that for ξ and η tending separately to zero there exists no finite limit of the functions P, V, and R. It turns out, however, that there exists a positive number β, depending on γ and γ_1, such that as $\xi, \eta \to 0$ the leading terms in the asymptotic representations of the functions P, V, and R have the form

$$
\begin{aligned}
P &= P\left(\frac{\xi}{\eta^\beta}\right), \\
V &= V\left(\frac{\xi}{\eta^\beta}\right), \\
R &= R\left(\frac{\xi}{\eta^\beta}\right).
\end{aligned} \tag{4.13}
$$

(The justification for this will be given below.) If this is so, then the limiting motion is self-similar, because $\xi/\eta^\beta = r/Bt^\alpha$, where $B = (E/\rho_0)^{(1-\beta)/5} R_0^\beta$ and $\alpha = (2/5)(1-\beta)$. The class of self-similar solutions of the gas-dynamic equations to which the limiting solution (4.13) of this problem belongs was indicated by Bechert (1941) and Guderly (1942), and considered later by Sedov (1945) and other authors.

For later analysis it is convenient to renormalize the self-similar variable and take it in the form

$$
\zeta = \text{const}\,\frac{\xi}{\eta^\beta} = r\left(\frac{At^2}{\rho_0}\right)^{-\frac{1-\beta}{5}}, \quad A = \sigma E R_0^{\frac{5\beta}{1-\beta}}, \tag{4.14}
$$

where the constant parameter σ is chosen so that at the front of the shock wave the value of the self-similar variable will be equal to unity, $\zeta = 1$. Then the asymptotic law of propagation of the shock wave can be written in the form

$$r_f = \left(\frac{A}{\rho_0}\right)^{\frac{1-\beta}{5}} t^{\frac{2(1-\beta)}{5}} \tag{4.15}$$

The variables ξ and η can also be made to tend to zero in another way: for fixed r and t, E tends to infinity or zero, and R_0 to zero. However, in order to obtain the same asymptotics as for our non-self-similar solution at large times, the product $ER_0^{5\beta/(1-\beta)}$ must remain constant:

$$ER_0^{\frac{5\beta}{1-\beta}} = \text{const.} \tag{4.16}$$

Thus for $\gamma_1 \neq \gamma$ the self-similar limiting solution corresponds not to a point explosion, i.e., not to the release of a finite amount of energy at the initial instant at the center of the explosion, but to the release within a finite region of radius R_0 of an amount of energy E that tends to zero or infinity, depending on the sign of β, as $R_0 \to 0$.

For given γ and γ_1, we can determine the parameter β or, what is the same thing, α, in either of two ways. First, we can follow, for example numerically, the evolution of a non-self-similar solution of the original problem up to its transition to a self-similar asymptotics. Second, we can use the fact that the self-similar asymptotics itself is a solution of the gas-dynamic equations satisfying certain conditions, and attempt to construct that solution and simultaneously to determine the exponent α.

We choose the second way to begin with. Thus we seek the desired limiting solution in the form

$$p = \rho_0 \frac{r^2}{t^2} P(\zeta, \gamma, \gamma_1), \quad \rho = \rho_0 R(\zeta, \gamma, \gamma_1), \\ v = \frac{r}{t} V(\zeta, \gamma, \gamma_1). \tag{4.17}$$

We substitute this solution into (4.4). We obtain, following the usual technique (Guderley, 1942; Sedov, 1959) one first-order equation,

$$\frac{dz}{dV} = -\frac{z}{V-\alpha}\left\{\frac{[2(V-1)+3(\gamma-1)V][(V-\alpha)^2 - z]}{[(3V-\varkappa)z - V(V-1)(V-\alpha)]} + \gamma - 1\right\}, \tag{4.18}$$

where

$$\varkappa = \frac{2(1-\alpha)}{\gamma}, \quad z = \frac{\gamma P}{R},$$

and two other first-order equations:

$$\frac{d\ln\zeta}{dV}=\frac{z-(V-\alpha)^2}{V(V-1)(V-\alpha)+(\varkappa-3V)z}, \tag{4.19}$$

$$(V-\alpha)\frac{d\ln R}{d\ln\zeta}=-3V-\frac{V(V-1)(V-\alpha)+(\varkappa-3V)z}{z-(V-\alpha)^2}. \tag{4.20}$$

Thus if the necessary solution of (4.18) is constructed, the solution of (4.19) and (4.20) is reduced to quadratures.

It is essential here that the desired solution of (4.18) must pass through two points: the image of the shock-wave front and the image of the center of symmetry. Substituting (4.17) and (4.15) into (4.9), we find

$$P(1,\gamma,\gamma_1)=\frac{2\alpha^2}{\gamma_1+1},\quad V(1,\gamma,\gamma_1)=\frac{2\alpha}{\gamma_1+1},$$
$$R(1,\gamma,\gamma_1)=\frac{\gamma_1+1}{\gamma_1-1}. \tag{4.21}$$

Hence the image of the front in the V, z plane will be the point

$$V=\frac{2\alpha}{\gamma_1+1},\qquad z=\frac{2\alpha^2\gamma(\gamma_1-1)}{(\gamma_1+1)^2}. \tag{4.22}$$

The image of the center of symmetry $\zeta=0$ in the V, z plane will be a singular point of equation (4.18) of saddle type,

$$V=\frac{2(1-\alpha)}{3\gamma},\qquad z=\infty. \tag{4.23}$$

Here we use the condition of no influx of matter or energy at the center of the explosion for $t>0$. Furthermore the self-similar variable ζ must increase monotonically from zero to one in the course of moving from the image of the center of symmetry to the image of the front. In general it is impossible to satisfy these conditions for arbitrary α; we cannot pass a solution of a first-order equation through two arbitrary points. We shall see, however, that there exists one exceptional value of α for which this is possible. Thus we have again arrived at a nonlinear eigenvalue problem: to construct a solution of the first-order equation (4.18), passing through the two points (4.22) and (4.23), and to determine the value of the parameter α for which such a solution exists—that is, the eigenvalue of the problem.

4. Qualitative Investigation of a Nonlinear Eigenvalue Problem

We consider the phase portrait–the picture of the integral curves in the first quadrant–which is the part of the V, z plane of interest to us. In the case $\gamma < 2$, $1 \leqslant \gamma_1 \leqslant 2\gamma + 1$, the phase portrait is shown in Fig. 4.1, where the curves numbered 1, 2, and 3 correspond respectively to the equations

$$z = -\gamma V(V - \alpha) \tag{4.24}$$

(which is the locus of points of the shock front), and

$$\begin{gathered} z = (V - \alpha)^2, \\ z = V(V - 1)(V - \alpha)(3V - \varkappa)^{-1}. \end{gathered} \tag{4.25}$$

The points of intersection of the curves (4.25) are singular points of (4.18). For $\gamma_1 < 2\gamma + 1$ all these singular points are situated below the curve (4.24). One can show, using standard techniques of the qualitative theory of differential equations (which will be demonstrated in detail in Chapter 6 on another example), that for such γ_1 one can find an α and moreover for each pair γ, γ_1 only one, such that point M, which is the image of the shock front, and point N, which is the image of the center of symmetry, lie on one integral curve–the separatrix of two families of integral curves–the single curve passing through

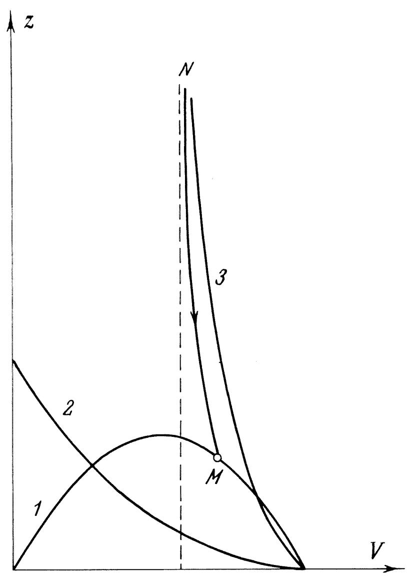


Figure 4.1. Phase portrait: picture of the integral curves of the first-order equation (4.18) for $1 < \gamma_1 < 2\gamma + 1$. The number 1 denotes the curve $z = -\gamma V(V - \alpha)$, the numbers 2 and 3 denote the curves $z = (V - \alpha)^2$ and $z = V(V - 1)(V - \alpha)(3V - \kappa)^{-1}$.

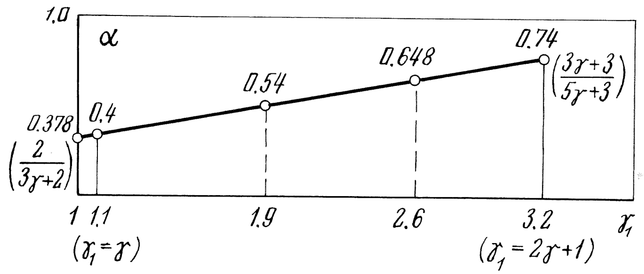


Figure 4.2. Dependence of the eigenvalue α on the effective adiabatic exponent γ_1 at the wave front for $\gamma = 1.1$ For $\gamma_1 < \gamma$, $\alpha < 2/5$; for $\gamma_1 > \gamma$, $\alpha > 2/5$; for $\gamma_1 = \gamma$ (adiabatic strong explosion), $\alpha = 2/5$; for $\gamma_1 = 2\gamma + 1$, $\alpha = (3\gamma + 3)/(5\gamma + 3)$ (detonation with variable speed of propagation of the detonation wave; the Chapman-Jouget condition is satisfied).

the image of the center of symmetry, which is a singular point of saddle type for (4.18). The graph of the function $\alpha(\gamma_1)$ is shown for $\gamma = 1.1$ in Fig. 4.2; it is a monotonically increasing curve passing through the points

$$\alpha = \frac{2}{3\gamma+2}, \quad \gamma_1 = 1;$$
$$\alpha = \frac{2}{5}, \quad \gamma_1 = \gamma; \tag{4.26}$$
$$\alpha = \frac{3\gamma+3}{5\gamma+3}, \quad \gamma_1 = 2\gamma + 1.$$

For $\gamma_1 > 2\gamma + 1$ the singular point (of nodal type) is located above the curve (4.24) as shown in Fig. 4.3, and one consequently has an interval of possible values of α. It will be shown that for such γ_1 the solution of the original non-self-similar problem also becomes nonunique. Hence to isolate a unique solution for such γ_1 requires an additional condition for the non-self-similar problem too.

The points (4.26) are of special interest. The first of them corresponds to motion with complete loss of thermal energy at the front of the wave; the compression at the front (the ratio of the density just behind the front to the original density of the gas) obtained in this case is infinite, the relative speed of the gas and the front is equal to zero. The second point corresponds to an ordinary strong explosion. The third point is very curious; it corresponds to an influx of energy at the front and satisfaction of the so-called Chapman–Jouget condition: the gas speed relative to the front is equal to the local speed of sound. The speed of sound at the front of the shock wave is in fact equal to

$$\sqrt{\gamma p_f/\rho_f} = D\sqrt{2\gamma(\gamma_1 - 1)}/(\gamma_1 + 1),$$

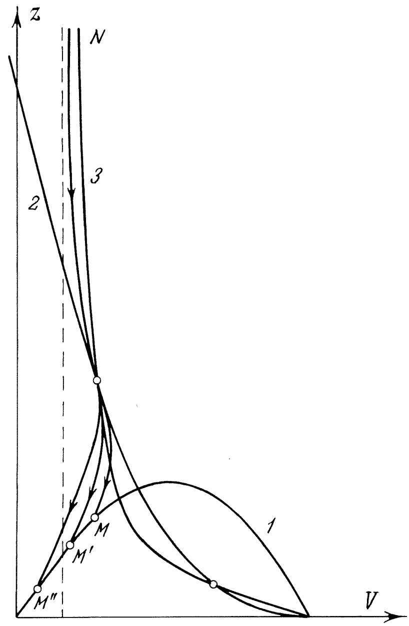


Figure 4.3. Phase portrait: picture of the integral curves of (4.18) for $\gamma_1 > 2\gamma + 1$. A singular point of nodal type is located above curve 1.

where D is, as before, the speed of the front. Now the speed of the gas relative to the front is given by $|v_f - D| = (\gamma_1 - 1)D/(\gamma_1 + 1)$. For $\gamma_1 = 2\gamma + 1$ these two speeds coincide, and the image of the front in the V, z plane coincides with a singular point of nodal type, lying on the curve (4.24) for this γ_1, so that the integral curve sought in this case joins two singular points of (4.18)–a saddle point, which is the image of the center of symmetry, and a nodal point, which is the image of the front.

Thus, for energy loss at the front ($\gamma_1 < \gamma$) and small energy release at the front ($\gamma < \gamma_1 < 2\gamma + 1$) the solution under investigation differs rather slightly in its characteristics from the solution corresponding to an ordinary strong explosion ($\gamma_1 = \gamma$): a singular point of nodal type is situated under the locus of points of the front, the motion in the perturbed domain is everywhere subsonic, etc. For $\gamma_1 = 2\gamma + 1$ the solution constructed represents a motion of detonation type, but with variable speed of propagation of the front

$$r_f = At^{\frac{3\gamma+3}{5\gamma+3}};$$

the speed of sound is achieved at the front of the wave, a node being the image of the front. Here the influx of energy at the front and the temperature at the front turn out to depend on the time. It is remarkable that for $\gamma_1 = 2\gamma + 1$ the solution obtained is not uniquely determined, so the usual solution to the problem of a spherical detonation wave (Zel'dovich, 1942; Zel'dovich and Kompaneets, 1960), corresponding to $\alpha = 1$, i.e., to constant speed of propagation of the wave, constant influx of energy at the front, and constant temperature at the front, also satisfies all the conditions of the problem posed. For $\gamma_1 > 2\gamma + 1$ the speed of sound at the front of the shock wave becomes smaller than the speed of the gas relative to the front. Hence for $\gamma_1 > 2\gamma + 1$ one should give another supplementary condition at the front of the shock wave, and without such a condition the solution of the problem formulated above becomes nonunique.

Thus, a self-similar solution is constructed which can be an intermediate asymptotics of the solution of the original non-self-similar problem. Only the constant A remains undetermined or, what is the same, the dimensionless constant σ. In the case $\gamma_1 = \gamma$ the quantity $A = \sigma E$ and the constant σ are found from the law of conservation of total energy

$$\mathcal{E} = 4\pi \int_0^\infty \left(\rho \frac{v^2}{2} + \frac{p}{(\gamma - 1)} \right) r^2 \, dr = \text{const},$$

which is also valid for the non-self-similar stage of the motion. For $\gamma_1 \neq \gamma$ there is no such conservation law; the equation for the conservation of energy assumes, as was already noted, the nonintegrable form

$$\frac{d\mathcal{E}}{dt} = -4\pi r_f^2 D \frac{(\gamma - \gamma_1)}{(\gamma - 1)(\gamma_1 - 1)} \rho_0 \frac{p_f}{\rho_f} \neq 0. \tag{4.27}$$

5. Numerical Calculations

The only method of determining the constant σ is to follow the evolution of the solution of the non-self-similar problem to a self-similar intermediate asymptotics, provided it converges to it. To elucidate this a numerical experiment was formulated. The following problem was solved. There is an unbounded space filled with gas. At the initial instant, outside a sphere of radius R_0 the density of the gas is constant and equal to ρ_0 and the pressure is equal to zero. Inside the sphere the distribution of flow properties, the pressure p, velocity v, and density ρ of the gas, correspond to the solution of the ordinary problem of a strong explosion for the same energy E and the same values of the other parameters at some instant $t = t_0 > 0$. Thus we assume that for $-t_0 \leqslant t < 0$ there occurs an ordinary strong explosion without radiation and release of energy, and then at $t = 0$ radiation or energy release at the front is switched on. For the subsequent evolution of the motion the properties of the flow in the region of continuous motion are described by the system (4.4) of equations for adiabatic motion of a gas. At the front of the shock wave the conditions have the form (4.9)–the same condition as in the ordinary problem of a strong explosion, but we emphasize again that the effective adiabatic exponent γ_1 in the conditions at the front differs from the adiabatic exponent γ in the region of continuous motion, which figures in the equation of entropy conservation. Moreover, the condition of absence of influx of matter and energy at the center of the explosion for $t > 0$ is to be satisfied.

The results of numerical calculation are presented in Figs. 4.4–4.6 and in Tables 4.1 and 4.2. The calculations were performed for two values of γ, one of them close to unity: $\gamma = 1.1$, and the other close to two: $\gamma = 1.9$. In each case calculations were done for various values of γ_1 in the interval of unique determination of a solution, $1 \leqslant \gamma_1 < 2\gamma + 1$. Along with the quantities p, ρ, v, and r_f the quantity $\mathcal{E}$, the total energy of the gas in the perturbed domain, was calculated. The basic result is that the solution quickly approaches the self-similar asymptotics of the form (4.17) considered above. Here the constant α turned out to depend only on γ and γ_1 [the graph of the relation $\alpha(\gamma_1)$ for $\gamma = 1.1$ is shown in Fig. 4.2] and not on the initial conditions (different t_0, E and ρ_0 having been taken). The constant A turned out to depend also on the initial conditions (the initial time t_0, initial energy E, and initial density ρ_0). The approach to self-similar conditions is shown in the graph of the dependence of $\ln r_f(t)$ on $\ln t$, which rapidly approaches the straight line $\ln r_f(t) \sim \alpha \ln t$; in the graph of the dependence of $\ln |d\mathcal{E}/dt|$ on $\ln t$, which also rapidly approaches the line $\ln |d\mathcal{E}/dt| \sim (5\alpha - 3) \ln t$ ($\gamma = 1.9, \gamma_1 = 1.1$; see Fig. 4.4); and also in the graphs of the dependence of the quantity p/p_f on r/r_f for different instants of time (Figs. 4.5a and 4.5b).

For comparison a numerical solution was also evaluated for the nonlinear eigenvalue problem formulated above for a system of ordinary equations. Namely, the system of ordinary equations (4.18)–(4.20) was solved numerically for the initial conditions (4.21), where the exponent α was calculated by a trial-and-error method so as to satisfy the condition of no influx of matter or energy

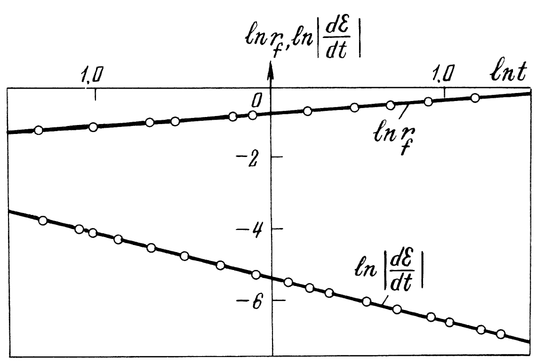
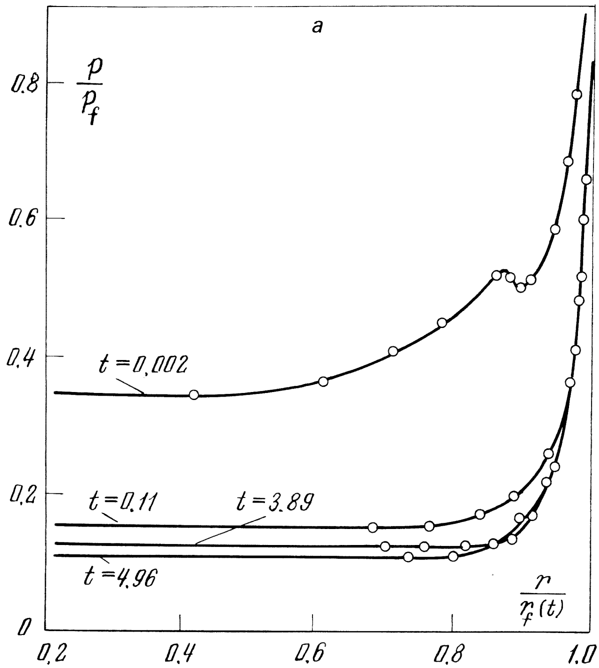


Figure 4.4. Dependence of ln $r_f(t)$ and ln $\mathcal{E}(t)$ on ln t, obtained by numerical solution of the non-self-similar problem, which rapidly approaches the straight line corresponding to a self-similar intermediate asymptotics.


a
$\frac{\rho}{\rho_f}$
0.8
0.6
0.4
0.2
0
t=0.002
t=0.11
t=3.89
t=4.96
$\frac{r}{r_f(t)}$
0.2
0.4
0.6
0.8
1.0

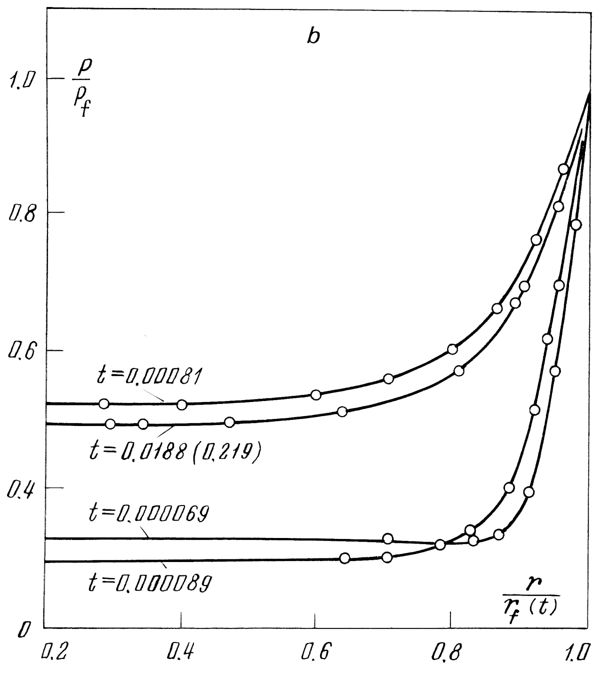

b
$\frac{\rho}{\rho_f}$
1.0
0.8
0.6
0.4
0
t=0.00081
t=0.0188 (0.219)
t=0.000069
t=0.000089
$\frac{r}{r_f(t)}$
0.2
0.4
0.6
0.8
1.0

TABLE 4.1

		α	
γ	γ_1	non-self-similar problem	eigenvalue problem
1.1	1.9	0.54	0.54
1.9	1.1	0.29	0.28

at the center for $t > 0$. Calculation was stopped when the quantity $P\zeta^2 = Rz\zeta^2/\gamma$ near $\zeta = 0$ could be considered constant to within an accuracy of 1%. Results of the comparison of values of the exponent α obtained by numerical calculation of stabilization of the solution to the non-self-similar problem for the system of partial differential equations and by computation of the nonlinear eigenvalue problem are given in Table 4.1.

Comparison of the eigenfunctions from the nonlinear eigenvalue problem with the limiting distributions obtained after stabilization of the solution to the non-self-similar problem also indicates good agreement; the relative deviation does not exceed 2% (cf. Table 4.2).

Thus, numerical integration for the initial conditions we have taken confirms that the asymptotic solution of the original non-self-similar problem is actually the self-similar solution considered at the beginning of the chapter.

As with the self-similar solution considered in Chapter 3, this self-similar solution is distinguished by two properties. First, the exponent α of time in the expression for the self-similar variable cannot be found from similarity considerations, but requires for its determination the solution of a nonlinear eigenvalue problem, i.e., it is found from the condition of the existence of a self-similar solution not in the small but in the large. Further, the whole solution is determined here only to within a constant appearing in the self-similar variable, which can be found only by matching the self-similar intermediate asymptotics to a non-self-similar solution of the original problem; here there is no integral conservation law whose use directly determines the value of this constant from the initial data of the original non-self-similar problem.

The self-similar solution considered here was obtained by Barenblatt and Sivashinskii (1970) [cf. also Barenblatt and Zel'dovich (1971), (1972)]. The numerical calculations were carried out by Andrushchenko, Barenblatt, and Chudov (1975). Sapunkov and Oppenheim and coauthors obtained by another method a self-similar solution of the problem considered in this section for the special case $\gamma_1 = 2\gamma + 1$ (satisfying the Chapman–Jouguet condition) (Sapunkov,

Figure 4.5. Dependence of p/p_f on r/r_f, obtained by numerical solution of the non-self-similar problem (Fig. 4.5a, $\gamma = 1.9$ and $\gamma_1 = 1.1$; Fig. 4.5b, $\gamma = 1.1$ and $\gamma_1 = 1.9$), which rapidly approaches the dependence corresponding to a self-similar intermediate asymptotics.

TABLE 4.2

Limiting distributions for the non-self-similar problem				Solutions of the nonlinear eigenvalue problem			
r/r_f	v/v_f	ρ/ρ_f	p/p_f	r/r_f	v/v_f	ρ/ρ_f	p/p_f
0.9655	0.9237	0.7333	0.8544	0.964	0.9206	0.7242	0.8493
0.9562	0.9046	0.6794	0.8237				
0.927	0.8476	0.5412	0.7434	0.928	0.8496	0.5451	0.7458
0.895	0.7912	0.4313	0.6781	0.892	0.7866	0.4227	0.6731
0.8835	0.7726	0.3996	0.6592				
0.8594	0.7356	0.3424	0.6252	0.856	0.731	0.3352	0.6211
0.8195	0.6803	0.2695	0.5826	0.820	0.6817	0.2701	0.5832
0.7899	0.6498	0.2276	0.5590	0.7841	0.6375	0.2201	0.5552
0.7573	0.6067	0.1900	0.5389	0.7481	0.5974	0.1804	0.5342
0.7397	0.5879	0.1723	0.5300				
0.6807	0.5296	0.1253	0.5076	0.6761	0.526	0.1219	0.5063
0.6098	0.4665	0.0842	0.4911	0.6041	0.4627	0.0816	0.4902
0.4908	0.3700	0.04052	0.4778	0.4961	0.3758	0.04198	0.4781
0.4562	0.3431	0.0317	0.4659	0.4601	0.3482	0.0327	0.476
0.3817	0.2863	0.0178	0.4735	0.3881	0.294	0.0188	0.4733
0.2616	0.1958	0.0052	0.4721	0.264	0.204	0.0053	0.4715
0.2206	0.1651	0.0029	0.4720	0.224	0.1765	0.003	0.4712

1967; Oppenheim, Kuhl, and Kamel, 1972; Oppenheim, Lundström, Kuhl, and Kamel, 1971, 1972).

6. The Problem of an Impulsive Load

A problem of the same type that has for our purposes very instructive peculiarities is that of an impulsive load, studied in the papers of C. von Weizsäcker, Ya. B. Zel'dovich, and their associates (von Weizsäcker, 1954; Hain and Hörner, 1954; Häfele, 1955; Meyer, 1955; Zel'dovich, 1956; Adamskii, 1956; Zhukov and Kazhdan, 1956). To illustrate those peculiarities we briefly present the problem here; a more detailed account is given in the monograph of Zel'dovich and Raizer (1967).

We suppose that space is divided into two halves by an impenetrable plane wall at $x = 0$ (x being the coordinate measured normal to the wall). The half-space $x \geqslant 0$ is occupied by a quiescent ideal gas of density ρ_0 at zero pressure; in the half-space $x \leqslant 0$ there is a vacuum. At the initial instant $t = 0$ a pressure $p = p_0$ is created on the right side of the wall (due for example to an explosion), which varies according to a certain law $p = p_0 f(t/\tau)$ up to some time $t = \tau$, after which the wall disappears instantaneously. The problem consists in investigating the motion arising for $t > \tau$, which obviously develops as follows. A plane shock wave $x = x_f(t)$ propagates to the right in the quiescent gas. In some region behind the shock the compressed gas continues to advance to the right. At the plane $x = x_0(t)$ the instantaneous speed of the gas particles becomes equal to zero, and all gas particles situated to the left of this plane move to the left; there occurs an expansion into the vacuum of the gas compressed by the shock wave.

The solution of this problem reduces to the solution of the same system of gas-dynamic equations as in the previous problem, but now for the plane case:

$$\begin{gathered} \partial_t v + v\,\partial_x v + \partial_x p/\rho = 0, \\ \partial_t \rho + \partial_x \rho v = 0, \\ \partial_t (p/\rho^\gamma) + v\partial_x (p/\rho^\gamma) = 0. \end{gathered} \tag{4.28}$$

The boundary conditions at the shock wave $x = x_f(t)$ are the same as in the problem of a strong explosion:

$$\begin{gathered} \rho_f (v_f - D) = -\rho_0 D, \\ \rho_f (v_f - D)\, v_f + p_f = 0, \\ \rho_f (v_f - D)\left[\frac{v_f^2}{2} + \frac{p_f}{(\gamma - 1)\,\rho_f}\right] + p_f v_f = 0, \end{gathered} \tag{4.29}$$

where $D = dx_f/dt$. The initial conditions at time $t = \tau$ correspond to a vacuum for $x < 0$, and for $x > 0$ to the state of motion that has developed at time $t = \tau$ in the half-space initially filled with quiescent gas of density ρ_0 at zero pressure due to maintenance on the boundary during the time interval τ of a pressure varying according to the law $p(0, t) = p_0 f(t/\tau)$.

It is evident that the density, pressure, and speed of the gas depend on the following dimensional quantities:

$$t,\ p_0,\ \rho_0,\ \tau,\ x, \tag{4.30}$$

and that the coordinate of the shock front depends on all these quantities except the last. Applying the standard procedure of dimensional analysis, we obtain

$$x_f(t) = \sqrt{p_0/\rho_0}\,\tau \xi_f(\Pi_1), \tag{4.31}$$

$$\left.\begin{aligned} \Pi_\rho &= \Phi_\rho(\Pi_1,\ \Pi_2), \\ \Pi_p &= \Phi_p(\Pi_1,\ \Pi_2), \\ \Pi_v &= \Phi_v(\Pi_1,\ \Pi_2). \end{aligned}\right\} \tag{4.32}$$

Here

$$\Pi_1 = t/\tau, \quad \Pi_2 = x/\sqrt{p_0/\rho_0}\,\tau,$$
$$\Pi_\rho = \rho/\rho_0, \quad \Pi_p = p/p_0, \quad \Pi_v = v/\sqrt{p_0/\rho_0}. \tag{4.33}$$

As is evident, the solution of the problem posed turns out to be non-self-similar. This results from the fact that the problem contains a characteristic time τ and a characteristic length scale $(p_0/\rho_0)^{1/2}\tau$.

7. Numerical Calculations. Self-Similar Asymptotics

Numerical calculations reveal, however, that the solution of the formulated problem has a remarkable property. Namely, the dependence of the coordinate of the wavefront on time rapidly (i.e., after some time interval of the order of τ after the start of the process of expansion into vacuum) approaches a power-law asymptotics, so that

$$\xi_f(\Pi_1) = \xi_0(\gamma)\,\Pi_1^\alpha, \tag{4.34}$$

where $\xi_0(\gamma)$ is some function of γ and the exponent α also depends on γ. Further, the density at the front rapidly approaches a constant value, and the pressure and speed of the gas at the front rapidly approach the power laws

$$\frac{p_f}{p_0} \sim \Pi_1^{-2(1-\alpha)}, \quad \frac{v_f}{\sqrt{p_0/\rho_0}} \sim \Pi_1^{-(1-\alpha)}. \tag{4.35}$$

Finally, it turns out that if one constructs the distributions of density, pressure, and speed in relative coordinates, taking x_f as scale of length and p_f, ρ_f, and v_f as scales for the flow properties, then those distributions just as rapidly become independent of time (cf. Fig. 4.6). In other words, it turns out that the solution of the problem rapidly approaches the self-similar asymptotics

$$\xi_f = \xi_0(\gamma)\,\Pi_1^\alpha, \quad \Phi_\rho = \Phi_{1\rho}(\Pi_2\Pi_1^{-\alpha}),$$
$$\Phi_p = \Pi_1^{-2(1-\alpha)}\Phi_{1p}(\Pi_2\Pi_1^{-\alpha}), \Phi_v = \Pi_1^{-(1-\alpha)}\Phi_{1v}(\Pi_2\Pi_1^{-\alpha}). \tag{4.36}$$

We note that the approach of the solution to the self-similar asymptotics does not occur uniformly in the whole region of motion $-\infty < x \leqslant x_f(t)$ but only close to the front $x_f(t)$, in a region that is larger the longer the time since the start of the expansion. Further, numerical calculation (Zhukov and Kazhdan,

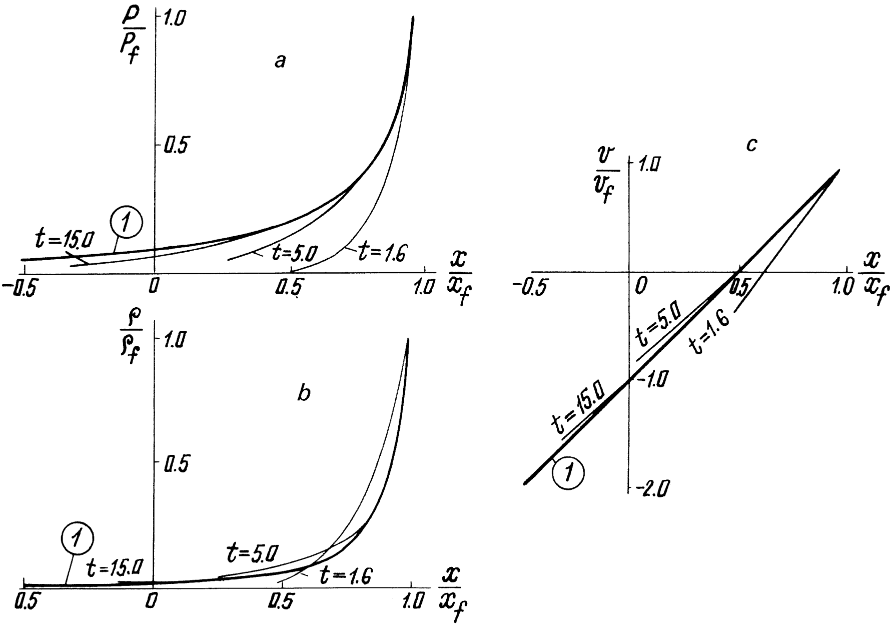


Figure 4.6. Dependence of (a), p/p_f, (b) ρ/ρ_f, and (c) v/v_f on x/x_f, obtained by numerical solution of the non-self-similar problem of an impulsive load for $\gamma = 1.4$, which rapidly approaches the dependence corresponding to a self-similar intermediate asymptotics (curve 1). (From Zhukov and Kazhdan, 1956.)

1956) has shown that the solution approaches a self-similar asymptotics of the form (4.36) with one and the same exponent α independent of whether the pressure at the wall in the time interval τ is constant or changes according to various laws.

8. Self-Similar Limiting Solution

It is natural to try to construct the limiting self-similar solution directly. Again we seek it in the class of solutions of Bechert (1941) and Guderley (1942):

$$p = \rho_0 \frac{x^2}{t^2} P(\xi), \quad \rho = \rho_0 R(\xi), \quad v = \frac{x}{t} V(\xi), \qquad (4.37)$$
$$x_f = At^\alpha, \quad \xi = x/At^\alpha.$$

Here A and α are constants. For the functions P, V, and R we obtain a system of ordinary differential equations, which reduce to one first-order ordinary differential equation

$$\frac{dz}{dV}=\frac{z}{\Delta}\{[2(V-1)+(\gamma-1)V](V-\alpha)^2-$$
$$-(\gamma-1)V(V-1)(V-\alpha)-[2(V-1)+\varkappa(\gamma-1)]z\}, \tag{4.38}$$
$$\Delta=(V-\alpha)[V(V-1)(V-\alpha)+(\varkappa-V)z],$$

with

$$\varkappa=\frac{2(1-\alpha)}{\gamma}, \qquad z=\frac{\gamma P}{R}, \tag{4.39}$$

and to two other first-order equations

$$\frac{d\ln\xi}{dV}=\frac{z-(V-\alpha)^2}{V(V-1)(V-\alpha)+(\varkappa-V)z}, \tag{4.40}$$

$$\frac{d\ln R}{d\xi}=-\frac{V(V-1)(V-\alpha)+(\varkappa-V)z}{(V-\alpha)[z-(V-\alpha)^2]}-\frac{V}{V-\alpha}. \tag{4.41}$$

Thus if the desired solution of (4.38) is known, the solution of (4.40), (4.41) reduces to quadratures.

The desired solution of (4.38) must pass through two singular points: the image of the shock front

$$V=\frac{2\alpha}{\gamma+1}, \qquad z=\frac{2\alpha^2\gamma(\gamma-1)}{(\gamma+1)^2} \tag{4.42}$$

and the image of the free boundary, a singular point of saddle type

$$V=\varkappa, \quad z=\infty. \tag{4.43}$$

Here the variable ξ must increase monotonically upon moving from the singular point (4.43) to the image of the shock front (4.42). Thus the mathematical problem turns out in this case to be close to the modified problem of a strong explosion that was considered at the beginning of this chapter.† We have again arrived at the necessity of drawing an integral curve of a first-order equation of the same type through two points, one of which is a saddle-type singular point.

†We recall that the problem of an impulsive load was solved significantly earlier than the modified problem of a strong explosion.

TABLE 4.3

γ	1.0	1.1	7/5	5/3	2.8	∞
α	1/2	0.569	3/5	0.611	0.627	0.642

This is impossible in general but, as in the previous problem, one can show that for each γ there exists a value of α, an eigenvalue of the problem, for which the integral curve of (4.38) passing through the image of the shock front goes into the saddle, the image of the free boundary. Values of α corresponding to various γ in the full range $1 \leqslant \gamma \leqslant \infty$ are given in Table 4.3. It is seen that for all γ in the interval $1 < \gamma < \infty$ we have the inequality

$$\frac{1}{2} < \alpha < \frac{2}{3}.$$

As in the previous problem, it turns out here that values of the exponent α determined by direct construction of a limiting self-similar solution of the problem of an impulsive load agree well with values obtained by numerical calculation of an asymptotic solution of the non-self-similar problem.

Evidently the limiting self-similar solution (4.37) is determined by direct construction only to within a constant A; comparison of (4.37) and (4.34) gives

$$A = \xi_0(\gamma)\sqrt{p_0/\rho_0}\,\tau^{1-\alpha}. \tag{4.44}$$

Thus if we want to get just the same asymptotics by reducing the duration τ of the impulse acting on the gas, we must correspondingly increase the pressure according to the law

$$p_0 = \text{const}\ \tau^{-2(1-\alpha)}. \tag{4.45}$$

9. Laws of Conservation of Energy and Momentum in the Problem of an Impulsive Load

The mass of gas involved at each moment in the flow through unit area of the boundary is finite. Hence there are in the problem conservation laws of momentum and energy, which are valid also in the non-self-similar stage of the motion. The idea naturally occurs of using those laws to determine the exponent α and the constant A of the limiting self-similar solution, as was done in Chapter 2 for the self-similar solutions of the first kind considered there.

The gas is initially quiescent and at zero pressure, so its momentum and energy are zero. The total momentum J of the gas involved in the motion is equal at any instant to the impulse of the pressure load:

$$J = \beta p_0 \tau, \quad \beta = \int_0^1 f(\lambda)\, d\lambda. \tag{4.46}$$

Hence we obtain the relation

$$\beta p_0 \tau = \int_{-\infty}^{x_f} \rho v\, dx. \tag{4.47}$$

As time increases the solution tends to a self-similar one. Passing to the limit under the integral sign, we substitute into (4.47) the expressions for the density and speed from the self-similar solution (4.37) and find

$$\beta p_0 \tau = \rho_0 A^2 t^{2\alpha - 1} \int_{-\infty}^{1} R(\xi)\, V(\xi)\, d\xi. \tag{4.48}$$

Since the integral on the right side is obviously independent of time it is necessary, in order for the left side to be independent of time, to have the relation $\alpha = 1/2$, after which it would turn out that one could find the constant A from (4.48).

However the work per unit area performed by the load on the gas is equal to

$$\int_0^{\tau} p(0,\ t)\, v(0,\ t)\, dt = \delta p_0^{3/2} \rho_0^{-1/2} \tau, \tag{4.49}$$

where δ is a numerical constant. But the energy of the gas entering the motion is zero because its speed and pressure are equal to zero. Hence the energy of the gas involved in the motion is at any instant equal to the work performed by the impulsive load:

$$\delta p_0^{3/2} \rho_0^{-1/2} \tau = \int_{-\infty}^{x_f} \rho \left(\frac{v^2}{2} + \frac{p}{(\gamma - 1)\rho} \right) dx. \tag{4.50}$$

Again passing to the limit under the integral sign and substituting the expressions for speed, density, and pressure from the limiting self-similar solution, we obtain

$$\delta p_0^{3/2} \rho_0^{-1/2} \tau = \rho_0 A^3 t^{3\alpha - 2} \int_{-\infty}^{1} R \left(\frac{V^2}{2} + \frac{P}{(\gamma - 1) R} \right) \xi^2\, d\xi. \tag{4.51}$$

At first glance it follows from this that $\alpha = 2/3$ and that (4.51) allows one also to determine the constant A. Thus a paradox arises, consisting in the fact that the exponents α of the self-similar variable determined from the laws of conservation of momentum ($\alpha = 1/2$) and of energy ($\alpha = 2/3$) do not agree with each other or with the exponent α ($1/2 < \alpha < 2/3$) determined by direct construction of a limiting self-similar solution.

The resolution of this paradox is trivial, and at the same time instructive. The fact is that the integral in the momentum equation (4.48) is equal to zero, and the integral in the energy equation (4.51) is equal to infinity, so that from these relations it is impossible to determine either the exponent or the constant A. The passage to the limit under the integral sign in the conservation laws (4.47) and (4.50) was itself inadmissible, because the convergence of the integrands to the limit is nonuniform over the domain of integration.

In fact the limiting self-similar motion is obtained by passage to the limit over the entire domain $-\infty < x \leqslant x_f$ with the duration τ of the impulse tending to zero and the pressure on the boundary tending to infinity according to the law $p_0 = \text{const}\,\tau^{-2(1-\alpha)}$. Here the total momentum $\beta p_0 \tau$ tends to zero like const $\tau^{2\alpha-1}$, and the energy $\delta p_0{}^{3/2} \rho_0{}^{-1/2} \tau$ to infinity like const $\tau^{3\alpha-2}$, so that (we recall that α lies between 1/2 and 2/3) the self-similar motion has zero momentum and infinite energy. Further, the self-similar solution is limiting for the solution of the original non-self-similar problem with finite p_0 and τ, and t tending to infinity. However, as was already mentioned, the convergence to the limiting solution is nonuniform in the domain $-\infty < x \leqslant x_f$. The momentum of the region of compression $x_0(t) \leqslant x \leqslant x_f(t)$ grows indefinitely with time. The momentum of the region of expansion $-\infty < x \leqslant x_0(t)$ has a negative sign and also grows indefinitely with time in absolute value. Their algebraic sum, equal to $\beta p_0 \tau$, becomes ever smaller compared with the momentum of each of the regions mentioned; it is different from zero only because of the departure of the motion from the self-similar.

We now consider the energy $\mathcal{E}$ of any region $x_1(t) = \xi_1 A t^\alpha \leqslant x \leqslant x_f(t)$ in which the motion becomes close to self-similar starting from some instant of time:

$$\mathcal{E} = \int_{x_1}^{x_f} \rho \left(\frac{v^2}{2} + \frac{p}{(\gamma - 1)\rho} \right) dx$$

$$= \rho_0 A^3 t^{3\alpha - 2} \int_{\xi_1}^{1} R \left(\frac{V^2}{2} + \frac{P}{(\gamma - 1) R} \right) \xi^2 \, d\xi. \tag{4.52}$$

It is evident that the energy $\mathcal{E}$ tends to zero with increasing t, so that the contribution of the self-similar region to the bulk energy becomes ever less in time, and the basic contribution to the energy is determined by the motion close to the free boundary, where it always remains non-self-similar no matter how much time has passed since the start of the motion.

5

Classification of Self-Similar Solutions

1. Complete and Incomplete Self-Similarity

In Chapters 2–4 several instructive and fundamentally different self-similar problems were considered. In the problems of the propagation of strong thermal and strong blast waves and the problem of an instantaneous point source of heat the situation turned out to be relatively simple. Namely, for them there exists some completely schematized degenerate statement of the problem (energy release at a point, initial temperature and pressure equal to zero). Considering this statement of the problem and applying the procedure of dimensional analysis to it in the standard way, we can reveal self-similarity of the solution, construct the self-similar variables, and thanks to the existence of some integral obtain the solution in finite form.

Deeper consideration shows, however, that this simplicity is illusory, and that for example in making the assumption of pointwise release of energy we have, as is said, gone to the brink of an abyss. In fact changing the formulation of the problem apparently only slightly, in such a way that it would seem all similarity considerations must preserve their validity, we arrived at a contradiction; it turned out that in the modified problems the required solutions simply do not exist. More detailed analysis showed that in trying to find solutions of the modified problems by the same standard procedure, starting from the formulation of a degenerate problem, it turned out that the very statement of the question was improper. What we actually needed was not exact solutions of the simply formulated degenerate problems, corresponding to instantaneous removal at a point of a finite mass of fluid or the instantaneous release at a point of a finite amount of energy. We are interested rather in the asymptotics of solutions of the nondegenerate problems for large times. We applied dimensional analysis to the nondegenerate problems, the existence and uniqueness of whose solutions are either strictly proved or evoke no doubt; the nondegenerate problems naturally turned out to be non-self-similar. The passage to the limit as the supplementary parameter that makes the problem non-self-similar tends to zero turned out to be irregular: in one case the limit is equal to zero or infinity, depending on the additional conditions on the problem; in the other case a limit simply does not exist. Nevertheless, in both cases meaningful intermediate asymptotics exist, and moreover they are self-similar. It was revealed that these asymptotics are precisely what we actually need. It turns out that the removal of mass in the problem of filtration in an elasto-plastic porous medium and the release of energy by a strong explosion with loss or deposition of energy at the front cannot be considered pointwise: changing the size of the region within which the initial energy release or removal of fluid mass occurs we must, in order to obtain a proper asymptotics of the solution to the original nondegenerate problem for large times, increase or decrease the amount of the mass removed or energy released in such a way that a certain "moment" of the initial distribution of mass or energy remains constant. It is essential that the power to which the length appears in the expression for this moment is not given in advance, and that in principle it is impossible to determine it from dimensional considerations; it must be found in the course of solving the problem of determining the self-similar asymptotics. Thus, we have encountered the existence of self-similar solutions of two types.

It might seem that the difference in types of self-similar solutions is connected with the availability or absence in the problem under consideration of an integral conservation law that is valid also in the non-self-similar stage. However the problem of an impulsive load considered in Chapter 4 shows that this is not so; the reason lies in the character of the transition from the non-self-similar solution to the self-similar asymptotics.

We now give a formal classification of self-similar relationships and, in particular, of self-similar solutions to problems of mathematical physics. We recall (cf. Chapter 1), that according to the Π-theorem, any relationship among $n+1$ dimensional quantities of the form

$$a=f(a_1, \ldots, a_k, \quad a_{k+1}, \ldots, a_n) \tag{5.1}$$

can be expressed in the form

$$\Pi=\Phi(\Pi_1, \ldots, \Pi_{n-k}), \tag{5.2}$$

where the parameters $\Pi, \Pi_1, \ldots, \Pi_{n-k}$ are dimensionless:

$$\Pi=\frac{a}{a_1^p a_2^q \ldots a_k^r},$$
$$\Pi_1=\frac{a_{k+1}}{a_1^{p_{k+1}} \ldots a_k^{r_{k+1}}}, \quad \ldots, \quad \Pi_{n-k}=\frac{a_n}{a_1^{p_n} \ldots a_k^{r_n}}. \tag{5.3}$$

It is assumed here that the governing parameters $a_1, \ldots, a_k$ have independent dimensions and that the dimensions of the remaining determining parameters

$a_{k+1}, \ldots, a_n$ can be expressed in terms of the dimensions of the parameters $a_1, \ldots, a_k$.

We now consider some governing parameter a_{k+i}. In traditional discussions "on a physical level" that parameter is considered essential, i.e., actually governing a phenomenon, if the value of the corresponding dimensionless governing parameter Π_i is not too large and not too small; let us say, as a particular convention, that it lies between 0.1 and 10. Otherwise it is assumed that the influence of this parameter can be neglected.

This argument is actually valid if there exists a finite nonzero limit of the function Φ in (5.2) as the parameter Π_i tends to zero or infinity with the other similarity parameters remaining constant, which is certainly not necessarily the case in general. In fact, even somewhat more is required: the function Φ must converge sufficiently rapidly to a limit as Π_i tends to zero or infinity, so that for $\Pi_i < 0.1$ or $\Pi_i > 10$ the function Φ will assume values sufficiently close to that limit. If these conditions are actually satisfied, then for sufficiently small or sufficiently large Π_i the function Φ in (5.2) can, to the required accuracy, be replaced by a function of one less argument:

$$\Pi = \Phi_0(\underbrace{\Pi_1, \ldots, \Pi_{i-1}, \Pi_{i+1}, \ldots, \Pi_{n-k}}_{n-k-1 \text{ arguments}}), \tag{5.4}$$

where $\Phi_0(\Pi_1, \ldots, \Pi_{i-1}, \Pi_{i+1}, \ldots, \Pi_{n-k})$ is the limit of the function $\Phi(\Pi_1, \ldots, \Pi_{i-1}, \Pi_i, \Pi_{i+1}, \ldots, \Pi_{n-k})$ as Π_i tends to zero (or infinity). In such cases we speak of *complete self-similarity*† of the phenomenon with respect to the parameter Π_i.

However, in the general case the function Φ obviously need not tend, for increase or decrease in the similarity parameter Π_i, to a limit and moreover a finite and nonzero one, so that in general the parameter a_{k+i} can remain essential no matter how small or large the value of the corresponding dimensionless parameter Π_i.

Suppose, for example, that as $\Pi_i \to 0$ or $\Pi_i \to \infty$ the function Φ converges to zero or infinity. It is clear that in this case the quantity Π_i remains essential, no matter how small or large it may be: replacing the function Φ in (5.2) by its limiting value as $\Pi_i \to 0$ or ∞ leads to the vacuous relation $\Pi = 0$ or $\Pi = \infty$. Hence it is impossible in general in this case simply to delete Π_i from the governing parameters and to replace the functions f and Φ in (5.1) and (5.2) by functions with one less argument; one does not have complete self-similarity in the parameter Π_i. Nevertheless, here too there exists a situation where one can decrease the number of arguments and obtain a relation of the form (5.4). Namely,

†The concept of self-similarity, first introduced by A. A. Gukhman as long ago as 1928, corresponds in the terminology adopted here to complete self-similarity with respect to a parameter.

suppose that there exists a number α, such that as $\Pi_i \to 0$ or $\Pi_i \to \infty$ the function Φ has the power-law asymptotic representation

$$\Phi = \Pi_i^\alpha \Phi_1 (\Pi_1, \Pi_2, \ldots, \Pi_{i-1}, \Pi_{i+1}, \ldots, \Pi_{n-k}) + o(\Pi_i^\alpha), \quad (5.5)$$

where the function Φ_1 again depends on $n-k-1$ arguments, and the second summand is arbitrarily small compared with the first for sufficiently small (or large) Π_i. In this case for sufficiently small (or sufficiently large) Π_i we have, to within the required accuracy, a relation of the form (5.4):

$$\Pi^* = \Phi_1 \underbrace{(\Pi_1, \Pi_2, \ldots, \Pi_{i-1}, \Pi_{i+1}, \ldots, \Pi_{n-k})}_{n-k-1 \text{ arguments}}, \quad (5.6)$$

where

$$\Pi^* = \Pi \Pi_i^{-\alpha} = \frac{a}{a_1^{p-\alpha p_{k+i}} a_2^{q-\alpha q_{k+i}}, \ldots, a_k^{r-\alpha r_{k+i}} a_{k+i}^\alpha}. \quad (5.7)$$

Thus in this case the relation (5.1) can again be written in terms of a function of $n-k-1$ arguments, just as in the case of complete self-similarity; but firstly the form of the dimensionless parameter Π^* can no longer be obtained from considerations of dimensional analysis, and secondly the argument a_{k+i} appears in it and hence does not cease to be essential.

Furthermore suppose that two dimensionless similarity parameters Π_i and Π_j are small or large, but as Π_i and Π_j tend independently to zero or infinity the function Φ tends to zero, infinity, or no limit at all. In this case, just as in the preceding one, the parameters Π_i and Π_j remain essential no matter how small or large they may be, and consequently the corresponding dimensional parameters a_{k+i} and a_{k+j} remain essential. There is however, here too, an exceptional case where the number of arguments of the function Φ in (5.2) is reduced, and we get a relation of the form (5.4). Thus suppose that there exist numbers α and β such that, as Π_i and Π_j tend to zero or infinity, the function Φ has the power-law asymptotic representation

$$\Phi = \Pi_i^\alpha \Phi_2 (\Pi_j/\Pi_i^\beta, \Pi_1, \ldots, \Pi_{i-1}, \Pi_{i+1}, \ldots, \Pi_{j-1}, \Pi_{j+1}, \ldots, \Pi_{n-k}) + o(\Pi_i^\alpha), \quad (5.8)$$

where the last summand is arbitrarily small compared with the first for sufficiently small (or large) Π_i, Π_j. In this case, for sufficiently small (or large) Π_i, Π_j, we have a relation of the form (5.4):

$$\Pi^* = \Phi_2 (\Pi^{**}, \Pi_1, \Pi_2, \ldots, \underbrace{\Pi_{i-1}, \Pi_{i+1}, \ldots, \Pi_{j-1}, \Pi_{j+1}, \ldots, \Pi_{n-k})}_{n-k-1 \text{ arguments}}, \quad (5.9)$$

where as before Π^* is defined by the relation (5.7), and Π^{**} by the relation

$$\Pi^{**}=\Pi_j\Pi_i^{-\beta}=\frac{a_{k+j}}{a_1^{p_{k+j}-\beta p_{k+i}}\ldots a_k^{r_{k+j}-\beta r_{k+i}}a_{k+i}^{\beta}}. \tag{5.10}$$

It is evident that in this case, as in the case of complete self-similarity, (5.1) can be described by a function of $n-k-1$ arguments; but now the form of the two dimensionless parameters Π^* and Π^{**} cannot be obtained from considerations of dimensional analysis, because these considerations cannot in principle give the quantities α and β. Moreover, the argument a_{k+i} appears in the parameter Π^*, and both the arguments a_{k+i} and a_{k+j} appear in the parameter Π^{**}, which thus cannot cease to be essential. Essentially analogously one can distinguish subclasses where three or more dimensionless parameters tend to zero or infinity and there exists a power-law asymptotics. Thus in these exceptional cases, despite the fact that there is no complete self-similarity in the similarity parameters Π_i, Π_j and the governing parameters a_{k+i}, a_{k+j} remain essential no matter how small or large the values of the similarity parameters Π_i, Π_j, there is a decrease in the number of arguments of the function Φ that defines the relationship in which we are interested, and we get a relation of the form (5.4) just as in the case of complete self-similarity. We speak in such cases of *incomplete self-similarity* of a phenomenon with respect to the parameters Π_j, Π_i.

The conclusion at which we have arrived is entirely natural: if the values of certain dimensionless parameters are small or large, then there are three possibilities:

(1) The limits of the corresponding functions Φ as $\Pi_i \to 0$ or ∞ exist and are finite and nonzero. The corresponding governing parameters, the dimensional a_{k+i} and dimensionless Π_i, can be excluded from consideration, and the number of arguments of the function Φ decreases. All the similarity parameters can be determined by means of the regular procedure of dimensional analysis. This case corresponds to complete self-similarity of the phenomenon with respect the similarity parameters† Π_i.

(2) No finite limits exist for the functions Φ as $\Pi_i \to 0$ or ∞, but one of the exceptional cases indicated above holds‡. In this case the number of arguments of the functions Φ can be decreased, but not all the parameters Π, Π_i can be

†We recall that phenomena are called self-similar if the distributions of their properties at different instants of time are obtained from one another by a similarity transformation. This concept should not be confused with the concept of self-similarity with respect to a similarity parameter.

‡The question can arise, why does one regard as exceptional only asymptotic representations of the power-law forms (5.5) and (5.8); is it impossible to take out in front of the function Φ other functions of Π_i, for example, $\log \Pi_i$? In fact, in this case one no longer gets relations among power-law combinations of the governed and the governing parameters of the form (5.4) with a smaller number of arguments. The quantities Π, Π_i, Π_j are power-law combinations of dimensional parameters, and only products of their powers give upon multiplication power-law combinations of the same form.

obtained from dimensional analysis; and the governing parameters $a_{k+i}, \ldots$ remain essential no matter how small (or large) the corresponding similarity parameters. This case corresponds to incomplete self-similarity with respect to the similarity parameters Π_i.

(3) No finite limits exist for the functions Φ as $\Pi_i \to 0$ or ∞, and the indicated exceptions do not hold. This case corresponds to lack of self-similarity of the phenomenon with respect to the similarity parameters Π_i.

It was already remarked that no matter how large (or small) the values of the parameters Π_i, one gets in this case no relation of the form (5.4) between power-law combinations of the governing and governed parameters with a smaller number of arguments for the function Φ. It is useful to distinguish the special case when for large (or small) values of the parameters Π_i one of them "separates," though not in power-law form. This means that for such values of the similarity parameters the function Φ can be represented in the form

$$\Phi = \Psi(\Pi_i)\,\Phi_3 + \text{ a small quantity},$$

where Ψ is a function of Π_i that is not a power law, for example $\log \Pi_i$, and the number of arguments of the function Φ_3 is less than $n-k$.

The difficulty is that *a priori*, until we obtain a non-self-similar solution of the complete nondegenerate problem,† we do not know with which case we are dealing, irrespective of whether or not we have an explicit mathematical formulation of the problem. Hence one can only recommend assuming in succession each of the possible situations for small (or large) similarity parameters—complete self-similarity, incomplete self-similarity, lack of self-similarity—and then comparing the relations obtained under each assumption with data from numerical calculations, experiments, or the results of analytic investigations.

2. Self-Similar Solutions of the First and Second Kinds

We now consider some problem of mathematical physics that describes a certain phenomenon and has a unique solution; let the quantity a be an unknown in this problem and the quantities $a_1, \ldots, a_n$ be independent variables and parameters appearing in the equations and in the boundary, initial, and other conditions that determine unique solutions.

Self-similar solutions are always solutions of degenerate problems, which are obtained if certain parameters a_{k+i} and the dimensionless parameters Π_i corresponding to them assume zero or infinite values. They are simultaneously exact solutions of degenerate problems and asymptotic (generally intermediate asymptotic) representations of solutions of wider classes of nondegenerate non-self-similar problems as the indicated parameters tend to zero or infinity.

†This, as a rule, does not happen. If the complete solution of the problem is known, there is no need to apply the methods of dimensional analysis.

It is clear that if an asymptotics is self-similar, and if the self-similar variables are power-law monomials, then one of the first two cases of the alternative in the last section must hold. Depending on which of those two cases holds for the self-similarity, self-similar solutions are divided into solutions of the first and second kind.

Self-similar solutions of the first kind are obtained when the passage to the limit from the non-self-similar nondegenerate problem to the self-similar degenerate problem is regular, in the sense that there is complete self-similarity with respect to the similarity parameters that make the problem nondegenerate and its solution non-self-similar. Expressions for all the self-similar variables, independent as well as dependent, can be obtained here by applying dimensional analysis.

Self-similar solutions of the second kind are obtained in the case when the degeneration of the original problem is irregular, in the sense that there is an incomplete self-similarity with respect to the similarity parameters indicated above. Then expressions for the self-similar variables cannot in general be obtained from dimensional considerations.

In the direct construction of self-similar solutions of the second kind, the determination of the exponent in the self-similar variables leads to a nonlinear eigenvalue problem. The constant multiplier A appearing in the self-similar variables is left undetermined in the direct construction of self-similar solutions of the second kind. The constant A can be found by following, for example by means of numerical calculations, the entire process of evolution of a solution of the nondegenerate problem into a self-similar asymptotics.

If the constant A can be found from integral conservation laws, this means that for an appropriate choice of governing parameters the problem can be reformulated and reduced to a problem of the first kind. For example, the classical problems of a heat source and a strong explosion can be represented as self-similar solutions of the second kind if one chooses the governing parameters of the nondegenerate pre-self-similar problem "unluckily". The possibility of obtaining solutions of these problems as self-similar solutions of the first kind is connected with the choice as governing parameters of the energy of the explosion and the bulk heat, which by virtue of the corresponding integral conservation laws do not vary with time.

There are also possible self-similar solutions of non-power type, for which the self-similar variables are no longer represented by power monomials. These solutions owe their existence to the special case indicated in (3) of the previous section. Instructive examples of such solutions are provided by the limiting self-similar solutions that will be considered in Chapter 7.

Examples of self-similar solutions of the first kind are given by the solutions considered in Chapter 2 to the problems of the propagation of strong thermal and strong blast waves and the problem of an instantaneous heat source. In fact, we shall return first to the solution of the problem of an instantaneous heat source—the self-similar solution of the equation

$$\partial_t u = \varkappa \partial^2_{xx} u \tag{5.11}$$

under the conditions

$$u(x,\ 0) \equiv 0,\ x \neq 0; \int_{-\infty}^{\infty} u(x,\ 0)\,dx = Q; \quad u(\infty,\ t) \equiv 0. \tag{5.12}$$

If we pass from the degenerate conditions (5.12), corresponding to the concentration at the initial instant of a finite amount of heat at a point, to the conditions

$$u(x,\ 0) = \frac{Q}{l} u_0\left(\frac{x}{l}\right), \quad \int_{-\infty}^{\infty} u_0(\zeta)\,d\zeta = 1; \quad u(\infty,\ t) \equiv 0 \tag{5.13}$$

[$u_0(\zeta)$ being an even function which rapidly decreases monotonically with increasing absolute value of the argument] corresponding to concentration at the initial instant of the same amount of heat in a region of finite size l, then the solution will cease to be self-similar. Dimensional analysis gives for it

$$u = f(t,\ \varkappa,\ Q,\ l,\ x); \quad \Pi = \Phi(\Pi_1,\ \Pi_2), \tag{5.14}$$

where

$$\Pi_1 = x/\sqrt{\varkappa t}, \quad \Pi_2 = l/\sqrt{\varkappa t}, \quad \Pi = u\sqrt{\varkappa t}/Q.$$

For Π_2 tending to zero, i.e., t tending to infinity or, what is the same thing, the region of initial concentration of heat contracting to a point, the function Φ converges to a finite limit. In fact by reducing to dimensionless form known relations from the theory of heat conduction [cf. Carslaw and Jaeger (1960); Petrovskii (1967)] it is easy to prove that

$$\Phi = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} u_0(\zeta)\, e^{-(\Pi_1 - \Pi_2\zeta)^2/4}\, d\zeta.$$

As $\Pi_2 \to 0$ the function Φ converges to $(1/2\pi^{1/2})e^{-\Pi_1^2/4} = \Phi(\Pi_1, 0)$. Hence the self-similarity is complete with respect to the parameter Π_2 that makes the problem nondegenerate, and for sufficiently small Π_2 one can, with any given accuracy, replace the function Φ in (5.14) by $\Phi(\Pi_1, 0) = \Phi_0(\Pi_1)$. But the function Φ_0 corresponds to the solution of a degenerate problem, which is already self-similar.

Complete self-similarity makes it possible not only to obtain expressions for the self-similar variables, as was demonstrated in Chapter 2, but also to give meaningful similarity laws. Suppose we want to determine the law of decay of the maximum temperature. The maximum of the temperature is obviously achieved at $x = 0$, so that for the degenerate problem (the heat at the initial instant being concentrated at the point $x = 0$) we find

$$u_{\max}=f(t,\ \varkappa,\ Q),\quad \Pi_{\max}=\frac{u_{\max}\sqrt{\varkappa t}}{Q}=\text{const}, \tag{5.15}$$

whence

$$u_{\max}=\text{const}\,\frac{Q}{\sqrt{\varkappa t}}. \tag{5.16}$$

Such an argument is valid in the present case because for the nondegenerate problem

$$u_{\max}=f(t,\ \varkappa,\ Q,\ l),\quad \Pi_{\max}=\Phi_{\max}(\Pi_2)=\Phi(0,\ \Pi_2), \tag{5.17}$$

and as $\Pi_2 \to 0$ the function $\Phi_{\max}$ converges to a finite limit equal to $1/2\pi^{1/2}$.

The situation is entirely analogous for both strong thermal and strong explosion waves. Thus passing in the problem of strong explosion waves from the degenerate formulation of the problem, corresponding to the release of energy at a point, to the nondegenerate formulation of the problem, corresponding to the release of energy in a sphere of finite radius R_0, we obtain for the pressure, density, and radius of the shock wave:

$$\begin{gathered}
p=\left(\frac{E^2\rho_0^3}{t^6}\right)^{1/5}\Phi_p(\Pi_1,\ \Pi_2),\qquad \Pi_1=r\Big/\left(\frac{Et^2}{\rho_0}\right)^{1/5},\\
\Pi_2=R_0\Big/\left(\frac{Et^2}{\rho_0}\right)^{1/5},\qquad \rho=\rho_0\Phi_\rho(\Pi_1,\ \Pi_2),\\
v=\left(\frac{Et^{-3}}{\rho_0}\right)^{1/5}\Phi_v(\Pi_1,\ \Pi_2),\quad r_f=\left(\frac{Et^2}{\rho_0}\right)^{1/5}\Phi_f(\Pi_2).
\end{gathered} \tag{5.18}$$

For Π_2 tending to zero, i.e. the region of initial release of energy contracting to a point, all the functions Φ_p, Φ_ρ, Φ_r, Φ_f tend to finite nonzero limits. (This fact has not been proved analytically, but it is verified by numerical calculations and there is no reason to doubt it.) Thus self-similarity is complete with respect to the parameter Π_2 that makes the problem nondegenerate, and for sufficiently small Π_2 we can with any given accuracy replace the functions Φ_p, Φ_ρ, Φ_v in (5.18) by $\Phi_p(\Pi_1, 0) = \Phi_{p0}(\Pi_1)$, $\Phi_\rho(\Pi_1, 0) = \Phi_{\rho 0}(\Pi_1)$, $\Phi_v(\Pi_1, 0) = \Phi_{v0}(\Pi_1)$, respectively, and the function $\Phi_f(\Pi_2)$ by the constant $C = \Phi_f(0)$. But the functions $\Phi_{\rho 0}$, Φ_{p0}, Φ_{v0} and the constant of proportionality C in the formula for the radius of the shock wave correspond to the self-similar solution of the degenerate problem (cf. Chapter 2).

Complete self-similarity in this case also allows one to obtain meaningful similarity laws, for example for the characteristics of the motion at the shock front:

$$p_f = \text{const}_1 \left(\frac{E^2 \rho_0^3}{t^6} \right)^{1/5},$$

$$v_f = \text{const}_2 \left(\frac{Et^{-3}}{\rho_0} \right)^{1/5},$$

$$r_f = \text{const}_3 \left(\frac{Et^2}{\rho_0} \right)^{1/5}.$$

Examples of self-similar solutions of the second kind are the solutions of the modified problems of an instantaneous heat source and a strong explosion and the problem of an impulsive load that were considered in Chapters 3 and 4.

The solution of the modified problem of an instantaneous heat source is a self-similar asymptotic solution for large t of the problem with the same initial conditions (5.13) but now for the modified equation

$$\begin{aligned} \partial_t u &= \varkappa \partial^2_{xx} u \quad (\partial_t u \geqslant 0), \\ \partial_t u &= \varkappa_1 \partial^2_{xx} u \quad (\partial_t u \leqslant 0) \end{aligned} \tag{5.20}$$

which is a nonlinear heat-conduction equation for $\varkappa_1 \neq \varkappa$. For this solution dimensional analysis gives

$$u = f(t,\ Q,\ l,\ \varkappa,\ \varkappa_1,\ x), \quad \Pi = \Phi(\Pi_1,\ \Pi_2,\ \Pi_3), \tag{5.21}$$

where

$$\Pi_1 = x/\sqrt{\varkappa t}, \quad \Pi_2 = l/\sqrt{\varkappa t}, \quad \Pi_3 = \varkappa_1/\varkappa,$$
$$\Pi = u\sqrt{\varkappa t}/Q.$$

The general case with $\varkappa_1 \neq \varkappa$ and $\Pi_3 \neq 1$ differs from the classical case with $\varkappa_1 = \varkappa$ and $\Pi_3 = 1$ in that for $\Pi_3 \neq 1$ and $\Pi_2 \to 0$ the function Φ in (5.21) no longer converges to a finite nonzero limit, but converges to zero or infinity, depending on whether the similarity parameter Π_3 is larger or smaller than one. Here one has the first type of incomplete self-similarity with respect to a parameter: as $\Pi_2 \to 0$,

$$\Phi \to \Pi_2^{\alpha} \Phi_1(\Pi_1,\ \Pi_3),$$

where α is some number depending on Π_3 and equal to zero for $\Pi_3 = 1$, positive for $\Pi_3 > 1$, and negative for $\Pi_3 < 1$.

In accord with what was said above, the relation (5.21) can be written for small Π_2 in the self-similar form

$$\Pi^* = \Phi_1(\Pi_1, \Pi_3), \qquad \Pi^* = \frac{u\,(\varkappa t)^{\frac{1+\alpha}{2}}}{Ql^{\alpha}}, \tag{5.22}$$

but now the dependent self-similar variable Π^* can no longer be found by dimensional considerations; the constant α is found by solving a nonlinear eigenvalue problem, and the full solution is found to within a multiplicative constant. Moreover the length l that makes the original problem non-self-similar appears explicitly in this self-similar variable. From (5.22) we get in particular the law of decay of the maximum of the quantity u:

$$\Pi^*_{\max} = \Phi_1(0, \Pi_3) = F(\Pi_3); \qquad u_{\max} = \frac{Ql^{\alpha}}{(\varkappa t)^{\frac{1+\alpha}{2}}} F(\Pi_3). \tag{5.23}$$

It is essential that although this law also has power form, it is impossible to obtain it by applying dimensional analysis. The situation is that the law of attenuation of the quantity $u_{\max}$ is determined by the dimensions of the contant Ql^{α}. These dimensions are unknown in advance, and are determined after the construction of a self-similar solution in the large, i.e., from the solution of a nonlinear eigenvalue problem. However, since we are dealing in the present case with an incomplete self-similarity of the first type, the independent self-similar variable, in this case Π_1, is found from dimensional analysis. Hence, in particular, for the law of propagation of the "unloading wave"–the boundary between regions with different κ–one gets the similarity law

$$x_0(t) = \xi_0(\Pi_3)\sqrt{\varkappa t}, \tag{5.24}$$

which can be established from dimensional considerations.

We turn to the modified problem of strong explosion waves. The desired self-similar solution is a self-similar asymptotics for large t of the solution to the equations of adiabatic motion of a gas with adiabatic exponent γ under conditions on the strong shock wave in which the effective isentropic exponent $\gamma_1 \neq \gamma$ enters, and the initial conditions corresponding to the release at the initial instant of energy E within a sphere of radius R_0.

The application of dimensional analysis to this nondegenerate problem again leads us to (5.18), where it is understood that among the arguments of the functions Φ_p, Φ_ρ, Φ_v, and Φ_f also appear the constant parameters $\Pi_3 = \gamma$ and $\Pi_4 = \gamma_1$.

However, in the case $\gamma_1 \neq \gamma$ there is no complete self-similarity as $\Pi_2 \to 0$ in the similarity parameter Π_2 that makes the problem nondegenerate ; the functions Φ_p, Φ_ρ, Φ_v, and Φ_f do not tend to finite nonzero limits. Here the functions Φ_p, Φ_v, and Φ_f converge to zero or infinity as $\Pi_2 \to 0$ depending on whether γ_1 is smaller or larger than γ, and the function Φ_ρ in general does not converge to any limit. Actually, we have here incomplete self-similarity of the second type: as $\Pi_1 \to 0$ and $\Pi_2 \to 0$,

$$\begin{aligned}
&\Phi_p \to \Pi_2^{2\beta}\Phi_{p2}(\Pi_1\Pi_2^{-\beta}),\\
&\Phi_\rho \to \Phi_{\rho 2}(\Pi_1\Pi_2^{-\beta}),\\
&\Phi_v \to \Pi_2^{\beta}\Phi_{v2}(\Pi_1\Pi_2^{-\beta}),\\
&\Phi_f \to \text{const}\,\Pi_2^{\beta}.
\end{aligned} \tag{5.25}$$

Hence in accord with what was said above, (5.18) can, for small Π_1 and Π_2, again be written in the self-similar form

$$\begin{aligned}
&\Pi_p^* = \Phi_{p2}(\Pi^{**}), \quad \Pi_\rho^* = \Phi_{\rho 2}(\Pi^{**}),\\
&\Pi_v^* = \Phi_{v2}(\Pi^{**}), \quad \Pi_f^* = \text{const},
\end{aligned} \tag{5.26}$$

where

$$\begin{aligned}
&\Pi_p^* = \frac{p}{\left(\dfrac{E^2\rho_0^3}{t^6}\right)^{1/5}\Pi_2^{2\beta}}, \qquad \Pi_\rho^* = \frac{\rho}{\rho_0},\\
&\Pi_v^* = \frac{v}{\left(\dfrac{Et^{-3}}{\rho_0}\right)^{1/5}\Pi_2^{\beta}}, \qquad \Pi_f^* = \frac{r_f}{\left(\dfrac{Et^2}{\rho_0}\right)^{1/5}\Pi_2^{\beta}},\\
&\Pi^{**} = \Pi_1\Pi_2^{-\beta} = \frac{r}{\left(\dfrac{Et^2}{\rho_0}\right)^{\frac{1-\beta}{5}} R_0^{\beta}}.
\end{aligned} \tag{5.27}$$

However, now not only the dependent self-similar variables Π_p^*, Π_v^*, but also the independent self-similar variable, cannot be determined from dimensional considerations, since the constant β is unknown in advance and is found by solving a nonlinear eigenvalue problem. Moreover, the radius R_0 of the sphere within which the release of energy takes place at the initial moment appears explicitly in all the self-similar variables. From (5.26) we get similarity laws for the pressure and speed at the front of the shock wave and for the radius of the shock wave:

$$
\begin{aligned}
p_f &= \mathrm{const}_1\, \rho_0 \left(\frac{E}{\rho_0}\right)^{\alpha} R_0^{2\beta} t^{-2(1-\alpha)},\\
v_f &= \mathrm{const}_2 \left(\frac{E}{\rho_0}\right)^{\alpha/2} R_0^{\beta} t^{-(1-\alpha)},\\
r_f &= \mathrm{const}_3 \left(\frac{Et^2}{\rho_0}\right)^{\alpha/2} R_0^{\beta},\\
\alpha &= 2(1-\beta)/5.
\end{aligned}
\tag{5.28}
$$

As is evident, despite the fact that these laws have power form, it is impossible to get them by dimensional analysis. The situation is that the dimensions of the constant $A = \sigma E R_0^{5\beta/(1-\beta)}$ that determines these laws can be found only after solving a nonlinear eigenvalue problem, to which the construction of a self-similar solution in the large reduces.

The examples considered are instructive ones. When we turn to the solution of a certain problem, and in particular to the search for its self-similar solutions, we do not know in advance to which type belong the solutions of the degenerate formulation of the problems. Comparison of the ordinary and modified formulations of the problems considered above shows that the situation can be rather insidious: from the point of view of the possibility of applying dimensional analysis these problems do not differ from one another superficially. Hence, for example, it is extremely tempting to begin by obtaining similarity laws without turning to the solution of the equations. Arguing in the usual fashion, we might assume in the modified problems that since the initial removal of mass or release of energy occurs in a small region, the size of that region is inessential, i.e., assume complete self-similarity in the similarity parameter that corresponds to the initial length. From this would follow automatically the similarity laws (5.16) and (5.19) corresponding to complete self-similarity. But as a matter of fact, as we have seen, the similarity laws here are quite different, although still power laws. Therefore it is necessary to keep in mind that it is a very strong hypothesis to assume the unimportance of certain parameters that make the problem nondegenerate (in the examples considered, the lengths l and R_0). These governing parameters may be essential and self-similarity nevertheless hold. Distinguishing among possible cases of self-similarity requires, in fact, a sufficiently deep mathematical investigation, which is unattainable in serious nonlinear problems. Therefore in obtaining self-similar solutions or similarity laws on the basis of dimensional analysis one should take care to verify, if only by means of numerical calculations, that the solutions or similarity laws found actually reflect the asymptotic behavior of the solutions of the problem considered. The situation is immeasurably more complicated if a mathematical formulation of the problem is lacking; in this case, to verify the basic assumptions one must turn to experiments. The examples considered in the following chapters confirm the necessity of such precautions.

6

Self-Similar Solutions and Progressive Waves

1. Solutions of Progressive-Wave Type

In various problems of mathematical physics an important role is played by invariant solutions of the *progressive-wave* type. These are solutions for which the distributions of properties of the motion at different times are obtained from one another by a translation rather than by a similarity transformation as in the case of self-similar solutions. In other words, one can always choose a moving Cartesian coordinate system such that the distribution of properties of a motion of progressive-wave type is stationary in that system. One can reduce to a consideration of progressive waves the study of the structure of shock-wave fronts in gas dynamics [cf. Kochin, Kibel', and Roze (1964); Zel'dovich and Raizer (1967)] and in magneto-hydrodynamics [cf. Kulikovskiy and Lyubimov (1965): Kulikovskii and Lyubimov (1960a, b)], the structure of flame fronts [cf. Zel'dovich (1948)], the investigation of solitary and periodic waves in a plasma and on the surface of a heavy fluid [cf. Jeffrey and Kakutani (1972); Karpman (1975)], and many other problems. In recent years many processes have been studied, including the effects of the propagation of plasma fronts in electrical, electromagnetic, and light (laser) fields, the so-called waves of discharge propagation (Velikhov and Dykhne, 1965; Raizer, 1968; Bunkin, Konov, Prokhorov, and Fedorov, 1969). These processes also lead to the consideration of solutions of progressive-wave type [cf. Raizer (1977)].

In accord with the definition given above, solutions of progressive-wave type can be expressed in the form

$$v = V(x - X(t)) + V_0(t). \tag{6.1}$$

Here v is the property of the phenomenon being considered; x is the spatial Cartesian coordinate, an independent variable of the problem; t is another independent variable, for simplicity identified with time; and $X(t)$ and $V_0(t)$ are time-dependent translations along the x and v axes. In particular, if the properties of the process do not depend directly on time, so that the equations governing the process do not contain time explicitly, the progressive wave propagates uniformly:

$$v = V(x - \lambda t + c) + \mu t. \tag{6.2}$$

Here λ, μ, ν, and c are constants, where λ and μ represent the speeds of translation along the x and v axes. For an important class of steady progressive waves the distribution of properties in the wave remains unchanged in time, so that $\mu = 0$, and

$$v = V(x - \lambda t + c). \tag{6.3}$$

In particular, steady progressive waves describe the structure of shock waves and flames.

Solutions of progressive-wave type are closely connected with self-similar ones. Thus in (6.1) we set

$$\begin{aligned} v &= \ln u, \quad t = \ln \tau, \quad V_0(t) = \ln u_0(\tau), \\ V &= \ln U, \quad x = \ln \xi, \quad X(t) = \ln \xi_0(\tau), \end{aligned} \tag{6.4}$$

and obtain a representation of the progressive wave in the self-similar form

$$u = u_0(\tau)\, U\left(\xi/\xi_0(\tau)\right). \tag{6.5}$$

In particular, the expression (6.2) for a uniformly propagating progressive wave reduces to the self-similar form with power self-similar variables that has been encountered repeatedly in previous chapters,

$$u = B\tau^{\mu} U(\xi/A\tau^{\lambda}) \tag{6.6}$$

(where A and B are constants).

The simple connection noted here between self-similar solutions and progressive waves is well known; it has been used to simplify the study of some self-similar solutions [see for example Staniukovich (1960)]. Surprisingly, however, the connection between the classification of self-similar solutions presented in the preceding chapter and the known classification of steady progressive waves has long remained unnoticed.

In fact, as already recalled, steady progressive waves describe the structure of the front of shock waves, flames, and analogous regions of rapidly changing density, speed, and other properties of the motion that are described by surfaces of discontinuity when dissipative processes are neglected. One distinguishes two types of such fronts [see for example Sedov (1971)]. For one type (shock waves in a compressible gas, detonation waves, etc.) the speed of propagation of the front is found from conservation laws of mass, momentum, and energy only. The structure of such a front is adapted to the conservation laws in the sense that for one and the same speed of propagation of the front, dictated by the conservation laws, its thickness can be different depending on the character of the dissipative processes in the transition region and the magnitudes of the dissipative coefficients. Of course analysis of the structure of shock waves allows one to reject unrealizable situations such as shock waves of rarefaction, for which it is impossible to construct the structure, but basically the speed of propagation of the front is determined independently of the structure of the transition region.

For fronts of the second type (a flame, gaseous discharge, etc.) conservation laws become insufficient for the determination of the speed of the front. The speed of the front in waves of the second type is found as some eigenvalue in the course of constructing the structure of the front, that is, a solution of progressive-wave type of the equations describing the dissipative processes in the transition region.

It turns out that this classification of progressive waves corresponds to the classification of self-similar solutions discussed above. Here we consider the simplest examples of steady progressive waves of both types, after which we shall see the correspondence of the two classifications.

2. Burgers Shock Wave—Steady Progressive Wave of the First Kind

The Burgers equation

$$\partial_t v + v\partial_x v = \nu\partial_{xx}^2 v \tag{6.7}$$

is a successful though rather simplified mathematical model of the motion of a viscous, compressible gas. Here v is the speed, ν the kinematic viscosity, x the spatial coordinate, and t the time. If the viscous term is neglected, (6.7) assumes the form of the simplest model equation of gas dynamics,

$$\partial_t v + v\partial_x v = 0. \tag{6.8}$$

This last equation has a solution of the type of a uniformly propagating shock wave: $v = V(\zeta)$, $\zeta = x - \lambda t + c$, where $V(\zeta)$ is a step function, equal to v_1 for $\zeta > 0$ and to v_2 for $\zeta \leqslant 0$, with $v_1 < v_2$. The value of the speed of propagation $\lambda = \lambda_0$ is obtained from the law of conservation of momentum at the front of the discontinuity, corresponding to (6.8):

$$-\lambda_0(v_1 - v_2) + \frac{v_1^2 - v_2^2}{2} = 0, \tag{6.9}$$

whence we find

$$\lambda_0 = \frac{v_1 + v_2}{2}. \tag{6.10}$$

We now take into account the dissipative processes, that is, the viscosity, and return to (6.7). We construct a solution of the Burgers equation of progressive-wave type: $v = V(\zeta)$, $\zeta = x - \lambda t + c$. We have, substituting this expression for v into (6.7),

$$-\lambda \frac{dV}{d\zeta} + V \frac{dV}{d\zeta} = \nu \frac{d^2V}{d\zeta^2}, \tag{6.11}$$

whence, integrating and using the condition $V = v_1$ at $\zeta = \infty$, we find

$$\nu \frac{dV}{d\zeta} = -\lambda (V - v_1) + \frac{V^2 - v_1^2}{2}. \tag{6.12}$$

To satisfy the condition at the left endpoint, $V(-\infty) = v_2$, it is necessary to take

$$\lambda = \frac{v_1 + v_2}{2} = \lambda_0, \tag{6.13}$$

after which the solution is obtained in the form

$$\frac{\zeta}{\nu} = \frac{2}{v_2 - v_1} \ln \frac{v_2 - V}{V - v_1}. \tag{6.14}$$

This solution describes the structure of the transition region on the length scale $\nu/(v_2 - v_1)$ that is characteristic for this region. We see that the condition $v_2 > v_1$ imposed above is essential, since a solution describing the structure of the transition region of a wave with $V(-\infty) = v_2 < v_1 = V(\infty)$ does not exist. In fact with (6.13) taken into account, (6.12) assumes the form

$$\nu \frac{dV}{d\zeta} = -\frac{(v_2 - V)(V - v_1)}{2}. \tag{6.15}$$

Since V lies between v_1 and v_2, the right side of (6.15) is always negative, and the left side is negative only for $v_2 > v_1$.

Solutions of progressive-wave type with $\lambda = \lambda_0$ serve as asymptotic representations of the solutions of initial-value problems for the Burgers equation with initial data of transitional type,

$$v(x,\ 0) \equiv v_2 \quad (x \leqslant a); \quad v_1 < v(x,\ 0) < v_2 \ (a < x < b); \\ v(x,\ 0) \equiv v_1 \quad (x \geqslant b), \tag{6.16}$$

where a and b ($a < b$) are arbitrary real numbers, and the function $v(x, 0)$ is monotonically nonincreasing: $\partial_x v(x, 0) \leqslant 0$. This was proved rigorously by Oleinik (1957). As is evident, in the present case the value of the speed of propagation λ_0 is obtained from a conservation law and is independent of the structure of the wave, that is of the viscosity ν. As (6.14) shows, the viscosity determines only the spatial scale of the transition region, that is, "the width of the front."

The situation is completely analogous for shock waves in gases and detonation waves: the speed of propagation of these waves is determined from the laws of conservation of mass, momentum, and energy alone, and does not require for its determination any consideration of the wave structure. The latter determines only the width of the transition region.

3. Flame—Steady Progressive Wave of the Second Kind

We now consider progressive waves of the other type, for which the speed of propagation cannot be found from conservation laws alone, but is determined by analysis of the structure.

A rigorous mathematical investigation of progressive waves in nonlinear problems with dissipation was first undertaken in the fundamental work of Kolmogorov, Petrovskii, and Piskunov (1937), carried out in connection with a biological problem concerning the speed of propagation of a gene that has an advantage in the struggle for existence. To describe the structure of the transition zone near the border of the domains of habitation of genes of both types they obtained the nonlinear diffusion equation

$$\partial_t v - \varkappa \partial^2_{xx} v = F(v), \tag{6.17}$$

where v is the concentration of gene, and $F(v)$ is a continuous function that is differentiable the necessary number of times, defined in the interval $0 \leqslant v \leqslant 1$ and having, in accord with the physical meaning of the problem, the following properties:

$$F(0) = F(1) = 0; \quad F(v) > 0 \quad (0 < v < 1); \\ F'(0) = \alpha > 0; \quad F'(v) < \alpha \quad (0 < v < 1). \tag{6.18}$$

Under these conditions (6.17) has a solution of progressive-wave type, $v = V(\zeta)$, $\zeta = x - \lambda t + c$, satisfying the conditions $v(-\infty) = 1$, $v(\infty) = 0$ for all speeds of propagation greater than or equal to $\lambda_0 = 2(\varkappa\alpha)^{1/2}$ and arbitrary c. It is most essential that among these solutions only the solution corresponding to the lowest point in the range of possible speeds of propagation can be an asymptotic representation as $t \to \infty$ of solutions of the initial-value problem with conditions, just as in (6.16), of transitional type:

$$v(x, 0) \equiv 1 \quad (x \leqslant a); \quad 0 < v(x, 0) < 1 \quad (a < x < b); \\ v(x, 0) \equiv 0 \quad (x \geqslant b). \tag{6.19}$$

In other words, it turns out that direct consideration of solutions of progressive-wave type gives a continuous "spectrum" of possible speeds of propagation $\lambda \geqslant \lambda_0 = 2(\varkappa\alpha)^{1/2}$, but only the solution corresponding to the lowest point $\lambda = \lambda_0$ of this spectrum can be an asymptotic solution as $t \to \infty$ of the initial-value problem with conditions of transitional type; the remaining progressive waves are unstable. The quantity λ_0 determines the required speed of propagation of the gene that has an advantage in the struggle for existence.

We consider here in more detail the rather similar problem of thermal flame propagation in gaseous mixtures (Taffanel, 1913, 1914; Daniell, 1930; Zel'dovich and Frank-Kamenetskii, 1938a, b; Zel'dovich, 1948). We formulate the simplest schematization of the problem. Suppose that in the course of a reaction a component of a gaseous mixture, whose concentration we denote by n, is annihilated. The reaction rate q, that is, the mass of combustible matter annihilated in unit volume in unit time depends on the concentration n and the temperature u. We introduce the notation

$$q = \frac{1}{\tau} \Phi(n, u), \tag{6.20}$$

where Φ is a function having the dimensions of density, and τ is a constant of the dimensions of time–the characteristic time of the reaction–a quantity that is ordinarily very small. It is known from physical chemistry that the temperature dependence of the reaction rate is very strong: a small change in temperature greatly changes the reaction rate. We shall assume that the reaction is irreversible, so that $\Phi \geqslant 0$. Furthermore the original state of the gaseous mixture, $n = 1$, $u = u_1$, is assumed to be uniform and stable. For this, it is sufficient that the function $\Phi(n, u)$ be equal to zero not only for the initial temperature $u = u_1$ but also in some interval of temperature $u_1 \leqslant u \leqslant u_1 + \Delta$ close to it (the meaning of this condition will be elucidated below). It is obvious also that the reaction does not take place in the absence of combustible matter. Thus, it is assumed that the function $\Phi(n, u)$ satisfies the conditions

$$\Phi(n, u) \geqslant 0; \quad \Phi(n, u) \equiv 0, \quad 0 \leqslant n \leqslant 1, \ u_1 \leqslant u \leqslant u_1 + \Delta, \\ \Phi(0, u) = 0. \tag{6.21}$$

The speed of the gas motion due to the spreading of flame is small compared with the speed of sound; therefore we can neglect the compressibility of the gas and assume that the density of the gaseous mixture depends only on the temperature†: $\rho = \rho(u)$. Finally, the reaction is assumed to be exothermic: combustion yields a release of heat. We denote by Q the thermal effect of the reaction, that is, the amount of heat released upon combustion of a unit mass of combustible gas. In accord with what has been said, the system of basic equations of motion of the mixture of combustible gas and the products of combustion formed in the course of the reaction can be written in the form

$$
\begin{gathered}
\partial_t \rho v_i + \partial_\alpha \rho v_i v_\alpha = -\partial_i p, \\
\partial_t \rho + \partial_\alpha \rho v_\alpha = 0, \\
\rho = \rho(u), \\
\partial_t \rho n + \partial_\alpha \rho n v_\alpha = \partial_\alpha \rho D \partial_\alpha n - \frac{1}{\tau}\Phi(n, u), \\
\partial_t \rho \sigma u + \partial_\alpha \rho \sigma u v_\alpha = \partial_\alpha k \partial_\alpha u + \frac{Q}{\tau}\Phi(n, u).
\end{gathered}
\tag{6.22}
$$

The first three equations are the usual equations of motion of an incompressible fluid with density depending on temperature (where repeated Greek indices α indicate summation from $\alpha = 1$ to $\alpha = 3$). In addition there are the equation of balance of mass of combustible gas and the energy equation. In these equations the v_i are the components of the velocity vector of the mixture, p is the gas pressure, k the coefficient of thermal conductivity, D the diffusion coefficient, and σ the specific heat at constant pressure, and we will assume the last two coefficients to be constant.

The problem under consideration has two length scales that differ greatly in magnitude: the inner scale $L_1 = (D\tau)^{1/2}$ characterizing the size of the region in which the processes of chemical reaction, diffusion, and heat transfer occur, and the outer scale $L_2 = L$: the size of the container or combustion chamber, the diameter of the burner, etc. In view of the great disparity in these two scales it is natural to apply to this problem the method of matched asymptotic expansions (Van Dyke, 1975; Cole, 1968; Lagerstrom and Casten, 1972). We first consider the "outer" asymptotic expansion of the solution, that is, we change to dimensionless variables in which we take the outer scale L as the length scale and $L/(D/\tau)^{1/2}$ as the time scale. Then the equations of balance of mass of com-

†Without loss of generality we can neglect the difference between the density of the combustible gas and the density of the combustion products at the same temperature, so that the density of the mixture does not depend on the concentration.

bustible gas and of energy, the last two equations of (6.22), assume the form

$$
\begin{aligned}
&\varepsilon\,(\partial_\theta \rho n + \partial_\alpha \rho n V_\alpha) = \varepsilon^2 \partial_\alpha \rho \partial_\alpha n - \Phi\,(n,\ u),\\
&\varepsilon\,(\partial_\theta \rho c u + \partial_\alpha \rho c u V_\alpha) = \varepsilon^2 \partial_\alpha \frac{k}{D}\,\partial_\alpha u + Q\Phi\,(n,\ u),
\end{aligned}
\tag{6.23}
$$

where θ is the dimensionless "slow" time $\theta = t(D/\tau)^{1/2}/L$; $V_\alpha = v_\alpha/(D/\tau)^{1/2}$; the operator ∂_α is taken in the dimensionless spatial variables associated with L, and $\epsilon^2 = L_1^2/L^2 \ll 1$. Thus everywhere except in the narrow regions in which the gradients of temperature and concentration are large (of order $1/\epsilon$) we can assume the reaction rate to be equal to zero. The intermediate transition regions must necessarily be narrow, of relative width not greater than ϵ, since the changes of temperature and concentration in them are bounded and the gradient is of order $1/\epsilon$. Hence it follows that the whole region occupied by the gas splits (Fig. 6.1) into regions (1) of cold unburnt gas, where the reaction has not yet started since the gas has not yet managed to warm up; regions (2) occupied by the hot products of combustion, where the reaction no longer continues since all the combustible matter there has burned; and narrow transition regions where the combustion reaction is going on and transport processes–diffusion and heat transfer–are taking place. If we pass to the limit $\epsilon \to 0$, that is, to the first outer approximation, the transition regions become discontinuity surfaces on which occur jumps in speed, density, temperature, and concentration, but not in pressure. The speed of propagation through the gas of the discontinuity surface–the flame–(the normal speed of the flame) is not determined by the equations of motion and the conditions of balance of mass, momentum, and energy on the discontinuity surface. One can regard this quantity as a physical-chemical constant that is defined independently, for example as determined from experiment. Thus one obtains a closed system of relations of the so-called gas-dynamic theory of combustion [cf. Landau and Lifschitz (1959)]. For an analytic determination of the normal speed of propagation of the flame we must turn to the "inner" asymptotic expansion of the solution and consider the phenomena in the transition zone, taking $L_1 = (D\tau)^{1/2}$ as characteristic length scale and τ as characteristic time. We choose the direction of the normal

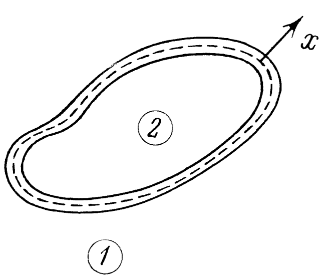


Figure 6.1. The region of motion splits into (1) a region occupied by the cold combustible mixture, (2) a region occupied by the products of combustion, and a narrow transition region in which the chemical reaction and the processes of diffusion and thermal conduction are occurring.

to the median of the transition zone (Fig. 6.1) as the direction of the coordinate x, reckoned from the median, and change to the dimensionless variable $\xi = x/(D\tau)^{1/2}$ referred to the scale L_1. Because of the narrowness of the transition zone, only derivatives with respect to ξ are of order unity. Derivatives with respect to the other spatial variable which would be of order unity in the outer scale L, are negligibly small in the new scale.

Keeping only leading terms in the equations for the balance of combustible matter and energy, we write these equations in the form

$$\begin{aligned} \rho\partial_\vartheta n + \rho V \partial_\xi n &= \partial_\xi \rho \partial_\xi n - \Phi(n, u), \\ \rho\sigma\partial_\vartheta u + \rho\sigma V \partial_\xi u &= \partial_\xi \frac{k}{D} \partial_\xi u + Q\Phi(n, u). \end{aligned} \tag{6.24}$$

Here $V = v_x/(D/\tau)^{1/2}$, v_x being the component of the velocity of the mixture along the x axis, and $\vartheta = t/\tau$ is the "faster" time, i.e., the dimensionless time referred to the scale τ. Furthermore, to the same approximation the equation of conservation of mass is written in the form

$$\partial_\vartheta \rho + \partial_\xi \rho V = 0. \tag{6.25}$$

We shall seek a solution of (6.24)-(6.25) of progressive-wave type:

$$n = N(\zeta), \quad u = U(\zeta), \quad V = V(\zeta), \quad \zeta = \xi - \lambda\vartheta + c, \tag{6.26}$$

where c is an arbitrary constant and λ is the speed of propagation of the progressive wave, which is unknown and subject to determination. Substituting (6.26) into (6.24) and (6.25), and keeping in mind that $\rho = \rho(U(\zeta)) = R(\zeta)$, we obtain for the determination of the unknown functions N, U, and V the system of ordinary differential equations

$$\begin{aligned} -\lambda R \frac{dN}{d\zeta} + RV \frac{dN}{d\zeta} &= \frac{d}{d\zeta} R \frac{dN}{d\zeta} - \Phi(N, U), \\ -\lambda R\sigma \frac{dU}{d\zeta} + RV\sigma \frac{dU}{d\zeta} &= \frac{d}{d\zeta} \frac{k}{D} \frac{dU}{d\zeta} + Q\Phi(N, U), \\ -\lambda \frac{dR}{d\zeta} + \frac{d}{d\zeta} RV &= 0. \end{aligned} \tag{6.27}$$

Integrating the last equation, we get

$$-\lambda R + RV = \text{const}. \tag{6.28}$$

The distributions of temperature, concentration, and velocity in the transition zone must satisfy the obvious boundary conditions: on one side of the transition zone, where it borders on the fresh combustible mixture, combustion has not yet begun, the gas is at rest, and its temperature is prescribed. On the other side of the transition zone combustible matter is fully burnt. According to standard asymptotic procedure, in view of the smallness of the inner scale L_1 compared with the outer L, the first boundary condition should be imposed at $\zeta = \infty$ and the second at $\zeta = -\infty$:

$$N(\infty)=1, \quad U(\infty)=u_1, \quad N(-\infty)=0, \quad V(\infty)=0. \tag{6.29}$$

Substituting these conditions into (6.28), we reduce this relation to the form

$$\lambda(\rho_0-R)+RV=0. \tag{6.30}$$

Here ρ_0 is the density of the fresh combustible mixture. Substituting (6.30) into the first two equations of (6.27), we reduce them to the form

$$\begin{aligned} -\lambda\rho_0\frac{dN}{d\zeta} &= \frac{d}{d\zeta}\rho\frac{dN}{d\zeta}-\Phi(N,\ U), \\ -\lambda\rho_0\frac{dU}{d\zeta} &= \frac{d}{d\zeta}\frac{k}{D\sigma}\frac{dU}{d\zeta}+\frac{Q}{\sigma}\Phi(N,\ U). \end{aligned} \tag{6.31}$$

It is known from physical chemistry that if the combustible matter and the products of combustion have close molecular weights, one can assume the quantity $k/\rho\sigma D$ to be equal to unity. Under this assumption, multiplying the first equation of (6.31) by Q/σ and adding the two equations we find that the system (6.31) has the integral

$$\frac{QN(\zeta)}{\sigma}+U(\zeta)=\text{const}, \tag{6.32}$$

which is called in the theory of combustion the Lewis-von Elbe similarity law for the fields of concentration and temperature. From (6.26) we find const $= u_1 + Q/\sigma$, and from this and (6.32) we obtain

$$U(-\infty)=u_1+Q/\sigma=u_2. \tag{6.33}$$

Using the similarity law (6.32), one can decompose (6.31) and reduce it to a single equation for the temperature,

$$\lambda\frac{dU}{d\zeta}+\frac{d}{d\zeta}\frac{\rho(U)}{\rho_0}\frac{dU}{d\zeta}+\Psi(U)\frac{\rho_0}{\rho(U)}=0. \tag{6.34}$$

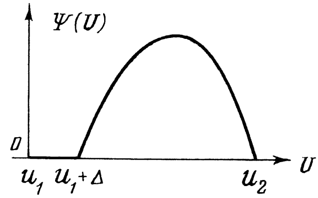


Figure 6.2. The function $\Psi(U)$ vanishes in some interval close to $U = u_1$.

Here the notation is

$$\Psi(U) = \frac{Q}{\rho_0 \sigma} \Phi\left(1 - \frac{\sigma}{Q}(U - u_1),\ U\right) \frac{\rho(U)}{\rho_0}. \tag{6.35}$$

By assumption, $\Phi(n, u) \equiv 0$ for $u_1 \leqslant u \leqslant u_1 + \Delta$.† Hence, and since $\rho(U)$ is positive and bounded, $\Psi(U)$ is identically equal to zero in the interval $u_1 \leqslant U \leqslant u_1 + \Delta$, vanishes for $U = u_2$, and is positive for $u_1 + \Delta < U < u_2$ (Fig. 6.2). Setting $p = (\rho(U)/\rho_0)\, dU/d\zeta$ and taking U as independent variable, we reduce (6.34) to the form

$$p \frac{dp}{dU} + \lambda p + \Psi(U) = 0. \tag{6.36}$$

It follows from (6.26) that the solution of (6.36) of interest to us satisfies the obvious condition that the heat flux vanishes on the boundaries of the transition zone:

$$\frac{dU}{d\zeta} = 0 \text{ for } \zeta = \pm\infty. \tag{6.37}$$

From this and (6.26) we obtain boundary conditions for the function $p(U, \lambda)$:

$$p = 0, \quad U = u_1; \quad p = 0, \quad U = u_2. \tag{6.38}$$

4. Nonlinear Eigenvalue Problem

We have again, as in the case of self-similar solutions of the second kind, a nonlinear eigenvalue problem: (6.36) is an equation of the first order and (6.38) gives two boundary conditions. We shall show, following Zel'dovich (1948), that

†As remarked above, this condition guarantees the stability of the original state. In fact, let us set $\partial_\xi n$ and $\partial_\xi u$ identically equal to zero in (6.24), so that n and u depend only on time. Then, if the assumed condition is satisfied, a small change in the temperature of the gas mixture does not cause a reaction to start.

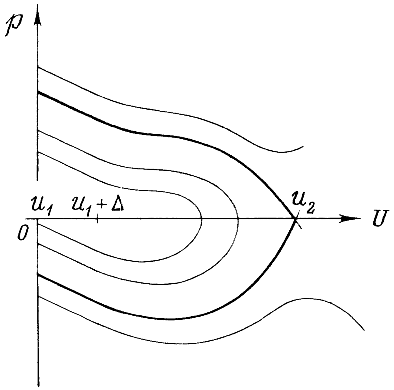


Figure. 6.3. Phase portrait: picture of the integral curves of the first-order equation (6.36).

there exists a unique eigenvalue λ for which the desired solution exists. We consider the phase portrait of (6.36) in the region of interest to us in the U, p plane (Fig. 6.3). At $U = u_2$ and $p = 0$, (6.36) has a singular point of saddle type. Through this singular point pass two separatrices with slopes $-\lambda/2 \pm [\lambda^2/4 - \Psi'(u_2)]^{1/2}$; since $\Psi'(u_2) < 0$, the slope of one of the separatrices is positive, the other negative. It is clear that only the separatrices can satisfy the second equation of (6.38). Furthermore for $\lambda = 0$ (6.36) can be integrated in finite form: solutions satisfying (6.38) for $U = u_2$ have the form

$$p = \pm \sqrt{2\int_U^{u_2} \Psi(u)\, du}\,, \tag{6.39}$$

so that the ordinates of the points of intersection of the corresponding integral curves with the vertical axis are

$$p_1 = \sqrt{2\int_{u_1}^{u_2} \Psi(u)\, du} > 0, \quad p_2 = -\sqrt{2\int_{u_1}^{u_2} \Psi(u)\, du} < 0. \tag{6.40}$$

We now consider the function $q(U, \lambda) = \partial_\lambda p$ for all solutions of (6.36) satisfying the second condition of (6.38). It is clear that $q(u_2, \lambda) \equiv 0$ since $p(u_2, \lambda) \equiv 0$. Differentiating (6.36), we get for the function q the equation

$$\frac{dq}{dU} = -1 + \frac{\Psi(U)}{p^2}\, q. \tag{6.41}$$

Close to the point $U = u_2$ the separatrices behave, according to the above, like $p = (U - u_2)\{-(\lambda/2) \pm [(\lambda^2/4) - \Psi'(u_2)]^{1/2}\}$. Differentiating with respect to λ, we find that the corresponding curves $q(U, \lambda)$ behave close to $U = u_2$ like $q = K(U - u_2)$, where

$$K = \left\{ -\frac{1}{2} \pm \frac{\lambda}{2\sqrt{\lambda^2 - 4\Psi'(u_2)}} \right\} < 0$$

for both separatrices, that is, $q > 0$ for $U < u_2$. Furthermore there cannot be an intersection of the curve $q(U, \lambda)$ with the axis $q = 0$ at some intermediate point between u_1 and u_2, because at a point of intersection one would have $dq/dU = -1$, which is geometrically impossible. Thus, $q(u_1 + \Delta, \lambda) > 0$. But for $u_1 \leqslant U \leqslant u_1 + \Delta$ we have $\Psi(U) \equiv 0$, and from this and (6.41) we get $q(u_1, \lambda) = q(u_1 + \Delta, \lambda) + \Delta > \Delta$. Since

$$p(u_1, \lambda) = p(u_1, 0) + \int_0^\lambda q(u_1, \lambda)\, d\lambda > -\sqrt{2\int_{u_1}^{u_2} \Psi(u)\, du} + \lambda\Delta, \tag{6.42}$$

it follows that one can find a value $\lambda = \lambda_0$, and moreover only one, such that the lower separatrix reaches the point $p = 0$, $U = u_1$, that is, satisfies all the conditions of the problem.

Thus, the existence and uniqueness of the solution of the nonlinear eigenvalue problem is proved. Using the methods developed by Kolmogorov, Petrovskii, and Piskunov (1937), Kanel' (1962) proved that this solution is an asymptotic representation as $t \to \infty$ of the solutions of a certain naturally defined class of initial-value problems with transitional type conditions. We note that in the problem of gene propagation as well as in the problem of the theory of flame propagation, direct construction of a solution of progressive-wave type $u = U(\xi - \lambda\vartheta + c)$ determines the solution to within the constant c. This last constant can be found only by matching an invariant solution with a noninvariant solution of the original problem. Here it is obvious that no matter what intermediate state of the system $V(\xi, \vartheta)$, $u(\xi, \vartheta)$, $n(\xi, \vartheta)$ we have taken as initial, the value of the constant c is unchanged. In this sense the constant c is an integral of the equations of the problem considered [cf. Lax (1968)].

The eigenvalue λ_0 obtained, when expressed in the original dimensional variables, also determines the speed of flame propagation:

$$m = \lambda_0 \sqrt{\frac{D}{\tau}}. \tag{6.43}$$

Furthermore, since the "faster" time was involved in the problem of determining the speed of the progressive wave, it is clear that in the natural outer time scale passage to the asymptotics occurs in fact very quickly, and the "preasymptotic" evolution of the solution is of no value.

We have demonstrated here the well-known fact that there exist two types of steady progressive waves. As was already mentioned, "external" conservation laws suffice to determine the speed of propagation of waves of the first kind, but they are insufficient to determine the speed of propagation of waves of the second kind, and it is necessary there to invoke the internal structure of the waves. The speed of propagation of progressive waves of the second kind is determined by the condition for the existence of internal structure in the large, that is, the condition of existence of a solution of progressive-wave type to the equations of motion in the transition region that satisfies the boundary conditions on the boundaries of this region.†

This situation corresponds exactly to the classification of self-similar solutions considered above. In fact, for a solution of type

$$u = U(x - \lambda t + c),$$

we again set $x = \ln \xi$, $t = \ln \tau$, $c = -\ln A$. Then this solution can be written, as we have seen already, in the form

$$U(x - \lambda t + c) = U\left(\ln \frac{\xi}{A\tau^\lambda}\right) = U_1\left(\frac{\xi}{A\tau^\lambda}\right), \tag{6.44}$$

that is, in self-similar form. It is obvious that the classification of solutions of progressive-wave type formulated above carries over into the language of self-similar solutions. In particular, the exponent λ in the expression for the self-similar variable corresponds to the speed of propagation in solutions of progressive-wave type. Therefore the classification of solutions of progressive-wave type into solutions for which the speed of propagation can be found from conservation laws at the shock front alone, and solutions for which this speed is obtained from the conditions for existence in the large of the inner structure corresponds to the classification of self-similar solutions into solutions of the first and second kinds. The correspondence of self-similar solutions and progressive waves will be used more than once in what follows.

†Sometimes (see below) the speed of propagation is determined nonuniquely upon consideration of the structure. This means that it depends on the initial conditions of the original problem, the asymptotic solution of which serves as the progressive wave.

7

Self-Similarity and Transformation Groups

1. Dimensional Analysis and Transformation Groups

Dimensional analysis, as already mentioned, has a transparent group character. Group considerations can turn out to be useful also in those cases when dimensional analysis becomes insufficient to establish self-similarity and determine self-similar variables.

We recall, first of all, the definition of a transformation group. Suppose we have a set of transformations with k parameters,

$$x'_\nu = f_\nu(x_1, \ldots, x_n; \quad A_1, \ldots, A_k), \tag{7.1}$$

where the $f_\nu (\nu = 1, \ldots, n)$ are smooth functions of their variables in some domain. One says that this set forms a k-parameter group of transformations if the following conditions are satisfied:

(1) Among the transformations (7.1) there exists the identity transformation.

(2) For each transformation of (7.1) there exists an inverse transformation that also belongs to the set (7.1).

(3) For each pair of transformations, a transformation A with parameters $A_1, \ldots, A_k$ and a transformation B with parameters $B_1, \ldots, B_k$, a transformation C with parameters $C_1, \ldots, C_k$ that also belongs to the set (7.1) is uniquely determined by the relations

$$C_i = C_i(A_1, \ldots, A_k; \quad B_1, \ldots, B_k)$$

such that successive realization of the transformations A and B is equivalent to the transformation C. The transformation C is called the product of transformations A and B.

Dimensional analysis is based on the Π-theorem, which was considered in detail in Chapter 1. This theorem allows one to express a function of n variables as a function of dimensional quantities

$$a = f(a_1, \ldots, a_k, a_{k+1}, \ldots, a_n) \tag{7.2}$$

in terms of a function of $n-k$ variables (k being the number of governing parameters with independent dimensions) that represents the relationship (7.2) in the form of a relation among dimensionless quantities:

$$\Pi=\Phi(\Pi_1,\ \Pi_2,\ \ldots,\ \Pi_{n-k}),$$

where

$$\Pi=\frac{a}{a_1^p a_2^q \ldots a_k^r},$$

$$\Pi_1=\frac{a_{k+1}}{a_1^{p_{k+1}} a_2^{q_{k+1}} \ldots a_k^{r_{k+1}}}, \ldots.$$

We note now that for any positive $A_1, \ldots, A_k$ the similarity transformation of the governing parameters with independent dimensions,

$$a_1'=A_1a_1,\quad a_2'=A_2a_2,\ \ldots,\ a_k'=A_ka_k, \tag{7.3}$$

can be obtained by changing from the original system of units of measurement to some other system of units of measurement belonging to the same class. Here the values of the remaining parameters $a, a_{k+1}, \ldots, a_n$ vary in accord with their dimensions in the following way:

$$\begin{aligned} a' &= A_1^p A_2^q \ldots A_k^r a, \\ a_{k+1}' &= A_1^{p_{k+1}} A_2^{q_{k+1}} \ldots A_k^{r_{k+1}} a_{k+1}, \\ &\ldots\ldots\ldots\ldots \\ a_n' &= A_1^{p_n} A_2^{q_n} \ldots A_k^{r_n} a_n. \end{aligned} \tag{7.4}$$

Direct verification shows easily that the transformations (7.3), (7.4) form a k-parameter group. The quantities $\Pi, \Pi_1, \ldots, \Pi_{n-k}$ remain unchanged for all transformations of the group (7.3), (7.4), i.e., they are invariants of this group. Thus, the Π-theorem is a simple consequence of the principle of invariance of relations with physical meaning among dimensional quantities of the form (7.2) with respect to the group of similarity transformations of the governing parameters with independent dimensions (7.3), (7.4). In fact, if invariance holds, all such relations must be representable in the form of relations between invariants of the group (7.3), (7.4). The number of independent invariants of the group is obviously smaller than the total number of governing and governed parameters by the number k of parameters of the group.

The invariance of the formulation, and hence of the solution, of any physically meaningful problem with respect to the group of transformations (7.3), (7.4) is necessary. It can turn out, however, that there exists a richer group with respect to which the formulation of the problem considered is invariant. Then the number of arguments of the function Φ in the universal (invariant) relation obtained after applying the Π-theorem in its own right should be reduced by the number of parameters of the supplementary group. Here the solution can turn out to be self-similar and the self-similar variables can be determined as a result of using the invariance with respect to the supplementary group, although this self-similarity is not implied by dimensional analysis (which exploits invariance with respect to the group of similarity transformations of quantities with independent dimensions). We consider some instructive examples.

2. The Boundary Layer on a Flat Plate

The boundary-layer problem for high-Reynolds-number flow past a semi-infinite plate placed along a uniform stream (the Blasius problem) leads to the system of equations (Kochin, Kibel', and Roze, 1964; Landau and Lifschitz, 1959)

$$u\partial_x u+v\partial_y u=\nu\partial^2_{yy}u, \quad \partial_x u+\partial_y v=0 \tag{7.5}$$

(x and y being the longitudinal and transverse coordinates, u and v the corresponding velocity components, and ν the kinematic coefficient of viscosity) under the boundary conditions

$$u(0, y)=U, \quad u(x, \infty)=U, \quad u(x, 0)=v(x, 0)=0 \tag{7.6}$$

(U being the constant speed of the exterior flow). By comparison the full Navier–Stokes equations governing the momentum balance for a plane viscous flow at arbitrary Reynolds number have the form

$$u\partial_x u + v\partial_y u = -\frac{1}{\rho}\,\partial_x p + \nu(\partial^2_{xx}u + \partial^2_{yy}u),$$

$$u\partial_x v + v\partial_y v = -\frac{1}{\rho}\,\partial_y p + \nu(\partial^2_{xx}v + \partial^2_{yy}v).$$

Among the governing parameters ν, x, U, and y in the boundary-layer problem, only two have independent dimensions: $[\nu]=L^2T^{-1}$, $[x]=L$, $[U]=LT^{-1}$, $[y]=L$. Hence direct application of dimensional analysis gives

$$\begin{gathered} u=U\Phi_1(\Pi_1,\ \Pi_2), \quad v=U\Phi_2(\Pi_1,\ \Pi_2), \\ \Pi_1=\xi=Ux/\nu, \quad \Pi_2=\eta=Uy/\nu, \end{gathered} \tag{7.7}$$

and (7.5)-(7.6) reduce to the form

$$\begin{gathered}\Phi_1\partial_\xi\Phi_1+\Phi_2\,\partial_\eta\Phi_1=\partial^2_{\eta\eta}\Phi_1,\\ \partial_\xi\Phi_1+\partial_\eta\Phi_2=0,\\ \Phi_1(0,\ \eta)=\Phi_1(\xi,\ \infty)=1,\quad \Phi_1(\xi,\ 0)=\Phi_2(\xi,\ 0)=0,\end{gathered}\tag{7.8}$$

so that direct application of dimensional analysis does not give any simplification of the problem. Now let $\Phi_1(\xi, \eta)$ and $\Phi_2(\xi, \eta)$ be a solution of (7.8), which by hypothesis exists and is unique. Simple verification shows that for any positive α the functions $\Phi_1(\alpha^2\xi, \alpha\eta)$, $\alpha\Phi_2(\alpha^2\xi, \alpha\eta)$ also satisfy the equations and all the conditions of the boundary-layer problem, although not of the full Navier–Stokes problem. Thus the formulation of the boundary-layer problem has turned out to be invariant with respect to the one-parameter group of transformations

$$\Phi_1'=\Phi_1(\xi,\ \eta),\quad \Phi_2'=\alpha^{-1}\Phi_2(\xi,\ \eta),\quad \xi'=\alpha^2\xi,\quad \eta'=\alpha\eta.$$

In view of the assumed uniqueness, the solution too must be invariant with respect to the same group of transformations, i.e., for any α the functions Φ_1 and Φ_2 must satisfy the relations

$$\Phi_1(\xi,\ \eta)=\Phi_1(\alpha^2\xi,\ \alpha\eta),\quad \Phi_2(\xi,\ \eta)=\alpha\Phi_2(\alpha^2\xi,\ \alpha\eta).\tag{7.9}$$

Since in (7.9) α can be taken equal to any positive number, we get, setting $\alpha = 1/\xi^{1/2}$,

$$\begin{gathered}\Phi_1(\xi,\ \eta)=\Phi_1\left(1,\ \frac{\eta}{\sqrt{\xi}}\right)=f_1\left(\frac{\eta}{\sqrt{\xi}}\right)=f_1\left(\frac{y}{\sqrt{\nu x/U}}\right),\\ \Phi_2(\xi,\ \eta)=\frac{1}{\sqrt{\xi}}\Phi_2\left(1,\ \frac{\eta}{\sqrt{\xi}}\right)=\frac{1}{\sqrt{\xi}}f_2\left(\frac{\eta}{\sqrt{\xi}}\right)=\sqrt{\frac{\nu}{Ux}}\,f_2\left(\frac{y}{\sqrt{\nu x/U}}\right).\end{gathered}\tag{7.10}$$

Thus, the self-similarity of the solution to the problem is proved and expressions for the self-similar variables are obtained, however no longer as a result of dimensional considerations, but only as a result of invariance of the formulation of the problem with respect to a group of transformations that is broader than the group of similarity transformations of the quantities with independent dimensions.

The example just considered is instructive in that the application of more general groups of transformations can here be given the form of a use of dimensional analysis, and this device turns out to be useful in many cases. Namely, we shall use different units to measure length in the x-direction and length in the y-direction, i.e., we introduce two dimensions of length, L_x and L_y. This is possible for the boundary-layer equations, in contrast to the full Navier–Stokes equations. (In the Navier–Stokes equations the term $\nu\partial^2_{yy}u$ appears in sum with the term $\nu\partial^2_{xx}u$, and if we measure x and y in different units these terms will have different dimensions.) Here it is necessary to take

$$\begin{aligned} &[u]=[U]=L_xT^{-1}, \quad [\nu]=L_y^2T^{-1}, \\ &[v]=L_yT^{-1}, \quad [x]=L_x, \quad [y]=L_y, \end{aligned} \tag{7.11}$$

so that both in the boundary-layer equations and in the boundary conditions of the problem all terms will have identical dimensions. Thus among the governing parameters no longer two but rather three have independent dimensions, and the single independent dimensionless similarity parameter will be

$$\Pi_1'=\frac{y}{\sqrt{\nu x/U}}=\zeta, \tag{7.12}$$

whence follows also the self-similarity of the solution of the problem:

$$u=Uf_1(\zeta), \qquad v=\sqrt{\frac{\nu U}{x}}\,f_2(\zeta).$$

Introducing the new function $\varphi(\zeta)=\int_0^\zeta f_1(\zeta)\,d\zeta$, we easily get from (7.5) and (7.6)

$$f_2=\frac{1}{2}(\zeta\varphi'-\varphi),$$

$$\varphi\varphi''+2\varphi'''=0, \quad \varphi(0)=\varphi'(0)=0, \quad \varphi'(\infty)=1,$$

i.e., a nonlinear boundary-value problem for a third-order ordinary differential equation. For the drag R of a section of unit width and length l of the flat plate in a uniform stream of velocity U we get from the previous relations, using the results of the numerical calculation of the function φ,

$$\begin{aligned} R&=2\int_0^l(\sigma_{xy})_{y=0}\,dx=2U\sqrt{\frac{U}{\nu}}\,\rho\nu\int_0^l f_1'(0)\frac{dx}{\sqrt{x}}= \\ &=\frac{4U^{3/2}\sqrt{l}}{\sqrt{\nu}}\,\rho\nu\varphi''(0)=1.328\rho\sqrt{\nu lU^3}. \end{aligned}$$

Here $(\sigma_{xy})_{y=0}$ is the shear stress on the plate. For more details see Kochin, Kibel', and Roze (1964); Landau and Lifschitz (1959); or Schlichting (1968).

Introducing the dimensionless parameter $\Pi = R/\rho U^2 l$ corresponding to R, we get

$$\Pi = \frac{1.328}{\sqrt{\mathrm{Re}}}, \quad \mathrm{Re} = \frac{Ul}{\nu}.$$

We note in passing that one can also look at this well-known relation as an incomplete self-similarity in Reynolds number. In fact, the drag R is determined by the following quantities: the length l of the section of the plate, the viscosity ν and density ρ of the fluid, and the velocity U of the stream. Application of the standard procedure of dimensional analysis gives

$$\Pi = \Phi\,(\mathrm{Re}).$$

For the high-Reynolds-number properties of the boundary layer there is no complete self-similarity with respect to Reynolds number, since there does not exist a nonzero limit of the function $\Phi = 1.328\ \mathrm{Re}^{-1/2}$ as $\mathrm{Re} \to \infty$. Hence the relations

$$\Pi = \mathrm{const}, \quad R = \mathrm{const}\,\rho U^2 l$$

that would have to hold in the case of complete self-similarity in the Reynolds number must not be expected, no matter how high the Reynolds number. Nevertheless, one has the relation

$$\Pi^* = \frac{R}{\rho U^{3/2}(l\nu)^{1/2}} = \mathrm{const} = 1.328,$$

corresponding to incomplete self-similarity: the parameter Π^* cannot be obtained from dimensional analysis and contains the dimensional parameter ν whose explicit introduction into the problem violates self-similarity.

3. Limiting Self-Similar Solutions

So-called limiting self-similar solutions,† i.e., solutions of the form $e^{\alpha t} f(xe^{\beta t})$, in which both the length scale and the scale of the governed quantity depend exponentially on the time, serve as interesting examples of the use of more general group considerations. We consider these solutions for the equation of nonlinear heat conduction,

$$\partial_t u = \varkappa \partial_{xx}^2 u^{n+1}, \quad n > 0. \tag{7.13}$$

†Sometimes these solutions are called limiting to self-similar, which is unfortunate in our view, since they themselves are also self-similar.

An appropriate degenerate problem is considered in the semi-infinite domain $x \geqslant 0$ for $t > -\infty$. We seek a solution of (7.13) that satisfies the conditions

$$u(x, -\infty) = 0, \quad u(0, t) = u_0 e^{\sigma t}. \tag{7.14}$$

Application of dimensional analysis gives, as is easy to prove using the standard procedure,

$$u = u_0 \Phi \left[x/(\varkappa\sigma^{-1}u_0^n)^{1/2}, \ \sigma t\right]. \tag{7.15}$$

We now observe that the problem formulated is invariant also with respect to the group of transformations of translation in time. This means that if $u(x, t, u_0, \sigma)$ is a solution of (7.13)-(7.14), then $u(x, t - \tau, u_0 e^{\sigma\tau}, \sigma)$ is also a solution of the same problem for any real τ. In fact, substituting $t' = t + \tau$ in (7.13) and (7.14), we get the same problem for the determination of u as a function of the variables x and t', but in place of u_0 will appear $u_0' = u_0 e^{\sigma\tau}$. Hence, from the uniqueness of the solution and from (7.15) it follows that for any τ,

$$\begin{aligned} u(x, t) &= u_0 \Phi \left[x/(\varkappa\sigma^{-1}u_0^n)^{1/2}, \ \sigma t\right] = u(x, t') = \\ &= u_0 e^{\sigma\tau} \Phi \left[x/(\varkappa\sigma^{-1}u_0^n e^{n\sigma\tau})^{1/2}, \ \sigma t - \sigma\tau\right]. \end{aligned}$$

But this means that the function Φ satisfies the invariance relation

$$\Phi(\xi, \eta) = e^{\sigma\tau}\Phi\left(\xi e^{-n\sigma\tau/2}, \ \eta - \sigma\tau\right) \tag{7.16}$$

for any τ. Setting $\tau = \eta/\sigma$ we get

$$\Phi(\xi, \eta) = e^{\eta}\Phi\left(\xi e^{-n\eta/2}, 0\right) = e^{\eta} f\left(\xi e^{-n\eta/2}\right), \tag{7.17}$$

from which now follows the self-similarity of the solution of (7.13)-(7.14):

$$u = u_0 e^{\sigma t} f\left[x/(\varkappa\sigma^{-1}u_0^n e^{n\sigma t})^{1/2}\right]. \tag{7.18}$$

The name of these solutions–limiting self-similar–is explained in the following way. Equation (7.13) has a family of self-similar solutions of ordinary power-law form satisfying the conditions

$$u(x, t_0) = 0, \quad u(0, t) = \mu(t - t_0)^{\alpha} \quad (t > t_0). \tag{7.19}$$

It is easy to show, using the standard procedure of dimensional analysis, that these solutions can be expressed in the form

$$u = \mu(t - t_0)^{\alpha} f_{\alpha}\left(x/[\varkappa\mu^n (t - t_0)^{\alpha n + 1}(\alpha + 1)]^{1/2}\right) \tag{7.20}$$

[the factor $(\alpha+1)$ having been introduced in the self-similar variable for convenience], where the function f_α is a solution of the equation

$$\frac{d^2 f_\alpha^n}{d\xi^2}+\frac{1}{2}\xi\frac{df_\alpha}{d\xi}-\frac{\alpha}{\alpha+1}f_\alpha=0 \tag{7.21}$$

that satisfies the conditions $f_\alpha(0)=1, f_\alpha(\infty)=0$ and is continuous, has continuous derivative $df_\alpha^n/d\xi$, and is in fact identically equal to zero for ξ greater than some $\xi_0(\alpha)<\infty$. We now choose $t_0=-\alpha/\sigma$, where σ is a constant having the dimensions of inverse time, and we let α tend to infinity while keeping $\mu(\alpha/\sigma)^\alpha$ constant and equal to u_0. It is easy to see that the power $(t-t_0)^\alpha=(\alpha/\sigma)^\alpha(1+\sigma t/\alpha)^\alpha$ here tends to an exponential, and the solution (7.20) tends to the solution (7.18) as its limit.

Solutions of the form $e^{\alpha t}f(xe^{\beta t})$ have appeared in various problems of mechanics, starting with the paper of Goldstein (1939) devoted to the theory of the boundary layer. The group analysis of these solutions given above and the explanation of their limiting character was carried out by Barenblatt (1954).

4. Rotation of Fluid in a Cylindrical Container

An instructive example of a self-similar solution for which considerations of dimensional analysis are insufficient for establishing the self-similarity is provided by the remarkable problem of S. L. Sobolev of small perturbations of a rotating fluid in a cylindrical container (Sobolev, 1954). The equation for pressure perturbation in this problem has, as Sobolev showed, the form

$$\partial_{tt}^2\Delta p+\omega^2\partial_{zz}^2 p=0. \tag{7.22}$$

Here t is the time, z is the coordinate measured along the axis of rotation, $\Delta=\partial_{\rho\rho}^2+(1/\rho)\,\partial_\rho+\partial_{zz}^2$ is the Laplace operator, $\rho=(x^2+y^2)^{1/2}$, x and y are rectangular coordinates in the plane perpendicular to the axis of rotation, and ω is the angular velocity of the rotation.

By the first fundamental solution of (7.22) is meant the solution satisfying the initial conditions

$$p(x,\ y,\ z,\ 0)=\frac{Q}{r},\qquad \partial_t p(x,\ y,\ z,\ 0)=0, \tag{7.23}$$

where $Q=$ const and $r=(\rho^2+z^2)^{1/2}$. The desired solution depends on the governing parameters ρ, Q, z, ω, and t, whose dimensions are

$$[Q]=[p]\,L,\quad [\rho]=[z]=L,\quad [\omega]=T^{-1},\quad [t]=T. \tag{7.24}$$

Furthermore dimensional analysis gives, as is easy to show,

$$p=\frac{Q}{\rho}\Phi(\xi,\ \eta), \qquad \xi=\frac{\rho}{z}, \qquad \eta=\omega t. \tag{7.25}$$

Substituting (7.25) into (7.22) and (7.23) and integrating, taking into account the regularity of the solution at the axis of rotation,† we reduce those relations to the form

$$\begin{aligned}&\xi^2\partial^3_{\xi\eta\eta}\Phi+\xi^2\partial_\xi\Phi+\xi\partial_\xi\left(\frac{1}{\xi}\,\partial^2_{\eta\eta}\Phi\right)=0,\\ &\Phi(\xi,\ 0)=\frac{\xi}{\sqrt{1+\xi^2}},\qquad \partial_\eta\Phi(\xi,\ 0)=0.\end{aligned} \tag{7.26}$$

Since the combination $\zeta=\xi/(1+\xi^2)^{1/2}$ appears on the right side of one of the conditions, it is convenient to take it as independent variable (it would appear automatically if we introduced r instead of z as the governing parameter), and to denote $\Phi(\xi,\eta)$ by $\Psi(\zeta,\eta)$. Then (7.26) assumes the form

$$\begin{aligned}&\zeta^2\partial_\zeta\Psi+\partial^2_{\eta\eta}\zeta\partial_\zeta(\Psi/\zeta)=0,\\ &\Psi(\zeta,\ 0)=\zeta,\qquad \partial_\eta\Psi(\zeta,\ 0)=0.\end{aligned} \tag{7.27}$$

If $\Psi(\zeta,\eta)$ is a solution of the problem then, as is easily verified, $\alpha^{-1}\Psi(\alpha\zeta,\alpha^{-1}\eta)$ also satisfies all the conditions of the problem for arbitrary positive α. By virtue of the uniqueness of the solution, it follows from this that the function $\Psi(\zeta,\eta)$ satisfies the invariance relation

$$\Psi(\zeta,\ \eta)=\alpha^{-1}\Psi(\alpha\zeta,\ \alpha^{-1}\eta) \tag{7.28}$$

for any $\alpha>0$. We now set $\alpha=1/\zeta$ and obtain

$$\Psi(\zeta,\ \eta)=\zeta\Psi(1,\ \zeta\eta)=\zeta\Xi(\zeta\eta), \tag{7.29}$$

i.e., the function $\Psi(\zeta,\eta)$ can be represented by a function of one variable. Substituting (7.29) into (7.25) and returning to the original variables, we get

$$p=\frac{Q}{r}\,\Xi\left(\frac{\omega\rho t}{r}\right), \tag{7.30}$$

so that the first fundamental solution of Sobolev's equation (7.22) actually turns out to be self-similar. The substitution of (7.30) into (7.22) and (7.23)

†The requirement of regularity includes also the vanishing of $\partial^2_{\eta\eta}\Phi(0,\eta)$, the coefficient of the "cylindrical" generalized function $\Delta(1/\rho)$, which is obtained when (7.25) is substituted into (7.22). The solution (7.31) satisfies this condition.

easily allows one to determine an expression for the function Ξ in terms of Bessel functions of order zero:

$$\Xi = J_0(\omega\rho t/r), \quad p = \frac{Q}{r} J_0(\omega\rho t/r). \tag{7.31}$$

The use of invariance with respect to a wider group for the proof of self-similarity and the determination of self-similar variables can here too be given the form of an application of dimensional analysis; the simple device applied below is also often useful. Namely, we write (7.27) in the form

$$\begin{gathered} \zeta^2\partial_\zeta\Psi + \lambda^2\partial^2_{\eta\eta}\zeta\partial_\zeta(\Psi/\zeta) = 0, \\ \Psi(\zeta, 0) = \mu\zeta, \quad \partial_\eta\Psi(\zeta, 0) = 0, \end{gathered} \tag{7.32}$$

and temporarily forget that the quantities Ψ, ζ, η are dimensionless, and λ and μ are equal to one. On the contrary, we assume that ζ has some dimensions Z, η has dimensions H, and Ψ has dimensions $[\Psi]$. Then in order that all terms of (7.32) have identical dimensions, it is necessary that the dimensions of λ and μ be the following:

$$[\mu] = [\Psi]Z^{-1}, \quad [\lambda] = ZH. \tag{7.33}$$

The solutions Ψ, as follows from (7.32), can depend only on ζ, η, λ, μ, whence we obtain by means of the standard procedure of dimensional analysis,

$$\Psi = \mu\zeta\Phi\left(\frac{\lambda}{\zeta\eta}\right) = \mu\zeta\Xi\left(\frac{\zeta\eta}{\lambda}\right). \tag{7.34}$$

Setting $\mu = \lambda = 1$, we again get (7.29).

The examples given above show how establishing the invariance of a problem with respect to a certain group of continuous transformations allows one to decrease the number of arguments of the function, just as do considerations of dimensional analysis, which are based on invariance with respect to a subgroup of the group of similarity transformations. Therefore of fundamental value is the idea developed by Birkhoff† (1960) of generalized inspectional analysis of the equations of mathematical physics, i.e., the idea of looking for groups with respect to which the equations of some physical phenomenon are invariant, and also for solutions that are invariant with respect to those groups. There naturally arises the question of an algorithm for seeking a maximally broad group of transformations with respect to which a given system of differential equations

†In Birkhoff's book (1960) his predecessors, in particular, T. A. Ehrenfest-Afanassjewa, are carefully cited.

is invariant. The basic idea here belongs to Sophus Lie; in recent times a series of results of general character, and applications to particular systems of equations met with in various problems of mechanics and physics, have been given by Ovsyannikov (1962) and Bluman and Cole (1974). We refer to the books cited for an account of the general approach and numerous examples. The account given above had as its goal to demonstrate the general idea of using wider groups in the search for self-similar solutions in instructive examples, and to indicate the use in a series of cases of the possibility of formal application of the standard technique of dimensional analysis in working with more general groups. It is clear that dimensional analysis can be applied even without knowing the mathematical formulation of the problem. It would appear that invariance with respect to a more general group than the group of similarity transformations of quantities with independent dimensions can be used only if one has a mathematical formulation of the problem. As a matter of fact this is not so, and invariance with respect to wider groups can also be suggested by physical considerations.

In conclusion we note that the consideration of self-similar solutions as intermediate asymptotic representations is closely connected with singular-perturbation methods, which have been widely developed and applied in the last twenty years (Van Dyke, 1975; Cole, 1968; Lagerstrom and Casten, 1972). Namely, self-similar solutions are inner or outer asymptotics of the solutions of the complete problem, depending on which of the scales of the independent variable is taken as the basis for analysis of the intermediate asymptotics. Therefore the determination of the constants appearing in a self-similar solution of the second kind can, in a number of cases, be achieved by matching the self-similar solution with a supplementary asymptotics.

8

The Spectrum of Exponents in Self-Similar Variables

In determining the exponent of time in the expression for self-similar variables in self-similar solutions of the second kind or, what is the same, the speed of propagation for solutions of progressive-wave type, we have arrived at peculiar eigenvalue problems for nonlinear operators. These problems are by nature close to classical eigenvalue problems for linear differential operators, and for them too there arises the question of the structure of the spectrum—the set of eigenvalues.

We recall the well-known problem of a vibrating string

$$\partial^2_{tt}u = \partial^2_{xx}u + q(x)\,u \tag{8.1}$$

(u being the displacement, x the coordinate measured along the string, and t the time) under the conditions of fixed ends:

$$u(0,\ t) = u(l,\ t) = 0 \tag{8.2}$$

(l being the length of the string). Separating variables, we seek a solution in the form

$$u = \exp\left(i\sqrt{\lambda}t\right)\Psi(x,\ \lambda). \tag{8.3}$$

For the determination of $\Psi(x, \lambda)$ we thus obtain the boundary-value problem

$$\begin{gathered}\Psi''(x,\ \lambda) + (\lambda + q(x))\,\Psi(x,\ \lambda) = 0,\\ \Psi(0,\ \lambda) = \Psi(l,\ \lambda) = 0.\end{gathered} \tag{8.4}$$

In general, for arbitrary λ, there exists no nontrivial solution of this boundary-value problem. However, there are exceptional values of λ, eigenvalues, for which nontrivial solutions of the boundary-value problem exist. These eigenvalues form a set (the spectrum) having a certain structure: discrete, continuous, mixed, etc., depending on the properties of the function $q(x)$.

One can look at all this somewhat differently. Equations (8.1) and (8.2) are invariant with respect to the two-parameter transformation group

$$u' = \alpha u, \quad t' = t + \beta, \quad x' = x. \tag{8.5}$$

This means that substituting (8.5) into (8.1) and (8.2), we obtain again the same problem in the variables u', x', t' for arbitrary group parameters–the constants α and β. In separating variables we actually look for solutions that are invariant with respect to some one-parameter subgroup of this group. The subgroup corresponds to the following relation between the parameters α and β:

$$\alpha = \exp\left(-i\sqrt{\lambda}\beta\right), \tag{8.6}$$

and the invariant solution has the form (8.3). The eigenvalues λ determining the subgroup are found from the condition that there exist an invariant solution of the form (8.3) in the large, i.e., a solution satisfying (8.2).

The situation is completely analogous for solutions of progressive-wave type. In order that such a solution exist, the equations and the boundary conditions must be invariant with respect to the two-parameter group of translational transformations

$$x' = x + \alpha, \quad t' = t + \beta, \quad u' = u. \tag{8.7}$$

Here too in finding a solution of progressive-wave type we seek a one-parameter subgroup of this transformation group corresponding to $\alpha = \lambda\beta + \text{const}$, where λ is an eigenvalue, and a solution, invariant with respect to that subgroup:

$$u(x', t') = u(x, t).$$

The eigenvalues λ that extract from the basic group a one-parameter subgroup are also determined by the condition of existence of an invariant solution in the large, i.e., the satisfaction of the boundary conditions by an invariant solution of the equations. In this case too, the spectrum of eigenvalues can have varied character. Thus, in the problem considered in the previous chapter of the propagation of a gene, it is continuous and semibounded: $\lambda \geqslant \lambda_0$. In the problem of flame propagation the spectrum consists of one point. There is a peculiar situation of the remarkable Korteweg–de Vries equation, which arose initially in the theory of surface waves on shallow water, and was later encountered in numerous applications to various other problems [cf. Jeffrey and Kakutani (1972); Karpman (1975)]:

$$\partial_t u + u\partial_x u + \beta\partial^3_{xxx} u = 0. \tag{8.8}$$

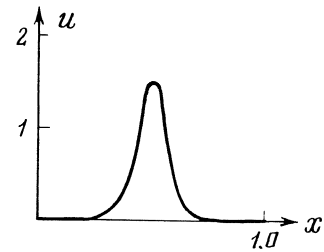


Figure 8.1. Solitary wave–soliton.

Here in the theory of waves u is, to within a constant factor, the horizontal velocity component, which is constant in the present approximation over the channel depth, $\beta = c_0 h^2/6$, $c_0 = (gh)^{1/2}$, g is the acceleration of gravity, h the undisturbed depth of the fluid layer, t the time, and x the horizontal coordinate in a system moving with speed c_0 relative to fluid at rest at infinity. An analogous equation is valid also in the corresponding approximation for the elevation of the free surface over its undisturbed level. Equation (8.8) has solutions of progressive-wave type, the so-called solitons (Fig. 8.1),

$$u = u(\zeta) = u_0 \cosh^{-2}\left[\sqrt{u_0/12\beta}\,\zeta\right], \tag{8.9}$$

where $\zeta = x - \lambda t + c$ and $u_0 = 3\lambda$.

The solution (8.9) satisfies the conditions

$$u(\infty) = u(-\infty) = 0 \tag{8.10}$$

for any $\lambda > 0$; the spectrum of eigenvalues λ is continuous and semi-bounded: $\lambda \geqslant 0$. There is, however, an essential difference between the continuous spectrum in the problem of gene propagation and in this problem. In the first problem only the lowest point $\lambda = \lambda_0$ of the spectrum satisfies the requirement that the solution of the initial-value problem with initial data of transitional type tends to the given solution of progressive-wave type as $t \to \infty$; for all other λ this is not so, and therefore the corresponding solutions are unstable. For the Korteweg–deVries equation a remarkable discovery was made by Gardner, Green, Kruskal, and Miura (1967) [see also Lax (1968)†]: as $t \to \infty$ for large positive x the solution of the initial-value problem is, for initial data $u(x, 0)$ that decrease sufficiently rapidly at $x = \pm\infty$, represented asymptotically (Fig. 8.2) by a finite sum of solutions of the form (8.9):

$$u \sim \sum_{n=1}^{N} 2|\mu_n| \cosh^{-2}\left\{\sqrt{\frac{|\mu_n|}{6\beta}}\left(x - \frac{2}{3}|\mu_n|\, t + c_n\right)\right\}, \tag{8.11}$$

†The so-called L, A–pair technique invented in the latter paper permitted this result to be understood from a more general point of view and analogous results to be obtained for other equations.

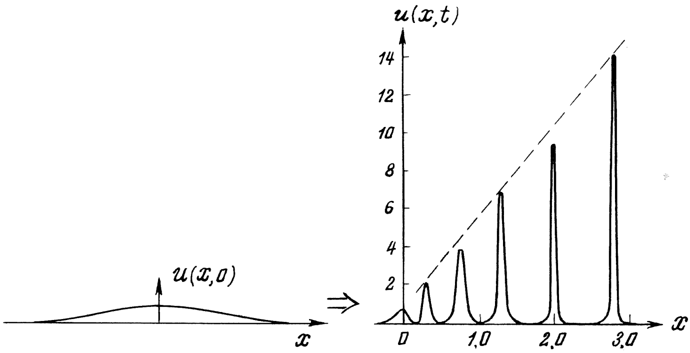


Figure 8.2. Initial elevation of the free surface of a heavy fluid in a shallow channel generates a finite series of solitary waves (solitons).

where the μ_n are the discrete eigenvalues of the Schrödinger operator with the potential equal to $-u(x, 0)$:

$$\frac{d^2\Psi}{dx^2}+\frac{1}{6\beta}\left[\mu+u(x,\ 0)\right]\Psi=0,\quad \Psi(\pm\infty)=0. \tag{8.12}$$

Hence any solution of soliton type can be an intermediate asymptotics of the solution of an initial-value problem as $t\to\infty$, but exactly which one it is will be determined by the initial conditions.

For self-similar solutions too there is an analogous situation. In fact, for a self-similar solution to exist it is necessary that the equations and boundary conditions of the degenerate problem be invariant with respect to some group—a subgroup of the group of similarity transformations of the independent and dependent variables. In searching for self-similar solutions one seeks a subgroup of that group and a solution that this subgroup leaves invariant. Here too, eigenvalues λ arise naturally that are determined by the condition of existence in the large of an invariant (self-similar) solution. The examples considered above illustrate what has been said. Thus in the problem considered in Chapter 3 of the spreading of a mass of fluid under filtration in an elasto-plastic porous medium, the basic equation

$$\partial_t u=\varkappa\partial^2_{xx}u\quad(\partial_t u\geqslant 0),\qquad \partial_t u=\varkappa_1\partial^2_{xx}u\quad(\partial_t u\leqslant 0) \tag{8.13}$$

and the condition at infinity

$$u(\infty,\ t)=0\quad(t\geqslant 0), \tag{8.14}$$

and also the condition of continuous matching, with continuous x-derivative, at the point $x=x_0(t)$ where the quantity $\partial_t u$ vanishes, are all invariant with respect to the two-parameter group

$$u' = \alpha u, \quad t' = \beta^2 t, \quad x' = \beta x, \tag{8.15}$$

which is a subgroup of the three-parameter group of similarity transformations $u' = A_1 u$, $t' = A_2 t$, $x' = A_3 x$. We seek a one-parameter subgroup of the group (8.15) for which $\alpha = \beta^{2\lambda}$, and also a self-similar solution $u = t^\lambda f(x/t^{1/2})$ that remains invariant under this one-parameter subgroup. The parameter λ, the eigenvalue, is determined from the condition of the existence of a self-similar solution in the large; the spectrum turns out to be discrete, and for $\epsilon = \varkappa_1/\varkappa$ not too large—even consisting of one point.

Analogously, in the problem of a strong explosion with loss or deposition of energy at the wave front (Chapter 4), the equations of spherically symmetric adiabatic motion of an ideal gas, the conditions at a strong shock wave, the condition $v(0, t) = 0$, and the condition $\rho(\infty, t) = \rho_0$ at infinity are all invariant with respect to the two-parameter group of transformations

$$p' = \alpha^2 p, \quad \rho' = \rho, \quad v' = \alpha v, \quad r' = \alpha\beta r, \quad t' = \beta t. \tag{8.16}$$

We seek a one-parameter subgroup of this group for which $\alpha = \beta^\lambda$ and also a self-similar solution

$$\rho = \rho_0 R\left(\frac{r}{t^{1+\lambda}}\right), \quad p = \rho_0 \frac{r^2}{t^2} P\left(\frac{r}{t^{1+\lambda}}\right), \quad v = \frac{r}{t} V\left(\frac{r}{t^{1+\lambda}}\right) \tag{8.17}$$

that is invariant with respect to this subgroup. As we have seen, the spectrum of eigenvalues λ, determined by the condition of existence of a self-similar solution in the large, turns out for $\gamma < 2$ to consist of one point.

In the problem of a converging strong shock wave, first considered by Guderley (1942), which also leads to self-similar solutions of Bechert–Guderley type (8.17) but with other boundary conditions, the spectrum for values of the adiabatic exponent $\gamma > \gamma_0 \cong 1.87$ turns out [cf. Brushlinskii and Kazhdan (1963)] to be continuous and semibounded. There is a conjecture due to I. M. Gel'fand according to which the intermediate asymptotics of the non-self-similar problem as $t \to 0$ (the time of collapse) selects the lowest point of the spectrum, just as in the problem of propagation of a gene, but the question actually remains open, since numerical calculations have not confirmed this conjecture.

There is an instructive self-similar interpretation of the result presented above for the Korteweg–deVries equation (8.8). If we set $x = \ln \xi$, $t = \ln \tau$, the equation (8.8) can be rewritten in the form

$$\tau\partial_\tau u + \xi u \partial_\xi u + \beta\left(\xi^3 \partial^3_{\xi\xi\xi} u + 3\xi^2 \partial^2_{\xi\xi} u + \xi \partial_\xi u\right) = 0. \tag{8.18}$$

The solution of progressive-wave type (8.9) here assumes the self-similar form

$$u = \frac{12\lambda}{2 + \eta^{\sqrt{\lambda/\beta}} + \eta^{-\sqrt{\lambda/\beta}}}, \qquad \eta = \frac{\xi}{A\tau^{\lambda}}. \tag{8.19}$$

Here, $A = e^{-c}$ is constant. We note that the expression on the right side of (8.19) is not small only for η of order unity; it is small if η is either large or small. The spectrum of eigenvalues λ, obtained by direct construction of solutions of progressive-wave type, is continuous and semibounded: $\lambda \geqslant 0$. The result of Gardner, Green, Kruskal, and Miura (1967) presented above is expressed in the following way in the self-similar interpretation: an asymptotic solution of the initial-value problem for (8.18) as $\tau \to \infty$ and large ξ can be represented in the form

$$u \sim \sum_{n=1}^{N} 12\lambda_n \left\{2 + \left(\frac{\xi}{A_n \tau^{\lambda_n}}\right)^{\sqrt{\frac{\lambda_n}{\beta}}} + \left(\frac{\xi}{A_n \tau^{\lambda_n}}\right)^{-\sqrt{\frac{\lambda_n}{\beta}}}\right\}^{-1}. \tag{8.20}$$

Thus the initial distribution $u(\xi, 0)$, which by assumption decreases sufficiently rapidly as $\xi \to 0$ or ∞, determines N positive constants $\lambda_1, \ldots, \lambda_N$ and N positive constants $A_1, \ldots, A_N$, and selects N intervals in ξ. Here, inside each of the intervals $\xi = O(\tau^{\lambda_n})$, the asymptotics of the solution is self-similar and has the form

$$u \sim 12\lambda_n \left\{2 + \left(\frac{\xi}{A_n \tau^{\lambda_n}}\right)^{\sqrt{\frac{\lambda_n}{\beta}}} + \left(\frac{\xi}{A_n \tau^{\lambda_n}}\right)^{-\sqrt{\frac{\lambda_n}{\beta}}}\right\}^{-1}. \tag{8.21}$$

Outside of the intervals mentioned the solution u is small: $u = o(1)$. Here it is significant that in the self-similar asymptotics not only do the constants A_n depend as usual on the initial conditions of the original nondegenerate problem, but so also do the exponents λ_n in the expressions for the self-similar variables. We meet an analogous situation later in considering self-similar decay of isotropic turbulence. This example once again emphasizes the insufficiency in the general case of dimensional analysis for determining the exponents in the self-similar variables.

The examples given above demonstrate the variety of possible structures of the spectrum of nonlinear eigenvalue problems that arise in the construction of self-similar solutions.

9

Stability of Self-Similar Solutions

The statement of the problem of stability of self-similar solutions is distinguished by certain peculiarities. In the present chapter a general approach to the investigation of the stability of self-similar and other invariant solutions is outlined and illustrated by several instructive examples. Presentation of the whole mass of concrete results that have been accumulated on these topics would go far beyond the scope of this book.

A simple example will immediately introduce us to the heart of the matter. The equation

$$\partial_t u = \varkappa \partial^2_{xx} u + f(u), \tag{9.1}$$

where $f(u)$ is bounded together with its first derivative and satisfies the conditions

$$\begin{gathered} f(u) \equiv 0, \quad u_1 \leqslant u \leqslant u_1 + \Delta; \quad f(u_2) = 0, \\ f(u) > 0, \quad u_1 + \Delta < u < u_2, \end{gathered} \tag{9.2}$$

is the simplest model of thermal flame propagation (u is the temperature), if the density of the gaseous mixture is constant and the concentration of combustible matter and the temperature at any moment are connected by the Lewis–von Elbe similarity law; this follows readily from what was presented in Chapter 6. Equation (9.1) has a solution of progressive-wave type,

$$u = U(\zeta), \quad \zeta = x - \lambda t + c, \tag{9.3}$$

where c is an arbitrary constant, and the speed of propagation λ is uniquely determined by solving a nonlinear eigenvalue problem: the equation

$$\lambda \frac{dU}{d\zeta} + \varkappa \frac{d^2 U}{d\zeta^2} + f(U) = 0, \tag{9.4}$$

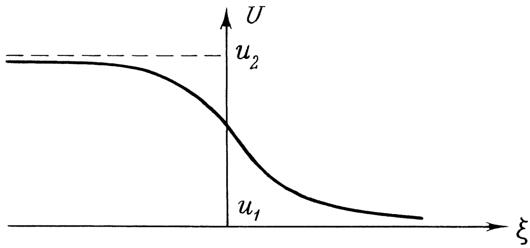


Figure 9.1. The temperature distribution in a progressive wave is monotonic.

obtained by substituting (9.3) into (9.1) with the conditions

$$U(-\infty)=u_2, \quad U(\infty)=u_1. \tag{9.5}$$

It is easy to show that the solution (9.3) is a monotonically decreasing function (Fig. 9.1). In fact, the function $U(\zeta)$ cannot have a minimum lying between $U=u_1$ and $U=u_2$, because at this point one would have to have $dU/d\zeta=0$, $d^2U/d\zeta^2>0$, $f(U)>0$, which is impossible by virtue of (9.4). Neither can the function have a maximum within these same limits, because there would also have to exist a minimum between $U=u_1$ and $U=u_2$, which is impossible by the preceding.

The stability of (9.3) has paramount importance. In fact, as has already been remarked more than once, the invariant solution (9.3) is of physical interest first of all as an asymptotic representation of a certain class of solutions to the nondegenerate initial-value problem for (9.1) with some initial data of transitional type. If this solution were unstable, so that a small perturbation imposed on the temperature distribution at some moment would lead to a large deviation in the temperature distribution at later moments, then the solution would be physically meaningless.

Here it is necessary however, to define stability and instability precisely. Suppose that at some moment $t=t_0$ the temperature distribution is determined by the relation

$$u(x, t_0)=U(x-\lambda t_0+c)+\delta\varphi(x), \tag{9.6}$$

where δ is a small parameter and $\varphi(x)$ is a finite function, i.e., one equal to zero outside some finite interval, so that the temperature distribution corresponds to the solution of progressive-wave type already considered plus a small local addition. At first glance, a natural definition of stability of the progressive wave would appear to be the following: if the solution of any "perturbed" initial-value problem of the given type (9.6) can be represented for $t > t_0$ in the form

$$u(x, t) = U(\zeta) + w(\zeta, t), \tag{9.7}$$

where the function $w(\zeta, t)$ tends to zero as $t \to \infty$, then the original solution is stable; otherwise it is not.

Just such a definition of stability was adopted by Rosen (1954), who arrived at the conclusion that instability is possible in the problem considered, indicated some approximate criteria for stability of solutions, etc. In fact, such a definition of stability is insufficient, and must be replaced by another one. This circumstance turns out to be essential for the statement of the problem of the stability of progressive waves and of self-similar and in general invariant solutions; it was clarified by Barenblatt and Zel'dovich (1957).

Actually, as was shown in Chapter 7, a solution of progressive-wave type is invariant under a one-parameter group of translations with respect to the coordinate and time. Hence the solution (9.3) is determined by (9.4) and (9.5) up to a constant. Consequently a definition of the stability of a progressive wave must also have the corresponding invariance. If in fact the perturbed solution tends not to the original unperturbed solution as $t \to \infty$, but to a shifted one (Fig. 9.2), then there is no reason to consider this transition as an instability. Thus an invariant definition of the stability of the progressive wave (9.3) consists in the following: this solution is stable if one can find a constant a such that the solution of the perturbed problem can be represented for $t > t_0$ in the form

$$u(x, t) = U(\zeta + a) + w(\zeta, t), \tag{9.8}$$

where $w(\zeta, t)$ tends to zero as $t \to \infty$; otherwise the solution is considered unstable.

In what follows we shall restrict the investigation of stability to the linear approximation. For small δ, the quantity a must be small. Expanding $U(\zeta + a)$ in series and restricting ourselves, in accord with our adoption of the linear ap-

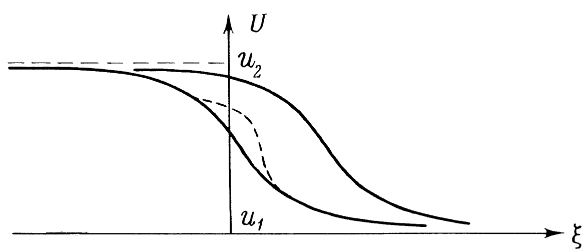


Figure 9.2. A perturbed solution tends to a shifted unperturbed one and there is no reason to consider this an instability.

proximation, to the first term of the expansion, we reformulate the definition of stability (9.8) thus: if one can find a constant a, tending to zero along with δ, such that the solution of the perturbed problem can be represented in the form

$$u(x,\ t)=U(\zeta)+aU'(\zeta)+w(\zeta,\ t), \tag{9.9}$$

where $w(\zeta, t) \to 0$ as $t \to \infty$, then the unperturbed solution is stable.

We shall prove the stability of the progressive wave (9.3), in the sense indicated, for any $f(u)$ satisfying (9.2). In (9.1) we set $u(x, t) = U(\zeta) + \delta v(\zeta, t)$. Discarding terms smaller than first order in δ, and using the fact that $U(\zeta)$ satisfies (9.4), we obtain for $v(\zeta, t)$ the equation

$$\partial_t v-\lambda\partial_\zeta v=\varkappa\partial^2_{\zeta\zeta}v+f'\left(U(\zeta)\right)v. \tag{9.10}$$

Applying the method of separation of variables, we construct a solution of the initial-value problem for (9.10), with arbitrary initial distribution of the perturbation $v(\zeta, 0)$ vanishing outside some finite interval, in the form of a Fourier series (or integral)

$$v=\sum_{n=1}^{\infty} c_n e^{-\mu_n t}\Psi(\zeta,\ \mu_n), \tag{9.11}$$

where the function $\Psi(\zeta, \mu_n)$ is the nth eigenfunction of the operator defined by the equation

$$\lambda\frac{d\Psi}{d\zeta}+\varkappa\frac{d^2\Psi}{d\zeta^2}+\left[\mu+f'\left(U(\zeta)\right)\right]\Psi=0 \tag{9.12}$$

and the conditions of tending to zero faster than any power $|\zeta|$ for $\zeta = \pm\infty$. Here μ_n is the nth eigenvalue.

The coefficients c_n in (9.11) are determined by expanding the initial condition in series with respect to the functions $\Psi(\zeta, \mu_n)$. According to (9.11), if it could be shown that all the eigenvalues μ_n are nonnegative, then the stability of the progressive wave in the sense of (9.9) would be proved.

We now note that differentiating (9.4) with respect to ζ,

$$\lambda\frac{d}{d\zeta}\left(\frac{dU}{d\zeta}\right)+\varkappa\frac{d^2}{d\zeta^2}\left(\frac{dU}{d\zeta}\right)+f'(U)\frac{dU}{d\zeta}=0\ldots, \tag{9.13}$$

we find that $dU/d\zeta$ satisfies (9.12) for $\mu = 0$. Observing further that $dU/d\zeta$ tends exponentially to zero as $\zeta \to \pm\infty$ we see that $dU/d\zeta$ coincides to within a factor with the eigenfunction corresponding to $\mu = 0$. From the proof of the monotonicity of the function $U(\zeta)$ given above it follows that $dU/d\zeta$ does not vanish for finite ζ. But an eigenfunction of the operator (9.12) has as many zeroes as its ordinal number.† Therefore $dU/d\zeta$ corresponds to the smallest eigenvalue. Since the corresponding eigenvalue is equal to zero, there are no negative eigenvalues in the problem; all $\mu_n \geqslant 0$. As for $\mu_0 = 0$, this eigenvalue does not spoil the stability, since the corresponding eigenfunction is equal to $dU/d\zeta$ to within a constant factor, and the contribution of this eigenfunction corresponds to just a shift of the progressive wave. Thus the stability of the progressive wave (9.3) in the sense formulated is proved.‡

It is clear that under assumptions that are sufficiently broad for our purposes the considerations presented have completely general meaning. In particular, they are easily reformulated to apply to the stability of self-similar solutions.

As we have seen, self-similar solutions are determined by direct construction to within some constant A, which for self-similar solutions of the first kind is found from conservation laws, and for solutions of the second kind can be found only by following the evolution of a non-self-similar solution, since the conservation laws here assume nonintegrable form.

By definition, *a self-similar solution is stable if the solution of any perturbed problem with sufficiently small perturbations can be represented in the form of a self-similar solution corresponding to a constant A' that has generally speaking changed, plus some additional term whose ratio to the unperturbed solution tends to zero as $t \to \infty$.*§

Relying on this definition, we proceed to investigate the stability of the solution to the modified problem of a heat source that was considered in Chapter 3 (Kerchman, 1971).

As shown in Chapter 3, using numerical computation, a self-similar intermediate asymptotics of the solution to the initial-value problem for the equation

$$\partial_t u = \varkappa \partial^2_{xx} u \quad (\partial_t u \geqslant 0), \quad \partial_t u = \varkappa_1 \partial^2_{xx} u \quad (\partial_t u \leqslant 0) \tag{9.14}$$

is obtained, for $\varkappa_1 \neq \varkappa$, in the form of a self-similar solution of the second kind,

$$u = \frac{A}{(\varkappa t)^{(1+\alpha)/2}} f(\xi, \varepsilon), \quad \xi = \frac{x}{\sqrt{\varkappa t}}, \quad \varepsilon = \frac{\varkappa_1}{\varkappa}, \tag{9.15}$$

where the function $f(\xi, \varepsilon)$ can be expressed in terms of parabolic cylinder functions. For the self-similar solution (9.15), for $|x| < \xi_0(\varkappa t)^{1/2}$ the derivative

†In fact (9.12) reduces to self-adjoint form if in it one sets $\Psi = e^{-(\lambda/2\varkappa)\zeta}\varphi$. But the factor $e^{-(\lambda/2\varkappa)\zeta}$ does not vanish, and for a self-adjoint operator the property formulated holds, as is well known.

‡We emphasize that the argument given proves the stability of a flame only under the assumptions indicated. In particular, if the similarity of the fields of concentration and temperature does not hold (for example, for the burning of gunpowder), some instability arises.

§For convenience the non-self-similar variable is identified with t.

$\partial_t u < 0$, but for $|x| > \xi_0(\varkappa t)^{1/2}$, $\partial_t u > 0$, so that the change of coefficient in (9.14) occurs for

$$x = \pm \xi_0 \sqrt{\varkappa t}. \tag{9.16}$$

The constants α and ξ_0 are found from the set of equations

$$D_{\alpha+2}\left(\frac{\xi_0}{\sqrt{2}}\right) = 0, \quad M\left(-1 - \frac{\alpha}{2}, \frac{1}{2}, \frac{\xi_0^2}{4\varepsilon}\right) = 0. \tag{9.17}$$

We now consider, in accord with the general procedure for analytically investigating the stability of a solution, the perturbed initial-value problem, for which the initial condition at $t = t_0$ can, without loss of generality, be written in the form

$$u(x, t_0) = \left[Af\left(\frac{x}{\sqrt{\varkappa t_0}}, \varepsilon\right) + \delta v_0\left(\frac{x}{\sqrt{\varkappa t_0}}\right)\right](\varkappa t_0)^{-(1+\alpha)/2}. \tag{9.18}$$

Here δ is a small parameter, and the function $v_0(\xi)$ vanishes outside some finite interval in ξ. In the linear approximation, for $t > t_0$,

$$u(x, t) = \frac{1}{(\varkappa t)^{(1+\alpha)/2}} \left[Af(\xi, \varepsilon) + \delta v(\xi, \tau)\right], \tag{9.19}$$

where $v(\xi, \tau)$ is the perturbation. [Instead of the time t it is convenient to take $\tau = \ln(t/t_0)$ as independent variable for the perturbation.] The surfaces $x_1(t)$ and $x_2(t)$ on which $\partial_t u$ vanishes are also shifted, so that

$$x_1 = (\xi_0 + \beta_1(\tau))\sqrt{\varkappa t}, \quad x_2 = -(\xi_0 + \beta_2(\tau))\sqrt{\varkappa t}. \tag{9.20}$$

The perturbation is not necessarily symmetric, so $\beta_1(\tau) \neq \beta_2(\tau)$. Substituting the perturbed solution (9.19)-(9.20) into the basic equation, we obtain an equation for the perturbation for $\xi > 0$ in the form

$$\begin{aligned}
&\partial_\tau v = \varepsilon \partial^2_{\xi\xi} v + \frac{\xi}{2}\partial_\xi v + \frac{1+\alpha}{2} v \quad (0 \leqslant \xi \leqslant \xi_0),\\
&\partial_\tau v = \varepsilon \partial^2_{\xi\xi} v + \frac{\xi}{2}\partial_\xi v + \frac{1+\alpha}{2} v + \frac{\varepsilon - 1}{\delta} A \frac{d^2 f}{d\xi^2}\\
&\qquad (\xi_0 < \xi \leqslant \xi_0 + \beta_1(\tau)),\\
&\partial_\tau v = \partial^2_{\xi\xi} v + \frac{\xi}{2}\partial_\xi v + \frac{1+\alpha}{2} v \quad (\xi_0 + \beta_1(\tau) \leqslant \xi < \infty),
\end{aligned} \tag{9.21}$$

and an analogous equation for $\xi < 0$.

Furthermore, from (9.19) we get an expression for the derivative $\partial_t u$ of the perturbed solution:

$$\partial_t u = \frac{1}{(\varkappa t)^{(1+\alpha)/2}} \frac{1}{t} \left\{ -\frac{1+\alpha}{2} [A f(\xi, \varepsilon) + \delta v(\xi, \tau)] - \frac{1}{2} \xi [A f'(\xi, \varepsilon) + \delta \partial_\xi v] + \delta \, \partial_\tau v \right\}.$$

Setting $\xi = \xi_0 + \beta_1(\tau)$ in this relation, linearizing and keeping in mind that $f''(\xi_0) = 0$ and $\partial_t u = 0$ for $\xi = \xi_0 + \beta_1(\tau)$, we obtain

$$\delta \left[\partial_\tau v - \frac{1+\alpha}{2} v - \frac{\xi}{2} \partial_\xi v \right] - \left(1 + \frac{\alpha}{2}\right) A f'(\xi_0) \beta_1(\tau) = 0,$$

whence it follows that the displacement of the boundary is proportional to the small parameter δ. Linearizing (9.21), we get for $v(\xi, \tau)$ the linear equation

$$\begin{aligned} \partial_\tau v &= \varepsilon \partial^2_{\xi\xi} v + \frac{1+\alpha}{2} v + \frac{\xi}{2} \partial_\xi v \quad (|\xi| \leqslant \xi_0), \\ \partial_\tau v &= \partial^2_{\xi\xi} v + \frac{1+\alpha}{2} v + \frac{\xi}{2} \partial_\xi v \quad (|\xi| \geqslant \xi_0). \end{aligned} \tag{9.22}$$

At $\xi = \xi_0$ the functions v and $\partial_\xi v$ must be continuous. In fact, from the second equation of (9.21) we get, integrating with respect to ξ from $\xi = \xi_0$ to $\xi = \xi_0 + \beta_1(\tau)$,

$$\varepsilon \partial_\xi v \Big|_{\xi = \xi_0}^{\xi = \xi_0 + \beta_1(\tau)} = \int_{\xi_0}^{\xi_0 + \beta_1(\tau)} \left[\partial_\tau v - \frac{\xi}{2} \partial_\xi v - \frac{1+\alpha}{2} v \right] d\xi - \frac{\varepsilon - 1}{\delta} A \frac{df}{d\xi} \Big|_{\xi = \xi_0}^{\xi = \xi_0 + \beta_1(\tau)}$$

The quantities under the integral sign on the right side are bounded, and by the preceding, $\beta_1(\tau)$ is of order δ, so that the entire integral is of order δ. Furthermore, for $\xi = \xi_0$ the quantity $f''(\xi, \varepsilon)$ vanishes, so the second term on the right side is also of order δ; and from this and the linearity of the approximation follows the continuity of $\partial_\xi v$ at $|\xi| = |\xi_0|$. The continuity of v is proved by multiplying by ξ and using the same kind of integration and subsequent estimates.

A solution of the initial-value problem for the perturbation is sought in the form

$$v(\xi, \tau) = \sum_{n=0}^{\infty} c_n e^{-\mu_n \tau} \Psi(\xi, \mu_n), \tag{9.23}$$

where the function $\Psi(\xi, \mu_n)$ is an eigenfunction of the operator determined by the equations

$$\varepsilon \frac{d^2\Psi}{d\xi^2} + \frac{\xi}{2}\frac{d\Psi}{d\xi} + \frac{1+\alpha+2\mu}{2}\Psi = 0 \quad (|\xi| < \xi_0),$$
$$\frac{d^2\Psi}{d\xi^2} + \frac{\xi}{2}\frac{d\Psi}{d\xi} + \frac{1+\alpha+2\mu}{2}\Psi = 0 \quad (|\xi| > \xi_0), \tag{9.24}$$

and the conditions of vanishing more rapidly than any power of $|\xi|$ for $\xi = \pm\infty$:

$$\Psi(\pm\infty, \mu_n) = 0 \tag{9.25}$$

(μ_n being the nth eigenvalue of this operator). Furthermore, the functions $\Psi(\xi, \mu_n)$ together with their first derivatives with respect to ξ are continuous at $\xi = \xi_0$. One proves by standard methods that the spectrum of the resulting eigenvalue problem is discrete.

It is convenient to consider separately the symmetric Ψ_1 and antisymmetric Ψ_2 eigenfunctions of the operator (9.24)-(9.25). A symmetric solution of (9.24) satisfying (9.25) must be representable in the form (cf. Chapter 3)

$$\Psi_1 = C_1 \exp\left(-\frac{\xi^2}{8\varepsilon}\right)\left[D_{\alpha+2\mu}\left(\frac{\xi}{\sqrt{2\varepsilon}}\right) + D_{\alpha+2\mu}\left(-\frac{\xi}{\sqrt{2\varepsilon}}\right)\right]$$
$$(|\xi| \leqslant \xi_0), \tag{9.26}$$
$$\Psi_2 = C_2 \exp\left(-\frac{\xi^2}{8}\right) D_{\alpha+2\mu}\left(\frac{|\xi|}{\sqrt{2}}\right) \quad (|\xi| > \xi_0).$$

To determine the constants C_1 and C_2 we use the continuity of Ψ and $d\Psi/d\xi$ for $|\xi| = \xi_0$. Thus we get a system of homogeneous linear algebraic equations; the condition that the determinant of this system vanishes gives the characteristic equation

$$\Delta(\mu) = (\alpha+2\mu+1)\, D_{\alpha+2\mu}\left(\frac{\xi_0}{\sqrt{2}}\right) M\left(-1-\frac{1}{2}(\alpha+2\mu),\ \frac{1}{2};\ \frac{\xi_0^2}{4\varepsilon}\right) +$$
$$+ D_{\alpha+2\mu+2}\left(\frac{\xi_0}{\sqrt{2}}\right) M\left(-\frac{1}{2}(\alpha+2\mu),\ \frac{1}{2};\ \frac{\xi_0^2}{4\varepsilon}\right) = 0. \tag{9.27}$$

The quantities α and ξ_0 are determined as before by (9.17). Using these relations it is easy to show that $\mu_0 = 0$ is a root of (9.27). We now show that the other roots of this equation are positive. Equation (9.27) can be transformed into the form

$$\Delta(\mu)=\left(\frac{\xi_0}{\sqrt{2}}\right)D_{1+\alpha+2\mu}\left(\frac{\xi_0}{\sqrt{2}}\right)M\left(-1-\frac{1}{2}(\alpha+2\mu),\ \frac{1}{2};\ \frac{\xi_0^2}{4\varepsilon}\right)+$$
$$+D_{\alpha+2\mu+2}\left(\frac{\xi_0}{\sqrt{2}}\right)\left[M\left(-\frac{1}{2}(\alpha+2\mu),\ \frac{1}{2};\ \frac{\xi_0^2}{4\varepsilon}\right)-\right.$$
$$\left.-M\left(-1-\frac{1}{2}(\alpha+2\mu),\ \frac{1}{2};\ \frac{\xi_0^2}{4\varepsilon}\right)\right]=0. \tag{9.28}$$

It is known [cf. Abramowitz and Stegun (1964)], that the function $M(a+l, 1/2, x_0)$ is a monotonically increasing function of l for $l>0$ if x_0 is the smallest positive root of the equation $M(a, 1/2, x)=0$. If ζ_0 is the smallest positive root of the equation $D_{\alpha+2}(\zeta)=0$, then

$$D_{\alpha+2\mu+2}(\zeta_0)>0 \quad \text{for} \quad \mu<0.$$

Therefore $\Delta(\mu)>0$ for all negative μ, and there are thus no negative roots of (9.27).

Further, the antisymmetric solution has the form

$$\Psi_2=\begin{cases} C_3\exp\left(-\frac{\xi^2}{8\varepsilon}\right)\left[D_{\alpha+2\mu}\left(\frac{\xi}{\sqrt{2\varepsilon}}\right)-D_{\alpha+2\mu}\left(-\frac{\xi}{\sqrt{2\varepsilon}}\right)\right] \\ \qquad (0\leqslant|\xi|\leqslant\xi_0), \\ C_4\exp\left(-\frac{\xi^2}{8}\right)D_{\alpha+2\mu}\left(\frac{\xi}{\sqrt{2}}\right) \quad (\xi_0\leqslant|\xi|<\infty). \end{cases} \tag{9.29}$$

Its characteristic equation can be reduced to the relation

$$\Delta_1(\mu)=\frac{\xi_0}{\sqrt{2}}D_{\alpha+2\mu}\left(\frac{\xi_0}{\sqrt{2}}\right)M\left(-1-\frac{1}{2}(\alpha+2\mu-1),\ \frac{1}{2};\ \frac{\xi_0^2}{4\varepsilon}\right)+$$
$$+\varepsilon D_{1+\alpha+2\mu}\left(\frac{\xi_0}{\sqrt{2}}\right)\left[M\left(-\frac{1}{2}(\alpha+2\mu-1),\ \frac{1}{2};\ \frac{\xi_0^2}{4\varepsilon}\right)-\right.$$
$$\left.-M\left(-1-\frac{1}{2}(\alpha+2\mu-1),\ \frac{1}{2};\ \frac{\xi_0^2}{4\varepsilon}\right)\right]=0. \tag{9.30}$$

Comparison with (9.28) shows that the smallest root of this equation is equal to $\mu_1=1/2$. Subsequent investigation reveals that the smallest positive root of (9.28) is equal to $\mu_2=1$, and the corresponding root of (9.30) is equal to $\mu_3 = 3/2$. Thus, (9.19) and (9.23) show that a solution of the perturbed initial value problem can be written in the following form:

$$u(x,\ t) = \frac{1}{(\varkappa t)^{(1+\alpha)/2}}\left[(A+\delta c_0) f(\xi,\ \varepsilon) + \delta c_1 \left(\frac{t_0}{t}\right)^{1/2} \Psi\left(\xi,\ \frac{1}{2}\right) + \right.$$

$$\left. + \delta c_2 \left(\frac{t_0}{t}\right) \Psi(\xi,\ 1) + \delta c_3 \left(\frac{t_0}{t}\right)^{3/2} \Psi\left(\xi,\ \frac{3}{2}\right) + o\left(\left(\frac{t_0}{t}\right)^{3/2}\right)\right] \tag{9.31}$$

[the c_i being the coefficients of the expansion of the function $v_0(\xi)$ in a Fourier series with respect to the eigenfunctions of the operator (9.24)-(9.25)]. Thus the self-similar solution constructed in Chapter 3 turns out to be stable with respect to small perturbations. It is evident that in the present case the constant A also turns out to be shifted: $A' = A(1 + \delta c_0)$, so that the invariance we have taken in the definition of stability of self-similar solutions is used in this case too.

In the linear case $\varkappa_1 = \varkappa$, $\varepsilon = 1$, $\alpha = 0$ one gets the natural result on the stability of a self-similar solution of instantaneous heat-source type. The representation (9.31) of the solution of the perturbed initial-value problem in this case assumes the form

$$u(x,\ t) = \frac{1}{\sqrt{\varkappa t}}\left[(A+\delta c_0)\, e^{-\xi^2/4} + \delta c_1 \left(\frac{t_0}{t}\right)^{1/2} \frac{1}{2} \xi e^{-\xi^2/4} + \right.$$

$$\left. + \delta c_2 \left(\frac{t_0}{t}\right) \frac{\xi^2 - 2}{4}\, e^{-\xi^2/4} + \dots\right]. \tag{9.32}$$

The coefficients of (9.24) in the linear case (for $\varepsilon = 1$) are actually continuous, and that equation can be written in the form

$$\frac{d^2\Psi}{d\xi^2} + \frac{\xi}{2}\frac{d\Psi}{d\xi} + \frac{1+2\mu}{2}\Psi = 0. \tag{9.33}$$

For $\mu = 0$ a solution to this equation that satisfies the condition (9.25) of rapid convergence to zero at infinity is $e^{-\xi^2/4}$. This function does not vanish for any finite ξ; hence it is the zeroth eigenfunction, and $\mu = \mu_0 = 0$ is the zeroth eigenvalue. Furthermore, the derivative of $e^{-\xi^2/4}$ with respect to ξ, equal to $-(\xi/2)e^{-\xi^2/4}$, vanishes except at infinity only for $\xi = 0$, and it satisfies (9.33) for $\mu = 1/2$, and also the conditions at infinity. This is consequently the first eigenfunction, and $\mu = \mu_1 = 1/2$ is the first eigenvalue. Thus $\Psi(\xi, \mu_1) = -(\xi/2)e^{-\xi^2/4}$. Analogously the subsequent eigenvalues are $\mu_n = n/2$; $n = 0, 1, 2, \dots$, and the corresponding eigenfunctions are equal to the nth derivatives of $e^{-\xi^2/4}$.

It is easy to get the result for the linear case directly [Zel'dovich and Barenblatt (1958); cf. also Zel'dovich and Raizer (1967)], since in this case there exists an explicit representation of the solution to the perturbed initial-value problem.

The analysis completed above demonstrated that invariant solutions—of both the progressive-wave and self-similar type—are asymptotics of the solutions of a certain class of nondegenerate problems with noninvariant solutions.

10

Self-Similar Intermediate Asymptotics of Some Linear Problems in the Theory of Elasticity and the Hydrodynamics of Ideal Fluids

1. The Problem of the Equilibrium of an Elastic Wedge under the Action of a Concentrated Couple Applied as Its Tip

The consideration of linear problems is instructive: for them one can follow analytically the transition to self-similar asymptotics of the solutions of non-degenerate problems, and the transition at certain critical values of the parameter from self-similarities of the first kind to self-similarities of the second kind. For non-linear problems one has not been able to do this.

We shall analyze initially some problems of the theory of elasticity for plane strain, when the components of the elastic fields—stress tensors, deformation tensors, displacement vectors, etc.—are identical in all planes perpendicular to some direction. The equilibrium equations for plane strain have the form (Muskhelishvili, 1963)

$$\partial_r\sigma_{rr}+\frac{1}{r}\partial_\theta\sigma_{r\theta}+\frac{1}{r}(\sigma_{rr}-\sigma_{r\theta})=0,\quad \frac{1}{r}\partial_\theta\sigma_{\theta\theta}+\partial_r\sigma_{r\theta}+\frac{2}{r}\sigma_{r\theta}=0. \quad (10.1)$$

Here r, θ are polar coordinates in the deformation plane and σ_{rr}, $\sigma_{\theta\theta}$, $\sigma_{r\theta}$ are the corresponding components of the stress tensor. (In what follows, polar coordinates are just what we shall need.) These equations are identically satisfied by introduction of the Airy stress function Ψ:

$$\sigma_{rr}=\frac{1}{r}\partial_r\Psi+\frac{1}{r^2}\partial^2_{\theta\theta}\Psi,\quad \sigma_{\theta\theta}=\partial^2_{rr}\Psi,\quad \sigma_{r\theta}=-\partial_r\left(\frac{1}{r}\partial_\theta\Psi\right). \quad (10.2)$$

Hooke's law relates the components of the stress tensor to the derivatives of a single displacement vector, whence it follows that the three components of the

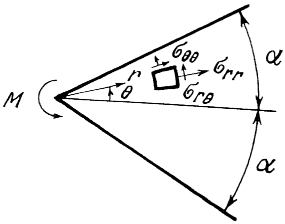


Figure 10.1. A wedge of opening angle 2α under the action of a couple of moment M applied at its tip.

stress tensor satisfy a certain integrability condition, the so-called compatibility relation. If we substitute into this relation the expressions (10.2) for the components of the stress tensor in terms of the stress function, we obtain for it the biharmonic equation:

$$\Delta\Delta\Psi = \left[\frac{1}{r}\,\partial_r\,(r\partial_r) + \frac{1}{r^2}\,\partial^2_{\theta\theta}\right]^2 \Psi = 0. \tag{10.3}$$

We begin with an instructive problem first considered by Carothers (1912) and Inglis (1922). Namely, we take (Fig. 10.1) an infinite wedge of opening angle 2α and at the tip of the wedge we apply a couple of moment M. The stress function Ψ that governs the stress distribution depends in this case on four parameters, M, r, θ, and α, whose dimensions in the class FLT are respectively F, L, 1, and 1. (The dimensions of the moment of a couple in plane problems of elasticity coincide with the dimensions of force, since in the problem one really deals with a couple per unit length.) By virtue of (10.2) the dimensions of the stress function also coincide with the dimensions of force. Hence the standard procedure of dimensional analysis leads to a relation

$$\Psi = M\Phi\,(\theta,\ \alpha). \tag{10.4}$$

Substituting (10.4) into (10.3), we get for $\Phi(\theta, \alpha)$ the ordinary differential equation

$$\Phi^{\mathrm{IV}} + 4\Phi'' = 0. \tag{10.5}$$

Furthermore the lateral faces of the wedge are free of stress over their entire extent:

$$\sigma_{\theta\theta}\,(r,\ \pm\alpha) = 0, \quad \sigma_{r\theta}\,(r,\ \pm\alpha) = 0,$$

and from this and (10.2) and (10.4) we get the boundary conditions for the function $\Phi(\theta, \alpha)$:

$$\partial_\theta \Phi(\pm\alpha, \alpha) = 0. \tag{10.6}$$

The equation and boundary conditions determine the solution to within a constant factor, which is found from the following consideration: we make a cut along a circle of arbitrary radius, calculate the total moment of the stresses acting on the cut, and equate the result to M—the cut-off part of the wedge must be in equilibrium. As a result the final expression for the stress function is obtained in the form

$$\Psi = \frac{M(2\theta\cos 2\alpha - \sin 2\theta)}{2(\sin 2\alpha - 2\alpha\cos 2\alpha)}, \tag{10.7}$$

and for the components of the stress field we have

$$\sigma_{rr} = \frac{2M\sin 2\theta}{(\sin 2\alpha - 2\alpha\cos 2\alpha)\,r^2}, \quad \sigma_{r\theta} = \frac{M(\cos 2\alpha - \cos 2\theta)}{(\sin 2\alpha - 2\alpha\cos 2\alpha)\,r^2}, \quad \sigma_{\theta\theta} = 0. \tag{10.8}$$

2. The Sternberg-Koiter Paradox. Intermediate Asymptotics of the Non-Self-Similar Problem

In the remarkable paper of Sternberg and Koiter (1958) attention was drawn for the first time to a strange property of the solution (10.7)-(10.8) just obtained: as the angle α approaches the value $\alpha = \alpha_* \approx 0.715\,\pi$ for which the denominator in (10.7)-(10.8) vanishes (which is perfectly admissible from the physical point of view), the stresses at all points of the wedge tend to infinity according to (10.8). In this connection the following question arises: is the singular self-similar solution of the degenerate problem (10.7) an asymptotics of some non-self-similar solution of the nondegenerate problem; in other words, does it have meaning?

In order to clarify this matter, Sternberg and Koiter considered for the same wedge the following nondegenerate problem (Fig. 10.2a). On finite segments of the lateral faces of the wedge $\theta = \pm\alpha$, $0 \leqslant r \leqslant r_0$, there is distributed according to some law a normal loading that is antisymmetric with respect to the axis of the wedge, and statically equivalent to a couple with moment M. The tangential stress is as before equal to zero everywhere on the faces of the wedge. Thus one has the conditions

$$\begin{gathered}\sigma_{\theta\theta}(r, \alpha) = -\sigma_{\theta\theta}(r, -\alpha) = p(r), \\ \sigma_{r\theta}(r, \alpha) = \sigma_{r\theta}(r, -\alpha) \equiv 0 \quad (0 < r < \infty),\end{gathered} \tag{10.9}$$

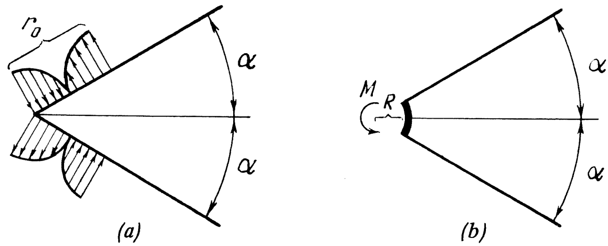


Figure 10.2. Nondegenerate problems of the elastic equilibrium of a wedge under the action of a couple of moment M: (a) forces distributed over the lateral faces of the wedge, (b) forces applied to a stiff ring segment of finite radius.

where $p(r)$ is a continuous function, identically equal to zero for $r \geqslant r_0$, and satisfying the conditions

$$\int_0^{r_0} p(r)\,dr = 0, \quad \int_0^{r_0} p(r)\,r\,dr = M/2. \tag{10.10}$$

Furthermore, to get a unique solution one imposes the additional regularity requirement of boundedness of the resulting force on any radial cut of the wedge:

$$\int_0^{\infty} \sigma_{\theta\theta}(r, \theta)\,dr < \infty, \quad \int_0^{\infty} \sigma_{r\theta}(r, \theta)\,dr < \infty. \tag{10.11}$$

To get a solution of the problem posed we apply the Mellin integral transformation in the variable r. As is well known [cf. Sneddon (1951)], the Mellin transform of a function and its inverse are given by the relations

$$\overline{f}(s) = \int_0^{\infty} f(r)\,r^{s-1}\,dr, \quad f(r) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \overline{f}(s)\,r^{-s}\,ds. \tag{10.12}$$

Applying the Mellin transformation to the biharmonic equation (10.3), we get for the transform of the stress function $\overline{\Psi}(s, \theta)$ the ordinary differential equation

$$\left[\frac{d^2}{d\theta^2} + s^2\right]\left[\frac{d^2}{d\theta^2} + (s+2)^2\right]\overline{\Psi}(s, \theta) = 0. \tag{10.13}$$

The stress field sought is antisymmetric, so the stress function must be also. The general antisymmetric solution of (10.13) has the form

$$\overline{\Psi}(s, \theta) = A(s)\sin s\theta + B(s)\sin(s+2)\theta. \tag{10.14}$$

Further, (10.9) with (10.2) taken into account can be written in the form

$$r^2\partial^2_{rr}\Psi(r, \pm\alpha) = \pm p(r) r^2,$$
$$\partial_r\left[\frac{1}{r}\partial_\theta\Psi(r, \pm\alpha)\right] = 0. \tag{10.15}$$

Applying the Mellin transformation to these conditions and integrating by parts (therefore it was necessary for us to multiply by r^2), we get the boundary conditions for the function $\overline{\Psi}(s, \theta)$:

$$\overline{\Psi}(s, \pm\alpha) = \pm\bar{p}(s)/s(s+1),$$
$$\frac{d\overline{\Psi}(s, \pm\alpha)}{d\theta} = 0. \tag{10.16}$$

Here

$$\overline{p}(s) = \int_0^{r_0} p(r) r^{s+1}\, dr. \tag{10.17}$$

From (10.14) and (10.16) we determine the constants $A(s)$ and $B(s)$; substituting the result into the inversion formula we obtain the solution for the stress function in the form

$$\Psi(r, \theta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\overline{p}(s)\,[s\cos s\alpha \sin(s+2)\theta - (s+2)\cos(s+2)\alpha \sin s\theta]\, r^{-s}\, ds}{s(s+1)\,[(s+1)\sin 2\alpha - \sin 2(s+1)\alpha]}. \tag{10.18}$$

The expressions for the components of the stress tensor are obtained from this by differentiation:

$$\sigma_{\theta\theta}(r, \theta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\overline{p}(s)\,[s\cos s\alpha \sin(s+2)\theta - (s+2)\cos(s+2)\alpha \sin s\theta]\, r^{-s-2}\, ds}{(s+1)\sin 2\alpha - \sin 2(s+1)\alpha},$$

$$\sigma_{rr}(r, \theta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\overline{p}(s)\,[(s+2)\cos(s+2)\alpha \sin s\theta - (s+4)\sin(s+2)\theta\cos s\alpha]\, r^{-s-2} ds}{(s+1)\sin 2\alpha - \sin 2(s+1)\alpha}, \tag{10.19}$$

$$\sigma_{r\theta}(r, \theta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\overline{p}(s)\,[(s+2)\cos s\alpha \cos(s+2)\theta - (s+2)\cos(s+2)\alpha \cos s\theta]\, r^{-s-2} ds}{(s+1)\sin 2\alpha - \sin 2(s+1)\alpha}.$$

It is evident that the integrands in (10.19) are meromorphic functions of the complex variable s, whose poles correspond to the zeros of the entire function

$$(s+1)\sin 2\alpha - \sin 2(s+2)\alpha = G(s, \alpha). \tag{10.20}$$

In the integrals (10.18)-(10.19) the abscissa of the line of integration $\operatorname{Re} s = c$ can be chosen arbitrarily within one band of regularity of the integrand. Which band of regularity to take is determined by the conditions imposed on the stress at infinity. The requirement of vanishing at infinity and of regularity of the stress, i.e., the satisfaction of (10.11) allows us to select the band of regularity containing the point $s = -1$. The representation of the solution by (10.18)-(10.19) is convenient for calculating asymptotics.

To calculate the integrals we must close the contour of integration, adding to the line $\operatorname{Re} s = c$ a semicircle of large radius on the right or the left, depending on whether we are interested in the asymptotics of the stress field for $r \to 0$ or $r \to \infty$, and then letting the radius of the circle tend to infinity. Thus the required integral is expressed in terms of the sum of the residues at the poles contained in the contour obtained, i.e., for the stress, at the points corresponding to the roots of the function (10.20). The principal terms in the asymptotic solution for $r \to \infty$ of interest to us are thus determined by the roots of (10.20) that have the smallest real parts.

Investigation of the roots of (10.20) shows that the situation changes at the value $\alpha = \alpha_* \approx 0.715\,\pi$ that makes the expression $\sin 2\alpha - 2\alpha \cos 2\alpha$ vanish. Namely, for $0 < \alpha < \alpha_*$ the root of (10.20) having the smallest real part is actually the simple root $s = 0$. For $\alpha = \alpha_*$ the root $s = 0$ becomes double: for $s = 0$, not only $G(s, \alpha)$ but also $G'(s, \alpha) = \sin 2\alpha - 2\alpha \cos 2(s+1)\alpha$ vanishes. Finally, for $\alpha_* < \alpha \leqslant \pi$ there appears a real simple negative root $s = \lambda(\alpha)$, where $\lambda(\alpha)$ varies monotonically from zero for $\alpha = \alpha_*$ to $-1/2$ for $\alpha = \pi$. Hence the principal terms in the expansion for $r \to \infty$ are different in these three cases:

(1) For $0 < \alpha < \alpha_*$ and $r \to \infty$,

$$\left.\begin{aligned}
\Psi &= \frac{M\,(2\theta \cos 2\alpha - \sin 2\theta)}{2\,(\sin 2\alpha - 2\alpha \cos 2\alpha)} + o(1),\\
\sigma_{rr} &= \frac{2M \sin 2\theta}{(\sin 2\alpha - 2\alpha \cos 2\alpha) r^2} + o(r^{-2}), \quad \sigma_{\theta\theta} = o(r^{-2}),\\
\sigma_{r\theta} &= \frac{M\,(\cos 2\alpha - \cos 2\theta)}{(\sin 2\alpha - 2\alpha \cos 2\alpha)\, r^2} + o(r^{-2}).
\end{aligned}\right\} \tag{10.21}$$

(2) For $\alpha = \alpha_*$ and $r \to \infty$,

$$\left.\begin{aligned}
\Psi &= \frac{M}{12\alpha_*^2 \sin 2\alpha_*}\left\{3\left(g(r_0)-\ln\frac{r}{r_0}\right)(2\theta\cos 2\alpha_* - \sin 2\theta) - \right.\\
&\quad \left. - 3\theta\cos 2\theta + 4\sin 2\theta - 5\theta\cos 2\alpha_* - 6\alpha_*\theta\sin 2\alpha_*\right\} + o(1),\\
\sigma_{rr} &= \frac{M}{12\alpha_*^2 \sin 2\alpha_* r^2}\left\{\left[12g(r_0) - 12\ln\frac{r}{r_0} - 1\right]\sin 2\theta + \right.\\
&\quad \left. + 12\theta\cos 2\theta - 6\theta\cos 2\alpha_*\right\} + o(r^{-2}),\\
\sigma_{\theta\theta} &= \frac{M(2\theta\cos 2\alpha - \sin 2\theta)}{4\alpha_*^2 \sin 2\alpha_* r^2} + o(r^{-2}),\\
\sigma_{r\theta} &= \frac{M}{12\alpha_*^2 \sin 2\alpha_* r^2}\left\{\left[6g(r_0) - 6\ln\frac{r}{r_0} + 1\right](\cos 2\alpha_* - \right.\\
&\quad \left. - \cos 2\theta) + 6\theta\sin 2\theta - 6\alpha_*\sin 2\alpha_*\right\} + o(r^{-2}),
\end{aligned}\right\} \quad (10.22)$$

where

$$g(r_0) = \frac{2}{M}\int_0^{r_0} p(r)\, r\ln\left(\frac{r}{r_0}\right) dr.$$

(3) For $\alpha_* < \alpha \leqslant \pi$ and $r \to \infty$,

$$\left.\begin{aligned}
\Psi &= \frac{\bar{p}(\lambda)\left[(\lambda+2)\cos(\lambda+2)\alpha\sin\lambda\theta - \lambda\cos\lambda\alpha\sin(\lambda+2)\theta\right]}{\lambda(\lambda+1)(\sin 2\alpha - 2\alpha\cos 2(\lambda+1)\alpha)\, r^{\lambda}} + \\
&\quad + \frac{M(2\theta\cos 2\alpha - \sin 2\theta)}{2(\sin 2\alpha - 2\alpha\cos 2\alpha)} + o(1),\\
\sigma_{\theta\theta} &= \frac{\bar{p}(\lambda)\left[(\lambda+2)\cos(\lambda+2)\alpha\sin\lambda\theta - \lambda\cos\lambda\alpha\sin(\lambda+2)\theta\right]}{[\sin 2\alpha - 2\alpha\cos 2(\lambda+1)\alpha]\, r^{\lambda+2}} + \\
&\quad + o(r^{-2}),\\
\sigma_{rr} &= \frac{\bar{p}(\lambda)\left[(\lambda+4)\cos\lambda\alpha\sin(\lambda+2)\theta - (\lambda+2)\cos(\lambda+2)\alpha\sin\lambda\theta\right]}{[\sin 2\alpha - 2\alpha\cos 2(\lambda+1)\alpha]\, r^{\lambda+2}} + \\
&\quad + \frac{2M\sin 2\theta}{(\sin 2\alpha - 2\alpha\cos 2\alpha)\, r^2} + o(r^{-2}),\\
\sigma_{r\theta} &= \frac{\bar{p}(\lambda)(\lambda+2)\left[\cos(\lambda+2)\alpha\cos\lambda\theta - \cos\lambda\alpha\cos(\lambda+2)\theta\right]}{(\sin 2\alpha - 2\alpha\cos 2(\lambda+1)\alpha)\, r^{\lambda+2}} + \\
&\quad + \frac{M(\cos 2\alpha - \cos 2\theta)}{(\sin 2\alpha - 2\alpha\cos 2\alpha)\, r^2} + o(r^{-2}),
\end{aligned}\right\} \quad (10.23)$$

where $\bar{p}(\lambda)$ is determined by (10.17) as before.

We now apply dimensional analysis to the original non-self-similar problem. We write the function $p(r)$ in the form

$$p(r)=\frac{M}{2r_0^2}\varphi\left(\frac{r}{r_0}\right). \tag{10.24}$$

It is evident that the solution Ψ is governed by the following quantities: M, r_0, r, θ, and α, whose dimensions are respectively F, L, L, 1, and 1. Consequently the standard procedure of dimensional analysis gives

$$\Psi=M\Phi\left(\frac{r}{r_0},\ \theta,\ \alpha\right). \tag{10.25}$$

The previous considerations by means of which we arrived at (10.4) were based on the implicit assumption that at large distances from the tip of the wedge the parameter r/r_0 is very large, and hence the length r_0 of the part of the lateral face of the wedge on which the loading was distributed is inessential.

The analysis just performed showed that this is actually so only for $0<\alpha<\alpha_*$. If $\alpha>\alpha_*$, then the size r_0 remains essential, no matter how far we have gone from the tip of the wedge. Nevertheless the asymptotics of the stress function, and hence also of all the components of the stress tensor, are self-similar; but this self-similarity is incomplete, not being determined by dimensional considerations.

In fact, in accord with the general procedure developed in Chapter 5, we assume that there exists a real number λ such that the function $\Phi(\eta, \theta, \alpha)$, where $\eta=r/r_0$, behaves like $\eta^{-\lambda}\Phi_1(\theta, \alpha)$ as $\eta\to\infty$ (i.e., as $r\to\infty$ or $r_0\to 0$). Then by virtue of (10.25) the limiting solution obtained by shrinking r_0 to zero, i.e., for $\eta\to\infty$, has the form

$$\Psi=\frac{Mr_0^\lambda}{r^\lambda}\Phi_1(\theta,\ \alpha). \tag{10.26}$$

It is clear here that if we want to get a correct asymptotics of the solution of the non-self-similar problem as $r/r_0\to\infty$ by shrinking r_0 to zero it is impossible to keep M constant; it should also tend to zero so that the product Mr_0^λ remains constant.

Let us substitute (10.26) into the biharmonic equation (10.3). We obtain for $\Phi_1(\theta, \alpha)$ the ordinary equation

$$\Phi_1^{IV}+\left[\lambda^2+(\lambda+2)^2\right]\Phi_1''+\lambda^2(\lambda+2)^2\Phi_1=0, \tag{10.27}$$

which coincides with (10.13) for $s=\lambda$. The solution of interest to us must be antisymmetric and must satisfy the conditions $\Phi_1=0$, $d\Phi_1/d\theta=0$ for $\theta=\pm\alpha$.

The last condition follows from the fact that the stress on the lateral faces of the wedge is zero. From these conditions we get an expression for Φ_1 to within a dimensionless constant factor β,

$$\Phi_1 = \beta\,[(\lambda+2)\cos(\lambda+2)\alpha\sin\lambda\theta - \lambda\cos\lambda\alpha\sin(\lambda+2)\theta], \tag{10.28}$$

and also the characteristic equation for determining λ,

$$(\lambda+1)\sin 2\alpha - \sin 2(\lambda+1)\alpha = 0, \tag{10.29}$$

which coincides with the vanishing of the function (10.20). As was said above, for $\alpha_* < \alpha < \pi$ (10.29) has a real negative root, and at the same time for $0 < \alpha < \alpha_*$ the smallest root of this equation is zero. Therefore for $0 < \alpha < \alpha_*$ the function $\Phi(\eta, \theta, \alpha)$ in the expression for the solution of the non-self-similar problem tends to a finite nonzero limit as $\eta \to \infty$ (as the region of application of the load shrinks to zero), and the dimensional considerations developed in Section 1 turn out to be applicable and lead to the correct final result. For $\alpha_* < \alpha \leqslant \pi$ however, the limiting solution can be written in the form

$$\Psi = \frac{A}{r^\lambda}\,[(\lambda+2)\cos(\lambda+2)\alpha\sin\lambda\theta - \lambda\cos\lambda\alpha\sin(\lambda+2)\theta], \tag{10.30}$$

$\lambda \neq 0$, where the constant $A = \beta M r_0^\lambda$ can no longer be determined if we seek a self-similar solution of the second kind directly. It can be found only if we follow the transition from a solution of the non-self-similar problem to a self-similar asymptotics. In fact the asymptotic representation of a solution of the non-self-similar problem for large r has, in the case $\alpha_* < \alpha \leqslant \pi$, a principal term which coincides with (10.30) [cf. (10.23)] if one takes

$$A = \frac{\int_0^{r_0} p(r)\, r^{\lambda+1}\, dr}{\lambda(\lambda+1)\,[\sin 2\alpha - 2\alpha\cos 2(\lambda+1)\alpha]}. \tag{10.31}$$

Thus the asymptotics of the solution obtained by shrinking the region of application of the loading on the lateral faces of the wedge to zero "remember" for $\alpha_* < \alpha \leqslant \pi$ not the ordinary moment of force, i.e., not the integral $\int_0^{r_0} p(r) r\, dr = M/2$, but a more complicated fractional-power moment of the system of forces acting on the lateral faces of the wedge. Here the power to which the radius appears in the moment depends on the opening angle of the wedge, and is determined by solving the eigenvalue problem for the linear equation (10.27) under the conditions that the solution and its derivative vanish at the endpoints of the interval.

The solution just considered is instructive in many respects. It contains the parameter α, the opening angle of the wedge. As is evident from the preceding analysis, for angles less than some critical value we can use the "naive" arguments of dimensional analysis, considering only the prescribed moment of forces acting on the wedge; we get a self-similar solution of the first kind, which is completely determined by direct construction with the help of dimensional analysis. For wedge angles larger than the critical one, "naive" considerations of dimensional analysis are not applicable, because it is impossible, for $\alpha > \alpha_*$, to delete r_0 from the list of governing parameters and to leave M there. Nevertheless, by shrinking to zero the region of application of the loading on the lateral faces of the wedge, we obtain in this case too a self-similar limiting solution. The attempt to construct this solution directly as a self-similar solution of the second kind determines the limiting solution, just as for any self-similar solution of the second kind, only to within a constant. The value of this constant can be obtained by matching the self-similar solution with a solution of the non-self-similar problem. It can be expressed, as carrying out the matching shows, in terms of some fractional moment of the stress distribution on the lateral faces of the wedge, but just which moment, i.e., to what power r, can be determined only after solving the problem; it is impossible to determine this power in advance from dimensional considerations. Finally, for the wedge angle equal to the critical one, dimensional considerations turn out to be meaningless; they do not lead to any simplification of the solution, and arguing about the smallness of the part on which loading is applied, leading to a degeneration of the problem, is not valid. In other words, self-similarity in the parameter η does not occur, no matter how large η may be.

Nevertheless, as (10.22) shows, the asymptotics of the solution is self-similar in this case, since the expression for $\Phi = \Psi/M$ can for large $\eta = r/r_0$ be written in the form

$$\Phi = \ln \eta \Phi_3(\theta).$$

This self-similarity, however, is not of power type and is itself no longer a solution.

It is obvious that it would be impossible to establish what was said above without knowing the non-self-similar solution of the complete nondegenerate problem. In nonlinear problems an analysis similar to that presented above is practically never possible; as already mentioned, one of the main reasons that we are generally interested in self-similar solutions of degenerate problems is the desire to obtain some idea of the structure of the solutions of complicated nondegenerate nonlinear problems. The example presented clearly demonstrates that it is insufficient simply to construct a self-similar solution; it is necessary to verify that this solution is an intermediate asymptotics for a certain at least restricted class of nondegenerate problems.

3. The Use of Self-Similar Solutions for Estimating the Stiffness of a Wedge

Recently Budiansky and Carrier (1973) carried out a very instructive investigation in connection with the same problem of a wedge on which a couple acts, concerning the application of self-similar solutions to estimating the bulk integral characteristics of the solution to nondegenerate problems. Namely, they considered (Fig. 10.2b) an elastic wedge truncated along a circular arc $r=R$ close to its tip and reinforced by an absolutely stiff ring segment at the cut $r=R$. Using the fact that the same equations of the plane theory of elasticity apply to the case of plane stress (thin plates) as to the case of a plane strain, Budiansky and Carrier considered a "generalized wedge" consisting of a tightly wound infinite helicoid. This makes it possible to consider the problem for arbitrary angles α, including those greater than π. A couple is applied to the ring segment with torque M (per unit thickness of the wedge). It is clear that the reinforced boundary rotates through some small angle Ω. Since the angle of rotation Ω of the reinforced boundary is proportional to the applied torque M per unit wedge thickness, it is natural to call the quantity M/Ω the torsional stiffness of the wedge. As considerations of dimensional analysis show, this quantity, which is governed by the shear modulus G, by Poisson's ratio ν, and the radius R of the circle along which the wedge is cut, is equal to $GR^2C(\nu)$, where the quantity $C(\nu)$ is called the dimensionless stiffness. The problem consists in obtaining a sufficiently reliable estimate for the dimensionless stiffness. Budiansky and Carrier used self-similar solutions for this effectively and instructively from a general point of view. They started from the principle of minimum complementary energy, proved in the theory of elasticity, according to which, among all virtual stress fields that vanish along the lateral faces of the wedge and have on the arc $r=R$ zero resultant force and torque equal to M, the actual stress field minimizes the stress energy (per unit thickness)

$$W=\frac{1}{2}\int_R^\infty\int_{-\alpha}^{\alpha}\sigma_{\mu\nu}\varepsilon_{\mu\nu}r\,dr\,d\theta \tag{10.32}$$

(with summation over repeated indices), where the components of the deformation tensor ε_{ij} are expressed in terms of the components of the stress tensor σ_{ij} by Hooke's law for a generalized state of plane stress (δ_{ij} being a unit tensor)

$$\varepsilon_{ij}=\frac{1}{2G}\left(\sigma_{ij}-\frac{\nu}{1+\nu}\sigma_{\gamma\gamma}\delta_{ij}\right). \tag{10.33}$$

The exact value of the total stress energy is evidently equal to $1/2M\Omega$. Consequently, if $\overline{W}$ is the energy corresponding to a certain virtual stress field, then

$$\overline{W} \geqslant W = \frac{1}{2} M\Omega = \frac{1}{2} \frac{M^2}{GR^2 C(\nu)}, \tag{10.34}$$

whence

$$C(\nu) \geqslant M^2/2GR^2\overline{W}. \tag{10.35}$$

The idea consists in using self-similar stress fields to obtain values of $\overline{W}$ as close as possible to the actual one, and by the same token to get good estimates for $C(\nu)$. Here the energy $\overline{W}$ is found from the relation

$$\overline{W} = -\frac{R}{2} \int_{-\alpha}^{\alpha} (\sigma_{rr}u_r + \sigma_{r\theta}u_\theta)_{r=R}\, d\theta, \tag{10.36}$$

obtained from the energy equation and the condition of rapid decrease of the stress at infinity. If we take as the virtual elastic field the field corresponding to Carothers' (1912) solution (10.7)-(10.8), then a simple but lengthy calculation according to the indicated recipe gives a lower bound for $C(\nu)$:

$$C(\nu) = \frac{16(\sin 2\alpha - 2\alpha \cos 2\alpha)^2}{4(\varkappa + 5)\alpha + 8\alpha \cos 4\alpha - (\varkappa + 7)\sin 4\alpha}. \tag{10.37}$$

Here $\varkappa = (3 - \nu)/(1 + \nu)$.

This estimate is represented by the dashed line in Fig. 10.3 for the case $\nu = 0$. It has obvious absurdities. For example, according to this estimate the

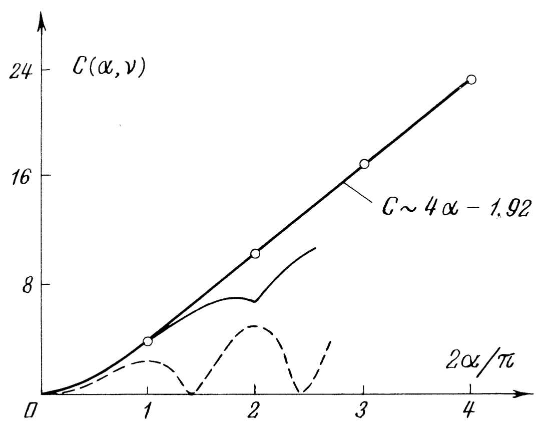


Figure 10.3. Dependence of the dimensionless stiffness of a wedge on its opening angle for $\nu = 0$.

wedge loses stiffness at the critical values at which the numerator of (10.37) vanishes, and long before the critical angles the stiffness starts to decrease as the angle α increases. The latter is obviously wrong since for any wedge an admissible stress field is given by that for a wedge of any smaller angle, extended as identically zero up to the boundary of the wedge, so that the stiffness is a non-decreasing function of the opening angle of the wedge: $dC/d\alpha \geqslant 0$. In fact, calculating the stiffness for $\alpha = \alpha_*$ on the basis of a non-self-similar solution given by the first terms of (10.22) and also belonging to the admissible stress fields, leads to a nonzero estimate of the stiffness. Furthermore, for angles not equal to the critical ones, admissible fields were taken to be represented by the sum of the solutions (10.7) and (10.30), where the coefficient A in (10.30) was chosen so as to minimize the stress energy $\overline{W}$. Also λ was taken as a real root of (10.29) giving minimal stress energy. The corresponding estimate for the stiffness, represented in Fig. 10.3 by a thin solid line, lies significantly higher than the previous estimate and, what is most essential, passes smoothly through the critical angles. However, this estimate is unsatisfactory for large α. For large α the "generalized" wedge must in fact behave like a closed elastic ring, for which $\sigma_{r\theta} = M/2\pi r^2$, $\sigma_{rr} = \sigma_{\theta\theta} = 0$ at all points and the stress energy is equal to

$$\frac{1}{2} M\Omega = 2\pi \int_R^\infty \frac{\sigma_{r\theta}^2}{2G} r\, dr = \frac{M^2}{8\pi R^2 G}; \tag{10.38}$$

consequently the dimensionless stiffness $C(\nu)$ is $C = M/\Omega G R^2 = 4\pi$. Thus for large α each increase of α by π must increase the dimensionless stiffness by 4π, so that $dC/d\alpha \sim 4$ and for large α we have for C the asymptotic formula

$$C = 4\alpha - \delta(\nu). \tag{10.39}$$

To obtain a very accurate estimate of the dimensionless stiffness, Budiansky and Carrier (1973) used the fact that for $\alpha = N\pi/2$, where N is an integer, all the roots of (10.29) are real and are expressed by a simple formula,

$$\lambda = \frac{m}{N} - 1, \tag{10.40}$$

where m is an integer. For the elastic field one can take an expression in the form of the sum of a large number of solutions of the type (10.30),

$$\Psi = \sum_{m=1\,(m\neq N)}^{K} \frac{A_m}{r^{\frac{m}{N}-1}} \left\{ \left(\frac{m}{N}+1\right) \cos\frac{(m+N)\pi}{2} \sin\left(\frac{m}{N}-1\right)\theta - \right.$$
$$\left. - \left(\frac{m}{N}-1\right) \cos\frac{(m-N)\pi}{2} \sin\left(\frac{m}{N}+1\right)\theta \right\} + \frac{M\left[2(-1)^N\theta - \sin 2\theta\right]}{2N\pi(-1)^{N+1}}, \tag{10.41}$$

where $K \geqslant N$, and m is positive. [The last term corresponds to the root that is equal to zero, and is expressed by just the same formulas as in (10.7).] The coefficients A_m are also obtained from a certain condition of minimality of the stress energy $\overline{W}$. The results of corresponding calculations are represented in Fig. 10.3 by the heavy solid line. It is evident that this estimate is considerably higher and, what is important, is compatible with (10.39). The calculation gave the following values for $\delta(\nu)$:

$$\delta(0) = 1.92; \quad \delta(0.25) = 1.68; \quad \delta(0.5) = 1.48.$$

This example shows that self-similar solutions can be used successfully to obtain estimates of the bulk characteristics of non-self-similar solutions of the nondegenerate problems. The investigation conducted by Budiansky and Carrier shows, however, that such use of self-similar solutions should be made very carefully. In fact, for example, for all $\alpha < \alpha_*$ the solution at large distances from the tip is well approximated by Carothers' solution. It would seem that this implies the possibility of using this solution to estimate the stiffness for $\alpha < \alpha_*$. This leads, however, to an unnatural decrease in stiffness, which is connected with the unsuitability of the self-similar solution for describing the stress field close to $r = R$, which makes an essential contribution to the stress energy.

4. The Flow of Ideal Incompressible Fluid In a Corner—Self-Similar Solution of the Second Kind

Even such a simple and generally known problem as plane flow of an ideal incompressible fluid in a corner gives an instructive example of self-similar solutions of the second kind. Thus let us attempt to construct, using dimensional analysis, a solution to the problem of flow past an infinite wedge of opening angle 2α with uniform velocity U at infinity (Fig. 10.4a). It is obvious that the flow is potential with $\mathbf{v} = \operatorname{grad} \varphi$, and the potential φ can depend only on the quantities U, r, θ, and α. The standard procedure of dimensional analysis leads to the relation $\varphi = Ur\Phi(\theta, \alpha)$. From the continuity equation for an incompressible fluid, $\operatorname{div} \mathbf{v} = 0$; and from the fact that the velocity field has a potential, we get the Laplace equation for the potential:

$$\Delta\varphi = \frac{1}{r}\,\partial_r r\,\partial_r \varphi + \frac{1}{r^2}\,\partial^2_{\theta\theta}\varphi = 0.$$

Substituting into this equation the relation $\varphi = Ur\Phi(\theta, \alpha)$, we find easily that $\Phi = A \cos\theta + B \sin\theta$, so that φ is found to be equal to the potential $\varphi = Ax + By$ of a uniform stream, which obviously does not satisfy the conditions on the lateral faces of the wedge.

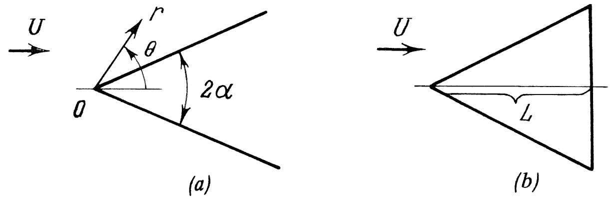


Fig. 10.4. Flow of an ideal incompressible fluid past a wedge: (a) infinite wedge, (b) wedge of finite length.

To resolve the contradiction we turn, following the general rule, to the nondegenerate problem of plane flow of an ideal fluid past a wedge of finite length (Fig. 10.4b). In this case, among the governing parameters there appears yet another, the length L of the wedge, and dimensional analysis gives

$$\varphi = Ur\Phi(\eta,\ \theta,\ \alpha), \quad \eta = r/L. \tag{10.42}$$

The assumption of complete self-similarity in η, i.e., of the existence of a finite limiting solution as $L \to \infty$—the solution of the problem for an infinite wedge—turns out to be invalid. In fact the asymptotic distribution of the flow field near the tip of the wedge can be represented in the form of a self-similar solution of the second kind. Namely, there exists a λ such that $\Phi(\eta, \theta, \alpha) = \eta^{\lambda}\Phi_1(\theta, \alpha)$ for small η, whence it follows that for small η, i.e., close to the tip of the wedge, or for large length L of the wedge, the limiting expression for the potential φ has the form $\varphi = UL^{-\lambda}r^{1+\lambda}\Phi_1(\theta, \alpha)$. Substituting this expression into the Laplace equation and solving, we obtain

$$\varphi = Ar^{1+\lambda}\cos[(\lambda+1)\theta+\gamma], \tag{10.43}$$

where $A = \beta UL^{-\lambda}$, β and γ being dimensionless constants. The eigenvalue λ is determined from the condition for existence of a self-similar solution in the large, in the present case from the condition that the circumferential component of velocity vanish on the lines $\theta = \alpha$, $\theta = \pi$, and $\theta = 2\pi - \alpha$, and only on these lines. Hence we find $\lambda = \alpha/(\pi - \alpha)$ and also $\gamma = -\pi\alpha/(1 - \alpha)$. The constant β can be found by matching with a solution of the original nondegenerate problem, which one obtains without difficulty by applying conformal mapping.

The examples given here are simple enough so that the effective construction of complete solutions to the nondegenerate problems is possible, and at the same time they exhibit clearly enough many of the complexities that can appear in nonlinear problems.

11

Self-Similarities of the First and Second Kind in the Theory of Turbulence. Homogeneous Isotropic Turbulence

1. The Problem of Turbulence

This chapter and the following one differ in that here self-similarities of the first and second kind will be established making essential use of experimental data and without turning to a mathematical formulation of the problem which, at the present time, is simply lacking for turbulence.

The problem of turbulence, to which these chapters are devoted, is considered with good reason the outstanding problem of contemporary classical physics. The phenomenon itself, as is well known, consists in the following. As we saw in Chapter 1, the basic similarity parameter that governs the global properties of the flow of an incompressible viscous fluid is the Reynolds number $\rho Ul/\mu$ (ρ being the density and μ viscosity of the fluid, U a characteristic speed, and l a characteristic length scale of the flow). When the Reynolds number passes through a certain critical value Re_{cr}, different for different flows (for example, for flow in a smooth cylindrical pipe of circular cross section, $\mathrm{Re}_{cr} \sim 10^3$; for flow in a boundary layer, $\mathrm{Re}_{cr} \sim 10^5$), the character of the flow changes suddenly and sharply. A stream that was regular, ordered, and laminar at subcritical values of the Reynolds number becomes essentially irregular. The flow for supercritical values of the Reynolds number undergoes sharp and disorderly changes in space and time, and the fields of flow properties, pressure, velocity, etc., can to a good approximation be considered stochastic, random. Such a condition of flow is called turbulent.

At the present time there exist only more or less likely conjectures, sometimes very interesting, regarding the origin of turbulence, but not having conclusive strength. Also there exists no complete mathematical description of developed turbulent flows. Under these circumstances, in all attempts to create theoretical models for certain classes of turbulent flows, similarity considerations occupy a primary place.

Together with the majority of investigators, we shall start from the fact that for velocities small compared with the speed of sound actual turbulent motion is described by the equations of motion of a viscous incompressible fluid, i.e., by the Navier–Stokes equations of momentum balance and the continuity equation, which in rectangular Cartesian coordinates can be written in the form (Kochin, Kibel', and Roze, 1964; Landau and Lifschitz, 1959)

$$\begin{gathered} \partial_t u_i + u_\alpha \partial_\alpha u_i = -\frac{1}{\rho}\partial_i p + \nu\,\Delta u_i, \\ \partial_\alpha u_\alpha = 0. \end{gathered} \tag{11.1}$$

Here the u_i are the components of the velocity vector, $\nu = \mu/\rho$ is the kinematic viscosity, p is the pressure, and one sums over repeated Greek indices from one to three.

To construct a solution of these equations corresponding to some concrete realization of a developed turbulent flow is impossible in view of their extreme instability. Hence, and also in view of the possibility noted above of considering the properties of a turbulent flow field as random, the description of turbulent flows is always given in statistical terms. As is known, [for details cf. Monin and Yaglom (1971, 1975)], a sufficiently complete description of a developed turbulent flow is given by a set of mean quantities

$$\langle u_i(\mathrm{x},\ t)\rangle,\ \langle p\,(\mathrm{x},\ t)\rangle \tag{11.2}$$

and moment tensors

$$\begin{aligned} B_{ijk\ldots} &= \langle u_i(\mathrm{x},\ t)\, u_j(\mathrm{x}_1,\ t)\, u_k(\mathrm{x}_2,\ t)\ldots\rangle, \\ B_{pij\ldots} &= \langle p\,(\mathrm{x},\ t)\, u_i(\mathrm{x}_1,\ t)\, u_j(\mathrm{x}_2,\ t)\ldots\rangle, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{11.3}$$

for all possible point systems: $\mathbf{x}, \mathbf{x}_1; \mathbf{x}, \mathbf{x}_1, \mathbf{x}_2; \mathbf{x}, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3; \ldots$.

Here the sign $\langle\ldots\rangle$ denotes the probability mean value. Taking probability mean values is used in theoretical work on turbulence as a natural method of averaging. In experimental practice, one uses volume or time means, the identification of these types of averaging with taking probability means being made on the basis of the so-called ergodic hypothesis.

The system of equations for the moments can be obtained by multiplying (11.1) by the velocity components at different points of the flow and subsequently averaging. This was first done by Keller and Friedmann (1924). However in the general case the resulting equations are so complicated to write that in that paper the equations themselves were not even written down; only the basic idea was indicated, and it was enumerated how many and what kind of equations are thus obtained. As a special case one gets the equations for the mean quantities first given in the fundamental paper of Reynolds (1895).

2. Homogeneous Isotropic Turbulence

Essential progress in the development of a statistical theory of turbulence occurred when Taylor (1935) introduced the idea of considering homogeneous isotropic turbulence. This idea gained fundamental significance after Kolmogorov (1941) predicted that in the small scales all developed turbulent flows (i.e., flows at large Reynolds numbers) have the properties of homogeneity and isotropy. A flow is called homogeneous and isotropic if all its moment tensors remain unchanged upon translation, rotation, or mirror reflection with respect to some plane of the system of points $\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2, \ldots$. (To be unchanged means that in a coordinate system arranged relative to the transformed system of points just as the original coordinate system was arranged relative to the original system of points, the values of the components of the tensor remain the same.) For a homogeneous isotropic flow the mean velocity vanishes, and the number of independent components of moment tensors is substantially reduced, as well as the number of quantities on which they depend. Thus in an arbitrary Cartesian coordinate system the components of the second-order moment tensor for the velocity field of a homogeneous isotropic flow are expressed in the following way:

$$B_{ij} = \langle u_i(\mathbf{x},\ t)\, u_j(\mathbf{x}+\mathbf{r},\ t)\rangle = (B_{LL} - B_{NN})\xi_i\xi_j/r^2 + B_{NN}\delta_{ij}. \qquad (11.4)$$

Here $r = |\mathbf{r}|$ is the distance between points, the ξ_i are the components of the radius vector $\mathbf{r}$ joining two points, t is the time, and

$$\begin{aligned} B_{LL}(r,\ t) &= \langle u_L(\mathbf{x},\ t)\, u_L(\mathbf{x}+\mathbf{r},\ t)\rangle, \\ B_{NN}(r,\ t) &= \langle u_N(\mathbf{x},\ t)\, u_N(\mathbf{x}+\mathbf{r},\ t)\rangle, \end{aligned} \qquad (11.5)$$

where u_L is the projection of the velocity vector in the direction of the radius vector $\mathbf{r}$ and u_N is the projection of the velocity vector in the direction normal to the radius vector $\mathbf{r}$. Because of the incompressibility of the flow, the quantities B_{LL} and B_{NN} are connected by the relation

$$B_{NN} = B_{LL} + (r/2)\,\partial_r B_{LL}. \qquad (11.6)$$

Thus, the second-order moment tensor for the velocity field is determined by a single scalar function of two scalar arguments $B_{LL}(r, t)$. The situation is analogous for the two-point third-order moment tensor

$$B_{ij,\,k} = \langle u_i(\mathbf{x},\ t)\, u_j(\mathbf{x},\ t)\, u_k(\mathbf{x}+\mathbf{r},\ t)\rangle,$$

which because of homogeneity, isotropy, and incompressibility can be expressed in terms of one component, a scalar function of the scalar arguments r and t, for example,

$$B_{LL,L}(r, t) = \langle u_L^2(\mathbf{x}, t)\, u_L(\mathbf{x}+\mathbf{r}, t)\rangle. \tag{11.7}$$

Similar reduction in the number of independent variables and independent components of moment tensors due to homogeneity, isotropy, and incompressibility hold also for moments of higher order.

We turn now to the Navier–Stokes equations, multiply them by different velocity components at successively increasing numbers of points, average, and use the symmetry relations following from the homogeneity and isotropy of the flow. Thus we obtain an infinite system of equations which is however not closed at any finite stage because of the presence of quadratic nonlinearity in the Navier–Stokes equations.

The first equation of this system, connecting the two-point second and third moments can be reduced to the form

$$\partial_t B_{LL}(r, t) = 2\nu r^{-4}\, \partial_r r^4\, \partial_r B_{LL} + r^{-4}\, \partial_r r^4 B_{LL,L}(r, t). \tag{11.8}$$

This relation, connecting two unknown functions, is called the Kármán–Howarth equation. It should be noted that in the fundamental paper of von Kármán and Howarth (1938) this equation was presented in another, less convenient form. It was first expressed in the form (11.8) by Loitsianskii (1945) and Millionshchikov (1939).

The problem in complete form consists of solving the infinite system of equations for given initial conditions on the moments; this is the so-called problem of the decay of homogeneous isotropic turbulence. As a matter of fact, we have at best only very general information concerning the initial conditions, and so are unable to give the complete initial distribution of the moments. Therefore the asymptotics of the solution for large t, which "remembers" only some basic properties of the initial conditions, is of particular interest. Under broad assumptions the asymptotics can be considered self-similar.

3. The Decay of Homogeneous Isotropic Turbulence for Negligibly Small Third Moments

If at some stage of the motion the contribution of the third moments in the Kármán–Howarth relation (11.8) is small, then this relation becomes a closed equation for the second moment $B_{LL}(r, t)$,

$$\partial_t B_{LL} = 2\nu r^{-4}\, \partial_r r^4\, \partial_r B_{LL}, \tag{11.9}$$

which coincides in form with the equation of heat conduction in five-dimensional space in the presence of central symmetry. Self-similar solutions of this equation were obtained in the same paper of von Kármán and Howarth (1938) [see also Sedov (1944, 1959)]; they have the form

$$B_{LL} = \frac{A}{(t-t_0)^n} f(\xi, n), \quad \xi = \frac{r}{\sqrt{\nu(t-t_0)}}, \tag{11.10}$$

where A, n, and t_0 are constants, and the function $f(\xi, n)$ satisfies the equation

$$\frac{d^2 f}{d\xi^2} + \left(\frac{4}{\xi} + \frac{\xi}{2}\right)\frac{df}{d\xi} + nf = 0 \tag{11.11}$$

under the conditions

$$f(0, n) = 1, \quad f(\infty, n) = 0, \tag{11.12}$$

the first of which is a normalization, and the second obtained from a natural assumption concerning the statistical independence of the velocities at infinitely distant points: $B_{LL}(\infty, t) = 0$. The function $f(\xi, n)$ so defined can be expressed, as is easy to prove [cf. Abramowitz and Stegun (1964)], in terms of a well-known special function, the confluent hypergeometric function $M(\alpha, \beta, z)$:

$$f = M(n, 5/2, -\xi^2/8). \tag{11.13}$$

The spectrum of the eigenvalues n that determine the rate of decay of the second-order moments turns out, upon direct construction of the self-similar solution (11.10), to be continuous: a solution of (11.11) under the conditions (11.12) exists for any $n > 0$. The value of n that is actually realized must be determined by the initial conditions of the nondegenerate problem, for which (11.10) is a self-similar intermediate asymptotics.

If the initial distribution $B_{LL}(r, 0)$ is such that the quantity

$$\Lambda_0 = \int_0^\infty r^4 B_{LL}(r, 0)\, dr \tag{11.14}$$

is finite and different from zero, i.e., $0 < \Lambda_0 < \infty$, then $n = 5/2$ and the asymptotics of the solution as $t \to \infty$ corresponding to such an initial distribution can be written in the form

$$B_{LL}(r, t) = \frac{\Lambda_0}{48\sqrt{2\pi\nu(t-t_0)^5}} \exp\left(-\frac{r^2}{8\nu(t-t_0)}\right). \tag{11.15}$$

Here the quantity

$$\Lambda = \int_0^\infty r^4 B_{LL}(r, t)\, dr \tag{11.16}$$

is an integral of the motion, analogous to the total amount of heat in the theory of heat conduction, i.e., is independent of time: $\Lambda \equiv \Lambda_0$. Loitsianskii (1945) has shown that under certain assumptions this quantity remains independent of time even when third moments are considered.

One can prove, using properties of the confluent hypergeometric functions, that the solutions (11.10) with $n > 5/2$ have the integral Λ equal to zero. These solutions are in a certain sense unstable with respect to the initial conditions. In fact, if perturbations of such solutions have, say, small but finite Λ_0, then for sufficiently large t just the contribution of the perturbations will govern the decay law, since it corresponds to the smallest $n : n = 5/2$. For this reason self-similar solutions with $n > 5/2$ are of rather little interest. On the other hand there is considerable interest in solutions with $n < 5/2$, for which $\Lambda = \infty$. These solutions can be represented in the form

$$B_{LL}(r,\ t) = \frac{\Lambda_0}{\sqrt{\nu(t-t_0)^5}}\left(\frac{\sqrt{\nu(t-t_0)}}{l}\right)^{5-2n} f\left(\frac{r}{\sqrt{\nu(t-t_0)}},\ n\right), \tag{11.17}$$

where Λ_0 and l are constants having dimensions L^7T^{-2} and L, respectively, where these constants are chosen so that $A = \Lambda_0\, l^{-(5-2n)}\nu^{5/2-n}$. It is evident that all these solutions with $n \neq 5/2$ are self-similarities of the second kind, "remembering" the characteristic length scale l of the initial distribution. (See Chapter 3, where in another problem a completely analogous situation was analyzed.) The situation is that the asymptotics of the dimensionless function $\Phi(\xi, \eta, \ldots)$, $\xi = r/[\nu(t-t_0)]^{1/2}$, $\eta = l/[\nu(t-t_0)]^{1/2}$, which appears upon applying dimensional analysis to the solution of the original non-self-similar problem has, for small η, the form

$$\Phi(\xi,\ \eta,\ \ldots) \cong \eta^{2n-5}\Phi_1(\xi,\ \ldots).$$

Therefore the characteristic length scale l of the initial distribution appears in the constant A governing the solution, but only in combination with Λ_0, and therefore does not spoil the self-similarity.

We note that the stage of development of homogeneous isotropic turbulence at which the third-order moments are negligibly small is sometimes called the final stage of decay. It is argued that at the final stage of decay the velocity is small and hence so are the third-order moments, which are of the order of the velocity cubed and therefore small compared with the second-order moments, which are of the order of the velocity squared. Such an argument is insufficient because the third moments appear in the basic Kármán–Howarth equation (11.8) with derivatives of one less order than the second moments, and each differentiation raises the rate of decay of the corresponding term. Actually, a stage at which the third-order moments are negligibly small can occur only at the start of the motion with a special choice of initial conditions.

4. The Decay of Homogeneous Isotropic Turbulence for Finite Third Moments

From the very first appearance in the papers of Taylor (1935) of the concept of homogeneous isotropic turbulent flow, one has attempted to model it by the decay of turbulence in wind and water tunnels. There is a detailed summary of this work in the paper of Gad-el-Hak and Corrsin (1974). One should note specially the careful experiments performed by Ling and his associates (Ling and Huang, 1970; Ling and Wan, 1972) in a water tunnel–a long channel of square section into which water was introduced through a passive or active grid of rods. In Fig. 11.1 are presented the arrangements of grids used in these experiments: passive (Fig. 11.1a) and active (Fig. 11.1b). In the active grid the rods are equipped with agitating bars that perform oscillating motions at various speeds and frequencies. In the work of Gad-el-Hak and Corrsin (1974) a different active grid (a "jet grid") was used; the rods of the grid were hollow and were provided with upwind or downwind controllable nozzles evenly distributed along each rod. Through the hollow rods and nozzles air was injected aι different rates into the flow. (In this work the experiments were performed in wind tunnels.) Thus in all these experiments turbulent pulsations were introduced into the flow by a grid, and then decayed as the fluid moved downstream. Here the pulsations of velocity become close to isotropic even at small distances from the grid. Figure 11.2 shows the results of measuring the ratios of the mean square fluctuations of the longitudinal and transverse components of velocity (Ling and Huang, 1970); it is evident that they are sufficiently close to unity. We see that if one takes as the time the quantity $t = x/U$ (U being the mean velocity of the flow and x the coordinate measured along the channel downstream from the grid), then the pattern of decay of turbulence along the channel corresponds sufficiently well to the scheme of decay of homogeneous isotropic turbulence in time. (The homogeneity was also specially checked by moving gauges in the crossflow planes x = const, by means of which velocities were measured.)

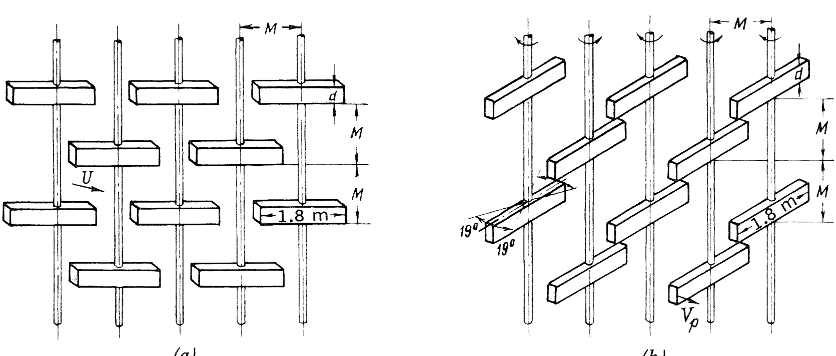


Figure 11.1 Turbulizing grids used by S. C. Ling and co-workers; (a) passive, (b) active.

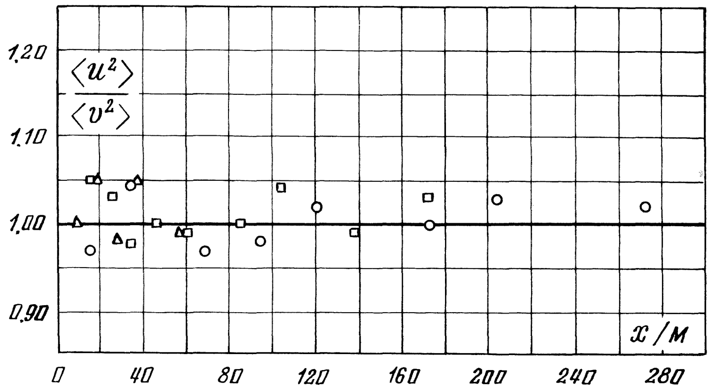


Figure 11.2. Velocity fluctuations in turbulent flow behind a grid are nearly isotropic. (○) $Re_M = 470$, $M = 1.78$ cm, $M/d = 2.8$, $U = 2.9$ cm/sec; (□) $Re_M = 940$, $M = 3.56$ cm, $M/d = 2.8$, $U = 2.9$ cm/sec; (△) $Re_M = 840$, $M = 3.18$ cm, $M/d = 5.0$, $U = 2.9$ cm/sec. [From Ling and Huang (1970)].

The statistical properties–the moment tensors of the turbulent motion under consideration–are thus governed by the mean velocity U of the flow, the characteristic length scale M of the grid, the thickness d of the rods, the viscosity coefficient ν, and the quantities r and $t-t_0$, where t_0 is the effective origin of time, about whose determination something will be said below. Furthermore, for active grids of the type used by Ling and Wan (1972) the moments are governed also by the speed V_p and frequency of oscillation ω of the tips of the agitating bars, and for the active grids used by Gad-el-Hak and Corrsin (1974) an additional governing parameter for the moment tensors is the injection ratio $J = Q_1/Q$ (Q_1 being the flux rate of gas supplied through the hollow rods of the grid and Q the flux rate of gas supplied to the grid).

Dimensional analysis gives for the two-point moments of second and third order

$$B_{LL} = \frac{\nu}{t-t_0}\,\Phi_{LL}\left(\xi,\ \eta,\ \frac{M}{d},\ \frac{MU}{\nu},\ \ldots\right), \tag{11.18}$$

$$B_{LL,L} = \left(\frac{\nu}{t-t_0}\right)^{3/2}\Phi_{LL,L}\left(\xi,\ \eta,\ \frac{M}{d},\ \frac{MU}{\nu},\ \ldots\right), \tag{11.19}$$

where Φ with indices is a dimensionless function of its dimensionless arguments,

$$\xi = \frac{r}{\sqrt{\nu(t-t_0)}},\quad \eta = \frac{M}{x-x_0} = \frac{M}{U(t-t_0)},$$

of the grid parameter M/d, the Reynolds number MU/ν of the grid, and also of the parameters characterizing the activity of the grid.

There is interest in considering the motion at sufficiently large distances from the grid, where $M/U(t-t_0) \ll 1$ and one can assume that the random details of the initial conditions at the grid no longer manifest themselves. The simplest assumption is that for $\eta \ll 1$ there is complete self-similarity in the parameter η. Such an assumption was introduced by von Kármán (von Kármán and Howarth, 1938), supposing that this assumption can be satisfied for large Reynolds numbers, when the influence of the viscosity is inessential. Under the assumption of complete self-similarity in η for $\eta \ll 1$, one must have at sufficiently large distances from the grid the relations

$$\frac{B_{LL}(r, t)}{B_{LL}(0, t)} = f\left(\xi, \frac{M}{d}, \frac{MU}{\nu}, \ldots\right), \quad B_{LL}(0, t) = \frac{A}{t-t_0}, \tag{11.20}$$

$$\frac{B_{LL,L}(r, t)}{B_{LL}^{3/2}(0, t)} = g\left(\xi, \frac{M}{d}, \frac{MU}{\nu}, \ldots\right). \tag{11.21}$$

Here A is a constant depending on the initial conditions at the grid. Equations (11.20) and (11.21) were proposed by Dryden (1943) and Sedov (1944).

Next in degree of complexity is the assumption of incomplete self-similarity in the variable η for $\eta \ll 1$. In this case one must have at large distances from the grid the relations

$$B_{LL}(r, t) = \frac{\nu M^\alpha}{U^\alpha (t-t_0)^{1+\alpha}} F\left(\xi, \frac{M}{d}, \frac{MU}{\nu}, \ldots\right), \tag{11.22}$$

$$\frac{B_{LL}(r, t)}{B_{LL}(0, t)} = f\left(\xi, \frac{M}{d}, \frac{MU}{\nu}, \ldots\right), \tag{11.23}$$

$$B_{LL}(0, t) = \frac{A}{(t-t_0)^{1+\alpha}}, \tag{11.24}$$

$$B_{LL,L}(r, t) = \frac{\nu^{3/2} M^\alpha}{U^\alpha (t-t_0)^{3/2+\alpha}} g\left(\xi, \frac{M}{d}, \frac{MU}{\nu}, \ldots\right), \tag{11.25}$$

$$\frac{B_{LL,L}(r,t)}{B_{LL}^{3/2}(0, t)} = B(t-t_0)^{\alpha/2} g\left(\xi, \frac{M}{d}, \frac{MU}{\nu}, \ldots\right). \tag{11.26}$$

Here A, B, and α are again constant quantities. The equality of the powers to which $\eta = M/U(t-t_0)$ appears in the expressions for $B_{LL}(r, t)$ and $B_{LL,L}(r, t)$ follows from the Kármán–Howarth equation (11.8) relating these quantities.

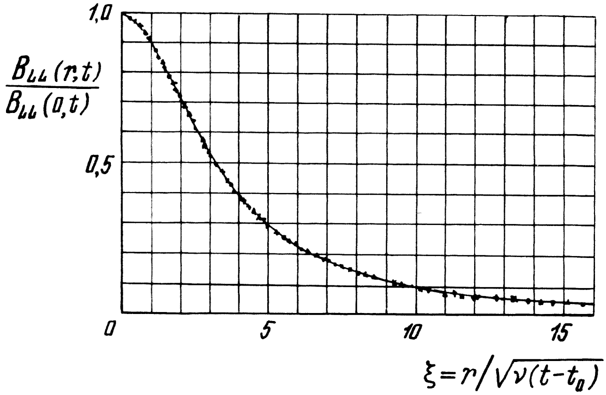


Figure 11.3. Correlation function f for flow behind a passive grid is self-similar: all experimental points lie on a single curve f(ξ). [From Ling and Huang (1970)].

We turn to the results of experiments. In Fig. 11.3 and Fig. 11.4 are shown the results of measuring the correlation function

$$f = \frac{B_{LL}(r, t)}{B_{LL}(0, t)}$$

in the cases of a passive grid (Ling and Huang, 1970) and an active grid (Ling and Wan, 1972) as a function of $\xi = r/[\nu(t - t_0)]^{1/2}$ (the effective origin t_0 being appropriately defined, see below). It is evident that in all cases the experimental points fall very closely onto a single curve, different for each different case. This

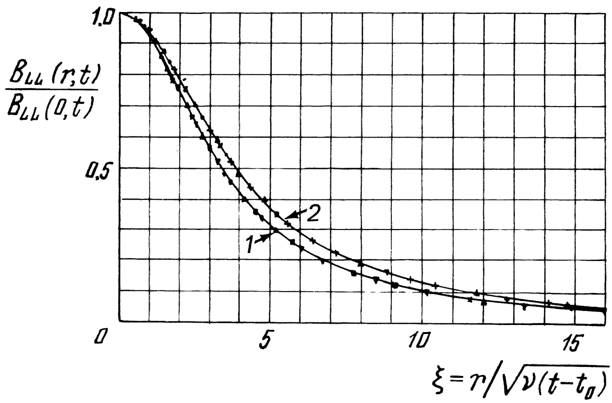


Figure 11.4. Correlation functions for flow behind an active grid are self-similar: all experimental points lie on a single curve f(ξ). Curve 1, $V_p/U = 3$; curve 2, $V_p/U = 17$. [From Ling and Huang (1970)].

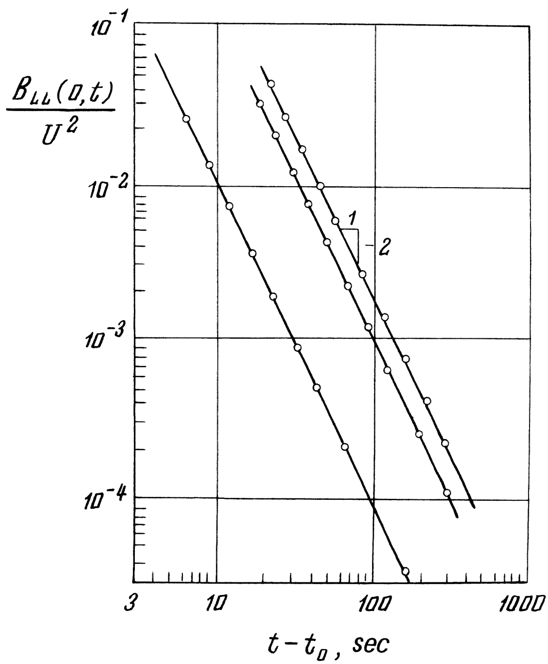


Figure 11.5. Moment $B_{LL}(0, t)$ behind a passive grid decays according to self-similar power law, but exponent is different from unity. Different curves correspond to different combinations of passive grids. [According to data of Ling and Huang (1970)].

confirms the self-similarity of the correlation function f, but does not determine the character of self-similarity of the moment tensors; it is evident from (11.20) and (11.23) that a corresponding result must hold in both cases, for complete as well as for incomplete self-similarity.

In Fig. 11.5 and Fig. 11.6 are shown results of measuring the quantity $B_{LL}(0, t)$ for, respectively, passive grids of different types (Ling and Huang, 1970) and active grids (Ling and Wan, 1972). It is evident that in all cases the decay, even at small distances, follows the self-similar power law

$$B_{LL}(0, t)=\frac{A}{(t-t_0)^n}, \quad n=1+\alpha. \tag{11.27}$$

The method for determining the effective origin of time t_0 is shown in Fig. 11.7. The power law of decay $B_{LL}(0, t)=\langle u^2\rangle \sim (t-t_0)^{-n}$ leads to the fact that for large t the quantity $[U^2/\langle u^2\rangle]^{1/n}$ must be a linear function of time, i.e., as a function of t it is represented by a straight line. Hence intersections with the axis of abscissas of straight lines drawn through the experimental points give the values of t_0.

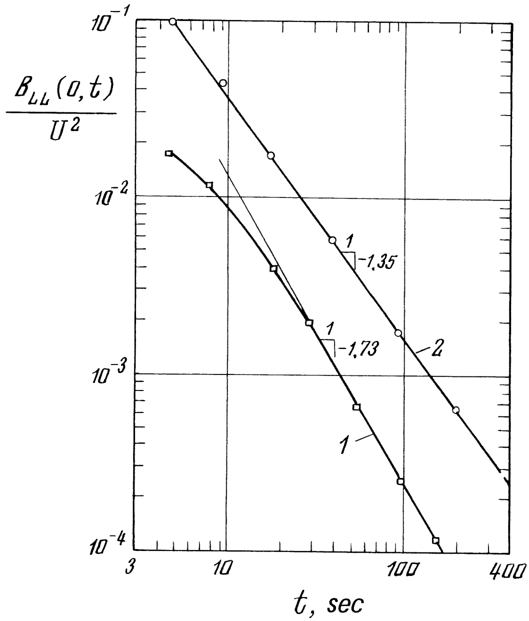


Figure 11.6. Moment $B_{LL}(0, t)$ behind an active grid decays according to self-similar power law, but exponent is different from unity. Curve 1 corresponds to $V_p/U = 3$, curve 2 to $V_p/U = 17$. [From Ling and Wan (1972)].

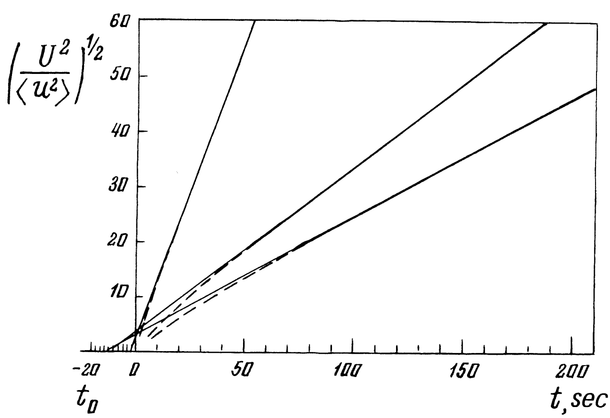


Figure 11.7. Determination of effective origin of time t_0 according to Ling and Huang (1970) for the case of a passive grid.

We see that in all cases the exponent α turns out to be different from zero: it is equal to unity for passive grids, 0.73 for an active grid with $V_p/U = 3$, and 0.35 for an active grid with $V_p/U = 17$. This exponent thus depends on the initial conditions, i.e., the conditions at the grid.

Table 11.1, taken from Gad-el-Hak and Corrsin (1974), contains results of the data processing of other experiments of various authors. In treating the variation of the quantity $B_{LL}(0, t)$ with t, the dependence was assumed to be a power law in accord with (11.27), and the exponents are given in the table. In some cases the turbulence was weakly anisotropic, so the decay exponents are presented for all three components of fluctuation velocity. The experiments reflected in the table were performed on active as well as passive grids. In the experiments on active grids, as already mentioned, a grid of hollow rods was used with nozzles through which air was injected into the flow. The dependence of the exponent on the injection ratio J is shown in Fig. 11.8.

It is evident that the exponents in the decay law depend on conditions at the grid (the Reynolds number of the grid and the characteristics of its activity: J, V_p/U, ω, etc.). The exponent α turns out to be equal to zero, i.e., the self-similarity of the decay turns out to be complete, only in the case of enormously large Reynolds numbers of the grid, reached by Kistler and Vrebalovich (1966).

Unfortunately, third-order moments have been measured by almost no one: one of the few papers up to now in which third moments are measured is that of Stewart (1951). In this paper the self-similarity of the correlation function is emphasized, and attention is specially given to the absence of a unique depen-

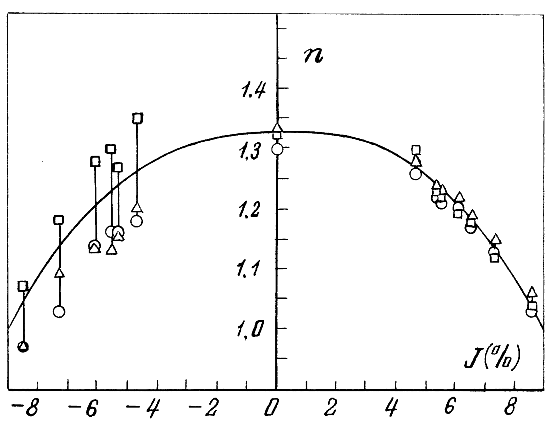


Figure 11.8. Dependence on injection rate J of exponent in law of decay of $B_{LL}(0, t)$ for active grid. Value $J > 0$ correspond to coflow injection, $J < 0$ to counterflow injection. For $J > 0$ decay is isotropic, exponent is different from unity. (○) lengthwise component of velocity; (△, □) transverse components of velocity. [From Gad-el-Hak and Corrsin (1974)].

TABLE 11.1[a]

Authors, type of grid	Reynolds number of grid, $R_M \times 10^{-4}$	Exponent in decay law, $n = 1 + \alpha$		
		$\langle u^2 \rangle$	$\langle v^2 \rangle$	$\langle w^2 \rangle$
Gad-el-Hak and Corrsin				
active grid with coflow injection through hollow rods				
injection ratio J (%): 8.55	4.18	1.03	1.06	1.04
7.32	4.83	1.13	1.15	1.12
6.55	5.36	1.17	1.19	1.18
6.12	5.72	1.19	1.22	1.20
5.56	6.26	1.21	1.22	1.23
5.34	6.50	1.22	1.24	1.23
4.70	7.34	1.26	1.28	1.30
the same grid with counterflow injection				
injection ratio J (%): 8.55	4.18	0.97	0.97	1.07
7.32	4.83	1.03	1.09	1.18
6.12	5.72	1.14	1.14	1.28
5.56	6.26	1.16	1.13	1.30
5.34	6.50	1.16	1.16	1.27
4.70	7.34	1.18	1.20	1.35
the same grid without injection	4.83	1.32	1.33	1.29
	5.72	1.32	1.31	1.34
	6.26	1.30	1.32	1.35
	6.50	1.28	1.35	1.32
	7.34	1.30	1.33	1.34
Corrsin				
passive biplane grid of round rods	0.85	1.30	1.22	–
	1.70	1.28	1.14	–
	2.60	1.35	1.16	–

Batchelor and Townsend				
passive biplane grid of round rods	0.55	1.13	–	–
	0.10	1.25	–	–
Baines and Peterson				
passive biplane grid of rods of square section	2.40	1.37	–	–
Tsuji and Hama				
passive biplane grid of round rods	3.30	1.35	–	–
Wyatt				
passive biplane grid of round rods	1.10	1.27	–	–
	2.20	1.27	–	–
	4.40	1.25	–	–
Kistler and Vrebalovich				
passive biplane grid of round rods	242.00	1.00	1.00	–
Uberoi				
passive biplane grid of round rods	2.90	1.20	1.20	–
Harris				
passive grid of parallel round rods	1.18	1.67	1.62	1.51
	1.15	1.52	1.60	1.74
	0.82	1.43	1.45	1.45
	2.64	1.14	1.45	1.25
	0.61	1.34	1.23	1.44
	1.82	1.43	1.33	1.39
Comte-Bellot and Corrsin				
passive biplane grid of rods of square section	1.70	1.29	1.28	–
	3.40	1.27	1.29	–
	6.80	1.25	1.27	–
	13.50	1.15	1.16	–
same, round rods	3.40	1.24	1.24	–
	3.60	1.26	1.27	–
same, grid of discs	3.40	1.32	1.52	–
	6.80	1.33	1.30	–
Guillon				
active grid with injection through hollow rods injection ratio $J = 4.40\%$	11.42	1.27	–	–

Continued

TABLE 11.1[a] (*Continued*)

Authors, type of grid	Reynolds number of grid, $R_M \times 10^{-4}$	Exponent in decay law, $n = 1 + \alpha$		
		$\langle u^2 \rangle$	$\langle v^2 \rangle$	$\langle w^2 \rangle$
Luxenberg and Wiskind				
active grid with injection through hollow rods				
injection ratio J (%): 0	0.65	1.31	–	–
2.35	0.69	1.03	–	–
2.98	0.65	1.08	–	–
3.87	0.65	1.35	–	–
4.33	0.65	1.69	–	–
Liu, Greber, and Wiskind				
active grid with injection through hollow rods				
injection ratio, J (%): 0	0.65	1.45	–	–
1.92	0.65	1.02	–	–
2.99	0.65	0.86	–	–
3.56	0.65	2.02	–	–
5.25	0.65	2.48	–	–
5.64	0.65	3.37	–	–

[a] From Gad-el-Hak and Corrsin (1974).

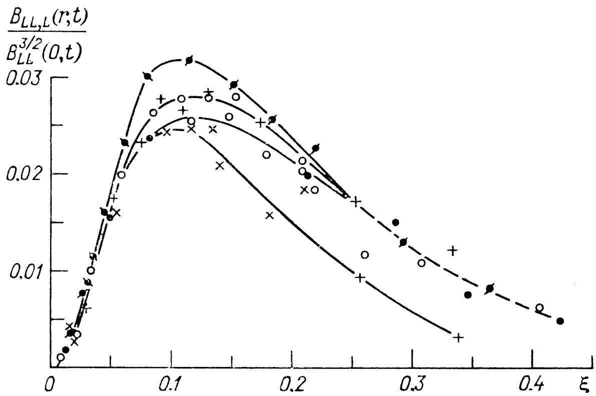


Figure 11.9. Unique dependence of the quantity $B_{LL,L}(r, t)/B_{LL}^{3/2}(0, t)$ on the self-similar variable is lacking–the curves for different moments of time do not coincide. $t = 0.041$ sec (×), 0.0615 sec (•), 0.123 sec (○), 0.184 sec (+), 0.246 sec (ɸ). [From Stewart (1951)].

dence of the quantity $B_{LL,L}(r, t)/B_{LL}^{3/2}(0, t)$ on the self-similar variable for different instants of time (Fig. 11.9). This conforms to incomplete self-similarity of the decay [cf. (11.26)] and would not hold for complete self-similarity.

Thus we have arrived at the conclusion that in experiments the decay of turbulence is self-similar even at small distances from the grid, but this self-similarity is of the second kind, so that the influence of the initial scale–the length scale of the grid–never vanishes, but because of the peculiarities of homogeneous isotropic turbulence appears only in combination with various parameters. The exponent in the law of decay cannot be determined from considerations of dimensional analysis, but is selected from the continuous spectrum of possible values by the initial conditions (the conditions at the grid)–the situation in principle being analogous to what we met above in considering the self-similar analogue of the Korteweg-deVries equation.

It is clear that the conclusions drawn relate only to the case of "weak" turbulence, for which "the Reynolds number of the turbulence" $\mathrm{Re}_T = vl/\nu$ (v being the scale of the fluctuation velocity and l the turbulence length scale) is sufficiently small. In the contrary case there is no reason to expect independence of Re_T, i.e., self-similarity of the decay. The analysis of self-similar decay of homogeneous isotropic turbulence presented above was given by Barenblatt and Gavrilov (1974).

5. Locally Isotropic Turbulence

The investigation of the local structure of turbulent flows of an incompressible viscous fluid at large Reynolds numbers in the papers of Kolmogorov

(1941, 1962) and Obukhov (1941, 1962) also furnishes very instructive examples of self-similar intermediate asymptotics of various types.

According to the basic hypothesis of Kolmogorov, at large Reynolds numbers hydrodynamic fields have the properties of local isotropy, homogeneity, and stationarity. Local isotropy and homogeneity mean that the moment tensors in which the relative velocities

$$\Delta_r \mathbf{u} = \mathbf{u}(\mathbf{x}+\mathbf{r},\ t) - \mathbf{u}(\mathbf{x},\ t) \tag{11.28}$$

appear must be homogeneous and isotropic. The condition of stationarity of the statistical properties of local fields means that the characteristic times of the local fields are much smaller than the characteristic times of variation of the basic flow.

Thus, as in the case of an ordinary homogeneous and isotropic turbulent flow, the tensor of second-order moments of the quantities $\Delta_r \mathbf{u}$ can be expressed in terms of one of its components, for example,

$$D_{LL} = \langle (u_L(\mathbf{x}+\mathbf{r},\ t) - u_L(\mathbf{x},\ t))^2 \rangle \tag{11.29}$$

(u_L being, as before, the component of the velocity vector $\mathbf{u}$ in the direction $\mathbf{r}$). The quantity D_{LL} depends on r, the modulus of the vector $\mathbf{r}$, and also on the kinematic viscosity of the fluid ν, the external scale Λ, and the energy transmitted per unit time to the fine-scale motions under consideration from the large-scale motions, which by virtue of stationarity is equal to the mean rate of viscous energy dissipation per unit volume $\langle\varepsilon\rangle$. Introducing in place of the viscosity the linear scale λ of the motions in which viscous dissipation occurs,

$$\lambda = \nu^{3/4} \langle\varepsilon\rangle^{-1/4}, \tag{11.30}$$

the so-called internal Kolmogorov scale, we have

$$D_{LL} = f(r,\ \langle\varepsilon\rangle,\ \lambda,\ \Lambda). \tag{11.31}$$

Dimensional analysis gives by the standard procedure

$$D_{LL} = \langle\varepsilon\rangle^{2/3} r^{2/3} \Phi\left(\frac{r}{\lambda},\ \frac{r}{\Lambda}\right). \tag{11.32}$$

The relationships valid in the so-called inertial range of scales, i.e., for $\lambda \ll r \ll \Lambda$, are intermediate asymptotics of (11.32) as $r/\lambda \to \infty$ but $r/\Lambda \to 0$. (For large Reynolds numbers, $\lambda \lll \Lambda$.) In the classical version of the Kolmogorov–Obukhov theory an assumption is implicitly made that is equivalent to the assumption of the existence of a finite nonzero limit of $\Phi(r/\lambda, r/\Lambda)$ as $r/\lambda \to \infty$

and $r/\Lambda \to 0$, i.e., of complete self-similarity in both parameters r/λ and r/Λ. Therefore for $\lambda \ll r \ll \Lambda$ we obtain the famous "two-thirds-power law" of Kolmogorov and Obukhov,

$$D_{LL} = C \langle \varepsilon \rangle^{2/3} r^{2/3}, \tag{11.33}$$

where C is a universal constant that must be equal to $\Phi(\infty, 0)$.

In fact, the existence of complete self-similarity in the variable r/Λ for small r/Λ evokes some doubts. The quantity ε is also a fluctuating one, and the contribution of the fluctuations of the rate of energy dissipation ε in scales larger than the scale of the "equilibrium range" $r \ll \Lambda$, which is the only one having local isotropy and homogeneity, can turn out to be essential. This question was raised in the first edition, published in 1944, of the book by Landau and Lifschitz (1959); it is discussed in detail by Monin and Yaglom (1975).

We therefore assume that there is complete self-similarity in the parameter r/λ for $r/\lambda \gg 1$ and incomplete self-similarity in the parameter r/Λ for $r/\Lambda \ll 1$, so that as $r/\lambda \to \infty$ as $r/\Lambda \to 0$, $\Phi(r/\lambda, r/\Lambda) \cong C_1(r/\Lambda)^\alpha$, where C_1 and α are universal constants. Then (11.32) gives

$$D_{LL} = C_1 \langle \varepsilon \rangle^{2/3} r^{2/3+\alpha} \Lambda^{-\alpha}. \tag{11.34}$$

But one gets just such a relation in the refined Komogorov–Obukhov theory that takes account of the influence of fluctuations of the energy dissipation. Here, from experimental data $\alpha = 0.04$, so that the dependence (11.34) actually differs from the two-thirds law only slightly, which does not decrease the theoretical interest in this difference.

There is also interest in the analysis of the following possibility. Turbulent flows at large Reynolds numbers contain vast numbers of vortical threads that permeate the whole mass of fluid and expand and contract in the course of their motion. No matter how large the Reynolds number may be, viscosity remains essential close to the nuclei of the threads. Therefore the influence of viscosity can turn out to be essential in the inertial range as well. If we make the additional assumption of incomplete self-similarity in the parameter r/λ for $r/\lambda \gg 1$ (reflecting the influence of viscosity), then (11.32) gives

$$D_{LL} = C_2 \langle \varepsilon r \rangle^{2/3} \left(\frac{r}{\lambda}\right)^\beta \left(\frac{r}{\Lambda}\right)^\alpha = C_2 \langle \varepsilon \rangle^{2/3+\beta/4} \nu^{-3\beta/4} r^{2/3+\alpha+\beta} \Lambda^{-\alpha}, \tag{11.35}$$

where C_2, α, and β are universal constants.

Thus in constrast to (11.34) the exponent of $\langle \varepsilon \rangle$ varies here, and what is important, the viscosity appears to some power. It would be interesting to compare this possibility with experiments.

12

Self-Similarities of the First and Second Kind in the Theory of Turbulence. Shear Flow

1. The Wall Region of a Turbulent Shear Flow

Together with homogeneous isotropic turbulent flow, whose similarity laws were considered in the preceding chapter, the investigation of shear flows is of fundamental significance. These are turbulent flows that are statistically stationary and homogeneous in the longitudinal direction, the mean velocity depending on only the transverse coordinate (Fig. 12.1). The simplest realization of such a flow is obtained in a pipe or channel far from the inlet, in the flow past a plate far from the leading edge, in the boundary layer of the atmosphere, etc. Close to the rigid wall bounding the flow one can consider the turbulent shear stress to be constant; it is natural to call the region in which this assumption is valid the wall region; in the atmosphere it is called the surface layer. The state of motion at some point in the wall region is governed by the turbulent shear stress τ, which is constant by assumption, the density ρ and kinematic coefficient of viscosity ν of the fluid, and also the distance z of the point under consideration from the wall, and some external length scale Λ: the diameter of the tube, the complete depth of the channel, the thickness of the boundary layer, etc.

Thus the gradient of mean velocity $\partial_z u$ at a given point depends on the following governing parameters,

$$\partial_z u = f(\tau, \rho, \nu, z, \Lambda), \tag{12.1}$$

the first three of which have independent dimensions. Applying the standard procedure of dimensional analysis, we get

$$\Pi = \Phi(\Pi_1, \Pi_2), \tag{12.2}$$

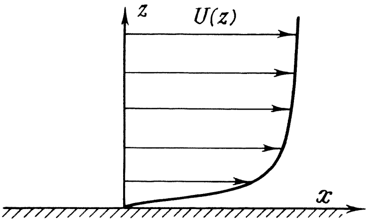


Figure 12.1. Shear flow.

where

$$\Pi = \frac{z\partial_z u}{u_*}, \quad \Pi_1 = \frac{u_* z}{\nu} = \mathrm{Re}_l, \quad \Pi_2 = \frac{u_* \Lambda}{\nu} = \mathrm{Re}_*. \tag{12.3}$$

The quantity $u_* = (\tau/\rho)^{1/2}$ appearing here with the dimensions of velocity is called the friction velocity, and the quantities $\Pi_1 = \mathrm{Re}_l$ and $\Pi_2 = \mathrm{Re}_*$ are called, respectively, the local and global Reynolds numbers. We note now that the value of the local Reynolds number, even at small distances from the wall, is very high. Thus, for example, in the flow of water ($\nu = 0.01$ cm^2/sec) with comparatively small friction velocity $u_* = 10$ cm/sec, the local Reynolds number at a distance of just one millimeter from the wall is equal to 100; and the global Reynolds number for a pipe of diameter 10 cm is 10,000. [The global Reynolds number differs from the Reynolds number usually used (cf. Chapter 1) in pipe hydraulics, which is based on the mean velocity and not on the friction velocity.]

It is natural in the first place to make the assumption of complete self-similarity of the flow in both the local and global Reynolds numbers (outside a small region in the immediate vicinity of the wall†). Under this assumption, which dates back to L. Prandtl, (12.2) gives

$$\Pi = z\partial_z u/u_* = \mathrm{const} = 1/\varkappa, \tag{12.4}$$

whence one gets the well-known universal (i.e., independent of global Reynolds number) logarithmic law of velocity distribution,

$$u = \frac{u_*}{\varkappa} \ln z + \mathrm{const}. \tag{12.5}$$

The quantity $\varkappa$, called the von Kármán constant, must under the assumption of complete self-similarity be a universal constant, independent of the Reynolds number.

†This is the very reason for considering the velocity gradient rather than the velocity itself. Unlike the gradient, the velocity depends also on the situation in the region immediately adjacent to the wall. In that region the assumption of self-similarity in Reynolds number is obviously invalid.

At first glance, the universal logarithmic law is confirmed satisfactorily by data from measurements of velocity distribution in smooth pipes and other analogous flows (Fig. 12.2). These data give for $\varkappa$ an estimate ≈ 0.4. However, more detailed analysis of the experimental data [cf. Hinze (1962); Tennekes (1968, 1973)] reveals a systematic dependence of the von Kármán constant on the global Reynolds number, i.e., a systematic though small deviation of the velocity distribution from the universal logarithmic law.

It is therefore natural to analyze the possible assumption (Barenblatt and Monin, 1976) of incomplete self-similarity of the flow in the local Reynolds number in the absence of self-similarity in the global Reynolds number. We repeat that the reason the influence of viscosity–i.e., of the Reynolds number– is preserved for large Reynolds number can be seen as follows. As is known, a developed turbulent flow contains a vast set of vortical threads permeating the moving fluid. With the growth of the Reynolds number the number of threads increases. No matter how large the Reynolds number may be, viscosity remains essential close to the "nucleus" of a thread, and thus its dynamic effect on the flow does not disappear.

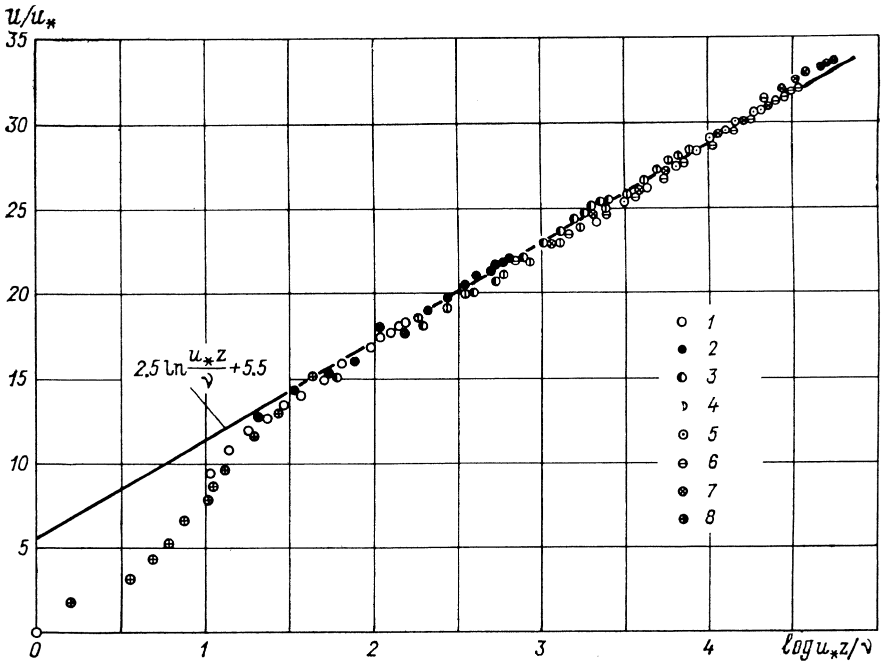


Figure 12.2. Universal logarithmic law is to the first approximation confirmed by the data from measurements of distributions of velocity in smooth pipes, in boundary layers over smooth plates, etc. (1) Re = 4.1×10^3; (2) Re = 2.3×10^4; (3) Re = 1.1×10^5; (4) Re = 4.0×10^5; (5) Re = 1.1×10^6; (6) Re = 2.0×10^6; (7) Re = 3.2×10^6 [(1)-(7) from Nikuradse]; (8) from Reichardt.

Under the assumption of incomplete self-similarity in the local Reynolds number, (12.2) gives

$$\frac{z\partial_z u}{u_*} = \left(\frac{u_* z}{\nu}\right)^\lambda \Phi(\mathrm{Re}_*), \tag{12.6}$$

where λ is a quantity also depending on the global Reynolds number. Integrating this relation and setting the constant of integration equal to zero in agreement with experiments, we get the power-law velocity distribution

$$u = \frac{u_*^{1+\lambda} z^\lambda}{\lambda \nu^\lambda \varkappa(\mathrm{Re}_*)} \tag{12.7}$$

with exponent depending on the global Reynolds number. Here one writes $\varkappa(\mathrm{Re}_*) = 1/\Phi(\mathrm{Re}_*)$. Power laws for the velocity distribution in various turbulent shear flows have been suggested for a long time as empirical relationships. They are well confirmed by numerous experiments, among which one should note the classical experiments on turbulent flows in smooth tubes of Möbius [cf. Schiller (1932) and Nikuradse (1932), and the comparatively recent excellent measurements of Laufer (1954)]. One recognizes [cf. Schlichting (1968); Hinze (1959)] that power laws for velocity distributions, with exponents depending on the global Reynolds number, are confirmed at least as well as the universal logarithmic law. In Fig. 12.3 are shown velocity distributions in smooth tubes according to the measurements of Nikuradse (1932), taken from the book of Schlichting (1968), which excellently confirm the power-velocity distribution over almost the entire section of the tube. Nevertheless the universal logarithmic law is considered to have, in contrast to the power law, a theoretical basis, whereas the power law is considered as simply an empirical relation. As a matter

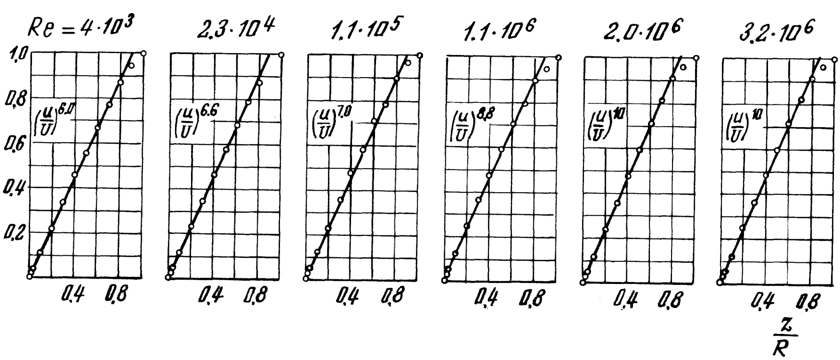


Figure 12.3. Measurements of Nikuradse (1932) confirm the power-law distribution of mean velocity over almost the entire section of the pipe. (U is the mean velocity on the axis of the pipe.)

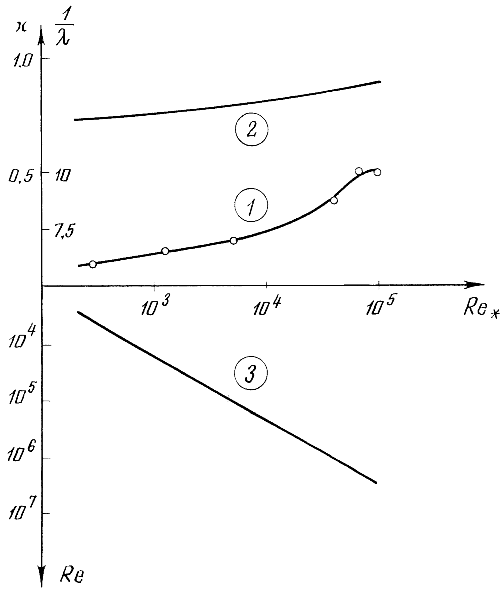


Figure 12.4. The dependences $\lambda(Re_*)$ (curve 1) and $\varkappa(Re_*)$ (curve 2), determined by measurements of velocity distributions in smooth pipes. Curve 3 shows the dependence of global Reynolds number on the ordinary Reynolds number based on mean velocity.

of fact, however, we have seen that the power law based on the assumption of incomplete self-similarity of the flow in the local Reynolds number is at least as well founded as the universal logarithmic law based on the assumption of complete self-similarity in that parameter.

The experimental data of the authors named above allow one to determine the relationship $\lambda(\mathrm{Re}_*)$ (Fig. 12.4, curve 1). The behavior of this curve for still larger values of Re_* remains unclear; for increasing Re_* does λ tend to zero or to some constant limit different from zero? The latter would mean that the universal logarithmic law, strictly speaking, does not hold in general even for very large Reynolds numbers and is only an approximate representation of the experimental data. The experiments mentioned allow one also to construct the relationship $\varkappa(\mathrm{Re}_*)$ also shown in Fig. 12.4 (curve 2). As is evident, for increasing Re_* the value of $\varkappa(\mathrm{Re}_*)$ changes comparatively little and to a first approximation it can be considered constant.

2. Similarity Laws for the Atmospheric Surface Layer

The surface layer of the atmosphere is usually modeled (Monin and Yaglom, 1971) by a turbulent flow that is statistically homogeneous horizontally and stationary, and is bounded below by a horizontal plane. The shear stress τ in the surface layer is also assumed to be constant. The essential difference from the flow in the wall region considered in Section 1 consists in the presence in the surface layer of thermal stratification–temperature inhomogeneity over the height of the layer. The stratification is stable if the temperature increases with height and unstable in the opposite case. Due to thermal inhomogeneity, a vertical displacement of fluid particles, produced by a vertical velocity fluctuation, is accompanied by work being done against the force of gravity or extracted (depending on whether the stratification is stable or not). This work is either taken from the turbulent energy or added to it, thus influencing the turbulence level, i.e., the transfer of heat, mass, and momentum, and consequently also the distribution of mean longitudinal velocity across the flow. The effectiveness of the influence of thermal stratification on the balance of turbulent energy is governed by the product of the coefficient of thermal expansion of the air and the acceleration of gravity, the so-called buoyancy parameter. The air in the atmospheric surface layer is usually considered a thermodynamically ideal gas, for which the coefficient of thermal expansion is equal to $1/T$, where T is the absolute temperature. The atmospheric surface layer is not thick, so the variation in mean pressure with the height of the layer under the action of the force of gravity and the corresponding variation in the density can be neglected. In general, the variations of density and absolute temperature in the surface layer are considered small, and their influence on the dynamics of the flow is taken into account only through the buoyancy, which governs the contribution of thermal stratification to the turbulent energy balance. Thus the state of motion at some point of the flow in the atmospheric surface layer is governed by the following quantities: (1) the friction velocity u_*; (2) the reference density ρ_0; (3) the dynamic temperature T_*, introduced by analogy with the friction velocity through the relation

$$T_* = -\frac{\langle w'T'\rangle}{u_*} \tag{12.8}$$

(where w' is the vertical velocity fluctuation, T' the temperature fluctuation, and the quantity $\langle w'T'\rangle$ coincides to within a constant factor with the vertical heat flux), the dynamic temperature being positive in the case of stable stratification ($\partial_z T > 0$) and negative for unstable stratification; (4) the buoyancy parameter $\beta = g/T_0$ (where g is the acceleration due to gravity and T_0 the reference temperature, which does not figure separately anywhere, since a change of temperature turns out to influence the flow dynamics only through the buoyancy parameter, i.e., in combination with the force of gravity); (5) the vertical coordinate z; (6) the kinematic viscosity ν; (7) the thermal diffusivity χ; and (8) the external geometric length scale Λ (the height of the atmospheric surface layer).

The standard procedure of dimensional analysis gives

$$\Pi_u = \frac{z\partial_z u}{u_*} = \Phi_u\left(\frac{z}{L_0}, \frac{u_* z}{\nu}, \frac{u_*\Lambda}{\nu}, \mathrm{Pr}\right), \tag{12.9}$$

$$\Pi_T = \frac{z\partial_z T}{T_*} = \Phi_T\left(\frac{z}{L_0},\ \frac{u_* z}{\nu},\ \frac{u_*\Lambda}{\nu},\ \mathrm{Pr}\right), \tag{12.10}$$

where $Pr = \nu/\chi$ is the Prandtl number and L_0 the thermal length scale†

$$L_0 = u_*^2/\beta T_*. \tag{12.11}$$

The existing theory of similarity of flows in the surface layer of the atmosphere, which owes its origin to the pioneering work of Prandtl (1932) and the work of Monin and Obukhov (1953, 1954), is based on the assumption of complete self-similarity of the flow in both Reynolds numbers—the local one $\mathrm{Re}_l = u_* z/\nu$ and the global one $\mathrm{Re}_* = u_*\Lambda/\nu$. The possibility of such an assumption, and consequently, of neglecting the dependence on Re_l and Re_* in (12.9) and (12.10) is usually argued on the basis of the very large values of both Reynolds numbers (for the local one, outside a small region close to the surface itself whose height does not exceed a few millimeters). Here the assumption of the existence of finite limits of the functions Φ_u and Φ_T as $\mathrm{Re}_l \to \infty$ and $\mathrm{Re}_* \to \infty$ is accepted implicitly. If the functions Φ_u and Φ_T tend to finite limits as $\mathrm{Re}_l \to \infty$ and $\mathrm{Re}_* \to \infty$ in accord with the assumption of complete self-similarity, then for sufficiently large Re_l and Re_* there must hold the universal similarity law, independent of the Reynolds numbers,

$$z\partial_z u/u_* = \Psi_u(z/L_0,\ \mathrm{Pr}), \tag{12.12}$$

$$z\partial_z T/T_* = \Psi_T(z/L_0,\ \mathrm{Pr}), \tag{12.13}$$

called in the literature the Monin–Obukhov similarity law. In the special case when thermal stratification of the flow disappears, we again arrive at Prandtl's universal logarithmic law, considered in Section 1 of this chapter.

The considerations presented in Section 1 show that even in the case of a thermally neutral flow one detects a weak dependence of the universal function on both Reynolds numbers. This weak dependence allows one to introduce the assumption of incomplete self-similarity of the flow in the local Reynolds number, which is apparently not contradicted by the experimental data on flows in smooth pipes, in the boundary layer of a plate, etc. It is natural to make a similar assumption for thermally stratified flows in the surface layer of the atmosphere (Barenblatt and Monin, 1976).

In the case of incomplete self-similarity in the local Reynolds number for thermally stratified shear flow, the similarity law can be written in the form

$$\frac{z\partial_z u}{u_*} = \left(\frac{u_* z}{\nu}\right)^{\lambda} \varphi_u\left(\frac{z}{L_0},\ \mathrm{Re}_*,\ \mathrm{Pr}\right), \tag{12.14}$$

†This definition of the thermal length scale follows Yaglom (1974) and is somewhat different from the conventional one.

$$\frac{z\partial_z T}{T_*}=\left(\frac{u_* z}{\nu}\right)^{\mu}\varphi_T\left(\frac{z}{L_0},\ \mathrm{Re}_*,\ \mathrm{Pr}\right), \tag{12.15}$$

where the exponents λ and μ depend on the global Reynolds number:

$$\lambda=\lambda\left(\frac{u_*\Lambda}{\nu}\right), \quad \mu=\mu\left(\frac{u_*\Lambda}{\nu}\right). \tag{12.16}$$

In principle a dependence of these exponents also on the Prandtl number is not excluded, but to a first approximation we neglect it. The general character of the relationship $\lambda(\mathrm{Re}_*)$ can be considered known from experiments in smooth pipes for global Reynolds numbers that are not too large. One can attempt to find the function $\mu(\mathrm{Re}_*)$ by considering the limiting regime of "windless convection" in the case of unstable stratification. For unstable stratification $L_0<0$ and $T_*<0$. In the limiting case of windless convection, the quantity u_* must in a definite sense drop out of the governing parameters. This means that for windless convection we have the limiting similarity law, corresponding to the case when u_* is sufficiently small, that the friction velocity drops out of the asymptotic expressions for the turbulent coefficients of viscosity and thermal diffusivity,

$$K=-\frac{\langle v'w'\rangle}{\partial_z u}=\frac{u_*^2}{\partial_z u}, \quad K_T=-\frac{\langle w'T'\rangle}{\partial_z T}=\frac{u_* T_*}{\partial_z T}. \tag{12.17}$$

At the same time, u_* must be sufficiently large that the local Reynolds number is nevertheless large (outside a small region close to the underlying surface) and it is possible to apply the asymptotic relations (12.14), (12.15). From these conditions, we get in the limiting flow considered,

$$\begin{gathered}\varphi_u \cong B_u(-\zeta)^{-\frac{1-\lambda}{3}}, \quad \varphi_T \cong B_T(-\zeta)^{-\frac{1-\mu}{3}}, \quad \zeta\to-\infty,\\ \zeta=z/L_0=z/(u_*^2/\beta T_*)=z/(-u_*^3/\beta\langle w'T'\rangle),\end{gathered} \tag{12.18}$$

or in dimensional form,

$$\partial_z u \cong B_u u_*^2 z^{-\frac{4}{3}(1-\lambda)}(Q\beta)^{-\frac{1-\lambda}{3}}\nu^{-\lambda}, \tag{12.19}$$

$$\partial_z T \cong B_T Q^{\frac{2+\mu}{3}} z^{-\frac{4}{3}(1-\mu)}\beta^{-\frac{1-\mu}{3}}\nu^{-\mu}. \tag{12.20}$$

Here B_u and B_T are dimensionless constants and $Q = \langle w'T' \rangle$. As the physical condition determining the function $\mu = \mu(\mathrm{Re}_*)$, we take the condition of boundedness of the turbulent Prandtl number α. We have

$$\alpha = \frac{K}{K_T} = \frac{\langle u'w' \rangle \, \partial_z T}{\langle w'T \rangle \, \partial_z u} = \frac{\varphi_T}{\varphi_u}\left(\frac{u_* z}{\nu}\right)^{\mu-\lambda} =$$

$$= \frac{\varphi_T}{\varphi_u}\left(-\frac{z}{L_0}\right)^{\mu-\lambda}\left(-\frac{u_* L_0}{\nu}\right)^{\mu-\lambda}. \tag{12.21}$$

(We recall that in the case being considered of unstable stratification the thermal length scale is negative.) The limiting case under study of small u_* corresponds to $\zeta \to -\infty$ and the satisfaction of the asymptotic relations (12.18); therefore as $\zeta \to -\infty$ φ_T/φ_u is proportional to $(-\zeta)^{(\mu-\lambda)/3}$, so that the condition of a bounded turbulent Prandtl number gives

$$\lambda = \mu. \tag{12.22}$$

This relation determines the dependence on the global Reynolds number of the second exponent in the similarity laws.

Unfortunately, the available experimental data relating to thermally stratified turbulent shear flows are very poor and do not at present allow one to make a simple judgment of the validity or invalidity of the assumption of incomplete self-similarity in the local Reynolds number.

3. Similarity Laws for Flow in a Turbulent Wall Region with Adverse Pressure Gradient

Here we consider, following the paper of Yaglom and Kader (1975), the flow in the wall region of a decelerating turbulent flow with adverse longitudinal pressure gradient. The flow is again assumed to be homogeneous in the longitudinal direction, so that the governing parameters of the flow are the friction velocity u_*, the density ρ, the coefficient of kinematic viscosity ν of the fluid, the distance z from the wall, and the external length scale Λ, in the present case the depth of the channel or the thickness of the boundary layer. All these quantities were governing parameters for the flow in the wall region in the absence of a longitudinal pressure gradient considered in Section 1 of this chapter. In addition, in the flow considered here there appears a new governing parameter, the longitudinal pressure gradient dp/dx, which by virtue of the homogeneity of the flow in the direction of the x axis, chosen along the wall, is independent of the longitudinal coordinate and is thus a constant. Since $[dp/dx] = ML^{-2}T^{-2}$ there appears in the flow in addition to the local and global Reynolds numbers

$\mathrm{Re}_l = u_* z/\nu$ and $\mathrm{Re}_* = u_* \Lambda/\nu$ the new similarity parameter

$$\frac{(dp/dx)\,\nu}{\rho u_*^3}. \tag{12.23}$$

The inverse of this parameter,

$$\frac{\rho u_*^3}{(dp/dx)\,\nu} = \mathrm{Re}_p, \tag{12.24}$$

plays the role of a Reynolds number based on the friction velocity, the kinematic viscosity, and the linear scale $\rho u_*^2/(dp/dx)$ corresponding to the adverse pressure gradient. It is clear that the relative magnitudes of all three Reynolds numbers are significant. We assume, keeping in mind the description of the available experimental data, that Re_p, though large, is nevertheless much less than the global Reynolds number. Now suppose we find ourselves in such a range of distances from the wall z that the local Reynolds number Re_l is much larger than unity, but still $\mathrm{Re}_l \ll \mathrm{Re}_p$. In this range of local Reynolds number the influence of the shear stress is more essential than the influence of the longitudinal pressure drop. Therefore everything said in Section 1 remains valid and there exists some basis for expecting incomplete self-similarity of the flow in the local Reynolds number, with the exponents no longer depending only on the global Reynolds number, but also on Re_p.

Also very instructive is the behavior of the flow in another intermediate range of local Reynolds numbers, where

$$\mathrm{Re}_p \ll \mathrm{Re}_l \ll \mathrm{Re}_*,$$

so that

$$\frac{\nu}{u_*} \ll \frac{\rho u_*^2}{dp/dx} \ll z \ll \Lambda. \tag{12.25}$$

In this range of distances from the wall the influence of the adverse pressure gradient is more essential than the influence of the shear stress, so it is natural to take as basic governing parameters with independent dimensions the quantities ρ, z, and dp/dx. Aside from these quantities, the properties of the flow can depend on the friction velocity u_*, the viscosity ν, and the external length scale Λ. Dimensional analysis gives

$$\frac{dU}{dz} = \sqrt{\frac{dp/dx}{\rho z}}\,\Phi(\Pi_1, \Pi_2, \Pi_3),$$

$$\Pi_1 = \frac{\Lambda}{z}, \quad \Pi_2 = \frac{u_*}{\sqrt{z\,(dp/dx)/\rho}}, \quad \Pi_3 = \frac{\nu}{z^{3/2}\sqrt{(dp/dx)/\rho}}. \tag{12.26}$$

By virtue of the relation (12.25) for the relative orders of magnitude, we have

$$\Pi_1 = \frac{\Lambda}{z} \gg 1, \quad \Pi_2 = \frac{u_*}{\sqrt{z\,(dp/dx)/\rho}} = \left(\frac{z}{\rho u_*^2/(dp/dx)}\right)^{-1/2} \ll 1,$$

$$\Pi_3 = \frac{\nu}{z^{3/2}\left(1/\rho\,(dp/dx)\right)^{1/2}} = \frac{\nu}{u_* \rho u_*^2/(dp/dx)} \left(\frac{z}{\rho u_*^2/(dp/dx)}\right)^{-3/2} \ll 1.$$

It is natural to start by making the assumption of complete self-similarity in all three dimensionless parameters appearing in the universal function Φ in (12.26). Then we must have

$$\frac{dU}{dz} = \sqrt{\frac{dp/dx}{\rho z}}\ \mathrm{const},$$

whence

$$U(z) = K\sqrt{\frac{(dp/dx)\,z}{\rho}} + K_1, \tag{12.27}$$

where K and K_1 must be universal constants. Equation (12.27) together with arguments equivalent to the assumption of complete self-similarity was given by Stratford (1959) and Townsend (1960).

The experiments of Newman (1951), performed in the boundary layer on a wing profile, have confirmed, as have also subsequent analogous experiments, that the range of velocity distributions described by the square-root law (12.27) exists, but the coefficient K turns out to be nonconstant (Fig. 12.5). This fact attracted attention and Perry and Schofield (1973) made a detailed statistical analysis of the data from a large number of experiments collected in the papers of the Stanford conference (1969). The results of this analysis are shown in Fig. 12.6 in the form of a histogram of the values of K, showing that the magnitude of K varies greatly (from ~3.5 to 13).

It is natural to assume that in the present case there is incomplete self-similarity in a similarity parameter. In fact, (12.26) can be written in the form

$$\frac{dU}{dz} = \sqrt{\frac{dp/dx}{\rho z}}\ \Psi(\Pi_1,\ \Pi_2',\ \Pi_3'), \tag{12.28}$$

where Π is equal to Λ/z, as before and

$$\Pi_2' = \Pi_1\Pi_2^{-2} = \frac{(dp/dx)\,\Lambda}{\rho u_*^2}, \quad \Pi_3' = \Pi_1\Pi_3^{-2/3} = \frac{\Lambda}{\nu^{2/3}}\left(\frac{1}{\rho}\frac{dp}{dx}\right)^{1/3}.$$

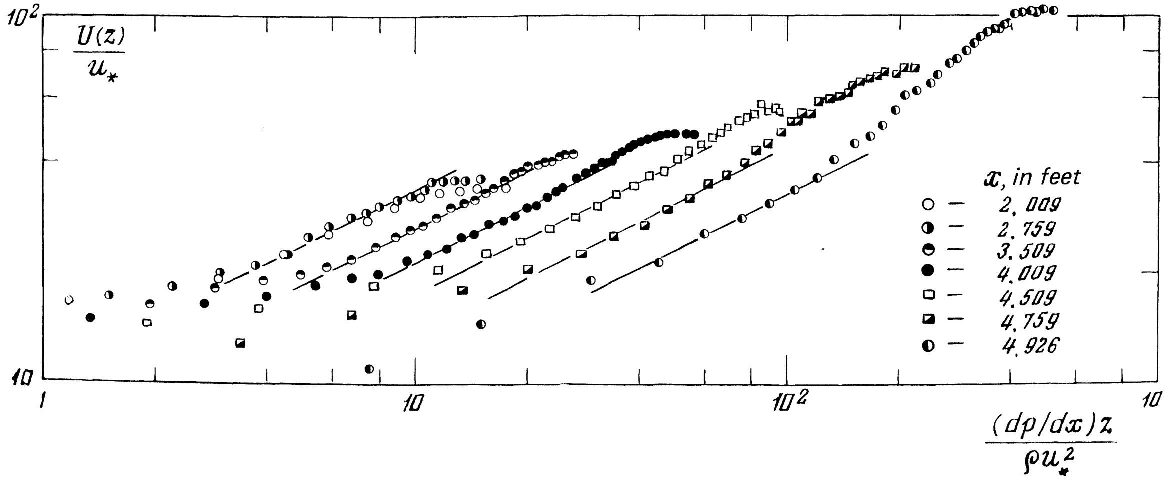


Figure 12.5. Experiments of Newman (1951) well confirm the square-root law (12.27) in a restricted range of values of the parameter $(dp/dx)z/\rho u_*^2$, but the coefficient K turns out to be nonconstant. The various points correspond to different values of the longitudinal coordinate of the section at which the velocity distribution was measured.

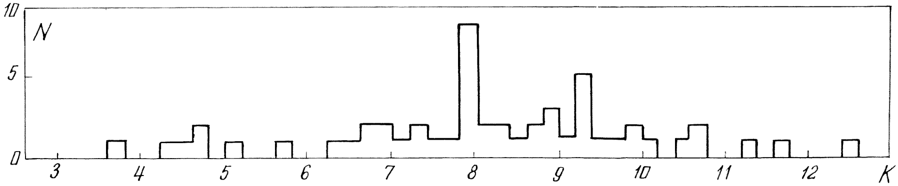


Figure 12.6. Histogram of values of K based on results submitted to the Stanford conference (1969), as analyzed by Perry and Schofield (1973).

Changing to the parameters Π'_2, Π'_3, is convenient, since these parameters relate to the flow in the large and do not contain the distance from the wall z, which changes from point to point. The inequality (12.25) shows that $\Pi'_2 \gg 1$ and $\Pi'_3 \gg 1$. It turns out that in (12.28) for $\Pi_1 \gg 1$, $\Pi'_2 \gg 1$, and $\Pi'_3 \gg 1$ there is complete self-similarity in the parameters Π_1 and Π'_3 and incomplete self-similarity in the parameter Π'_2, so that

$$\Psi = A\Pi_2'^{m}, \tag{12.29}$$

and from this and (12.28) it follows that

$$U(z) = 2A\left(\frac{(dp/dx)\,\Lambda}{\rho u_*^2}\right)^m \left(\frac{(dp/dx)\,z}{\rho}\right)^{1/2} + C_1, \tag{12.30}$$

where A, m, and C_1 must be universal constants.

Treating the same data presented at the Stanford conference (1969), Yaglom and Kader (1975) have confirmed (Fig. 12.7) the incomplete self-similarity and given the values of the parameters $A = 12.25$, $m = -1/3$, $C_1 = 0$. Thus the histogram of Fig. 12.6 reflects simply a distribution of ranges of variation of the parameter $(dp/dx)\Lambda/\rho u_*^2$, with which various authors who presented their contributions at Stanford have worked.

4. Unsteady Phenomena in the Viscous Layer of a Turblent Shear Flow

In recent years there have been published fundamental investigations of turbulent shear flows in the immediate vicinity of the wall, performed by basically two groups of American authors. [See Kline, Reynolds, Schraub, and Runstadler (1967); Corino and Brodkey (1969); Kim, Kline, and Reynolds (1971); Offen and Kline (1975); and the large bibliographies in these papers.] By skillful combination of methods of visualization (by hydrogen bubbles and tracing pigments) and thermoanemometric methods it was shown in these

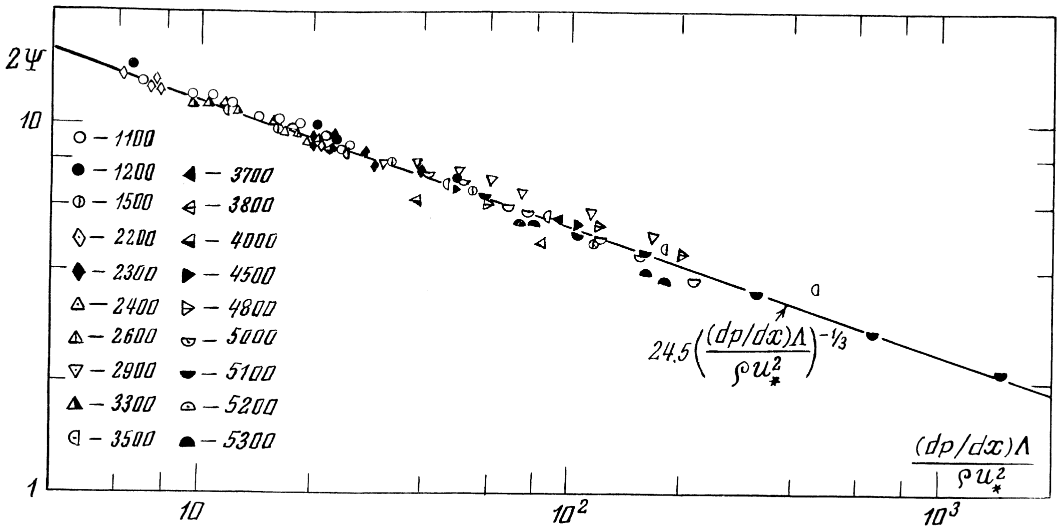


Figure 12.7. *Treatment of the same experimental data presented at the Stanford conference (1969), performed by A. M. Yaglom and B. A. Kader (1975), confirms incomplete self-similarity in the parameter $(dp/dx)\Lambda/\rho u_*^2$. The various symbols correspond to papers of different authors according to the nomenclature of the Stanford conference (1969).*

papers that turbulent flow close to the wall has a complicated, essentially unsteady and spatially inhomogeneous structure.

The question is of the phenomena in a viscous layer where the global characteristics of the flow are governed by the shear stress τ and the density ρ and kinematic viscosity ν of the fluid, and also by some external length scale Λ, for example the momentum thickness of the boundary layer. Thus if the kinematic properties are considered, they must be governed by only the friction velocity $u_* = (\tau/\rho)^{1/2}$, the external length scale Λ, and the kinematic viscosity ν. It turns out that in the range of thickness of the order of some tens of the characteristic linear scale of the viscous layer ν/u_* there arise with a statistically determined frequency local separations of the flow, as a result of which horseshoe-shaped vortices are generated, move deep into the flow, and in their own right stimulate the occurrence of new local separations. This generates a checkered pattern of longitudinal strips of the retarded flow. Interactions of the horseshoe vortices that arise among themselves leads to complicated phenomena of the type of local loss of stability of bursting character. As is convincingly shown in the papers of Kline and his associates, it is just these bursts that determine the generation of turbulence close to a rigid boundary in a turbulent shear flow. An illustration of the character of the local flows that arise is given by the photograph of Fig. 12.8, taken from Kim, Kline, and Reynolds (1971), which shows the twisting by these flows of originally vertical lines of hydrogen bubbles.

Despite the complicated character of the local flows in the viscous layer of a turbulent shear flow, some of their statistical characteristics are well described

Figure 12.8. In the wall region of a turbulent shear flow there exists a complicated unsteady and spatially inhomogeneous flow. In the photograph: twisting by a vortex of initially vertical lines of hydrogen bubbles. [From Kim, Kline, and Reynolds (1971)].

by similarity laws obtained by the method of dimensional analysis. We demonstrate this here with the example of determining the mean time T_B between bursts, i.e., the mean period of the cyclic process occurring close to the wall. This quantity can depend, according to the above, on the friction velocity u_*, the kinematic viscosity ν, and the external length scale Λ, whence, applying dimensional analysis, we obtain

$$T_B = \frac{\nu}{u_*^2}\,\Phi\left(\frac{u_*\Lambda}{\nu}\right). \tag{12.31}$$

The parameter $u_*\Lambda/\nu$ is very large, of order 100 or more; therefore it was natural to make the original assumption, equivalent to the hypothesis of complete self-similarity in this parameter. This gives

$$T_B = C\frac{\nu}{u_*^2}, \tag{12.32}$$

where C is a constant. The experimental data at first glance confirm the dependence (12.32) (cf. Fig. 12.9). However, an attempt to apply (12.32) to the

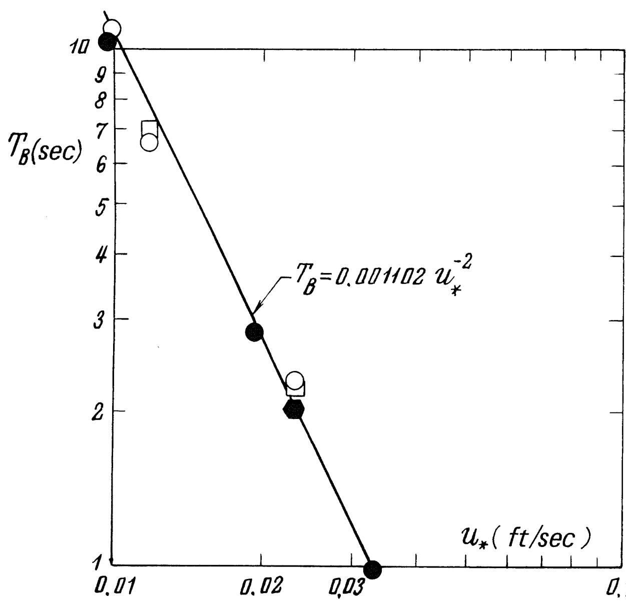


Figure 12.9. Experimental data at first glance confirm complete self-similarity of the dependence of the time T_B between bursts on the parameter $u_*\Lambda/\nu$ for large values of this parameter. [From Kim, Kline, and Reynolds (1971).]

experiments of B. J. Tu and W. W. Willmarth [cf. Rao, Narasimha, and Badri Narayanan (1971)], in which significantly higher friction velocities were achieved, led to errors of more than an order of magnitude. Actually, as was shown by Rao, Narasimha, and Badri Narayanan (1971), there is not complete self-similarity in the parameter $u_*\Lambda/\nu$. Consideration of more complete experimental data in that paper and by Kim, Kline, and Reynolds (1971) gave the relation

$$\Phi = 0.65\left(\frac{U_*\Lambda}{\nu}\right)^{0.73} \tag{12.33}$$

(see Fig. 12.10). Here U is the free-stream velocity and Λ the momentum thickness of the boundary layer. As is well known (cf. curve 3 in Fig. 12.4) the ratio u_*/U is a power-law function of the global Reynolds number. Indeed, using the formulas from Schlichting (1968) corresponding to the range of the "one-seventh-power law" we obtain

$$\frac{u_*\Lambda}{\nu} = 0.11\left(\frac{U\Lambda}{\nu}\right)^{0.875} ,$$

and from this and (12.33) we get

$$\Phi = 4.0\left(\frac{u_*\Lambda}{\nu}\right)^{0.83}$$

This relation reveals the incomplete self-similarity with respect to the parameter $u_*\Lambda/\nu$.

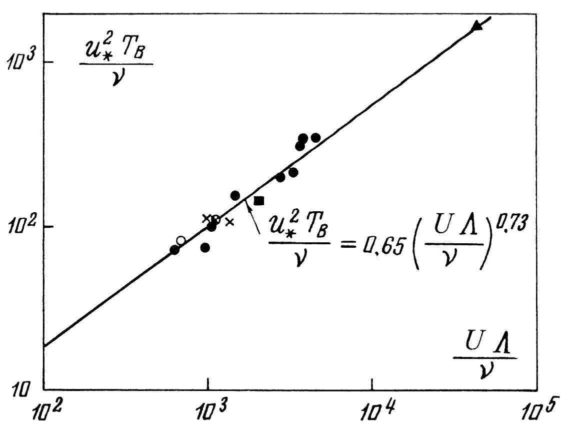


Figure 12.10. Experiments in a wider range of values of the parameter $U\Lambda/\nu$ show the presence of incomplete self-similarity in this parameter. [From Kim, Kline, and Reynolds (1971).]

The examples presented above show that incomplete self-similarity is met with far more frequently than was thought, and even in fields where so many investigators had worked that the similarity laws had seemed to be completely clarified.

5. The Regime of Limiting Saturation of a Turbulent Shear Flow Loaded with Sediment

We now consider the turbulent shear flow described in Section 1 in more detail, based on the semiempirical theory proposed by Kolmogorov (1942). Kolmogorov's theory is based on closing the equations of conservation of momentum and turbulent energy with the help of some similarity hypotheses.

The equation of momentum balance for a shear flow can be written in the form

$$-\rho \langle u'w' \rangle = \tau, \tag{12.34}$$

where u', w', respectively, are the fluctuations in the longitudinal and vertical velocity components, ρ the density of the fluid, and τ the shear stress; in (12.34) we neglect the contribution of viscous stresses in comparison with the turbulent Reynolds stresses. The equation of turbulent energy balance for a shear flow can be written in the form (Monin and Yaglom, 1971)

$$\langle u'w' \rangle\, \partial_z u + \partial_z \left\langle \left(\frac{p'}{\rho} + \frac{u'^2 + v'^2 + w'^2}{2} \right) w' \right\rangle + \varepsilon_t = 0. \tag{12.35}$$

Here v' is the fluctuation in the transverse velocity component, p' the fluctuation in pressure, and ε_t the mean rate of dissipation of turbulent energy per unit mass of fluid. Equation (12.35) reflects the simple fact that the local balance of turbulent energy consists of the generation of turbulent energy by the mean motion (the first term), diffusive influx of turbulent energy (the second term), and dissipation of turbulent energy into heat. For the problem of interest the transfer of turbulent energy by diffusion is small and we can neglect it.

We introduce the coefficient K of momentum exchange by the relation

$$\langle u'w' \rangle = -K\, \partial_z u. \tag{12.36}$$

We stress that for a shear flow (12.36) is simply a redesignation and does not involve any additional hypothesis.

Kolmogorov's (1942) hypothesis is that the momentum exchange coefficient and the rate of energy dissipation ε_t at a given point in the flow are governed by only the local values of the mean turbulent energy per unit mass,

$$b=\frac{1}{2}\langle u'^2+v'^2+w'^2\rangle ,$$

and the turbulence external length scale l. Dimensional analysis leads in the standard way to the relations

$$K=l\sqrt{b}, \quad \varepsilon_t=\gamma^4 b^{3/2}/l, \tag{12.37}$$

where by virtue of the indefiniteness of the scale to within a constant factor, the constant in the first relation can be taken to be equal to unity, and the constant γ is close to 0.5 by estimates from experimental data.

Substituting (12.36) and (12.37) into (12.34) and (12.35), and neglecting in the latter equation the term expressing the contribution of the diffusion of turbulent energy, we get a system of equations in the form

$$l\sqrt{b}\,\partial_z u=u_*^2, \quad l\sqrt{b}\,(\partial_z u)^2-\gamma^4 b^{3/2}/l=0, \tag{12.38}$$

where $u_* = (\tau/\rho)^{1/2}$ is the friction velocity.

This system is still not closed, since the turbulence length scale l is not yet defined. In accord with Section 1, in the wall region of a turbulent shear flow, where the friction velocity u_* is constant, the turbulence scale l depends on the friction velocity u_*, the kinematic viscosity ν, the vertical coordinate z, and the external length scale Λ. Dimensional analysis gives

$$l=z\Phi_l(\mathrm{Re}_l,\ \mathrm{Re}_*), \tag{12.39}$$

where as before $\mathrm{Re}_l = u_* z/\nu$ and $\mathrm{Re}_* = u_*\Lambda/\nu$ are the local and global Reynolds numbers. Under the assumption of complete self-similarity in both Reynolds numbers, the function Φ_l is identically equal to a constant which it is convenient to denote by $\varkappa\gamma$, γ being the constant introduced earlier, and $\varkappa$, a new constant called the von Kármán constant, so that

$$l=\varkappa\gamma z. \tag{12.40}$$

Substituting (12.40) into (12.38), we get

$$u=\frac{u_*}{\varkappa}\ln z+\mathrm{const}, \quad b=\frac{u_*^2}{\gamma^2}, \tag{12.41}$$

i.e., the logarithmic law of distribution of velocity (12.5), obtained earlier from more general considerations. If we assume that the self-similarity of the flow in the local Reynolds number is incomplete, then the relation for the turbulent length scale assumes the form

$$l = z\left(\frac{u_* z}{\nu}\right)^{-\lambda} \Phi_l(\mathrm{Re}_*), \tag{12.42}$$

and from this and (12.38) we find

$$\frac{z\partial_z u}{u_*} = \left(\frac{u_* z}{\nu}\right)^{\lambda} \Phi(\mathrm{Re}_*), \quad \Phi(\mathrm{Re}_*) = \frac{\gamma}{\Phi_l(\mathrm{Re}_*)}, \tag{12.43}$$

i.e., (12.6), obtained earlier from more general considerations.

We turn now to the consideration of a flow containing a load of small suspended particles. The volume and mass concentrations of particles are assumed to be very small (for example in rivers carrying a large quantity of alluvia, their volume and mass concentrations rarely exceed several ten-thousandths). Nevertheless, the dynamic action of the particles on the flow can turn out to be essential due to the vast influence of the force of gravity. Furthermore, the particles are assumed to be much smaller than the internal turbulence scale; therefore one can assume that the horizontal components of the instantaneous velocities of the particles and the fluid coincide, and the vertical ones differ by a constant quantity a, the velocity of free fall of the particles in the unbounded fluid.

We consider again the wall region of the flow (for example, the surface layer of the atmosphere or the bottom layer of a channel). The momentum equation in this region remains just the same as for the pure fluid, since the influence of the particles on the density of the mixture is negligibly small.

The equation of conservation of mass for the load is obtained by setting equal to zero the total flux of particles through unit horizontal area. This flux is the sum of the flux of turbulent transport of particles $\langle s'w'\rangle$ and the flux of settling particles $-as$, so that

$$\langle s'w'\rangle - as = 0. \tag{12.44}$$

Here s and s' are the mean volume concentration of particles and its fluctuation, respectively.

Finally, the equation of turbulent energy balance assumes, if one neglects the contribution of the diffusion of turbulent energy, the form

$$\langle u'w'\rangle\, \partial_z u + \varepsilon_t + \sigma \langle s'w'\rangle\, g = 0. \tag{12.45}$$

Here $\sigma = (\rho_p - \rho)/\rho$ (ρ_p being the density of particles) is the relative excess of the density of particles over the density of the fluid. The last term expresses the expenditure of turbulent energy on the turbulent suspending of particles by the flow. Despite the smallness of the concentration of particles in the flow, this term can have a significant value, since the force of gravity is very large and its influence can compensate for the smallness of the concentration.

Equation (12.45) can be put into the form

$$\langle u'w' \rangle \partial_z u (1 - \mathrm{Ko}) + \varepsilon_t = 0, \tag{12.46}$$

where the dimensionless parameter

$$\mathrm{Ko} = -\frac{\sigma g \langle s'w' \rangle}{\langle u'w' \rangle \partial_z u}, \tag{12.47}$$

called the Kolmogorov number, expresses the relative expenditure of turbulent energy on the suspending of particles by the flow. This parameter is a natural criterion for the dynamic activity of the load, i.e., the influence of the suspended particles on the dynamics of the flow. We introduce, in analogy with the coefficient of momentum exchange, the coefficient of load exchange according to the relation

$$\langle s'w' \rangle = -K_s \partial_z s, \tag{12.48}$$

and we assume that this coefficient too, as well as the coefficient of momentum exchange and the mean rate of dissipation, depends on only the local turbulent energy of a unit mass and the length scale of the turbulence, whence by dimensional analysis we get.

$$K_s = l\sqrt{b}\,. \tag{12.49}$$

(As an unimportant simplification we have taken the constant factor that makes the coefficient of load exchange different from the coefficient of momentum exchange equal to one.)

In the problem being considered of a loaded flow, in contrast to the flow of pure fluid, there appears an additional parameter, the Kolmogorov number Ko, so that for the external scale of turbulence we have

$$l = z\Psi(\mathrm{Re}_l,\ \mathrm{Re}_*,\ \mathrm{Ko}).$$

Under the assumption of complete self-similarity in the local and global Reynolds numbers the turbulent length scale can be expressed through a universal function of the Kolmogorov number,

$$l = \varkappa\gamma z \Phi_l(\mathrm{Ko}), \tag{12.50}$$

where $\Phi_l(0)$ is obviously equal to one. The turbulence scale decreases under the influence of the load, so the function Φ_l must decrease when its argument increases. Thus, under the assumptions made, the basic system of equations of a loaded turbulent shear flow in the wall region assumes the form

$$l\sqrt{b}\,\partial_z u = u_*^2, \quad l\sqrt{b}\,\partial_z s + as = 0, \quad b = \frac{u_*^2}{\gamma^2}(1-\mathrm{Ko})^{1/2},$$
$$l = \varkappa\gamma z\Phi_l(\mathrm{Ko}), \quad \mathrm{Ko} = \frac{\sigma g a s}{u_*^2\partial_z u}. \tag{12.51}$$

The system of equations (12.51) has some characteristic properties. First of all, it contains only the gradient of the velocity $\partial_z u$, and not the velocity itself. Furthermore, for the case of an unrestricted supply of particles on the underlying surface, in view of the back influence of the particles on the dynamics of the flow, we can anticipate the existence of a regime of flow in which the flow absorbs the maximum possible amount of the load for given friction velocity and other parameters. This regime, which we shall call the regime of limiting saturation, must be described by a singular solution of (12.51), which will be determined by the parameters appearing in the differential equations themselves. Thus the determination of the regime of limiting saturation does not require prescribing any boundary condition for the sediment concentration.

It is essential that the system (12.51) is invariant with respect to the transformation group

$$s = S/\alpha, \quad z = \alpha Z, \quad u = U \tag{12.52}$$

($\alpha > 0$ being the group parameter), so that substituting (12.52) into (12.51) we get in the variables S, U, Z the same system (12.51). Let the singular solution corresponding to the regime of limiting saturation determine the velocity gradient and load concentration by the relations

$$\partial_z u = f(z), \quad s = g(z). \tag{12.53}$$

But the singular solution is determined only by the system itself and therefore also must be invariant with respect to the group (12.52), i.e., it can be expressed in the form

$$\partial_Z U = f(Z), \quad S = g(Z).$$

Expressing U, S, and Z in terms of u, s, z, and α, we get for the functions f and g the functional equations

$$f(z)=\alpha f(\alpha z), \quad g(z)=\alpha g(\alpha z). \tag{12.54}$$

The solution of these functional equations is found to be elementary:

$$f=C_1/z, \quad g=C_2/z, \tag{12.55}$$

where C_1 and C_2 are constants subject to determination. Substituting into (12.51) the relations

$$\partial_z u=C_1/z, \quad s=C_2/z, \quad \mathrm{Ko}=\frac{\sigma g C_2}{C_1^2}=\text{const}, \tag{12.56}$$

we obtain

$$C_1=\frac{u_*}{\varkappa(1-\mathrm{Ko})^{1/4}\Phi_l(\mathrm{Ko})}=\frac{u_*}{\varkappa\omega}, \quad \omega=\frac{a}{\varkappa u_*}, \tag{12.57}$$

whence we find a finite equation for determining the Kolmogorov number Ko, which is constant in the regime of limiting saturation:

$$\omega=(1-\mathrm{Ko})^{1/4}\Phi_l(\mathrm{Ko}). \tag{12.58}$$

But Φ_l is a nonincreasing function of its argument, $\Phi_l(0)=1$, and Ko by its physical meaning lies between zero and one. Hence it follows that for $\omega>1$ there exists no root of (12.58), and for $\omega<1$ a unique root exists. Therefore a necessary condition for the existence of a regime of limiting saturation is

$$\omega=\frac{a}{\varkappa u_*}<1. \tag{12.59}$$

The physical meaning of (12.59) is transparent. In fact, the value of the friction velocity u_* is proportional to the mean square velocity fluctuation. Thus if the fluctuation is large, so that during the time of lifting of a fluid mass by the turbulent fluctuation the heavy particles inside it have no time to fall (the velocity a of free fall being relatively small), the particles come into the main core of the flow and become suspended in it. In the opposite case the particles are transported by the flow in the bottom layer, they do not reach the main part of the flow, and they do not influence the flow dynamics in the main part of the stream.

From the first equation of (12.56), taking into account (12.57), we get for $\omega<1$

$$u=\frac{u_*}{\varkappa\omega}\ln z+\text{const}. \tag{12.60}$$

This means that for the flow in the regime of saturation, which can be realized for $\omega < 1$, the velocity distribution remains logarithmic, just as in a pure fluid, but there somehow occurs a reduction of the von Kármán constant: instead of $\varkappa$ it becomes equal to $\varkappa\omega$. Therefore under the same external conditions (the same friction velocity) the flow accelerates under the action of particles in comparison with the flow of pure fluid.

Since the capture of particles by the flow is realized by turbulent fluctuations, the turbulent energy must decrease. Actually, the turbulent energy per unit mass for the regime of limiting saturation is equal to

$$b = b_0 (1 - \mathrm{Ko})^{1/2}, \tag{12.61}$$

where $b_0 = u_*^2/\gamma^2$ is the turbulent energy for the flow of pure fluid with the same friction velocity. But the resistance of the turbulent flow depends on the intensity of the fluctuations, so it turns out that the suspended particles decrease the turbulent resistance. It is clear that this conclusion is valid only under the conditions indicated above of horizontal or nearly horizontal flow, small volume and mass concentrations of particles, etc. Under such conditions a drag reduction in the flow and an apparent decrease in the von Kármán constant under the action of suspended particles have been observed repeatedly by experimenters (Vanoni, 1946; Einstein and Ning Chen, 1955).

The theory presented here of transport of particles by a turbulent flow was developed in the papers of Barenblatt (1953), Kolmogorov (1954), and Barenblatt (1955); the derivation of the equations for the regime of limiting saturation on the basis of group considerations was given by Barenblatt and Golitsyn (1974).

Epilogue

Methods of research based on dimensional analysis and similarity have once more become a subject of wide interest and have produced surprises connected mainly with incomplete self-similarity. The basic purpose of the present book is to discuss this group of problems.

Incomplete self-similarity opens up new possibilities for the analysis of experimental data; therefore there has at once arisen a demand for more precise experimental data for a series of problems that are considered classical. Thus incomplete self-similarity is being outlined in turbulent shear flow. We are at present still far from wishing to discard the universal logarithmic velocity profile in such a flow at very high Reynolds number. But it should be taken into consideration that precise measurements of velocity profiles under controlled conditions are surprisingly scanty, and that similar measurements are lacking in pipes at Reynolds numbers greater than 3.2×10^6. At the same time, existing results of measurements show that the velocity profiles deviate from the universal logarithmic law and agree well with a power law. We may hope that the concept of incomplete self-similarity will serve as a definite stimulus for experimenters, and that future experimental investigations will soon permit us to express a definite opinion regarding the presence or absence in a shear flow of incomplete self-similarity with respect to Reynolds number.

Incomplete self-similarities and self-similar solutions of the second kind are encountered in many problems. The simple general scheme and the examples mentioned in this book show the possibilities that arise in this connection. Perhaps they will help the reader who encounters a similar situation in his investigations.

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