Proposition 1. *For any matrices* $A \in \mathbb{R}_+^{n\times n}$, $B \in \mathbb{R}_+^{k\times k}$, $S \in \mathbb{R}_+^{n\times k}$, $S' \in \mathbb{R}_+^{n\times k}$, *and* A, B *are symmetric, the following inequality holds*

$$\sum_{i=1}^{n}\sum_{p=1}^{k}\frac{(AS'B)_{ip}S_{ip}^2}{S'_{ip}} \quad \geq \quad Tr(S^TASB) \tag{1}$$

Proof. Let $S_{ip} = S'_{ip}u_{ip}$. Using the explicit index, the difference $\Delta = LHS - RHS$ can be written as

$$\Delta \quad = \quad \sum_{i,j=1}^{n}\sum_{p,q=1}^{k}A_{ij}S'_{jq}B_{qp}S'_{ip}(u_{ip}^2 - u_{ip}u_{jq}) \tag{2}$$

Because, A, B are symmetric, this is equal to

$$\begin{aligned} &= \quad \sum_{i,j=1}^{n}\sum_{p,q=1}^{k}A_{ij}S'_{jq}B_{qp}S'_{ip}\left(\frac{u_{ip}^2+u_{jq}^2}{2} - u_{ip}u_{jq}\right) \\ &= \quad \frac{1}{2}\sum_{i,j=1}^{n}\sum_{p,q=1}^{k}(u_{ip}-u_{jq})^2 \geq 0 \end{aligned} \tag{3}$$

□