Towards a Notion of Defeasibility in Formal Concept Analysis

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1 Introduction

2 The Initial Case

2.1 A Defeasible Formal Context

We define a defeasible formal context a $\widetilde{\mathbb{K}} := (G, M, I, <)$ where G is a set of objects, M a set of attributes, I an incidence relation on $G \times M$ expressed as gIm for $g \in G, m \in M$ (exactly as we see with typical formal contexts); and, $<$ a strict partial order over G.

2.2 The Derivation Operators

Over the defeasible formal context $\widetilde{\mathbb{K}} = (G, M, I, <)$ we introduce two operators; $(\underline{X}^{\circ})$ and $(\underline{X}^{\circ})^{\circ}$, $X \subseteq M$. Before we define these operators, we introduce a definition of minimality over objects in G.

$$Min(A) := \{g \in A | \nexists h \in A, h < g\} \tag{1}$$

We can now define the respective operators, $A \subseteq G, B \subseteq M$.

$$\underline{A}^{\circ} := \{m \in M | \forall g \in Min(A), gIm\} \tag{2}$$
$$\underline{B}^{\circ} := \{g \in G | \forall m \in B, \nexists h \in Gh < g, gIm\} \tag{3}$$

(1) Describes the minimally ranked objects in G. From here, (2) describes an operation from a set of objects $A \subseteq G$ to the attributes shared by the minimal elements in A. (3) Is an operation from a set of attributes $B \subseteq M$ which results in the *minimal* set of objects which have all the attributes in B.

We can apply these operators twice, $(\underline{B}^{\circ})^{\circ}$ where $A \subseteq M$; this procedure would give the set of attributes shared by the minimal objects which satisfy the properties B. Conversely, $(\underline{A}^{\circ})^{\circ}$ where $A \subseteq G$ describes the minimal set of objects whichs satisfy the properties satisfied by all the objects in A.

For an example, consider the defeasible context $\widetilde{\mathbb{K}} = \{G, M, I, <\}$, with $o_i < o_j$ if $i < j$ for all objects.

	m_1	m_2	m_3	m_4
o_1	×	×		×
o_2	×	×	×	
o_3		×	×	×
o_4		×		×

Figure 1: Defeasible Formal Context

Given $A = \{m_2, m_3\}$, we have that $\underline{A}^{\circ} = \{o_1\}$, and $(\underline{A}^{\circ})^{\circ} = \{m_1, m_2, m_4\}$. Conversely, given $B = \{o_3, o_4\}$, $\underline{B}^{\circ} = \{m_2, m_3, m_4\}$, $(\underline{B}^{\circ})^{\circ} = \{o_3\}$.[1]

[1] I acknowledge that this second example could just be replaced with $Min(B)$, I will think more about this.

2.2.1 Properties of $(\underline{\cdot}^{\circ})^{\circ}$

nonmonotonic: $A \subseteq B \not\Rightarrow (\underline{A}^{\circ})^{\circ} \subseteq (\underline{B}^{\circ})^{\circ}$

Assume $(\underline{\cdot}^{\circ})^{\circ}$ were monotonic; then $A \subseteq B \Rightarrow (\underline{A}^{\circ})^{\circ} \subseteq (\underline{B}^{\circ})^{\circ}$.

From Figure 1, if we take the oo-operation on $A_1 = \{m_1, m_2\}$, we get $(\underline{A_2}^{\circ})^{\circ} = \{o_1\}^{\circ} = \{m_1, m_2, m_4\}$. Then, on $A_2 = \{m_1, m_2, m_3\}$, we have $(\underline{A_2}^{\circ})^{\circ} = \{o_2\}^{\circ} = \{m_1, m_2, m_3\}$. Obviously, we have $A_1 \subseteq A_2$ but $(\underline{A_1}^{\circ})^{\circ} \not\subset (\underline{A_2}^{\circ})^{\circ}$. Thus, the oo-operator is nonmonotonic.

extensive: $X \subseteq (\underline{X}^{\circ})^{\circ}$

Assume that $(\underline{\cdot}^{\circ})^{\circ}$ is not extensive; then, there exists some $X \not\subseteq (\underline{X}^{\circ})^{\circ}$. Let $Y \subseteq G$ such that $Y = \underline{X}^{\circ}$. Then $\underline{Y}^{\circ}$ is the set of properties shared by all minimal objects in Y (this minimal requirement is redundant since all elements of Y are minimal by construction). Observe that $\underline{Y}^{\circ} \equiv (\underline{X}^{\circ})^{\circ}$. Since $X \not\subseteq (\underline{X}^{\circ})^{\circ}$, there must be some $y \in Y$ such that $y^{\circ} \cap X \neq X$. That is, there is some object in Y which does not have all the attributes from X. However, this is a contradiction, since y would not be in Y since it is not the case that $\forall m \in X, yIm$ - which is our definition of $(\underline{\cdot}^{\circ})$.

idempotent: $(\underline{A}^{\circ})^{\circ} = (\underline{(\underline{A}^{\circ})^{\circ\circ}})^{\circ}$