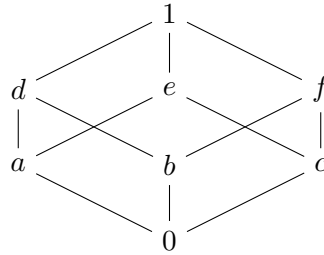


# Everything I know so far about [Formal Concept Analysis]

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# 1 Introduction

## 1.1 Lattices

A lattice  $\mathcal{C}$  is a poset s.t. for any pair  $(a, b) \in \mathcal{C}$ , the supremum  $a \wedge b$ , and infimum  $a \vee b$  exist. We extend this to a complete lattice, which has the requirement that for any subset  $\mathcal{D} \subseteq \mathcal{C}$  the supremum  $\bigvee \mathcal{D}$  and infimum  $\bigwedge \mathcal{D}$  exist.

### 1.1.1 Supremum and infimum

The supremum of two elements is defined as the *least upper bound* of those two elements. We can obviously extend this over more than two elements. Given the set  $S = \{1, 2, 3, 4, 5\}$ , where  $S \subset \mathbb{N}$ . The supremum,  $\bigvee S = 5$ ; similarly, the infimum  $\bigwedge S = 1$ .

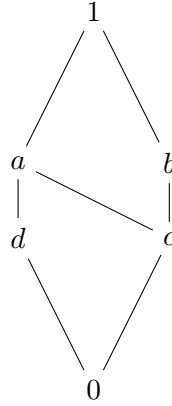


Figure 1:  $d \vee c = a$ ,  $b \wedge a = c$

When we are talking about lattices (or Hasse diagrams in general), we can refer to the supremum and infimum as the *meet* and *join*. Refer to 1.5 for more pertinent discussion regarding lattices.

We also have infimum (supremum) satisfying the following

- $x \wedge y = y \wedge x$  (commutativity)
- $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$  (associativity)
- $x \wedge x = x$  (idempotency - but arguably, reflexivity)

We use these later on to show that any finite lattice is a complete lattice.

There are also the following interesting properties:

- $a \vee 0 = a$
- $a \leq b \implies a \vee b = b$ <sup>1</sup>

## 1.2 Formal Contexts

A Formal Context is a triple  $\langle G, M, I \rangle$  where  $G$  refers to a set of objects,  $M$  to a set of properties, and  $I$  an incidence relation over  $G \times M$ .

We have derivation operators  $A'$  and  $B'$ ; for  $A'$ , where  $A \subseteq G$ , the derivation operator tells us which properties belong to the objects in  $A$ , the dual holds for properties and their objects. **Formally**,

**Definition 1.1**

$$A' := \{m \in M \mid \forall g \in A, gIm\}$$

$$B' := \{g \in G \mid \forall m \in B, gIm\}$$

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<sup>1</sup>This is useful because it enables to move from orders (posets) to lattices

We also have closure operators,  $A''$ , which works intuitively by applying the derivation operator on  $A$  ( $B$ ), which yields a set of properties. Then applying it again on  $A'$  ( $B'$ ), which yields back a set of objects (properties).

**Proposition 1.2** *For subsets  $A, B \subseteq G$  (defined dually for properties  $C, D \subseteq M$ ), we have*

- a.  $A \subseteq B \implies B' \subseteq A'$
- b.  $A \subseteq A''$
- c.  $A' = A'''$

For more natural discussion, *a* describes the behaviour that if we have two sets of objects  $A$  and  $B$ , where  $A \subseteq B$ ; then it follows that objects in  $A$  will have at *least* all the properties of objects in  $B$ .

### 1.3 Formal Concepts

Presume we are working with a formal context  $\langle G, M, I \rangle$ .

**Definition 1.3**  $(A, B)$  is a **formal concept** of our formal context iff  $A \subseteq G$ ,  $B \subseteq M$ ,  $A' = B$ , and  $B' = A$

$A$  is called the **extent**, and  $B$  is called the **intent**. We can refer to the set of all formal concepts of a formal context as a  $\mathcal{B}(G, M, I)$ .

### 1.4 Concept Hierarchies

When we think about concepts, we typically think of them in a structure (something like a taxonomy, or ontology). That is, we think of *sub* and *super* concepts. For example, Dog is a subconcept of Mammal; and so is Cat. However, Dog and Cat are not sub nor superconcepts of one another (so, in some sense we have a partial order here). Formal Concepts have the same idea.

**Definition 1.4** Let  $(A_1, B_1)$  and  $(A_2, B_2)$  be formal concepts of some  $\mathcal{B}(G, M, I)$ . We say that  $(A_1, B_1)$  is a **subconcept** of  $(A_2, B_2)$  (equivalently,  $(A_2, B_2)$  is a **superconcept** of  $(A_1, B_1)$ ) and use the  $\leq$  sign to express this. Thus we have

$$(A_1, B_1) \leq (A_2, B_2) : \iff A_1 \subseteq A_2 \iff B_2 \subseteq B_1.$$

### 1.5 The Basic Theorem

#### 1.5.1 I

The first part of the Basic Theorem tells us that given some formal concept  $(G, M, I)$ , the concept lattice of  $\mathcal{B}(G, M, I)$  is a complete lattice. So, constructing a lattice from a formal context, will always result in a complete lattice.

**From**  $\mathcal{B}(G, M, I)$ , let  $\{(A_t, B_t) | t \in T\} \subseteq \mathcal{B}(G, M, I)$  be some arbitrary subset of formal concepts. We can then define the supremum and infimum of this subset as:

$$\begin{aligned} \bigvee_{t \in T} (A_t, B_t) &:= \left( \left( \bigcup_{t \in T} A_t \right)', \left( \bigcap_{t \in T} B_t \right) \right) \\ \bigwedge_{t \in T} (A_t, B_t) &:= \left( \left( \bigcap_{t \in T} A_t \right), \left( \bigcup_{t \in T} B_t \right)' \right) \end{aligned}$$

Now, we want to show that there is some complete lattice  $\mathcal{V}$  that is isomorphic to the concept lattice given by  $\mathcal{B}(G, M, I)$ .

	Fur	Feathers	Scales	Lives on Land	Lays Eggs	Cold-Blooded
Mammals	×			×		
Birds		×		×	×	
Reptiles			×	×	×	×
Amphibians			×	×	×	×
Fish			×			×

Suppose we are interested in Mammals and Birds,  $\{Mammals\}' = \{Fur, LivesonLand\}$ ;  $\{Mammals\}'' = \{Mammals\}$ .  $\{Birds\}' = \{Feathers, LivesonLand, LaysEggs\}$ ;  $\{Birds\}'' = \{Birds\}$ .

Using the basic theorem from above, we can determine the supremum and infimum of our set of concepts. We have the set of concepts from above, lets call this  $S := (A_t, B_t)$ . We can calculate  $\bigvee S$  by

$$\begin{aligned} & \left( \bigcup_{t \in T} A_t \right)'', \left( \bigcap_{t \in T} B_t \right) \\ & \left( \{Mammals, Birds\}'', \{LivesonLand\} \right) \\ & \left( \{Mammals, Birds, Amphibians, Reptiles\}', \{LivesonLand\} \right) \end{aligned}$$

Dually, we can find the infimum  $\bigwedge S$ , given as: (which is the bottom element)

$$\begin{aligned} & \left( \bigcap_{t \in T} A_t, \left( \bigcup_{t \in T} B_t \right)'' \right) \\ & \left( \emptyset, (\{Fur, Feathers, LivesonLand, LaysEggs\})' \right) \\ & (\emptyset, \emptyset) \end{aligned}$$

The first part of the Basic Theorem, which we have just gone through, shows that for the concept lattice of a given formal context,  $\mathcal{B}(G, M, I)$ , we have an infimum and supremum for any arbitrary subset of concepts. This is just our definition of a complete lattice - so we have that any formal context has a complete lattice.

### 1.5.2 II

The second part of the basic theorem tells us that every complete lattice is *isomorphic* to a concept lattice. That is, given a complete lattice  $(\mathcal{L}, \leq)$ , we can find some  $\mathcal{B}(G, M, I)$  that is isomorphic to  $\mathcal{L}$ .

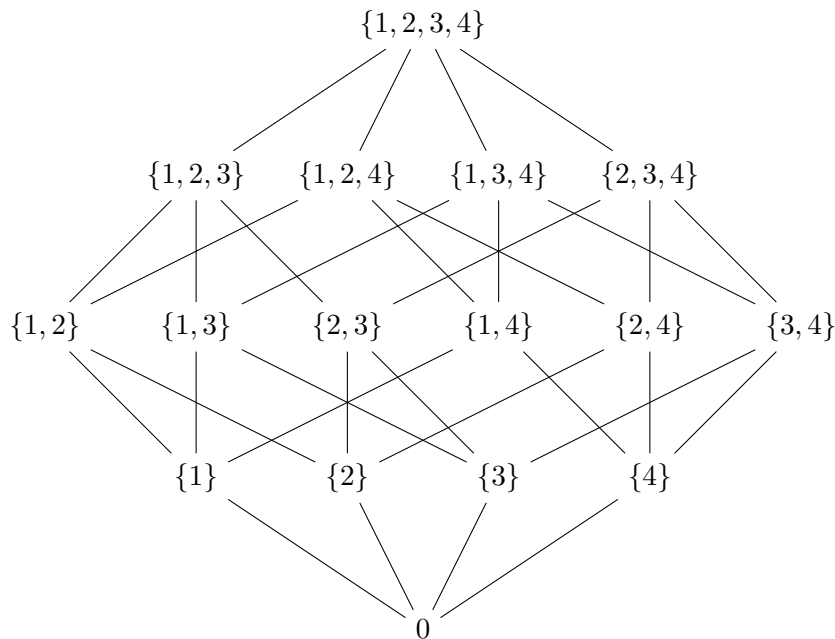


Figure 2: Example of a complete lattice of  $(\{1, 2, 3, 4\}, \subseteq)$

### Example 1.5

This is a complete lattice (it is finite, so it is complete); we can find a relatively simply formal context, which is isomorphic to the lattice: we need two sets, the set of supremum-irreducible elements, and the set of infimum-irreducible elements. For small lattices, we can do this via inspection - the supremum-irreducible (infimum-irreducible) elements are those which only have a single line below (above) the element.

Thus, our supremum-irreducible set is  $\{1, 2, 3, 4\}$ ;

and our infimum-irreducible set is  $\{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$ . The supremum-irreducible set represents the objects, and infimum-irreducible set represents the properties. The reason for this is intuitive: we start bottom-up with our maximally sub-concepts (remember, the distinction between properties and objects is a modelling problem), these basic elements are the ones which supremum-irreducible, i.e. they cannot be written as a super-concept of some other elements. Conversely, our infimum-irreducible elements are the most general (other than  $\top$ ) elements of our lattice, they cannot be written as the sub-concepts of elements.<sup>2</sup>

Table 1: The formal context for 2

	$\{1, 2, 3\}$	$\{1, 2, 4\}$	$\{1, 3, 4\}$	$\{2, 3, 4\}$
1	×	×	×	
2	×	×		×
3	×		×	×
4		×	×	×

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<sup>2</sup>We can of course write a infimum-irreducible element  $e$  as  $e \vee \top$ , but this is not useful.