Everything I know so far about [Formal Concept Analysis]

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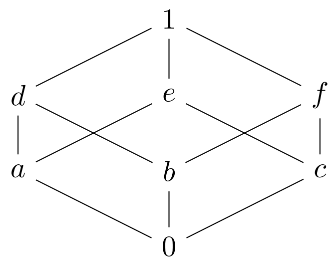

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1 Introduction

1.1 Lattices

A lattice $\mathcal{C}$ is a poset s.t.for any pair $(a, b) \in \mathcal{C}$, the supremum $a \wedge b$, and infimum $a \vee b$ exist. We extend this to a complete lattice, which has the requirement that for any subset $\mathcal{D} \subseteq \mathcal{C}$ the supremum $\bigvee \mathcal{D}$ and infimum $\bigwedge \mathcal{D}$ exist.

1.1.1 Supremum and infimum

The supremum of two elements is defined as the *least upper bound* of those two elements. We can obviously extend this over more than two elements. Given the set $S = \{1, 2, 3, 4, 5\}$, where $S \subset \mathbb{N}$. The supremum, $\bigvee S = 5$; similarly, the infimum $\bigwedge S = 1$.

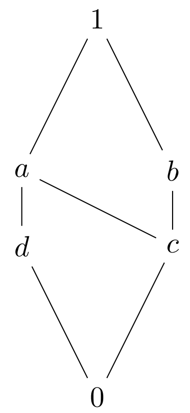

Figure 1: $d \vee c = a$, $b \wedge a = c$

When we are talking about lattices (or Hasse diagrams in general), we can refer to the supremum and infimum as the *meet* and *join*. Refer to 1.5 for more pertinent discusison regarding lattices.

We also have infimum (supremum) satisfying the following

- $x \wedge y = y \vee x$ (commutativity)
- $x \vee (y \vee z) = (x \vee y) \vee z$ (associativity)
- $x \vee x$ (idempotency - but arguably, reflexivity)

We use these later on to show that any finite lattice is a complete lattice.

There are also the following interesting properties:

- $a \vee 0 = a$
- $a \leq b \implies a \vee b = b$[1]

1.2 Formal Contexts

A Formal Context is a triple $\langle G, M, I \rangle$ where G refers to a set of objects, M to a set of properties, and I an incidence relation over $G \times M$.

We have derivation operators A' and B'; for A', where $A \subseteq G$, the derivation operator tells us which properties belong to the objects in A, the dual holds for properties and their objects. **Formally**,

Definition 1.1

$$A' := \{m \in M \mid \forall g \in A, gIm\}$$

$$B' := \{g \in G \mid \forall m \in B, gIm\}$$

[1]This is useful because it enables to move from orders (posets) to lattices

We also have closure operators, A'', which works intuitively by applying the derivation operator on A (B), which yields a set of properties. Then applying it again on A' (B'), which yields back a set of objects (properties).

Proposition 1.2 *For subsets $A, B \subseteq G$ (defined dually for properties $C, D \subseteq M$), we have*

a. $A \subseteq B \implies B' \subseteq A'$

b. $A \subseteq A''$

c. $A' = A'''$

For more natural discussion, *a* describes the behaviour that if we have two sets of objects A and B, where $A \subseteq B$; then it follows that objects in A will have at *least* all the properties of objects in B.

1.3 Formal Concepts

Presume we are working with a formal context $\langle G, M, I \rangle$.

Definition 1.3 *(A, B) is a **formal concept** of our formal context iff $A \subseteq G$, $B \subseteq M$, $A' = B$, and $B' = A$*

A is called the **extent**, and B is called the **intent**. We can refer to the set of all formal concepts of a formal context as a $\mathcal{B}(G, M, I)$.

1.4 Concept Hierarchies

When we think about concepts, we typically think of them in a structure (something like a taxonomy, or ontology). That is, we think of *sub* and *super* concepts. For example, Dog is a subconcept of Mammal; and so is Cat. However, Dog and Cat are not sub nor superconcepts of one another (so, in some sense we have a partial order here). Formal Concepts have the same idea.

Definition 1.4 *Let (A_1, B_1) and (A_2, B_2) be formal concepts of some $\mathcal{B}(G, M, I)$. We say that (A_1, B_1) is a **subconcept** of (A_2, B_2) (equivalently, (A_2, B_2) is a **superconcept** of (A_1, B_1)) and use the $\leq$ sign to express this. Thus we have*

$$(A_1, B_1) \leq (A_2, B_2) :\iff A_1 \subseteq A_2 (\iff B_2 \subseteq B_1).$$

1.5 The Basic Theorem

1.5.1 I

The first part of the Basic Theorem tells us that given some formal concept (G, M, I), the concept lattice of $\mathcal{B}(G, M, I)$ is a complete lattice. So, constructing a lattice from a formal context, will always result in a complete lattice.

From $\mathcal{B}(G, M, I)$, let $\{(A_t, B_t) | t \in T\} \subseteq \mathcal{B}(G, M, I)$ be some arbitrary subset of formal concepts. We can then define the supremum and infimum of this subset as:

$$\bigvee_{t \in T} (A_t, B_t) := \Big((\bigcup_{t \in T} A_t)'', (\bigcap_{t \in T} B_t) \Big)$$

$$\bigwedge_{t \in T} (A_t, B_t) := \Big((\bigcap_{t \in T} A_t), (\bigcup_{t \in T} B_t)'' \Big)$$

Now, we want to show that there is some complete lattice $\mathcal{V}$ that is isomorphic to the concept lattice given by $\mathcal{B}(G, M, I)$.

	Fur	Feathers	Scales	Lives on Land	Lays Eggs	Cold-Blooded
Mammals	×			×		
Birds		×		×	×	
Reptiles			×	×	×	×
Amphibians			×	×	×	×
Fish			×			×

Suppose we are interested in Mammals and Birds, $\{Mammals\}' = \{Fur, LivesonLand\}$; $\{Mammals\}'' = \{Mammals\}$. $\{Birds\}' = \{Feathers, LivesonLand, LaysEggs\}$; $\{Birds\}'' = \{Birds\}$.

Using the basic theorem from above, we can determine the supremum and infimum of our set of concepts. We have the set of concepts from above, lets call this $S := (A_t, B_t$. We can calculate $\bigvee S$ by

$$\Big(\bigcup_{t \in T} A_t)'', \bigcap_{t \in T} B_t\Big)$$

$$\Big(\{Mammals, Birds\}'', \{LivesonLand\}\Big)$$

$$\Big(\{Mammals, Birds, Amphibians, Reptiles\}, \{LivesonLand\}\Big)$$

Dually, we can find the infimum $\bigwedge S$, given as: (which is the bottom element)

$$\Big(\bigcap_{t \in T} A_t, (\bigcup_{t \in T} B_t)''\Big)$$

$$\Big(\emptyset, (\{Fur, Feathers, LivesonLand, LaysEggs\})\Big)$$

$$\Big(\emptyset, \emptyset\Big)$$

The first part of the Basic Theorem, which we have just gone through, shows that for the concept lattice of a given formal context, $\mathcal{B}(G, M, I)$, we have an infimum and supremum for any arbitrary subset of concepts. This is just our definition of a complete lattice - so we have that any formal context has a complete lattice.

1.5.2 II

The second part of the basic theorem tells us that every complete lattice is *isomorphic* to a concept lattice. That is, given a complete lattice $(\mathcal{L}, \leq)$, we can find some $\mathcal{B}(G, M, I)$ that is isomorphic to $\mathcal{L}$.

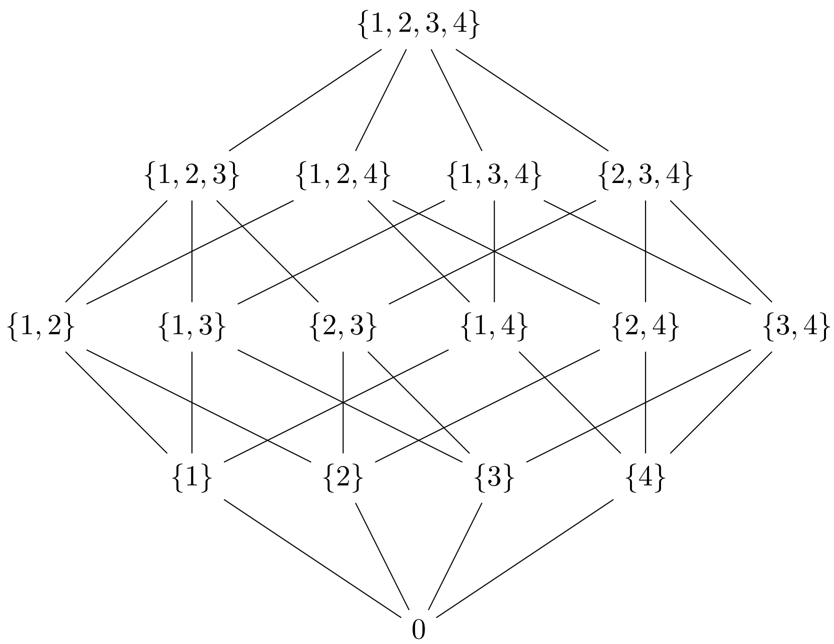

Figure 2: Example of a complete lattice of $(\{1, 2, 3, 4\}, \subseteq)$

Example 1.5

This is a complete lattice (it is finite, so it is complete); we can find a relatively simply formal context, which is isomorphic to the lattice: we need two sets, the set of supremum-irreducible elements, and the set of infimum-irreducible elements. For small lattices, we can do this via inspection - the supremum-irreducible (infimum-irreducible) elements are those which which only have a single line below (above) the element.

Thus, our supremum-irreducible set is $\{1, 2, 3, 4\}$;

and our infimum-irreducible set is $\{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$.

The supremum-irreducible set represents the objects, and infimum-irreducible set represents the properties. The reason for this is inuitive: we start bottom-up with our maximally sub-concepts (remember, the distinction between properties and objects is a modelling problem), these basic elements are the ones which supremum-irreducible, i.e. they cannot be written as a super-concept of some other elements. Conversely, our infimum-irreducible elements are the most general (other than $\top$) elements of our lattice, they cannot be written as the sub-concepts of elements.[2]

Table 1: The formal context for 2

	$\{1, 2, 3\}$	$\{1, 2, 4\}$	$\{1, 3, 4\}$	$\{2, 3, 4\}$
1	$\times$	$\times$	$\times$	
2	$\times$	$\times$		$\times$
3	$\times$		$\times$	$\times$
4		$\times$	$\times$	$\times$

[**How do we determine the supremum(infimum)-irreducible sets without inspection?**]

[2] We can of course write a infimum-irreducible element e as $e \vee \top$, but this is not useful.

2 Closure Systems

2.1 Defining a Closure System

If we are given a set M, a closure system $\mathcal{C}$ on M is a set of subsets of M ($\mathcal{C} \subseteq \mathcal{P}(M)$) for which:

- $M \in \mathcal{C}$
- if $D \subseteq \mathcal{C}$ then, $\bigcap D \in \mathcal{C}$

We might observe that the first point is ensured by the second point; this is because $\bigcap \emptyset = M$ - if we don't have this property, we have a contradiction. We might also observe that a closure system has a natural ordering; specifically, by subsumption. That is, $\mathcal{C}$ is a poset with $(\mathcal{C}, \subseteq)$.

2.2 Closure Operators

3 Implications

3.1 Introduction

If M is a set of attributes, we say that $A \to B$ is an implication over M; $A, B \subseteq M$. A set $D \subseteq M$ respects an implication $A \to B$ if i) $A \not\subseteq D$ or ii) $B \subseteq D$ - the same as material implication. We can represent this notion as $D \models A \to B$.

If we have a set of implications $L := \{I_1, \ldots, I_n\}$ we say that a subset $D \models L$ if it respects every implication in L: formally, $D \models L \iff \forall A \to B \in L, D \models A \to B$.

3.1.1 Implications in a Formal Context

Suppose we have a formal context $\mathcal{K} := (G, M, I)$ and $A \to B$ where $A, B \subseteq M$. We say that $K \models A \to B$ if $\forall g \in G, g' \models A \to B$. That is, a formal context is a model of some implication, if every object in G has a intent which is a model of that implication.

Given $K \models A \to B$, the following are equivalent:

$$1. \forall g \in G, g' \models A \to B$$

$$2. B \subseteq A''$$

$$3. A' \subseteq B'$$

(1) is obvious.

For (2), we need to show $K \models A \to B \iff B \subseteq A''$.

$\Rightarrow$ We want to show that $\forall g \in G, g' \models A \to B \implies B \subseteq A''$

1. Every object which has A in its intent, also has B in its intent.
2. This means, $\forall g \in G, A \subseteq g', B \subseteq g'$
3. So B is a subset of every object intent in G.
4. Thus, $B \subseteq A''$.

$\Leftarrow$ Show that $B \subseteq A'' \implies \forall g \in G, g' \models A \to B$.

1. A''' is the set of objects which have A in their intents
2. We know that all of these objects will have B in their intents too (assumption)
3. So, $\forall g \in G$ if $g \in A'''$ then $B \in g'$ and so $g' \models A \to B$.
4. $\forall g \in G$ and $g \notin A'''$, then $g' \models A \to B$ trivially.
5. $\forall g \in G, g' \models A \to B$

[We still need to do proof for (3)]

3.2 Sets of Implications

Let L be a set of implications over M, then

$$Mod(L) := \{T \in \mathcal{P}(M) | T \models L\}$$

$Mod(L)$ is the set of all subsets of M which are models of the set of implications L. This is a closure system on M. We can see that $M \in Mod(L)$ because for any implication $A \to B$ over M, $M \models A \to B$.

Proof: $M \models A \to B$ for any implication over M with $A, B \subseteq M$.

1. Let $A \to B$ be some implication over M.
2. Suppose $M \not\models A \to B$.
3. Then $A \subseteq M$ and $B \not\subseteq M$
4. Contradiction, since $A, B \subseteq M$

□

Proof: The intersection of any elements in $Mod(L)$ is also in $Mod(L)$.

1. Let $X, Y \in Mod(L)$,
2. Assume $X \cap Y \notin Mod(L)$,
3. Then, for some $A \to B \in L, X \cap Y \not\models A \to B$,
4. $X \cap Y \subseteq A$ and $X \cap Y \not\subseteq B$.
5. But $Mod(L)$ is such that $\forall m \in Mod(L), A \subseteq m \implies B \subseteq m$.
6. If $X \cap Y \subseteq A$ then $X \cap Y \subseteq B$.
7. Thus, $X \cap Y \in Mod(L)$

□

The above proves that $Mod(L)$ is a closure system on M. (From the definition in 2) If L is the set of all implications of a formal context, then $Mod(L)$ is the system of all concept intents.

Since $Mod(L)$ is a closure system, we have a closure operator $X \mapsto L(X)$, which can be described in two ways.

$X^{\mathcal{L}} := X \cup \bigcup\{B | A \to B \in \mathcal{L}, A \subseteq X\}$.

We then keep applying $X^{\mathcal{L}}$ until we get a fixed point $\mathcal{L}(X) := X^{\mathcal{L}\dots\mathcal{L}}$ with $\mathcal{L}(X)^{\mathcal{L}} = \mathcal{L}(X)$.

Alternatively, we define the closure operator as $\mathcal{L}(X) := \bigcap\{Y | X \subseteq Y \subseteq M, Y \in Mod(L)\}$

These two definitions are equivalent; the first one is an iterative definition, which says: given a subset of M, X - keep applying all the implications from $\mathcal{L}$ until we reach a fixed point. The second definition says that we define the closure of X as the intersection between all the models of $\mathcal{L}$ which contain X.

3.3 When does an implication follow from other ones?

Let $\mathcal{L}$ be a set of implications, and $A \to B$ an implication over M. How do we know if $\mathcal{L} \models A \to B$. Or, how do we know if an implication follows semantically from a set of implications? If every subset of $\mathcal{L}$ which respects $\mathcal{L}$ also respects $A \to B$.

Proof: $\mathcal{L} \models A \to B \iff B \subseteq \mathcal{L}(A)$

$\mathcal{L} \models A \to B \Rightarrow B \subseteq \mathcal{L}(A)$

1. Assume $B \not\subseteq \mathcal{L}(A)$
2. Then, $\exists Y \in Mod(\mathcal{L})$ s.t.$A \subset Y$ and $B \not\subseteq Y$.
3. $Y \not\models A \to B$ and $Y \in Mod(\mathcal{L})$
4. Contradiction, since $\mathcal{L} \models A \to B$

$B \subseteq \mathcal{L}(A) \Rightarrow \mathcal{L} \models A \to B$ by contrapositive.

1. Assume $\mathcal{L} \not\models A \to B$
2. Then, there exists some element $I \in Mod(\mathcal{L})$ with $A \subseteq I$ and $B \not\subseteq I$.
3. By inspection, $I \in \{Y|A \subseteq Y \subseteq M, Y \in Mod(\mathcal{L})\}$
4. Which would then mean that $B \not\subseteq \bigcap\{Y|A \subseteq Y \subseteq M, Y \in Mod(\mathcal{L})\}$

$\square$