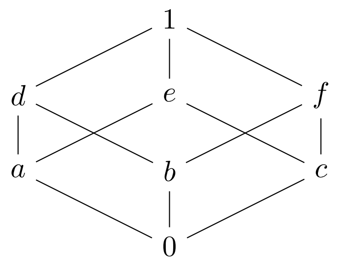
Everything I know so far about [Formal Concept Analysis]

Lucas Carr

May 19, 2024



Contents

1 Introduction

1.1 Lattices

A lattice $\mathcal{C}$ is a poset s.t.for any pair $(a, b) \in \mathcal{C}$, the supremum $a \wedge b$, and infimum $a \vee b$ exist. We extend this to a complete lattice, which has the requirement that for any subset $\mathcal{D} \subseteq \mathcal{C}$ the supremum $\bigvee \mathcal{D}$ and infimum $\bigwedge \mathcal{D}$ exist.

1.1.1 Supremum and infimum

The supremum of two elements is defined as the *least upper bound* of those two elements. We can obviously extend this over more than two elements. Given the set $S = \{1, 2, 3, 4, 5\}$, where $S \subset \mathbb{N}$. The supremum, $\bigvee S = 5$; similarly, the infimum $\bigwedge S = 1$.

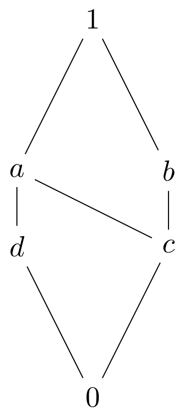


Figure 1: $d \vee c = a$, $b \wedge a = c$

When we are talking about lattices (or Hasse diagrams in general), we can refer to the supremum and infimum as the *meet* and *join*. Refer to 1.5 for more pertinent discusison regarding lattices.

We also have infimum (supremum) satisfying the following

- $x \wedge y = y \vee x$ (commutativity)
- $x \vee (y \vee z) = (x \vee y) \vee z$ (associativity)
- $x \vee x$ (idempotency - but arguably, reflexivity)

We use these later on to show that any finite lattice is a complete lattice.

There are also the following interesting properties:

- $a \vee 0 = a$
- $a \leq b \implies a \vee b = b$[1]

1.2 Formal Contexts

A Formal Context is a triple $\langle G, M, I \rangle$ where G refers to a set of objects, M to a set of properties, and I an incidence relation over $G \times M$.

We have derivation operators A' and B'; for A', where $A \subseteq G$, the derivation operator tells us which properties belong to the objects in A, the dual holds for properties and their objects. **Formally**,

Definition 1.1

$$A' := \{m \in M \,|\, \forall g \in A, gIm\}$$

$$B' := \{g \in G \,|\, \forall m \in B, gIm\}$$

[1]This is useful because it enables to move from orders (posets) to lattices

We also have closure operators, A'', which works intuitively by applying the derivation operator on A (B), which yields a set of properties. Then applying it again on A' (B'), which yields back a set of objects (properties).

Proposition 1.2 *For subsets $A, B \subseteq G$ (defined dually for properties $C, D \subseteq M$), we have*

a. $A \subseteq B \implies B' \subseteq A'$

b. $A \subseteq A''$

c. $A' = A'''$

For more natural discussion, *a* describes the behaviour that if we have two sets of objects A and B, where $A \subseteq B$; then it follows that objects in A will have at *least* all the properties of objects in B.

1.3 Formal Concepts

Presume we are working with a formal context $\langle G, M, I\rangle$.

Definition 1.3 *(A, B) is a **formal concept** of our formal context iff $A \subseteq G$, $B \subseteq M$, $A' = B$, and $B' = A$*

A is called the **extent**, and B is called the **intent**. We can refer to the set of all formal concepts of a formal context as a $\mathcal{B}(G, M, I)$.

1.4 Concept Hierarchies

When we think about concepts, we typically think of them in a structure (something like a taxonomy, or ontology). That is, we think of *sub* and *super* concepts. For example, Dog is a subconcept of Mammal; and so is Cat. However, Dog and Cat are not sub nor superconcepts of one another (so, in some sense we have a partial order here). Formal Concepts have the same idea.

Definition 1.4 *Let (A_1, B_1) and (A_2, B_2) be formal concepts of some $\mathcal{B}(G, M, I)$. We say that (A_1, B_1) is a **subconcept** of (A_2, B_2) (equivalently, (A_2, B_2) is a **superconcept** of (A_1, B_1)) and use the $\leq$ sign to express this. Thus we have*

$$(A_1, B_1) \leq (A_2, B_2) :\iff A_1 \subseteq A_2 (\iff B_2 \subseteq B_1).$$

1.5 The Basic Theorem

The first part of the Basic Theorem tells us that given some formal concept (G, M, I), the concept lattice of $\mathcal{B}(G, M, I)$ is a complete lattice. So, constructing a lattice from a formal context, will always result in a complete lattice.

From $\mathcal{B}(G, M, I)$, let $\{(A_t, B_t) | t \in T\} \subseteq \mathcal{B}(G, M, I)$ be some arbitrary subset of formal concepts. We can then define the supremum and infimum of this subset as:

$$\bigvee_{t \in T} (A_t, B_t) := \Big((\bigcup_{t \in T} A_t)'', (\bigcap_{t \in T} B_t) \Big)$$

$$\bigwedge_{t \in T} (A_t, B_t) := \Big((\bigcap_{t \in T} A_t), (\bigcup_{t \in T} B_t)'' \Big)$$

Now, we want to show that there is some complete lattice $\mathcal{V}$ that is isomorphic to the concept lattice given by $\mathcal{B}(G, M, I)$.