Defeasibility in Formal Concept Analysis

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1 Establishing Requirements

1.1 A defeasible formal context

A regular formal context $\mathbb{K}$ is a tripple (G, M, I) of objects G, attributes M, and an incidence relation $I \subseteq G \times M$. We define a defeasible formal context $\widetilde{\mathbb{K}}$ as a quadruple $(G, M, I, \leq)$ with G, M, I as before, and $\leq$ being a partial order on the objects G.

The ordering on G can be loosely thought of as expressing a preference on objects such that if $o_1 \leq o_2$ and $o_1 \neq o_2$, then o_1 is preferred to o_2, where $o_1, o_2 \in G$.

1.2 Minimisation

In formal context analysis, two operators are defined

$$A^{\uparrow} := \{m \in M | \forall g \in A, (g, m) \in I\} \qquad A \subseteq G$$

$$B^{\downarrow} := \{g \in G | \forall m \in B, (g, m) \in I\} \qquad B \subseteq M$$

Much of the time it is not necessary to distinguish between $\cdot^{\uparrow}$ and $\cdot^{\downarrow}$, and so A' and B' are used instead.

We introduce a variation on these two operators, which has the same set building behavior, but discards any objects which are not minimal with respect to the partial order given by $\widetilde{\mathbb{K}}$. The result is

$$Min(O) := \{g \in O | \nexists h \in O, h < g\} \qquad O \subseteq G$$

$$\underline{A^{\uparrow}} := (Min(A))^{\uparrow} \qquad A \subseteq G$$

$$\underline{B^{\downarrow}} := Min(B^{\downarrow}) \qquad B \subseteq M$$

When applied to a set of objects $A \subseteq G$, the minimisation operator removes all non-minimal elements from A, and then takes $A^{\uparrow}$. This gives us the set of attributes common to all minimal objects from A. Conversely, when applied to a set of atttributes $B \subseteq M$, the minimisation operator takes $B^{\downarrow}$, and removes all non-minimal objects, leaving the minimal objects which have all the attributes from B in their object intents.

Given a set of attributes $B \subseteq M$, we can determine $\underline{(\underline{B^{\downarrow}})^{\uparrow}}$ (although, this double minimisation is not strictly necessary, when applying the operator twice on an initial set of attributes $B \subseteq M$, we will omit the double minimisation, giving us $\underline{B''}$). This constructs the set of attribute which are common to all minimal objects which have B in their object intents. We make this clear with an example

1.2.1 Example

The context $\widetilde{\mathbb{K}} = (G, M, I, \leq)$ with the order over G being $\{o_1 < o_3, o_2 < o_3, o_3 < o_4, o_5 < o_4\}$

Given the set of attributes $Y :=$ {`laysEggs`, `canFly`}, our first application of the minimisation operator gives us $\underline{Y'} = Min(\{o_2, o_4\}) = \{o_2\}$. Applying the operator again, to the first result, we get $\underline{Y''} = \underline{\{o_2\}'} =$ {`feathered`, `laysEggs`, `aquatic`, `canFly`}. Plainly, this tells us that the most preferred objects which can fly and lay eggs, also are feathered and aquatic.

	feathered	laysEggs	aquatic	canFly	hasScales
o_1		×	×		×
o_2	×	×	×	×	
o_3		×	×		×
o_4	×	×		×	
o_5	×	×			

Figure 1: Defeasible Formal Context

1.3 Properties of the minimisation-operator

The $\cdot''$ operator in FCA is a closure-operator, so it is monotonic, extensive, and idempotent. In the following, we show that the minimisation-operator is nonmonotonic, extensive, and idempotent over sets of M in a defeasible context $\widetilde{\mathbb{K}} = (G, M, I, \leq)$.

Proposition 1.1 *The minimisation operator is nonmonotonic, extensive, and idempotent over sets of M.*

Proof: Nonmonotonicity, Extensivity, Idempotency of the minimisation operator.

Nonmonotonicity:

If $\underline{\cdot}''$ were monotonic, we would have $A \subseteq B \Rightarrow \underline{A}'' \subseteq \underline{B}''$, with $A, B \subseteq M$.

Let $\widetilde{\mathbb{K}}$ be a defeasible context with two objects, o_1 and o_2, attributes $\{m_1, m_2, m_3, m_4\}$, and an ordering $o_1 < o_2$. The intents of each objects are $\{m_1, m_2, m_4\}$ and $\{m_1, m_2, m_3\}$, respectively. If we take the sets $A = \{m_1, m_2\}$ and $B = \{m_1, m_2, m_3\}$ we have that $A \subseteq B$. However, $\underline{A}'' = \{m_1, m_2, m_4\}$ and $\underline{B}'' = \{m_1, m_2, m_3\}$, and so $\underline{A}'' \nsubseteq \underline{B}''$. Consequently, the minimisation operator is nonmonotonic.

Extensivity:

For $A \subseteq M$, extensivity requires that $A \subseteq \underline{A}''$.

We know that $A \subseteq A''$, since $\cdot''$ is extensive. So, for all $a \in A'$, we have that A is in a object intent $A \subseteq a'$. Since the minimisation operator simply reduces the size of A', then $\underline{A}' \subseteq A'$. So the earlier condition holds, that for all $a \in \underline{A}'$ the object intent is a superset of A, or $A \subseteq a'$. From this, it is easy to see that $A \subseteq \bigcap\{a' | a \in \underline{A}'\}$, and equivalently $A \subseteq \underline{A}''$.

Idempotency:

The proof for idempotency follows from extensivity. Given some $A \subseteq M$, $\underline{A}''$ represents attributes of the minimal objects satisfying A. If $\underline{A}'' \neq \underline{(\underline{A}'')'}'$, then we would have that there were some new minimal objects in $\underline{(\underline{A}'')'}$, these objects would all have A in their intents by extensivity. But this is a contradiction, since $\underline{A}''$ is defined to be the minimal objects with A in their intent, there can be no objects minimal $\underline{A}'$ which also have A in their intents.

□

2 Defeasible Implications

2.1 Normal implications

Conventionally, we define an implication $A \rightarrow B$ between two sets of attributes $A, B \subseteq M$. An implication is respected [1] in a formal context iff

- $B \subseteq A''$
- $A' \subseteq B'$

An example from Figure 1 would be that $\texttt{canFly} \rightarrow \texttt{feathered}$. The objects which satisfy the attribute `canFly`, namely $\{o_2, o_4\}$, are a subset of those which satisfy `feathered`, $\{o_2, o_4, o_5\}$. We do not have that $\texttt{feathered} \rightarrow \texttt{canFly}$ - this is obvious, and can be shown by $\texttt{canFly} \not\subseteq \texttt{feathered}''$

2.2 Borrowing the $\mid\!\sim$

In $\widetilde{\mathbb{K}} = (G, M, I, <)$, we define a defeasible implication $A \mid\!\sim B$, where $A, B \subseteq M$ as being respected in the $\widetilde{\mathbb{K}}$ iff

- $B \subseteq \underline{A}''$
- $\underline{A}' \subseteq B'$

The motivation behind this style of implication is to convey a type of consequence that may not exist in the entire context, but holds for at least the preferred object(s).

2.2.1 Incomparible objects

As defined, the $A \mid\!\sim B$ relation relies on the notion of minimal objects derived through the minimisation operation $\underline{A}'$. This raises (at least) two important questions; how do we deal with incomparible objects? And, what happens if the set of minimal objects is empty?

The first question is straightforward, and is actually handled in the definition of the operator found in subsection 1.3. If there are incomparible objects in $\underline{A}'$, we take the intersection of all object intents, giving us

$$\underline{A}'' := \cap\{a' | a \in \underline{A}'\}$$

If the set of minimal objects $\underline{A}'$ is empty, we say that $\underline{A}'' = M$; this holds vacuously - there are no objects in $\underline{A}'$, so (glossing over exact details) every object has every attribute.

From Figure 1, $\texttt{feathered} \mid\!\sim \texttt{canFly}$ is a valid defeasible implication in $\widetilde{\mathbb{K}}$. The set of minimal objects which satisfy `feathered` also satisfy `canFly`. Or, $\{\texttt{canFly}\} \subseteq \underline{\{\texttt{feathered}\}}''$. The point here is that while it is not the case that all feathered objects can fly, we still want to be able to say that the minimal (read: typical) feathered objects do in fact fly. Or, that typical feathered objects can fly.

2.3 Do defeasible implications hold through negation of antecedent?

Given the defeasible context $\widetilde{\mathbb{K}}$, with $o_1 < o_2$

	wheels	engine	drive
o_1		×	×
o_2	×	×	

Figure 2: Defeasible Formal Context

If we refer again to Figure 2, we could see that the implication $\{\texttt{wheels, engine}\} \mid\!\sim \{\texttt{drive}\}$ is respected by o_1, and $o_1 < o_2$, but $\widetilde{\mathbb{K}}$ does not respect $\{\texttt{wheels, engine}\} \not\mid\!\sim \{\texttt{drive}\}$. This might seem unintuitive, *the implication holds in our most preferred objects, after all.*

[1] being *respected* is analagous to being entailed

While this discomfort makes sense from the lens of classical logics, where we understand the material implication $\alpha \to \beta$ to be equivalent to $\neg\alpha \vee \beta$, when we use $\mid\!\sim$ we are shifting into the realm of conditional logics. Now, a conditional ($\mid\!\sim$) is assertable if its antecedent being true entails its consequent [1].

This is illustrated in [2], 4.2 Preferential Entailment, where preferential consequence relations are introduced:

> A preferential interpretation $\mathcal{P}$ is a tripple $\langle S, l, \prec \rangle$ with S being a set of states, l:S$\mapsto \mathcal{U}$ is a function mapping states to valuations, and $\prec$ a preorder on S. For every $\alpha \in \mathcal{L}$ and some $\mathcal{P}$, let $[\![\alpha]\!]^{\mathcal{P}} := \{s \in S, S \in \mathcal{P}, l(s) \Vdash \alpha\}$. In other words, let $[\![\alpha]\!]^{\mathcal{P}}$ be the set of all states in $\mathcal{P}$ such that the valuation associated with each state satisfies α.

And also,

> Given a preferential interpretation $\mathcal{P} = \langle S, l, \prec \rangle$ and $\alpha, \beta \in \mathcal{L}$, then $\mathcal{P}$ defines a preferential consequence relation $\mid\!\sim_{\mathcal{P}}$ such that $\alpha \mid\!\sim_{\mathcal{P}} \beta$ iff for any s minimal in $[\![\alpha]\!]^{\mathcal{P}}, s \Vdash \beta$.

3 Properties of Implication using minimisation operator

3.1 KLM Postulates

These are some of the KLM postulates, described in [2]. Reflexivity is another postulate, which has been left our for now for brevity. The *Or* postulate was also omitted, since there is no notion of disjunction in FCA.

$$A \mid\!\sim A \qquad \textbf{(Reflexivity)} \tag{1}$$

Proof: All we require is that $\underline{A}' \subseteq \underline{A}'$, obviously we have this. $\square$

$$\frac{A \leftrightarrow B, A \mid\!\sim C}{B \mid\!\sim C} \qquad \textbf{(Left-Logical Equivalence)} \tag{2}$$

Proof: Let $A, B, C \subseteq M$.

1. $\underline{A}' \subseteq C'$ (Given: $A \mid\!\sim C$)
2. $\underline{A}' = \underline{B}'$ (Given: $A \leftrightarrow B$)
3. $\underline{B}' \subseteq C'$ (From 1 and 2)
4. Therefore, $B \mid\!\sim C$ (By definition of $\mid\!\sim$)

$\square$

$$\frac{A \to B, C \mid\!\sim A}{C \mid\!\sim B} \qquad \textbf{(Right Weakening)} \tag{3}$$

Proof: Let $A, B, C \subseteq M$.

1. $A' \subseteq B'$ (Given: $A \to B$)
2. $\underline{C}' \subseteq A'$ (Given: $C \mid\!\sim A$)
3. $\underline{C}' \subseteq B'$ (Transitivity of $\subseteq$, from 1 and 2)
4. Therefore, $C \mid\!\sim B$ (By definition of $\mid\!\sim$)

$\square$

$$\frac{A \cup B \mid\!\sim C, A \mid\!\sim B}{A \mid\!\sim C} \qquad \textbf{(Cut)} \tag{4}$$

Lemma 3.1 *For any* $A, B \subseteq M$, $A' \cap B' = (A \cup B)'$.

Proof: We show $\subseteq$ in both directions.

$A' \cap B' \subseteq (A \cup B)'$:

Let $g \in A' \cap B'$

1. $\forall m \in A, (g, m) \in I$ (Definition of A')
2. $\forall m \in B, (g, m) \in I$ (Definition of B')
3. So, $\forall m \in A \cup B, (g, m) \in I$ (1,2)
4. $g \in (A \cup B)'$ (3 is definition of $(A \cup B)'$)

$(A \cup B)' \subseteq A' \cap B'$:

Let $g \in (A \cup B)'$

1. $\forall m \in A \cup B, (g, m) \in I$. (Definition of $(A \cup B)'$)
2. $\forall m \in A, (g, m) \in I$ (1, $A \subseteq A \cup B$)
3. $\forall m \in B, (g, m) \in I$ (1, $B \subseteq A \cup B$)
4. So, $g \in A' \cap B'$ (2,3)

□

$$\frac{A \mid\!\sim B, A \mid\!\sim C}{A \mid\!\sim (B \cup C)} \qquad \textbf{(And)} \tag{5}$$

Proof: Let $A \mid\!\sim B$ and $A \mid\!\sim C$ with $A, B, C \subseteq M$

1. $\underline{A}' \subseteq B'$ (Given: $A \mid\!\sim B$)
2. $\underline{A}' \subseteq C'$ (Given: $A \mid\!\sim C$)
3. $\underline{A}' \subseteq B' \cap C'$ (1,2)
4. $\underline{A}' \subseteq (B \cup C)'$ (3, Lemma 3.3)
5. $A \mid\!\sim B \cup C$ (From 4)

□

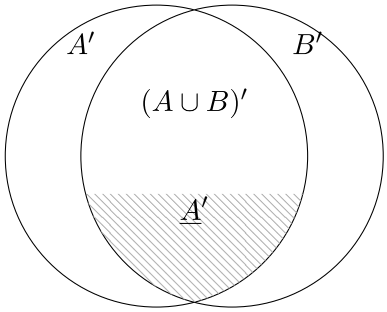
Lemma 3.2 *If* $A \mid\!\sim B$*, then* $\underline{A}' = \underline{(A \cup B)}'$

Proof:

1. $\underline{A}' \subseteq A'$ (Definition of minimisation)
2. $\underline{A}' \subseteq B'$ (Given that $A \mid\!\sim B$)
3. $\underline{A}' \subseteq A' \cap B'$ (1,2)
4. $\underline{A}' \subseteq (A \cup B)'$ (3, Lemma 3.3)
5. $(A \cup B)' \subseteq A'$ (Set Theory)
6. $\underline{A}' \subseteq (A \cup B)' \subseteq A'$ (3,4)

With this in mind, let g be some element of $\underline{A}'$. By definition, there is no element h in A' such that $h < g$. Then, because of 5, there is no element h in $(A \cup B)'$ such that $h < g$. From this, it follows that $\underline{A}' \subseteq \underline{(A \cup B)'}$.

What is left is to show that $\underline{(A \cup B)'} \subseteq \underline{A}'$. Let $g \in \underline{(A \cup B)'}$ and $g \notin \underline{A}'$. Since $\underline{A}' \subseteq \underline{(A \cup B)'}$, $\nexists h \in \underline{A}'$ such that $h < g$.



$\square$

$$\frac{A \mid\!\sim B, A \mid\!\sim C}{(A \cup B) \mid\!\sim C} \qquad \textbf{(Cautious Monotonicity)} \tag{6}$$

Proof: Let $A, B, C \subseteq M$.

1. $\underline{A}' \subseteq A'$ (Definition of minimisation operator)
2. $\underline{A}' \subseteq B'$ (Given: $A \mid\!\sim B$)
3. $\underline{A}' \subseteq C'$ (Given: $A \mid\!\sim C$)
4. $\underline{A}' \subseteq (A \cup B)'$ (1,2, Lemma 3.3)
5. $\underline{A}' \subseteq (A \cup C)'$ (1,3, Lemma 3.3)
6. $\underline{A}' = \underline{(A \cup B)'}$ (4, Lemma 3.5)
7. $\underline{A}' = \underline{(A \cup C)'}$ (5, Lemma 3.5)
8. $\underline{(A \cup C)'} \subseteq (A \cup C)'$ (Definition of minimisation)
9. $\underline{(A \cup B)'} \subseteq (A \cup C)'$ (6,7,8)
10. $(A \cup B) \mid\!\sim C$ (9 is definition of this implication)

$\square$

Remark 3.3 *Cautious monotonicity stipulates that when given new information, if that information could have been inferred before, then it should not be inconsistent with earlier inferences. Monotonicity requires that any new knowledge not invalidate existing knowledge; CM weakens this requirement to be that only pieces of knowledge inferred from the same existing knowledge should not invalidate one another.*

$$\frac{A \mid\!\sim B, A \not\mid\!\sim \neg C}{A \cup C \mid\!\sim B} \qquad \textbf{(Rational Monotonicity)} \tag{7}$$

4 Typical Concepts

The partial order over objects in a formal context was introduced as some kind of preference, or typicality between objects. It is quite intuitive that if there are typical objects, then there may be typical concepts too.

	m1	m2	m3
o_1	×		×
o_2		×	×
o_3	×	×	×
o_4		×	

Figure 3: Defeasible Formal Context, with the order $o_1 < o_2, o_3, o_4$

4.1 As sets of objects which agree with their minimals

An initial attempt to formalise the notion of a typical concept defines it as a pair of an extent and intent. Specifically, $((\underline{B}')'', (\underline{B}')')$, where $B \subseteq M$. [2] This characterisation of a *typical* is familiar, and has some desirable properties. It looks familiar because it is in fact a regular formal concept. If we have a set of objects from a formal context, $A \subset G$, then (A'', A') is a formal concept of this context. Since $\underline{B}', B \subseteq M$ can be thought of as simply a set of objects in G, the typical concept defined above is simply a regular concept derived from a starting point of minimal objects.

However, the intention behind introducing a notion of a typical concept would be to see what type of structure the set of typical concepts has for some defeasible context. That is, for some $\widetilde{\mathbb{K}} = (G, M, I, \preceq)$ the set of all regular concepts $\mathcal{B}(\widetilde{\mathbb{K}})$ is isomorphic to some concept lattice. The point of interest is, what structure does the set of typical concepts $\mathcal{T}(\widetilde{\mathbb{K}})$ have?

It turns out, that the set of typical concepts derived from the formal context $\widetilde{\mathbb{K}} = (G, M, I, \preceq)$ from Figure 4 is not a lattice.

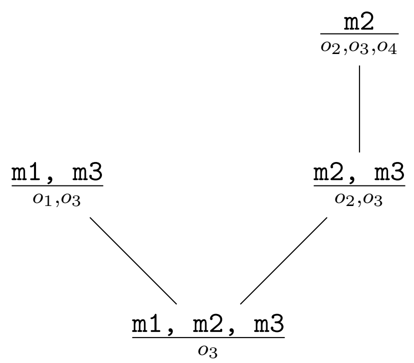


Figure 4: Hasse Diagram for $\mathcal{T}(G, M, I, \preceq)$

4.2 Restriction on the order

For a defeasible context $\widetilde{\mathbb{K}} = (G, M, I, \preceq)$, the partial ordering is restricted to only those orders such that for any $B \subseteq M$, $\underline{B}' \equiv (\underline{B}')''$ we avoid the issues in Figure 4. We also get an improved notation for a typical concept being, for some $B \subseteq M$, $(\underline{B}', (\underline{B}')')$.

4.3 Mapping from formal concepts to typical concepts

References

[1] P. Egré and H. Rott, "The Logic of Conditionals," in *The Stanford Encyclopedia of Philosophy*, Winter 2021 ed., E. N. Zalta, Ed. Metaphysics Research Lab, Stanford University, 2021.

[2] A. Kaliski, "An overview of klm-style defeasible entailment," 2020.

[2] This notation is unintuitive, but it will be done away with in the next section