Lifting Formal Concept Analysis to System-Z and Beyond

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1.2.1 Pearl

The following may, at times, be entirely plagiarised from [1]. It is for my own understanding.

Consider a set of rules $R = \{r : A_r \to B_r\}$ where A_r and B_r are sets of attributes and $\to$ is the normal attribute implication. In a classical sense, this implication is respected by another set of attributes, Y, in case $A \not\subseteq Y$ or $B \subseteq Y$. A stronger notion is that Y *verifies* $A \to B$ when $A \cup B \subseteq Y$. This is enforcing an intuitive understanding of conditionals, where the antecedent *must* be true - we avoid vacuous truths of implications. [1] Conversely, Y is said to falsify $A \to B$ when $A \subseteq Y$ and $B \not\subseteq Y$.

A new notion of *toleration* is introduced in the form of a *toleration relation*:

Definition 1: A set of rules $R' \subseteq R$ *tolerates* an individual rule r, denoted $T(r \mid R')$, if

$$\bigcup_{r' \in R'} (A'_r \cup B'_r) \cup \{A_r \cup B_r\}$$

is satisfiable.

What it means for an individual rule, r, to be tolerated by a set of rules R' is that there should be a model of R' which verifies r and does not falsify any $r' \in R'$. Shifting into the world of formal concept analysis: an implication, i, is tolerated by a set of implications I if there is an object g such that g' respects I and g' verifies i.

The next notion to be introduced is *consistency*,

Definition 2: A set R of rules is *consistent* if in every non-empty subset $R' \subseteq R$ there exists an r' such that R' tolerates r'.

$$\forall\, R' \subseteq R,\ \exists\, r' \in R',\ \text{such that}\ \ T(r' \mid R' - r') \tag{1}$$

Consistency is stronger than satisfiability: $\alpha \to \beta$ and $\alpha \to \neg\beta$ is satisfiable by $\neg\alpha$, although it is not consistent. Any ω that verifies $\alpha \to \neg\beta$ necessarily falsifies $\alpha \to \beta$ and vice versa. Implicit in the notion of consistency is that implications which are only ever true through negation of the antecedent do not align with our understanding of conditonals. [1]

Consistency gives rise to a natural ordering of the rules in R. Given a consistent R, identify every rule that is tolerated by R, and assign this rule a rank of 0.

Algorithm 1 Z-ordering

Input: A consistent set of rules R
Input: A tolerance relation T over R
Output: A tolerance partition $R_Z = (R_0, R_1, \ldots, R_k)$

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i := 0;
while R ≠ ∅ do
    R_i := {r ∈ R | (r | R) ∈ T};
    R = R \ R_i;
    i := i + 1;
return (R_0, R_1, ..., R_i)
```

References

[1] Judea Pearl. System z: a natural ordering of defaults with tractable applications to nonmonotonic reasoning. In *Proceedings of the 3rd Conference on Theoretical Aspects of Reasoning about Knowledge*, TARK '90, page 121–135, San Francisco, CA, USA, 1990. Morgan Kaufmann Publishers Inc.