Developing Non-monotonicity in Formal Concept Analysis

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1 INTRODUCTION

Formal concept analysis (FCA) provides a framework, grounded in lattice theory, for mathematically reasoning about *formal concepts* and their hierarchies [3, 4, 9]. The matter of *concepts* has largely been of Philosophical concern: the notion of a concept as the dualism between *intension* and *extension* has foundations in Aristotle's *Organon* and, much later on, in the *Logic of Port-Royal* [1, 9]. In this view, the extension of a concept contains to those "things" that one might refer to as instances of the concept. Dually, intension describes the meaning, or *sense*, of a concept.

Formal concept analysis—which adopts this view of concepts—introduces a *formal context* as the structure of data. This is a triple consisting of a finite set of objects G, attributes M, and a binary relation, $I \subseteq G \times M$, which indicates that a particular object has a respective property [3, 4]. A *formal concept* is a pair, made up of the formal concept extension and intension, respectively. The set of all concepts, when ordered by the *sub/super-concept* relation, form the *concept lattice* used for analysis.

Another important topic in FCA is the discovery of *implications* that pertain to, or are *respected* by, a context [3, 9]. Implications are used to express correspondence between (sets) of attributes. The notion of a context *respecting* an attribute implication is analogous to that of *entailment* in classical logic [4]. As such, *respecting* describes a monotonic notion of consequence.

Discussion and work on non-monotonic propositional, first-order, and description logic is a well established topic in artificial intelligence, [2, 5, 7, 8, 10]. However, there does not appear to be any effort to introduce this expressivity to the attribute logic of FCA.

2 BACKGROUND

The proceeding subsections provide an introduction to aspects of FCA, classical logic, and the KLM-framework for non-mononotonic reasoning which are particularly relevant to the scope of this research.

2.1 Formal Concept Analysis

The starting point of FCA is the *formal context*. This is a triple consisting of a finite set of objects, a finite set of attributes, and a relation which describes when an object g 'has' an attribute m.

Definition 2.1. A *formal context* is a triple, $\mathbb{K} = (G, M, I)$, where G is a finite set of objects, M is a finite set of attributes, and $I \subseteq G \times M$ is an incidence relation on the Cartesian product of G and M.

For contexts of reasonable size, a cross-table is frequently used as a visual aid. Each row represents an object, and each column an attribute. A '×' is used to indicate where an object has an attribute [3, 4]. See **??** for an example. For any subset of objects (attributes) we use a derivation operator to describe the attributes (objects) common to all members of that subset. These derivation operators form a antitone Galois connection between the power sets $\mathcal{P}(G)$ and $\mathcal{P}(M)$ [3].

Definition 2.2. In a context, $\mathbb{K} = (G, M, I)$, we define two *derivation operators*, both denoted by $(\cdot)'$, as follows:

$$A \subseteq G: \quad A' := \{m \in M \mid \forall g \in A : (g, m) \in I\}$$
$$B \subseteq M: \quad B' := \{g \in G \mid \forall m \in B : (g, m) \in I\}$$

In case of a singleton set, $\{g\}$, the braces are omitted. If g is an object, then g' describes the *object intent*; if g is an attribute, g' is the *attribute extent*. A derivation operator can be applied to the result of an earlier derivation operator, this is then referred to as a *double-prime operator*, and expressed as $(\cdot)''$. Any set derived from a double-prime operator is closed, and so each derivation operator describes a closure system on G and M, respectively [3, 4, 9]. As such, the double-prime operator $(\cdot)''$ satisfies:

$$\begin{array}{ll} \text{Monotonocity:} & \text{if } A_1 \subseteq A_2 \text{ then } A_1'' \subseteq A_2'' \\ \text{Idempotency:} & A'' = (A'')'' \\ \text{Extensivity:} & A \subseteq A'' \end{array}$$

Definition 2.3. A *formal concept* of a context, $\mathbb{K} = (G, M, I)$, is a pair (A, B) where $A \subseteq G$ and $B \subseteq M$ for which $A' = B$ and $B' = A$. Then A is the *concept extent* and B is the *concept intent*. The set of all concepts of a context is given by $\mathfrak{B}(G, M, I)$.

There is obvious redundancy in this definition of concepts: if (A, B) is a formal concept, it could equivalently be given by (A, A') or (B', B). Moreover, for an arbitrary set A of objects (*resp.* attributes), A' defines a concept intent (*resp.* extent), and A'' defines a concept extent (*resp.* intent) [4].

Definition 2.4. Let (G, M, I) be a formal context, then for every object, $g \in G$, we define a *object concept* as

$$\gamma g := (g'',\ g')$$

and for each attribute, $m \in M$, we define the *attribute concept* as

$$\mu m := (m',\ m'')$$

The idea of sub and super-concepts gives rise to a natural partial order on $\mathfrak{B}(G, M, I)$. Specifically, if (A_1, B_1) and (A_2, B_2) are concepts in $\mathfrak{B}(G, M, I)$, then (A_1, B_1) is a *sub-concept* of (A_2, B_2) iff $A_1 \subseteq A_2$ (equivalently, $B_2 \subseteq B_1$). Respectively, (A_2, B_2) would be a *super-concept* of (A_1, B_1) and we say that $(A_1, B_1) \leq (A_2, B_2)$ [3]. Then, $\mathfrak{B}(G, M, I)$ and the relation $\leq$ form a complete lattice called the *concept lattice*. [link to concept lattice in appendix]

Previously, it was mentioned that contexts of a reasonable size have a tidy representation in the form of a cross-table. In case of larger contexts, the concept lattice can instead be inferred from *attribute implications* [3, 4].

Definition 2.5. Let M be an arbitrary attribute-set. An *implication* over M takes the form $B_1 \rightarrow B_2$ where $B_1, B_2 \subseteq M$. Another set $C \subseteq M$, *respects* the implication iff $B_1 \nsubseteq C$ or $B_2 \subseteq C$. Then, $C \vDash B_1 \rightarrow B_2$. C *respects a set* of $\mathcal{L}$ implications if it respects every implication in $\mathcal{L}$.

Respecting an implication can be generalised to the notion of *validity* w.r.t a formal context.

Definition 2.6. For a formal context $\mathbb{K} = (G, M, I)$, and an implication $B_1 \rightarrow B_2$ over M, the implication is *valid* in the formal context ($\mathbb{K} \models B_1 \rightarrow B_2$) iff for every object $g \in G$, it is the case that the object intent g' respects the implication. Then, the following are equivalent:

$$\mathbb{K} \models B_1 \rightarrow B_2$$
$$B_1' \subseteq B_2'$$
$$B_2 \subseteq B_1''$$

If $\mathcal{L}$ is the set of all implications valid in a formal context, $\mathbb{K} = (G, M, I)$, then the concept intents can be given by $\{B \subseteq M \mid B \models \mathcal{L}\}$.

2.2 KLM-Style Non-monotonic Reasoning

In [10, 11], Shoham developed a semantic framework for introducing non-monotonicity, which he called *preferential reasoning*. The style of reasoning is based on the idea that an ordering can be imposed on the valuations of a knoweldege base, expressesing that some valuations are preferred to others. In the classical setting, $\alpha \vDash \beta$ means that all models of α are also models of β. A corrolary of this is that $\alpha \wedge \gamma \vDash \beta$, which implies monotonicity. Under preferential reasoning, $\alpha \vDash_{\sqsubset} \beta$ does not enforce the preferred models of $\alpha \wedge \gamma$ to be models of β, thus constituting a non-monotonic consequence [10].

Independently, Kraus, Lehmann, and Magidor introduced systems of non-monotonic reasoning as consequence relations, these have since been unified under the *KLM framework*. The central contribution of [7] is *system P*: a preferential consequence relation who's semantics are a variation of Shoham's preferential reasoning. A preferential consequence relation satisfies certain properties, called the *KLM postulates.*[1] Of particular interest is *cautious monotonicity*, which stipulates that new knoweldege—which could already be inferred—cannot invalidate previous conclusions. Put plainly, only genuinely new information should cause retraction of existing information [6, 7].

Later, Lehmann and Magidor [8] introduced the stronger *rational consequence relation*, which satisfies all the properties of P, with the addition of *rational monotonicity*. Rational monotonicity says that new information which is consistent with existing knoweldege should not lead to the retraction of prior inferences. *Ranked interpretations* provide a semantic characterisation of a rational consequence relation.

Definition 2.7. A *ranked interpretation* is a function $\mathsf{R} : \mathcal{U} \mapsto \mathcal{N} \cup \infty$ s.t. for every $n \in \mathcal{N}$ there exists a $u \in \mathcal{U}$ such that $\mathsf{R}(u) = n$. Then, there exists $u' \in \mathcal{U}$ s.t. $\mathsf{R}(u') = n'$ with $0 \leq n' < n$

[1]Further discussion of these postulates can be found in [7]

Following the intuition of preferential reasoning, for two valuations, v, v', v is regarded as being representative of a more typical (i.e., preferable, less exceptional) state of affairs if $\mathsf{R}(v) < \mathsf{R}(v')$ [8]. A ranked interpretation, R, defines a rational consequence relation $\mid\!\sim_{\mathsf{R}}$. For $\alpha \mid\!\sim_{\mathsf{R}} \beta$, we define $[\![\alpha]\!]^{\mathsf{R}} := \{u \in \mathcal{U} \mid u \Vdash \alpha\}$. Then, the conditional holds iff for all minimal valuations $v \in [\![\alpha]\!]^{\mathsf{R}}$, $v \Vdash \beta$.

What remains is to describe a process of non-monotonic entailment which is consistent with pattern of reasoning described by rational consequence relations.

3 RESEARCH QUESTIONS & OBJECTIVES

The aim of this work is to introduce KLM style non-monotonicity to the attribute logic underpinning FCA. Doing so creates two principle areas of interest. The first, and most obvious, concerns the notion of a *non-monotonic implication* in FCA. Secondly, a corollary of introducing a ranking to a context is that it a provides expressivity to develop *typical concepts*.

Concerning non-monotonic implications, we begin our work by finding a translation from the weaker *system P* to FCA. This system relies on being able to construct an ordering over the valuations of a knowledge base. We find that assuming a partial ordering over the set of objects in a formal context—implicitly providing a way to compare object intents—introduces a suitable structure for a semantic definition of preferential implications. Definition 2.6 can be altered to define a defeasible implication which is valid in a context iff the minimal objects in A' are a subset of B'.

4 MOTIVATION OF RESEARCH AREA

5 PROJECT PLAN

The period of October 2024 to January 2025 will be spent continuing the theoretical research with the aim of having results which collectively present a coherent, unified idea. More specifically, we aim to have looked into

(1) An algorithmic description of how to obtain a ranking over objects in a formal context
(2) An algorithmic definition for rational entailment in FCA
(3) A notion of *typical concepts* which includes a proof for lattice-like structure
(4) A notion of a *defeasible canonical basis*
(5) An algorithmic approach to determining typical concepts, analagous to *concept exploration*

5.1 Risks

5.2 Milestones, Deliverables, and Timeline

5.3 Gantt Chart

	2024	April	May	June	July	August	September	Oct	Nov	Dec
1	Reading									
2	**Learning**									
2.1	Learning FCA									
2.2	Learning KLM									
2.3	Learning other things									
3	**Development**									
3.1	Developing questions to answer									
3.2	Investigating questions									
3.3	**Writing**									
3.3.1	FCA Background									
3.3.2	NMR Background									
3.3.3	Defeasible FCA Implications									
3.3.4	Typical Concepts									
3.3.5	Attempt publish second paper									
4	**Events**									
4.1	Dresden									
4.2	SACAIR									
4.2.1	Paper Writing									
4.2.2	Paper Edits									
4.2.3	Preparing Talk									
4.2.4	Conferences									
4.3	Cape KR									
4.3.1	Preparing Talk									
4.3.2	Conference									

	2025	Jan	Feb	March	April	May	June	July	August	Sept	Oct	Nov
1	Reading											
2	Learning											
2.1	Learning FCA											
2.2	Learning KLM											
2.3	Learning other things											
3	Development											
3.1	Developing questions to answer											
3.2	Investigating questions											
3.3	Write Ups											
3.3.1	FCA Background											
3.3.2	NMR Background											
3.3.3	Defeasible FCA Implications											
3.3.4	Typical Concepts											
3.3.5	Attempt publish second paper											
4	**Events**											
4.1	Possible Semester in Paris											

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