Rationalr Concept Analysis

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Acknowledgements

Do not do this before you have finished

Abstract

Also, don't do this.

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Chapter 1

Mathematical Preliminaries

The ultimate interest of this dissertation is to introduce a style of non-monotonic reasoning, originally situated in propositional logic, into Formal Concept Analysis. We use the first chapter to recall some of the more foundational ideas relating to topics which will be developed further on. In addition, we use this chapter as an opportunity to clarify the notation which will be deployed throughout the remainder of this work.

This chapter begins by discussing notions in *order* and *lattice theory*, then moving onto *closures* and *Galois connections*, and concluding with a brief introduction to *propositional logic* and some more general notions relating to logic.

1.1 Order and Lattice Theory

In substantially different ways, the two areas on which this work is based—non-monotonic reasoning via preferential semantics, and Formal Concept Analysis—both depend on notions from the theory of mathematical orders. We recount some of the necessary, and unnecessary but interesting, ideas in the following subsections. The proceeding follow largely from the treatments by [Davey and Priestley, 2002] and [Bergman, 2015].

1.1.1 Orders

A *binary relation* R over the sets X and Y is a set of ordered pairs (x, y) where $x \in X$ and $y \in Y$. We may choose to express that $(x, y) \in R$ using infix notation and write xRy, which tells us that R relates x to y.

Certain binary relations, satisfying specific properties, occur frequently enough to warrant their own denomination. One such relation, which will be used in almost every section of this dissertation, is called a *partial order*.

Definition 1. A *partial order* is a binary relation $\preceq \subseteq X \times X$ that satisfies the following properties:

$$\text{(Reflexivity)} \quad x \preceq x \tag{1.1}$$

$$\text{(Antisymmetry)} \quad x \preceq y \text{ and } y \preceq x \quad \text{implies} \quad x = y \tag{1.2}$$

$$\text{(Transitivity)} \quad x \preceq y \text{ and } y \preceq z \quad \text{implies} \quad x \preceq z \tag{1.3}$$

for all $x, y, z \in X$.

Frequently, "preference" is used as a metonym for an order. In this context, writing "element x is preferred to y" should be interpreted to mean that $(x, y) \in \preceq$, or simply $x \preceq y$. The metaphorical use of "preference" will become more apparent in Chapter 3.

We write $x \npreceq y$ to indicate that (x, y) is not in the relation, and $x \prec y$ for the case where $x \preceq y$ and $x \neq y$. In the scenario where $x \npreceq y$ and $y \npreceq x$—i.e., that x and y are incomparable—we may write $x \parallel y$. From a partial order we can quite easily induce the notion of a *strict partial order*.

Definition 2. A *strict partial-order* is a binary relation $\prec \subseteq X \times X$ that satisfies:

$$\text{(Irreflexivity)} \quad x \nprec x \tag{1.4}$$

$$\text{(Asymmetry)} \quad x \prec y \quad \text{implies} \quad y \nprec x \tag{1.5}$$

$$\text{(Transitivity)} \quad x \prec y \text{ and } y \prec z \quad \text{implies} \quad x \prec z \tag{1.6}$$

for all $x, y, z \in X$.

An *ordered set* is a pair $(X, \preceq)$ with X being a set and $\preceq$ being an ordering on the elements of X. We make notation easier, and use $\mathbf{X}$ to denote the pair. We may reference the order relation associated with $\mathbf{X}$ by writing $\preceq_X$ in settings where there is ambiguity. If $\mathbf{Y}$ is a subset of $\mathbf{X}$, then $\mathbf{Y}$ inherits the order relation from $\mathbf{X}$; and so, for $x, y \in \mathbf{Y}$, $x \preceq_Y y$ if and only if $x \preceq_X y$. Ordered sets may be described visually by *Hasse diagrams* [Huth and Ryan, 2004].

(a) **A** (b) **B**

Figure 1.1: The Hasse diagrams of two partially ordered sets

As an illustrative example, from the ordered set in Subfigure 1.1a we read that $z \preceq x$, as there is a (strictly) upward path from z to x. In fact, it is clear that $z \preceq w, x, y, z$, or that "z is preferred to every other element in $\mathbf{A}$". We say such an element is *minimal*.

More formally, an element $x \in \mathbf{X}$ is *minimal* with respect to the ordering on its' container set if there exists no distinct element $y \in \mathbf{X}$ such that $y \preceq x$. Conversely, we say x is *maximal* if there exists no distinct $y \in \mathbf{X}$ where $x \preceq y$. Then x is the *minimum* element if $x \preceq y$ for all $y \in \mathbf{X}$; the dual notion of a *maximum* is defined as we might expect. In the ordered set $\mathbf{B}$ shown in Subfigure 1.1b, ω is the maximum element and the order has no minimum.

The following definition captures how we might describe certain mappings from one ordered set to another, and how these mappings relate to the orders of their respective sets.

Definition 3. Let $\mathbf{X}$ and $\mathbf{Y}$ be ordered sets with a mapping $\varphi : \mathbf{X} \to \mathbf{Y}$. We call φ an *order-preserving* (or, *isotone*) map if $x \preceq_X y$ implies $\varphi(x) \preceq_Y \varphi(y)$. It is an *order-embedding* if it is injective, and $x \preceq_X y$ if and only if $\varphi(x) \preceq_Y \varphi(y)$ for all $x, y \in \mathbf{X}$. Finally, φ is an *order-isomorphism*, or just *isomorphism*, if it is an order-embedding that is also surjective. The dual notion to an order-preserving map is an *order-reversing* (or, *antitone*) map. We say that two orders are *dually isomorphic*, or *order-anti-isomorphic*, if there is an order-reversing bijection between them.

1.1.2 Lattice Theory

Lattice theory studies partially ordered sets that behave well with respect to certain properties involving upper and lower bounds [Davey and Priestley, 2002]. Given an ordered set $\mathbf{X}$ and a subset $\mathbf{Y} \subseteq \mathbf{X}$, the set of upper bounds of $\mathbf{Y}$ is defined as

$$\mathbf{Y}^u := \{x \in \mathbf{X} \mid \forall y \in \mathbf{Y} : y \preceq x\},$$

and the set of lower bounds, $\mathbf{Y}^l$, is defined dually. If $\mathbf{Y}^u$ has a minimum element, then we call that element the *supremum* of $\mathbf{Y}$. Dually, if $\mathbf{Y}^l$ has a maximum element, then we call that element the *infimum* of $\mathbf{Y}$ (the supremum and infinum are also referred to as the *least upper bound* and *greatest lower bound*, respectively).

Instead of talking about the supremum of two elements $x, y \in \mathbf{X}$, we opt for the term *join* and write $x \vee y$, or $\bigvee \mathbf{Y}$. Instead of infimum, we say *meet* and write $x \wedge y$, or $\bigwedge \mathbf{Y}$. With these definitions of meets and joins in mind, we are able to define a lattice as:

Definition 4. Given a partially ordered set $\mathbf{L}$, we say that $\mathbf{L}$ is a *lattice* if for every pair $x, y \in \mathbf{L}$ the join $x \vee y$ and meet $x \wedge y$ exist, and are unique. Then, we call $\mathbf{L}$ a *complete lattice* if for every subset $\mathbf{M} \subseteq \mathbf{L}$ both $\bigvee \mathbf{M}$ and $\bigwedge \mathbf{M}$ exist in $\mathbf{L}$.

A lattice $\mathbf{L}$ is *bounded* if there exists an element $x \in \mathbf{L}$ such that $x \vee y = x$ for all $y \in \mathbf{L}$, and there exists an element $z \in \mathbf{L}$ with $z \wedge y = z$ for all $y \in \mathbf{L}$. Frequently we refer to such an x and y as a top and bottom element and denote them by $\top$ and $\bot$, respectively. It is not difficult to spot that every complete lattice is bounded—in fact this is true by definition of a complete lattice—as a corollary every finite lattice is also bounded.

Example 1. *The set of natural numbers $\mathbb{N}$ forms a complete lattice when ordered by divisibility, sometimes called the division lattice. In the division lattice, the join operation corresponds to the least common multiple, and the meet operation to the greatest common divisor. The bottom element of this lattice is* 1 *as it divides every other natural number, while the top element is* 0 *since it is divisible by all other naturals. Although the division lattice is infinite, it is indeed an example of a complete lattice.*

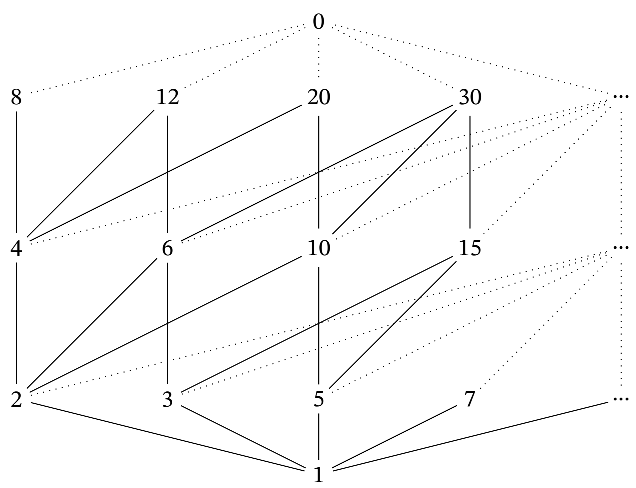

Figure 1.2: The Division Lattice $(\mathbb{N}, \preceq)$

1.1.2.1 Lattices as algebraic structures

Lattices may also be considered from an algebraic perspective. Although, as we will soon show, the algebraic and order-theroetic perspectives coincide, it is frequently beneficial for one's intuition to be able to switch between these two perspectives.

Consider a set L equipped with a binary operation $\vee$ that satisfies the following properties

$$\text{(Idempotence)} \quad x \vee x = x \tag{1.7}$$
$$\text{(Commutativity)} \quad x \vee y = y \vee x \tag{1.8}$$
$$\text{(Associativity)} \quad (x \vee y) \vee z = x \vee (y \vee z) \tag{1.9}$$

for all elements $x, y, z \in L$.

From the algebraic structure $(L, \vee)$ one can induce a *unique* partial order $\preceq$ on L by construction of the relation $\{(x, x \vee y) \mid x, y \in L\}$ [Bergman, 2015]. The structure $(L, \vee)$ is called an algebraic *join semilattice*, and $\preceq$ its *underlying partial order*. The relation can equivalently be described as "$x \preceq y$ if and only if $x \vee y = y$".

Example 2. *Consider the set $S = \{1, 2, 3\}$ and the union operation $\cup$. From $(\mathfrak{P}(S), \cup)$ we can induce the order relation characterised by the set $\{(X, X \cup Y) \mid X, Y \subseteq S\}$ (frequently, this kind of ordering is called the set inclusion order). From the Hasse diagram in Figure 1.3 representing this order, we observe that the $\cup$ binary relation corresponds to the $\vee$ described in Definition 4.*

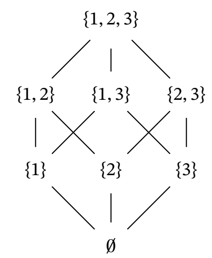

Figure 1.3: The underlying order of $(\mathfrak{P}(S), \cup)$
We may call the above lattice the *powerset lattice* or *boolean algebra* of S.

If we instead consider the structure $(L, \wedge)$ where $\wedge$ is a different binary operation on L satisfying the same properties Equation (1.7)–Equation (1.9), we can construct a partial order characterised by the set $\{(x, y) \mid x \wedge y = x \text{ and } x, y \in L\}$. The structure $(L, \wedge)$ is called a *meet semilattice*.

It is an obvious next step to wonder how these two algebraic structures might be related. Fix a set non-empty set L with the two binary operations, $\vee$ and $\wedge$, introduced in the prior paragraphs. We have already seen that the underlying order of the join semilattice can be described as $x \preceq y$ if and only if $x \vee y = y$, while the underlying order of the meet semilattice is described by $x \preceq y$ if and only if $x \wedge y = x$. These two partial orders are *compatible* under the following condition:

$$\text{(Compatibility)} \quad x \vee (x \wedge y) = x \qquad\qquad \text{(resp.)} \quad x \wedge (x \vee y) = x \tag{1.10}$$

Compatibility (which is sometimes also called *absorption*) is satisfied when the underlying orders of both semilattices refer to the same partial order. The property can be rewritten as $x \vee y = y$ if and only if $x \wedge y = x$. The formal definition of an algebraic lattice is given below:

Definition 5. The algebraic structure $(L, \vee, \wedge)$, where L is a set and $\vee, \wedge$ are two binary operations on L, is a *lattice* if both binary operations satisfy *idempotence, commutativity,* and *associativity* as well well as *absorption*.

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Lemma 1 (The Connecting Lemma). *If* $\mathbf{L}$ *is a lattice with* $x, y \in \mathbf{L}$*, then the following are equivalent:*

1. $x \preceq y$;
2. $x \vee y = y$;
3. $x \wedge y = x$.

Moving forward, we will no longer maintain a rigid distinction between these two perspectives on lattices; and rather, will tacitly adopt whichever viewpoint best suits the surrounding context. Lattices are central to Formal Concept Analysis, and will be visited again in Chapter 2.

1.2 Closures & Galois Connections

1.2.1 Closure Systems

We frequently encounter collections of elements that satisfy certain properties of interest. A fundamental question that might arise is whether our collection includes all such elements satisfying the specified properties. We aliken this to the more formal notions of *closures* and *closed sets*.

Definition 6. A subset X of S is *closed* under an operation φ if and only if the application of φ to elements of X always results in an element of X.

This condition can be expressed equivalently by stating that X is closed under the operation φ when $\varphi[X] \subseteq X$, where $\varphi[X]$ denotes the image of X under φ. This formulation suggests that a natural approach to finding the closure of any subset: systematically add all elements that result from applying the operation to existing elements in the set, continuing this process until no new elements are generated.

In some cases, consider the singleton set $\{1\}$ as a subset of the natural numbers with the operation '+2', this process may fail to reach a fixed point. It is not that the there is no set closed under the operation '+2', but rather that it will not be found with this approach.

In light of this, it is useful to abstract the idea to that of a *closure operator* and *closure system*:

Definition 7. A *closure operator* on a set S is a function $\mathsf{cl} : \mathfrak{P}(S) \to \mathfrak{P}(S)$ assigning each subset of S to its closure; cl satisfies the following properties for all $X, Y \subseteq S$.

$$\text{(Monotony)} \quad X \subseteq Y \quad \text{implies} \quad \mathsf{cl}(X) \subseteq \mathsf{cl}(Y) \tag{1.11}$$

$$\text{(Extensivity)} \quad X \subseteq \mathsf{cl}(X) \tag{1.12}$$

$$\text{(Idempotency)} \quad \mathsf{cl}(X) = \mathsf{cl}\big(\mathsf{cl}(X)\big) \tag{1.13}$$

Definition 8. A *closure system* on a set S is a family of subsets $\mathcal{C} \subseteq \mathfrak{P}(S)$ containing the set S itself and the intersection of any subsets of $\mathcal{C}$, and so if $\mathcal{D} \subseteq \mathcal{C}$ then $\bigcap \mathcal{D} \in \mathcal{C}$.

From Example 2, we know that under the inclusion order, the powerset of any set forms a complete lattice. Then, from Definition 8 it follows that a closure system on S is a meet subsemilattice of $\mathfrak{P}(S)$.

Apart from their nomenclature, it is not immediately obvious how closure operators and closure systems are related, the following theorem shows that they are indeed *cryptomorphic* [Caspard and Monjardet, 2003].

Theorem 1. *If $\mathcal{C}$ is a closure system on S, then $\varphi : \mathfrak{P}(S) \to \mathfrak{P}(S)$ where*

$$\varphi(Y) = \bigcap\{D \in \mathcal{C} \mid Y \subseteq D\} \quad \textit{for } Y \in \mathfrak{P}(S)$$

for $Y \subseteq S$, defines a closure operator on S. Conversely, if cl is a closure operator on S, then the collection

$$\{\mathsf{cl}(X) \mid X \subseteq S\}$$

defines a closure system on S.

Example 3. *Consider the partially ordered set $S = \{a, b, c, d\}$ where the order is given by $a \preceq b \preceq d$ and $a \preceq c \preceq d$. Let $\mathcal{D}$ denote the collection of all downsets in S, where a downset is a subset $X \subseteq S$ such that if $x \in X$ and $y \preceq x$, then $y \in X$. Then we have*

$$\mathcal{D} = \big\{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, d\}, \{a, b, c, d\}\big\}$$

We can verify that $\mathcal{D}$ forms a closure system: it contains S and remains closed under arbitrary intersections. For instance, $\{a, b, d\} \cap \{a, c\} = \{a\}$, which itself is a member of $\mathcal{D}$. We can then induce the closure operator $\psi : \mathfrak{P}(S) \to \mathcal{D}$ as

$$\psi(X) = \bigcap\{Y \in \mathcal{D} \mid X \subseteq Y\}$$

Theorem 2. *Let cl be a closure operator on the set S. Then the closure system given by $\{X \subseteq S \mid X = \mathsf{cl}(X) = X\}$ forms a complete lattice under the inclusion order, where meets and joins are given by*

$$\bigwedge_{i \in I} X_i = \bigcap_{i \in I} X_i,$$

$$\bigvee_{i \in I} X_i = \mathsf{cl}\Big(\bigcup_{i \in I} X_i\Big).$$

1.2.2 Galois Connections

Galois connections can be viewed as a generalisation of the Fundamental Theorem of Galois Theory; see [Bergman, 2015]. They are a useful way of describing correspondence between two sets and with a relation. We will spend a bit of time describing Galois connections, as they are fundamental to FCA, and closely related to ideas around closures.

Definition 9. Let $\mathbf{X}, \mathbf{Y}$ be ordered sets, then a pair of maps $\varphi : \mathbf{X} \to \mathbf{Y}$ and $\psi : \mathbf{Y} \to \mathbf{X}$ constitute an *antitone Galois connection* between $\mathbf{X}$ and $\mathbf{Y}$ if and only if they satisfy

$$x \preceq_X x' \quad \text{implies} \quad \varphi(x') \preceq_Y \varphi(x) \tag{1.14}$$

$$y \preceq_Y y' \quad \text{implies} \quad \psi(y') \preceq_X \psi(y) \tag{1.15}$$

$$x \preceq_X \psi\big(\varphi(x)\big) \quad \text{and} \quad y \preceq_Y \varphi\big(\psi(y)\big) \tag{1.16}$$

for all $x \in \mathbf{X}$ and $y \in \mathbf{Y}$.

In future, we will drop the *antitone*, and just write *Galois connection*. We recount some well-known propositions which are helpful for one's intuition, in that they provide several different perspectives on Galois connections.

Proposition 1. *Let (φ, ψ) refer to the same pair of maps constituting a Galois connection, then*

$$x \preceq_X \psi(y) \quad \text{if and only if} \quad y \preceq_Y \varphi(x) \tag{1.17}$$

for all $x \in \mathbf{X}, y \in \mathbf{Y}$.

Proposition 2. *Let (φ, ψ) refer to the same pair of maps constituting a Galois connection, then*

$$\varphi(x) = \varphi\Big(\psi\big(\varphi(x)\big)\Big) \quad \text{and} \quad \psi(y) = \psi\Big(\varphi\big(\psi(y)\big)\Big) \tag{1.18}$$

for all $x \in \mathbf{X}, y \in \mathbf{Y}$.

Proposition 3. *Let (φ, ψ) refer to the same pair of maps constituting a Galois connection, then*

$$x \mapsto \psi\big(\varphi(x)\big) \quad \text{and} \quad y \mapsto \varphi\big(\psi(y)\big)$$

are monotone, extensive and idempotent and thus constitute closure operators on **X** *and* **Y**, *respectively.*

So, we have it that two mappings that constitute a Galois connection will further define two closure operators by taking their compositions. By Theorem 2 each of these closure operators will induce a closure system on its' respective domain, between which the mappings are dually isomorphic.

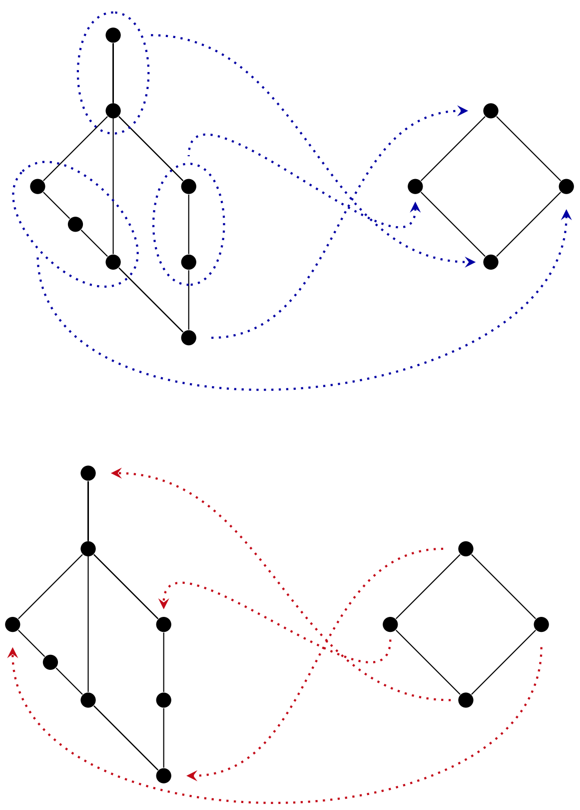


Figure 1.4: A Galois connection

That is, if we consider the map φ from the closure system on **X**, it is infact surjective (*onto*) the closure system induced for **Y**. Likewise, ψ is surjective to the closure system on **X**. Consider two elements x, x' of the closure system on $\mathbf{X}$ so that $\varphi(x) = \varphi(x')$. We may construct the equality $x = \psi(\varphi(x)) = \psi(\varphi(x')) = x'$ which shows that each map is additionally injective (*one-to-one*), and so also a bijection. That they are order-reversing can be seen from Proposition 1, and so we have a dual isomorphism.

Figure 1.4 demonstrates a rather contrived Galois connection between two partially ordered sets. Generally, Galois connections are interesting when the closure systems they induce are interesting in their own right. We will wait until Chapter 2 to see an example of this.

1.3 Propositional Logic

Propositional logic is a system for abstracting reasoning away from natural language. A *propositional statement* is a sentence like "Tralfamadorians have one eye". That is, something that can be assigned a value of *true* or *false* [Ben-Ari, 2012]. More complex propositions may be constructed recursively from simpler ones. The main interest is then to discover which true propositions follow—under some agreeable sense of what it means to *follow*—from others.

1.3.1 Syntax

Propositional atoms are the fundamental building blocks of a propositional language. They are indivisible statements that can either be true or false, and may be be combined with the boolean connectives $\{\neg, \vee, \wedge, \rightarrow\}$ to construct more complex statements, or *formulae*. We denote propositional atoms with lower-case Latin letters p, q, r, s, and t, and the set of all atoms will usually be denoted by $\mathcal{P}$, sometimes this is called the *universe of discourse*. Lower-case Greek letters $\alpha, \beta, \gamma, \phi, \psi$, and φ will be used to denote formulae, and corresponding upper-case Greek letters will usually represent sets of formulae.

Not all combinations of atoms and Boolean connectives result in meaningful expressions. For example, "$\wedge \rightarrow \phi$," which can be parsed as "Conjunction materially implies ϕ", has no discernible meaning. We distinguish between any formula and so-called *well-formed formulae*. The latter being constructions that are valid with respect to the rules of the grammar defined in Backus–Naur form below [Huth and Ryan, 2004].

$$\phi ::= p \mid (\neg\phi) \mid (\phi_1 \vee \phi_2) \mid (\phi_1 \wedge \phi_2) \mid (\phi_1 \rightarrow \phi_2) \mid (\phi_1 \leftrightarrow \phi_2) \tag{1.19}$$

We are entirely uninterested in formulae that are not well-formed, so we drop the suffix "well-formed" under the recognition that we shall never again mention the alternative [Huth and Ryan, 2004].

Any formula accepted by this grammar is said to be in the language $\mathcal{L}^{\mathcal{P}}$, although we may also drop the superscript where possible. To make reading this dissertation slightly more enjoyable, we may construct examples where we denote propositional atoms by monospaced text, enabling the expression of formulae suggesting a more interesting domain, such as $\texttt{tralfamadorians} \rightarrow \neg\texttt{human}$, which should be interpreted as proposing that "Tralfamadorians are not human" [Vonnegut, 1969].

1.3.2 Semantics

In the previous section we described the syntax of propositional logic and how the boolean connectives enable the construction of arbitrary formulae from atoms. The semantics of propositional logic are concerned with how meaning may be ascribed to these formulae. The aim is to provide a method to answer questions like "when is this formula true?" or, "if ϕ, ψ, φ are true, what else must be true?".

Propositional atoms were described as indivisible statements that can be assigned values of *true* or *false*. We now define a function that assigns a truth value to each atom in the set $\mathcal{P}$ of atoms.

Definition 10. A *valuation* is a function $u : \mathcal{P} \to \{\, true,\ false \,\}$ that assigns a truth value to each propositional atom.

Given a set of atoms $\mathcal{P} = \{p, q, r\}$, we write $\mathcal{U}$ to denote the set of all possible valuations. A valuation $u \in \mathcal{U}$ *satisfies* an atom $p \in \mathcal{P}$ if u maps p to *true*, and so we write $u \vDash p$. Otherwise, we write $u \nvDash p$ to indicate that u does not satisfy p (in this context, meaning u maps p to *false*) [Ben-Ari, 2012]. We might represent the valuation that maps p and q to true but r to false by $\{p, q, \bar{r}\}$.

This satisfaction relation can be extended beyond propositional atoms to include more complex formulae, as described in Subsection 1.3.1. For any $\phi, \psi \in \mathcal{L}$,

- $u \vDash \neg\phi$ if and only if $u \nvDash \phi$ (negation)
- $u \vDash \phi \vee \psi$ if and only if $u \vDash \phi$ or $u \vDash \psi$ (disjunction)
- $u \vDash \phi \wedge \psi$ if and only if $u \vDash \phi$ and $u \vDash \psi$ (conjunction)
- $u \vDash \phi \to \psi$ if and only if $u \vDash \neg\phi$ or $u \vDash \psi$ (material implication)
- $u \vDash \phi \leftrightarrow \psi$ if and only if $u \vDash \phi \to \psi$ and $u \vDash \psi \to \phi$ (material equivalence)

Following the idea of what it means for a formula to be satisfied, we have,

Definition 11. For a valuation $u \in \mathcal{U}$ and formula $\phi \in \mathcal{L}$ we call u a *model* of ϕ if and only if u satisfies ϕ. The set of all models of ϕ is constructed by $\hat{\phi} := \{u \in \mathcal{U} \mid u \vDash \phi\}$.

Satisfiability can be extended to sets of formulae in such a way that a valuation $u \in \mathcal{U}$ *satisfies* the set $\Phi := \{\phi_0, \dots, \phi_n\}$, and so is a model of it, when $u \vDash \phi_i$ for all $0 \leq i \leq n$, and we write $u \vDash \Phi$. If Φ has no model then it is *unsatisfiable* [Ben-Ari, 2012].

Definition 12. INSERT COMPACTNESS

1.3.3 Logical Consequence

The introduction of models at the end of Section 1.3 leads quite naturally into a discussion on the matter of *logical consequence*, which provides an answer—in terms of the semantics—to the question of when it is appropriate to infer one formula from another [Tarski, 1956b].

For example, if we know that "*Billy Pilgrim lived in Slaughterhouse 5*" and that "*The inhabitants of Slaughterhouse 5 survived the bombing of Dresden*". We may then hold the view that, as a consequence of these two pieces of knowledge, it would be sensible to infer that "*Billy Pilgrim survived the bombing of Dresden*". Of course, this inference being *sensible*—under some common concept of consequence—gives little insight into how logical consequence may be appropriately be formalised. Informally, it might help ones' intuition to try imagine a world where the first propositions are true while the final one false.

We place a brief moratorium on this discussion to (formally) introduce the notions of the *object* and *metalanguage*. In the scenario we have just described, the italicised sentences form a part of the object language: the language we use to model the world and represent information. The metalanguage facilitates reasoning about elements in the object language. That is, we can use the metalanguage to describe one sentence being a consequence of another (where these sentences are elements in the object language). Here, *consequence* is an element of the metalanguage [Ben-Ari, 2012, p 22].

In this case, both the object and metalanguage are comprised of English, which can certainly lead to confusion. The distinction is clearer in propositional logic: constructions derived from combinations of atoms and the boolean operators result in elements of the object language; while the metalanguage uses symbols like $\vDash$ and $\vdash$, which are introduced below.

Definition 13. Let Γ be a set of formulae and φ a formula in the language $\mathcal{L}$. We say that φ is a *logical consequence* of Γ, and write $\Gamma \vDash \varphi$, if and only if every model of Γ is a model of φ, or equivalently if $\hat{\Gamma} \subseteq \hat{\varphi}$.

This semantic account of consequence, due to Tarski [1956b], makes implicit the view that if φ is indeed a consequence of Γ, then it should not be possible for all the sentences (formulae) in Γ to be true while φ be false.

Example 4. *If we modelled the earlier example in propositional logic, we might initialise Billy Pilgrim with* b, *slaughterhouse 5 with* h, *and survived with* s. *It is then our aim to determine whether* $\{\mathsf{b} \to \mathsf{h},\ \mathsf{h} \to \mathsf{s}\} \vDash \mathsf{b} \to \mathsf{s}$. *From the satisfaction relation in Subsection 1.3.2 it can be derived that the models of* $\{\mathsf{b} \to \mathsf{h},\ \mathsf{h} \to \mathsf{s}\}$ *are precisely:*

$$\{\{\bar{\mathsf{b}},\ \bar{\mathsf{h}},\ \bar{\mathsf{s}}\}, \{\bar{\mathsf{b}},\ \bar{\mathsf{h}},\ \mathsf{s}\}, \{\bar{\mathsf{b}},\ \mathsf{h},\ \mathsf{s}\}, \{\mathsf{b},\ \mathsf{h},\ \mathsf{s}\}\}.$$

These are indeed all models of $\mathsf{b} \to \mathsf{s}$, *and so we answer the question in the affirmative.*

Consequence operators are functions over a given language that map sets of formulae to their consequences. They were introduced by Tarski [1956a] as the symbol $\mathcal{C}n$ representing a general idea that sets can be closed under a specified notion consequence. In future, we use $\mathcal{C}n$ to refer specifically to closure under logical (*Tarskian*) consequence; and so, application of the $\mathcal{C}n$ operator to a set Γ would yield $\mathcal{C}n(\Gamma) := \{\varphi \mid \Gamma \vDash \varphi\}$. Where we wish to reference closure under a different notion of consequence, suppose that of a Hilbert system, we will use a subscript to denote the system $\mathcal{C}n_{\mathcal{H}}$. To refer to the general notion of closure under *some* notion of consequence, we write $\mathcal{C}n_X$ [Citkin and Muravitsky, 2022, p. 4].

A *theory* (also called a *deductive system*, but we avoid this terminology) is a set of formulae Γ that equals its closure $\mathcal{C}n_X(\Gamma)$. The study of such operators is useful in its ability to reveal properties about the respective notion of consequence. For instance, it was observed [Tarski, 1956a] that $\mathcal{C}n$ satisfies the following properties:

$$\text{(Inclusion)} \quad \Gamma \subseteq \mathcal{C}n(\Gamma) \tag{1.20}$$

$$\text{(Idempotency)} \quad \mathcal{C}n(\Gamma) = \mathcal{C}n(\mathcal{C}n(\Gamma)) \tag{1.21}$$

$$\text{(Monotonicity)} \quad \Gamma \subseteq \Gamma' \Rightarrow \mathcal{C}n(\Gamma) \subseteq \mathcal{C}n(\Gamma') \tag{1.22}$$

Inclusion is a relatively easy property to justify: all that is required is that the consequences of some information includes at least that starting information. *Idempotency* requires that the supposed set of all consequences is in fact the set of *all* consequences.

1.3.4 Deductive Systems

Logical consequence, described in Subsection 1.3.3, offers a purely semantic account of how it might be inferred that one formula follows (logically) from another set thereof. In contrast, deductive systems answer this question syntactically by describing a system of *axiomata* and *rules of inference* that affirm the truth of one formula from the truth of others [Ben-Ari, 2012, p. 49].

Definition 14. A *deductive system* is a collection of axiomata and rules of inference. A *proof* in such a system is a sequence of formulae where each formula is either an axiom, or has been inferred by application of an inference rule to previous formulae in the sequence. The final formula in the sequence, ϕ, is called the *theorem* and is then *provable*, and so we write $\vdash \phi$.

A *theory* (or, *deductively closed theory*) in a deduction system is a set of formulae closed under application of axioms and inference rules of the system; and so a set of formulae Γ is a theory if it is equal to its deductive closure $\mathcal{C}n_{\mathcal{H}}(\Gamma) := \{\varphi \mid \Gamma \vdash \varphi\}$. As before, elements of a theory are called theorems.

Deductive systems offer some advantages over their semantic counterparts; particularly, when reasoning over large—possibly infinite—domains, logical consequence can become difficult. Moreover, semantic consequence provides little insight into the relationships between pieces of information that lead to the inferences we make; while the sequential nature of deduction systems trace a path describing this relationship [Ben-Ari, 2012].

1.3.4.1 Hilbert Systems

A propositional Hilbert system $\mathcal{H}$ is characterised by three axiom schemata,

$$\text{(Axiom 1)} \quad \vdash (\phi \to (\psi \to \phi)), \tag{1.23}$$
$$\text{(Axiom 2)} \quad \vdash (\phi \to (\psi \to \gamma)) \to ((\phi \to \psi) \to (\phi \to \gamma)), \tag{1.24}$$
$$\text{(Axiom 3)} \quad \vdash (\neg\phi \to \neg\psi) \to (\phi \to \psi), \tag{1.25}$$

and a single rule of inference:

$$\text{(Modus Ponens)} \quad \frac{\vdash \phi, \qquad \vdash \phi \to \psi}{\vdash \psi}. \tag{1.26}$$

The axiom schemata themselves are not axioms, but rather patterns containing meta-variables that, when uniformly substituted for formulae, result in an instantiated axiom. The turnstile symbol ($\vdash$) is the syntactic counterpart to the double-turnstile ($\models$) used for logical consequence. We frequently express that ϕ is a theorem by writing $\vdash \phi$ [Ben-Ari, 2012, p. 55].

As it stands, constructing proofs from instances of axiom schema and applications of modus ponens is a challenging ordeal. As an illustration, we provide the following Hilbert-style proof for the inference made in Example 4.

Example 5. *We demonstrate a Hilbert-style derivation of* $\mathsf{b} \to \mathsf{s}$ *from the assumptions* $\mathsf{b} \to \mathsf{h}$ *and* $\mathsf{h} \to \mathsf{s}$.

Proof.

1. $\vdash (\mathsf{b} \to \mathsf{h})$ *(Premise)*

2. $\vdash (\mathsf{h} \to \mathsf{s})$ *(Premise)*

3. $\vdash (\mathsf{h} \to \mathsf{s}) \to (\mathsf{b} \to (\mathsf{h} \to \mathsf{s}))$ *(Axiom 1)*

4. $\vdash (\mathsf{b} \to (\mathsf{h} \to \mathsf{s}))$ *(MP 2,3)*

5. $\vdash (\mathsf{b} \to (\mathsf{h} \to \mathsf{s})) \to ((\mathsf{b} \to \mathsf{h}) \to (\mathsf{b} \to \mathsf{s}))$ *(Axiom 2)*

6. $\vdash (\mathsf{b} \to \mathsf{h}) \to (\mathsf{b} \to \mathsf{s})$ *(MP 4,5)*

7. $\vdash (\mathsf{b} \rightarrow \mathsf{s})$ *(MP 1,6)*

□

Derived rules are introduced as a means to make it easier to spot the next step in a proof sequence. Of particular importance is the so called *deduction theorem*, which facilitates the construction of proofs that are conditioned on a hypothesis, without requiring that the hypothesis be an axiom.

$$\text{(Deduction Theorem)} \quad \frac{\Delta \cup \phi \vdash \psi}{\Delta \vdash \phi \rightarrow \psi} \tag{1.27}$$

As an illustration, the proof in Example 5 can be restated using the *transitivity* derived rule:

$$\text{(Transitivity)} \quad \frac{\Delta \vdash \varphi \rightarrow \psi, \qquad \Delta \vdash \psi \rightarrow \gamma}{\Delta \vdash \varphi \rightarrow \gamma} \tag{1.28}$$

Derived rules must be sound with respect to what can be proved by the three axioms and applications of modus ponens. That is, a derived rule should not enable one to make an inference that would not be possible without such a rule.

The fact that it was able to be shown that "Billy Pilgrim survived the bombing of Dresden" through both semantic and syntactic notions of consequence is not a coincidence. This correspondence between logical consequence and Hilbert-style deduction systems holds in general for propositional logic (and, in fact for many other systems with relatively limited expressive power). We capture this correspondence more formally through the notions of *soundness* and *completeness*.

Definition 15. Given a set of formulae Γ and one γ in $\mathcal{L}$, $\Gamma \vdash \gamma$ if and only if $\Gamma \vDash \gamma$. Where $\vDash$ and $\vdash$ are used in the context with which they have been introduced.

The proofs for Definition 15 are well-known, and can be found in [Ben-Ari, 2012].

1.3.5 Consequence Relations

In the preceding sections the discussions on consequence—be they either semantic or syntactic—have been concrete realisations of the more general idea of *consequence relations*. When discussing consequence relations we will abuse notation and denote the relation with "$\vdash$" (relying on the surrounding context to distinguish between consequence relations and syntactic derivations). Any particular consequence relation will be denoted by a subscript referencing the system.

Definition 16. A *consequence relation* $\vdash \subseteq \mathcal{L} \times \mathcal{L}$ is a binary relation over a formal language that satisfies

$$\text{(Reflexivity)} \quad \varphi \in \Gamma \quad \text{implies} \quad \Gamma \vdash \varphi \tag{1.29}$$

$$\text{(Monotonicity)} \quad \Gamma \vdash \varphi \text{ and } \Gamma \subseteq \Psi \text{ imply } \Psi \vdash \varphi \tag{1.30}$$

$$\text{(Cut)} \quad \Gamma \vdash \Delta, \Psi \text{ and } \Psi, \Phi \vdash \Omega \text{ imply } \Gamma, \Phi \vdash \Delta, \Omega \tag{1.31}$$

We can view a consequence relation as a set of ordered pairs $\{(\Gamma_0, \varphi_0), \ldots, (\Gamma_n, \varphi_n), \ldots\}$; in a pair (Γ, φ) Γ represents a premise, and φ a conclusion [Citkin and Muravitsky, 2022]. Then, the presence of said pair describes that φ is a consequence of Γ. The absence of such a pair is understood to mean that φ is not a consequence of Γ.

This is a much more significant abstraction on the matter of consequence, requiring no concrete proof system or semantics.

As we will see in Chapter 3, consequence relations provide a useful abstraction which allow a logic to be described in terms of certain properties, or *postulates*, corresponding to a certain desired behaviour. The properties are able to provide an intuition for the logic without the construction of a semantics, or deduction-system.

Chapter 2

Formal Concept Analysis

Formal Concept Analysis (FCA) [Wille, 1982; Wille, 1992; Ganter and Wille, 1999] is an approach to reasoning about *concepts* and corresponding *conceptual structures* in terms of lattice theory. It was first proposed by Wille in his monograph *Restructuring Lattice Theory* [Wille, 1982], which was an attempt to reestablish a connection between the theory and practice of lattices.

A core aspect of FCA's foundation is the mathematisation of the philosophical view of concepts, in which a they are understood as a unit comprising two parts: the *extension*, which is the collection of things said to be instances of the concept, and the *intension*, which contains those properties used to ascribe meaning to the concept [Duquenne, 1999].

2.1 Basic Notions in Formal Concept Analysis

Despite our omission of a formal introduction to the topic of concepts, one is readily able to grasp the concept of a 'person'. Many sense-making properties come to mind: *mortal*, *warm-blooded*, *mammalian*, and so on. Listing some of the instances included in the extension of this concept is not particularly challenging either—for example, *the author of this work*, or *the reader*. What is also clear is that exhaustively detailing a list of either the intension or extension is likely to be a challenging task, even when these lists are finite, although finiteness need not be a property.

This leads us to—what is usually the starting point of FCA—a structure called a *formal context*, or simply a context. A context restricts our ontological consideration to a reduced universe of discourse, as well as detailing the underlying structural relationship that exists between elements in this universe [Wille, 1982; Dau and Klinger, 2005].

Definition 17. A formal context $\mathbb{K} = (G, M, I)$ is a triple comprised of a set G of objects , a set M of attributes, and a binary relation $I \subseteq G \times M$ referred to as an 'incidence' relation. For an object-attribute pair $(g, m) \in I$ we may say "The object g *has* the attribute m".

We consider the set of objects G (from the German *Gegenstände*) as the extensional dimension of the context, whereas the set of attributes M (from the German *Merkmale*) represents the intensional dimension. However, the distinction between the extensional and intensional dimensions of the context is largely a matter of convention, or a means of improving intuition for when one is first introduced to FCA. There is no strict requirement around what these sets are made up of, or even that they be distinct. It is perfectly acceptable to have a context where the objects and attributes are the same set: for example, the context $(\mathbb{N}, \mathbb{N}, \texttt{divides})$ which describes the division relationship between natural numbers.

It should be noted that, while the presence of an object–attribute pair (g, m) in the incidence relation is interpreted as the object satisfying the respective attribute, FCA is usually concerned with *positive* information. The absence of an object-attribute pair from the relation is not interpreted to mean that the object has the negation of the attribute. We will discuss, in more depth, the rather troubling matter of negation and negative attributes in Section 2.2.

When the cardinalities of G and M are small, contexts may be represented as a cross-table such as in Figure 2.1. Each object is represented by a row in the table, each attribute by a column, and each pair in the incidence relation is marked with an '×' at the appropriate position [Ganter and Wille, 1999]. Given a context presented in this form, it is a trivial task to identify all the attributes that a particular object satisfies. One need only scan across the respective row and note where the marks appear. The resulting set of attributes is called the *object's intent*. The dual notion of an *attribute's extent* can be found by traversing down a column in the table.

Algebraic Structures	closure	associative	identity	inverse	commutative
magma	×				
semigroup	×	×			
monoid	×	×	×		
group	×	×	×	×	
abelian group	×	×	×	×	×
loop	×		×	×	
quasigroup	×			×	
groupoid		×	×	×	
category		×	×		
semicategory		×			

Figure 2.1: A formal context showing necessary properties of group-like structures.

Using this approach, we can see that the intent of `magma` is the singleton {`closure`}, while the extent of `closure` is the set {`magma`,`semigroup`,`monoid`,`group`,`ableian group`,`loop`,`quasigroup`}.

This approach to determining object extents and attribute intents becomes impractical when considering non-trivial contexts, or large sets of objects and attributes. The procedure can be formally described by the *derivation operators*:

Definition 18. Given a context (G, M, I), the *derivation operators* are two maps $(\cdot)^{\uparrow} : \mathfrak{P}(G) \to \mathfrak{P}(M)$ and $(\cdot)^{\downarrow} : \mathfrak{P}(M) \to \mathfrak{P}(G)$. Then, for any subsets $A \subseteq G$ and $B \subseteq M$,

$$A^{\uparrow} := \{m \in M \mid \forall g \in A,\ (g, m) \in I\}$$
$$B^{\downarrow} := \{g \in G \mid \forall m \in B,\ (g, m) \in I\}$$

The derivation operators describe a mapping from a subset $A \subseteq G$ of objects in a context to the corresponding subset of attributes that each (and every) object in A related to. The dual notion holds for starting with a subset of attributes. As an example, we might wish to determine the set of attributes satisfied by the objects {`group`,`groupoid`,`abelian group`}. Application of the $(\cdot)^{\uparrow}$ derivation operator to this set yields {`associative`,`identity`,`inverse`}.

To offer another perspective on the derivation operators, obvserve that the derivation of a set $A \subseteq G$ of objects is the intersection of each object intent, and so we have

$$\begin{aligned} A^{\uparrow} &= \bigcap \{a^{\uparrow} \mid a \in A\} \qquad && A \subseteq G \\ B^{\downarrow} &= \bigcap \{b^{\downarrow} \mid b \in B\} \qquad && B \subseteq M. \end{aligned}$$

In fact, we have already explored a more general perspective on the derivation operators in Section 1.2 through the notion of closure operators and Galois connections. The proposition below recontextualises properties of Galois connections in the setting of FCA.

Proposition 4. *Let (G, M, I) be a formal context and consider the subsets $X, X_1 \subseteq G$ of objects (resp. $Y, Y_1 \subseteq M$ of attributes) then*

$$X \subseteq X_1 \Rightarrow X_1^{\uparrow} \subseteq X^{\uparrow} \qquad \textit{(resp.)} \qquad Y \subseteq Y_1 \Rightarrow Y_1^{\downarrow} \subseteq Y^{\downarrow} \tag{2.1}$$

$$X \subseteq X^{\uparrow\downarrow} \qquad \textit{(resp.)} \qquad Y \subseteq Y^{\downarrow\uparrow} \tag{2.2}$$

$$X^{\uparrow} = X^{\uparrow\downarrow\uparrow} \qquad \textit{(resp.)} \qquad Y^{\downarrow} = Y^{\downarrow\uparrow\downarrow} \tag{2.3}$$

$$X \subseteq Y^{\downarrow} \quad \textit{if and only if} \quad X^{\uparrow} \supseteq Y \tag{2.4}$$

If we consider the powerset lattices of $\mathfrak{P}(G)$ and $\mathfrak{P}(M)$ equipped with their usual inclusion orders, then the derivation operators constitute a Galois connection on these sets: Equations (2.1) to (2.3) are just a rephrasing of Equations (1.14) to (1.16); while Equation (2.4) rephrases Proposition 1. Mirroring the discussion on Galois connections, we obtain the closure systems $\mathcal{G}$ and $\mathcal{M}$ on G and M, respectively. Each of these closure systems form a complete lattice where, recounting Theorem 2, meets and joins are given by:

$$\begin{aligned} \bigwedge_{i \in I} A_i &= \bigcap_{i \in I} A_i \quad \text{and} \quad \bigvee_{i \in I} A_i = \Big(\bigcup_{i \in I} A_i\Big)^{\uparrow\downarrow} \qquad && A \subseteq \mathcal{G} \\ \bigwedge_{i \in I} B_i &= \bigcap_{i \in I} B_i \quad \text{and} \quad \bigvee_{i \in I} B_i = \Big(\bigcup_{i \in I} B_i\Big)^{\downarrow\uparrow} \qquad && B \subseteq \mathcal{M} \end{aligned}$$

For a set of objects $A \subseteq G$, it follows directly from Equation (2.3) that $A^{\uparrow}$ is a member of the closure system on $\mathcal{M}$.

Corollary 1. *Given a context (G, M, I) the derivation of a set of objects $A \subseteq G$ (resp. attributes $B \subseteq M$) is always a member of the closure system $\mathcal{M}$ (resp. $\mathcal{G}$).*

We recall the discussion at the end of Subsection 1.2.2 where it was shown that a pair of mappings that form a Galois connection are dually isomorphic when their domains are restricted to the associated closure systems. With the aid of Figure 2.2, we note this correspondence between the lattices of closure systems $\mathcal{G}$ and $\mathcal{M}$.

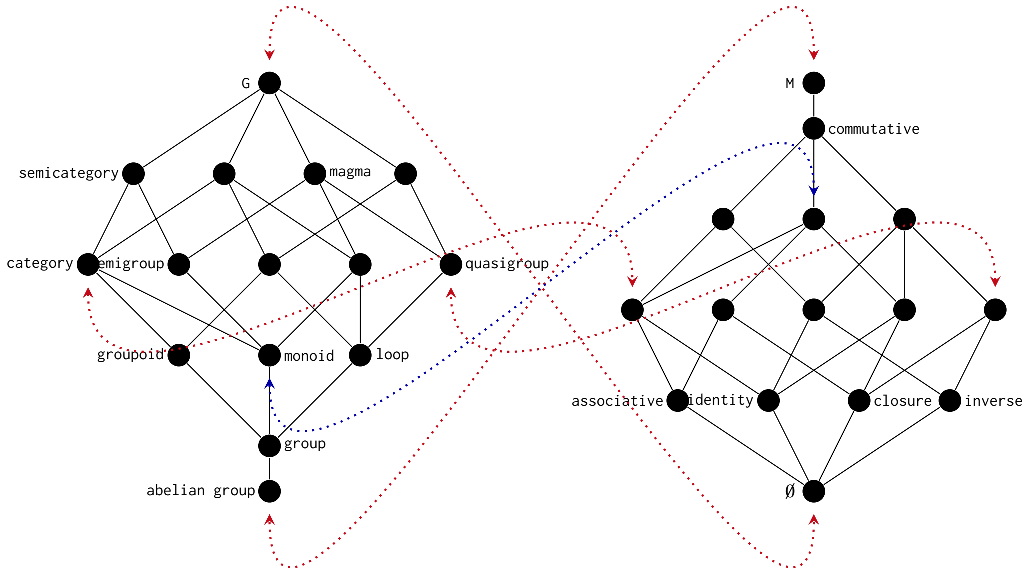


Figure 2.2: The lattices induced by the closure systems $\mathcal{G}$ and $\mathcal{M}$. Each node in a lattice represents the set comrpising of all labels reachable from below

Earlier, it was suggested that Galois connections are particularly interesting when the closure systems they induce are themselves interesting, and, future discussion of an example of such interesting closure systems was promised. Let us now give such an example, and in doing so provide some intution for why Galois connections are a useful way of modelling formal concepts, to be introduced immediately afterwards.

Consider the concept derived from the algebraic structure of a *monoid*: a set equipped with a binary operation that satisfies the properties of closure, associativity, and has an identity element. We should want to include in the extension of this concept all those algebraic structures that would be considered subclasses of monoids. These are those structures which satisfy the properties of a monoid (and possibly additional properties). Of course, a *monoid* should be a member of this extension. Another structure we should expect to see in this extension is that of a *group*, which satisfies the requirements while also having an inverse.

Now let us consider the concept derived from a *group*. We have just shown that a group is a subclass of monoid, so it should follow that the extension of a group must be a subset of the extension of a monoid. What of the intension? Suppose the intension of the concept of a monoid were not a subset of that of a group, then monoids would have to satisfy some property that groups did not. This creates a contradiction, as it would mean we could not properly consider a group to be a subclass of a monoid.

This reveals the relationship between concepts, such that when we move from the more general concept (monoid) to the more specific concept (group), the extension becomes smaller while the intension grows. The extension of group is contained within the extension of monoid, but the intension of monoid is contained within the intension of group. This inverse relationship between the extensions and intensions of concepts is precisely the kind of relationship described by a Galois connection.

With this in mind, it is finally an appropriate time to define what is meant by *formal concept*:

Definition 19. A *formal concept* of a context (G, M, I) is a pair (A, B) where $A \subseteq G$ and $B \subseteq M$ where $A^{\uparrow} = B$ and $B^{\downarrow} = A$. We call A the *concept extent* and B the *concept intent*. We write $\mathfrak{B}(G, M, I)$ to denote the set of all concepts of (G, M, I).

A concept is a pair of closed sets, each representing either the extensional or intensional perspective. Indeed, there is some degree of redundancy in this definition, as the extension fully determines the intension and vice verse [Ganter and Obiedkov, 2016]. However, this redundancy is beneficial for ones' intuition, and so we persevere with it.

A pleasing result of the definition of concepts is that, for any set $A \subseteq G$ of objects, the derivation $A^{\uparrow}$ will always define a concept intent (the intent of the concept constructed by $(A^{\uparrow\downarrow}, A^{\uparrow})$). For the same reason, the derivation of any set of attributes $B \subseteq M$ will always be a concept extent. In fact, this is fact is merely a rephrasing of Corollary 1 within the perspective of concepts.

To make this idea concrete, we reconsider the previous discussion where we (informally) described the concept of a monoid. It follows that the derivation $\{\texttt{monoid}\}^{\uparrow}$ yields the concept intent:

$$\{\texttt{closure}, \texttt{associative}, \texttt{identity}\}.$$

In turn, the derivation of the concept intent results in the set

$$\{\texttt{monoid}, \texttt{group}, \texttt{abelian group}\},$$

which is the concept extent, and so

$$\big(\{\texttt{monoid}, \texttt{group}, \texttt{abelian group}\}, \{\texttt{closure}, \texttt{associative}, \texttt{identity}\}\big)$$

is a formal concept of Formal Context 2.1.1.

The union of concept extents (*resp.* intents) does not in general yield another concept extent (*resp.* intent), whereas the intersection of concept extents (*resp.* intents) always yields a concept extent (*resp.* intent). Equivalently, the closure systems $\mathcal{G}$ and $\mathcal{M}$ are not closed under arbitrary unions, but they are closed under arbitrary intersections.

Proposition 5. *Let T be an indexing set, for every $t \in T$, let $A_t \subseteq G$ be a set of objects. Then,*

$$\Big(\bigcup_{t \in T} A_t\Big)^{\uparrow} = \bigcap_{t \in T} A_t^{\uparrow}$$

If we examine the lattices in Figure 2.2, we observe that the two derivation operators map to and from the concept intent and extent (shown by the dotted blue arrows). This observation suggests a more consolidated perspective: The two lattices can be unified into a single structure where elements of the lattice are precisely the pairs (A, B) where $A \in \mathcal{G}$ and $B \in \mathcal{M}$ and also $A^{\uparrow} = B$ and $B^{\downarrow} = A$. In other words, the lattice of concepts. We appropriately call this structure the *concept lattice* of a formal context, and write $\underline{\mathfrak{B}}(G, M, I)$ to refer to the set of all concepts equipped with this structure.

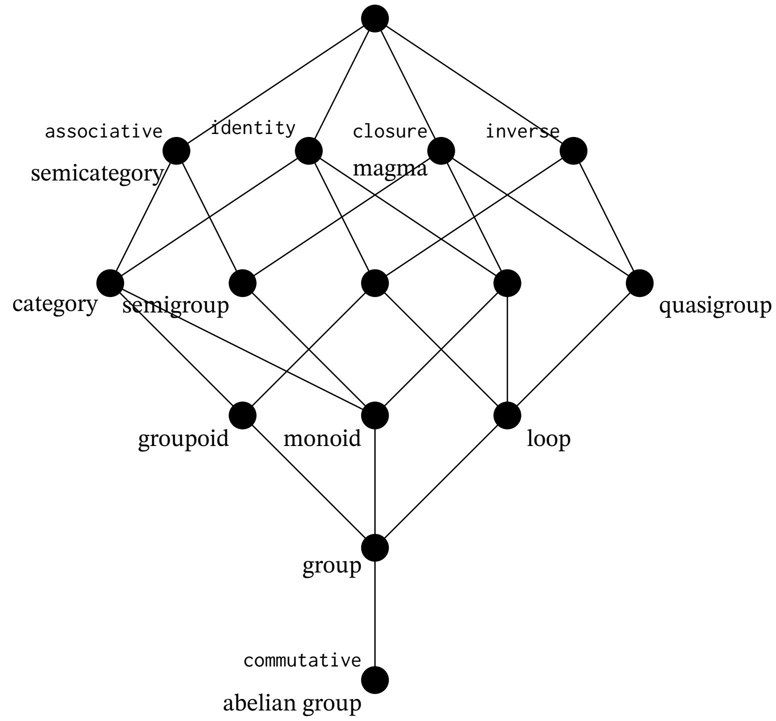


Figure 2.3: The concept lattice corresponding to Formal Context 2.1.1

Reading a concept lattice requires a small amount of getting used to: The labels above a node in the diagram correspond to attributes, while objects are labelled below a node. Each node represents a concept $C \in \mathfrak{B}(G,M,I)$. The labelled nodes in the concept lattice diagram correspond to either *object* or *attribute concepts*.

Definition 20. Given a context $\mathbb{K} = (G,M,I)$, the *object concepts* are defined for each object $g \in G$ as

$$\gamma g := (g^{\uparrow\downarrow}, g^{\uparrow}),$$

and the *attribute concepts* as

$$\mu m := (\mu^{\downarrow}, \mu^{\downarrow\uparrow})$$

for each $m \in M$.

Given a node in the diagram for a concept lattice, the extension of the concept is given by all those objects labelled at nodes for which there is a (strictly) downward path from the respective node. The concept intent is given by the labelled attributes with a strictly upward path.

The concept lattice is isomorphic to the closure system $\mathcal{G}$ of objects, and dually isomorphic to the closure system $\mathcal{M}$ of attributes. We now consider the first part of the fundamental result in FCA.

Theorem 3 (The Basic Theorem of Concept Lattices: Part one)**.** *The concept lattice* $\underline{\mathfrak{B}}(G,M,I)$ *of a formal* *context is a complete lattice in which meets and joins are given by*

$$\bigwedge_{t\in T}(A_t, B_t) = \Big(\bigcap_{t\in T} A_t,\ \big(\bigcup_{t\in T} B_t\big)^{\downarrow\uparrow}\Big)$$
$$\bigvee_{t\in T}(A_t, B_t) = \Big(\big(\bigcup_{t\in T} A_t\big)^{\uparrow\downarrow},\ \bigcap_{t\in T} B_t\Big)$$

Part one of the basic theorem describes how we might find the meet and join of concepts in the concept lattice. To make this clear, recall the isomorphism between $\mathcal{G}$ and $\underline{\mathfrak{B}}(G, M, I)$, which tells us that

$$A_0 \wedge A_1 \cong (A_0, B_0) \wedge (A_1, B_1)$$

for any $A_0, A_1 \in \mathcal{G}$ where (A_i, B_i) is the concept with extent A_i.

The isomorphism between the concept lattice $\underline{\mathfrak{B}}(G, M, I)$ and the lattice corresponding to the closure system $\mathcal{G}$ means that the meet of any two concepts can be found by considering the concept formed by the meet of their extensions with respect to $\mathcal{G}$. Of course, since $\mathcal{G}$ is a closure system, the meet operation on this lattice corresponds to taking the intersection of these extensions (and $\mathcal{G}$ is closed under arbitrary intersections). As there is a dual isomorphism between $\underline{\mathfrak{B}}(G, M, I)$ and $\mathcal{M}$, we can equivalently discover the meet of two concepts by considering the join operation on the concept intensions with respect to the lattice of $\mathcal{M}$. This corresponds to taking the closure of the union of concept intensions (cf. Theorem 2 for a reminder of why).

The join of two concepts—again, due to the dual isomorphism—corresponds to the meet operation on the concept extensions with respect to the lattice of $\mathcal{M}$ and can thus be found by taking the intersection of their intensions, under which $\mathcal{M}$ is closed. Or, by the join operation on $\mathcal{G}$, which corresponds to the closure of the union of extensions.

For another perspective, consider that each A_t is equivalent to $B_t^{\downarrow}$. By Proposition 5

$$\bigwedge_{t\in T}(A_t, B_t) = \Big(\bigcap_{t\in T} A_t,\ \big(\bigcup_{t\in T} B_t\big)^{\downarrow\uparrow}\Big)$$

can be restated as

$$\bigwedge_{t\in T}(A_t, B_t) = \Big(\big(\bigcup_{t\in T} B_t\big)^{\downarrow},\ \big(\bigcup_{t\in T} B_t\big)^{\downarrow\uparrow}\Big)$$

We may arrive at the concept lattice in another way, by considering the rather natural order on concepts which arises from the *subconcept–superconcept* relation. We had, earlier, discussed that 'group' is a subclass of 'monoid', and so the extension of the associated concept of a group should be a subset of that of a monoid. In line with this, we say that a concept (A_0, B_0) is a subconcept of another (A_1, B_1) if and only if $A_0 \subseteq A_1$. Dually, (A_1, B_1) is a superconcept of (A_0, B_0) if and only if $B_1 \subseteq B_0$. In this case we write $(A_0, B_0) \leq (A_1, B_1)$. When considering the set of concepts $\mathfrak{B}(G, M, I)$ with the relation defined by $\leq$, the resulting structure is the same concept lattice $\underline{\mathfrak{B}}(G, M, I)$ from before. As an illustration, consider the two selected concepts in Figure 2.3.

I think it is worth mentioning some discussion on how, as a result of the basic theorem, formal concept analysis is just a restructuring of lattice theory. I.e., it makes work on complete lattices more accessible. Worthwhile in terms of motivations behind formal concept analysis. Restructuring also relates to attribute logic, so this is a good place to do it.

2.2 Attribute Logic

This section explores several of the more *logic*-related ideas that present themselves in FCA. In light of Section 1.3, many of the notions here should be somewhat familiar. However, we follow the suggestion by Ganter [2025] where it is suggested that we think of logic, as it relates to FCA, not in terms of truth values, but rather attributes and conceptual meaning.

We begin by studying the *implication logic* of FCA, which is perhaps most fundamental. Afterwards, we will discuss a more expressive logic.

2.2.1 Implication Logic

It is often sufficient to study what is, in terms of expressivity, a fairly restricted logic. This is the *implication* logic of FCA. From the perspective of mathematical logic, it is analogous to prositional Horn logic.

An implication describes a correspondence between attributes, expressing the idea that "Whenever we see attribute m, we also see attribute n". Formally, we have:

Definition 21. An *implication* over a set M of attributes is written $A \to B$, where $A, B \subseteq M$. We call A the *premise* and B the *conclusion* of the implication.

The semantics of attribute implications quite closely resemble the material implications outlined in Section 1.3; however, are primarily interested in attribute implications as they relate to formal contexts, and so we define the semantics with this in mind.

Definition 22. Another set of attributes $C \subseteq M$ *respects* the implication if either $A \nsubseteq C$ or $B \subseteq C$, in which case we write $C \vDash A \to B$ and call C a *model* of the implication. If $\mathcal{T}$ is a set of implications, C is a model of $\mathcal{T}$ if it respects every implication in $\mathcal{T}$. A formal context $\mathbb{K} = (G, M, I)$ *respects* an implication $A \to B$ over M if and only if the intent $g^{\uparrow}$ of each object $g \in G$ respects the implication, in which case we write $\mathbb{K} \vDash A \to B$; this idea is extended to a set of implications as expected.

Recall that in FCA we are usually only interested in positive information, we cannot—at least for the time being—speak about attribute implications with disjunction or negation. In Subsection 2.2.2 we discuss how we might increase the expressivity of the logic.

The definition for when a context respects an implication admits a more concise representation: It is equivalent to the expression that $A^{\downarrow} \subseteq B^{\downarrow}$, or $B \subseteq A^{\downarrow\uparrow}$. Recalling Corollary 1, which asserts that both $A^{\downarrow}$ and $B^{\downarrow}$ are always concept extents, we have the following:

Proposition 6. *The implication $A \to B$ is respected by a context (G, M, I) if and only if the concept derived from A is a subconcept of the one derived from B. Formally, if $(A^{\downarrow}, A^{\downarrow\uparrow}) \leq (B^{\downarrow}, B^{\downarrow\uparrow})$ holds.*

For an illustration, we consider whether the implication $\{\texttt{inverse}, \texttt{associative}\} \to \{\texttt{identity}\}$ holds in the context of group-likes. Indeed, the concept lattice in Figure 2.3 confirms that

$$(\{\texttt{inverse}, \texttt{associative}\}^{\downarrow}, \{\texttt{inverse}, \texttt{associative}\}^{\downarrow\uparrow}) \leq (\{\texttt{identity}\}^{\uparrow}, \{\texttt{identity}\}^{\uparrow\downarrow}),$$

and so the implication holds. Reading implications off a concept lattice in this *bottom-up* manner is convenient, but there is a more significant connection between attribute implications and concept lattices. We explore this connection next.

2.2.1.1 Implication Inference

Given a set of implications, it is interesting to explore which other implications follow. This was exactly the subject of *Subsection* 1.3.3, and many of those ideas are recontextualised here.

Definition 23. An implication $A \to B$ *logically follows* from a set $\mathcal{T}$ of implications over M if and only if each attribute subset $C \subseteq M$ that respects $\mathcal{T}$ also respects $A \to B$, then we write $\mathcal{T} \vDash A \to B$.

A set $\mathcal{T}$ of attribute implications is called *closed*, or an *implication theory*, when it contains all implications that logically follow from it. It is *complete* with respect to some closure operator if every implication that holds in the respective closure system follows. Then, $\mathcal{T}$ is complete with respect to a context (G, M, I) if for every $A, B \subseteq M$ where $B \subseteq A^{\downarrow\uparrow}$ it follows that $\mathcal{T} \vDash A \to B$. In other words, when every implication that is respected by (G, M, I) follows from $\mathcal{T}$ [Ganter and Obiedkov, 2016; Ganter and Wille, 2024]. This correspondence can be formalised as:

Proposition 7. *If $\mathcal{T}$ is a set of implications over an attribute set M, then the set* $\mathsf{Mod}(\mathcal{T}) := \{A \subseteq M \mid A \vDash \mathcal{T}\}$ *is a closure system on $\mathfrak{P}(M)$.*

If $\mathcal{T}$ is the complete implication theory of (G, M, I) then the closure system $\mathsf{Mod}(\mathcal{T})$ *is the system $\mathfrak{P}(M)$ of concept intents.*

Indeed, a complete implication theory is sufficient to fully describe the structure of a concept lattice, or its corresponding formal context [Ganter and Wille, 2024]. There is a syntactic counterpart to the semantic notions of consequence, defined in terms of the Armstrong axioms [Armstrong, 1974].

Definition 24. The *Armstrong axioms* are defined as inference rules of the form:

$$\frac{}{A \to A}, \qquad \frac{A \to B}{A \cup C \to B}, \qquad \frac{A \to B, \quad B \cup C \to D}{A \cup C \to D} \tag{2.5}$$

where $A, B, C, D \subseteq M$.

The interpretation should be that, if the instantiation of an Armstrong axiom has its premise contained in a set $\mathcal{T}$ of implications, then the conclusion of the axiom may be inferred. To show that an implication is a consequence of some set one shows that there is a sequence of application of these rules that leads to the desired implication.

The following closure operator describes how a set of attributes can be closed under a set of implications.

Definition 25. Let $\mathcal{T}$ be a set of implications over a set of attributes M. For $X \subseteq M$, define the set:

$$X^{\mathcal{T}} = X \cup \bigcup \{B \mid (A \to B) \in \mathcal{T},\ A \subseteq X\},$$

which extends X by adding all conclusions of implications in $\mathcal{T}$ whose premises are contained within X. The sequence $X^{\mathcal{T}}, X^{\mathcal{T}\mathcal{T}}, \ldots$ will reach a fixed-point $\mathcal{C}n_{\mathcal{T}}(X) := X^{\mathcal{T}\ldots\mathcal{T}}$ where $\mathcal{C}n_{\mathcal{T}}(X) = \mathcal{C}n_{\mathcal{T}}(X)^{\mathcal{T}}$. This fixed point defines the *closure of X under $\mathcal{T}$*.

It is much simpler to consider this closure operator, instead of a sequence of Armstrong axioms, when determining whether a set of implications logically imply another.

Proposition 8. *An implication $A \to B$ is a logical consequence of a set of implications $\mathcal{T}$ over M if and only if $B \subseteq \mathcal{C}n_{\mathcal{T}}(A)$.*

Remark 1. It is important to recognise the connections that exist between the topics of discussion. We have established that an implication $A \to B$ holds in a context (G, M, I) when $B \subseteq A^{\downarrow\uparrow}$. Here, $(\cdot)^{\downarrow\uparrow}$ is one closure operator. By Definition 25 there is another closure operator, $\mathcal{C}n_{\mathcal{T}}(\cdot)$ which operators on the syntactic level. We have just shown that if there is a set of implications $\mathcal{T}$, then another implication $A \to B$ follows (holds) when $B \subseteq \mathcal{C}n_{\mathcal{T}}(A)$. If $\mathcal{T}$ is the complete implication theory of a context, then these two closure operators are identical.

Computationally, computing the closure under a set of implications is quite efficient. The LinClosure algorithm [Maier, 1983], and slightly modified by [Ganter and Obiedkov, 2016], describes a procedure which finds the closure of an attribute set with respect to a set of implications. The first block of the algorithm, lines 1–9, determines the size of the premise for each implication, the respective implication is then associated with each individual attribute in the premise. All implications with an empty premise are vacuously satisfied, and so their conclusions are added to X.

The result is a tally, for each implication, of how many attributes are required to satisfy the premise; and, for each attribute $m \in M$ a list of implications for which m appears in the antecdent. All attributes from X, alongside the conclusions of vacuously satisfied implications, are stored in *update*.

Algorithm 1 LinClosure

Input: An attribute set $X \subseteq M$ and a set $\mathcal{T}$ of implications over M.
Output: The closure of X with respect to $\mathcal{T}$.

```
1  for all A → B ∈ T do
2      count[A → B] := |A|
3      if |A| = 0 then
4          X := X ∪ B
5      end if
6      for all m ∈ A do
7          add A → B to list[a]
8      end for
9  end for
10 update := X
11 while update ≠ ∅ do
12     Choose m ∈ update
13     update := update \ {m}
14     for all A → B ∈ list[m] do
15         count[A → B] := count[A → B] − 1
16     end for
17     if count[A → B] = 0 then
18         add := B \ X
19         X := X ∪ add
20         update := update ∪ add
21     end if
22 end while
23 return X
```

The second block, lines 11–22, chooses an attribute $m \in update$ and for each of its' associated implications, given by *list*$[m]$, decrements the counter associated with the implications premise. If an implication's counter reaches zero, the conclusion (after removing any attributes which have already been dealt with, or will be dealt with in future) is added to *update*. The process repeats until *update* is empty, afterwhich the closed set is returned.

Proposition 9. *The complexity of* LinClosure *is* $O(|\mathcal{T}|)$*, and so the procedure is linear on the size of the* *implication set.*

2.2.1.2 Implication Bases

Definition 26. If $\mathcal{I}$ is a set of implications then $\mathcal{B}$ is an *implication basis* of $\mathcal{I}$ if and only if

1. $\mathcal{B}$ is sound with respect to $\mathcal{I}$, i.e. if $\mathcal{B} \vDash A \to B$ then $\mathcal{I} \vDash A \to B$
2. $\mathcal{B}$ is complete with respect to $\mathcal{I}$, i.e. if $\mathcal{I} \vDash A \to B$ then $\mathcal{B} \vDash A \to B$
3. $\mathcal{B}$ is non-redundant, i.e. $\nexists i \in \mathcal{I}$ such that $\mathcal{I} \setminus \{i\} \vDash i$

Soundness and completeness are obviously properties that an implication basis should satisfy. Given that checking logical consequence is linear on the size of the implication set, having non-redundant implications is computationally appealing.

Definition 27. A set $X \subseteq M$ is *pseudo-closed* if it is not closed, and each proper subset of X is itself pseudo closed. Formally,

1. $X \neq X^{\downarrow\uparrow}$
2. $Y \subset X$ is pseudo-closed for all $Y \subset X$

Example 6. *Give an the canonical base of the algebraic structures*

$$\left\{ \begin{array}{l} \{\texttt{commutative}\} \to \{\texttt{closure, identity, inverse, associative}\} \\ \{\texttt{inverse, associative}\} \to \{\texttt{identity}\} \end{array} \right\}$$

Examine Formal Context 6.0.1

2.2.2 Compound Attributes

The implication logic of FCA, despite giving rise to interesting results like the canonical basis, is quite limited. Certain settings require more expressive power. *Compound attributes* are introduced for this purpose. We use the lowercase Greek alphabet to denote compound attributes as they quite closely resemble propositional formulae, *atomic* (normal) attributes will continue to be denoted with lowercase Latin alphabet.

Definition 28. Let M be a set of attributes (as we might find in a formal context). A compound attribute is constructed by iterative application of the following:

- An *atomic* attribute $m \in M$ is a compound attribute
- If $\varphi \in M^+$ is a compound attribute, then so is $\neg\varphi$
- If $\Gamma \subseteq M^+$ is a set of compound attributes, then $\bigwedge \Gamma$ is a compound attribute

We write M^+ to denote the set of all compound attributes over the attribute set M.

The definition allows for arbitrarily complex compound attributes to be constructed. The semantics of these are given by their extension, relative to a formal context.

Definition 29. In a formal context (G, M, I), the extension of each compound attribute $\varphi \in M^+$ is written $\varphi^{\vDash}$ and defined inductively as follows:

- If φ is an atomic attribute (i.e., $\varphi \in M$) then $\varphi^{\models} := \varphi^{\downarrow}$
- If φ is the negation of another compound attribute $\gamma \in M^{+}$ then $\varphi^{\models} := G \setminus \gamma^{\models}$
- If φ is the result of a conjunction of a set of compound attributes $\bigwedge \Gamma$ then $\varphi^{\models} := \bigcap_{\gamma \in \Gamma} \gamma^{\models}$

For an object $g \in G$ and compound attribute $\varphi \in M^{+}$, we write $g \models \varphi$ if and only if g is in the extension of φ and we say g *satisfies* φ, otherwise $g \not\models \varphi$.

Other kinds of compound attributes which express useful intution, in particular the disjunction of compound attributes, have an equivalent construction using only Definition 28. If Γ is a set of compound attributes, then the disjunction of its elements $\bigvee \Gamma$ then its' extension $\bigcup_{\gamma \in \Gamma} \gamma^{\models}$ is the union of all extensions of compound attributes in the set. Such a compound attribute is equivalent to $\neg \bigwedge \Gamma$.

The proposed equivalence of $\bigvee \Gamma$ and $\neg \bigwedge \Gamma$ necessitates some further discussion; it is possible for two compound attributes to be equivalent in one formal context, but not in another. We might say that in the former case, these attributes are *locally* equivalent. *Global* equivalence is stronger, requiring that the equivalence hold for any context.

Definition 30. Two compound attributes are *extensionally equivalent*, or *locally equivalent*, with respect to a formal context if and only if they share the same extension. They are *globally equivalent* if and only if they would share the same extension in any possible context.

Example 7. *From the formal context described below we may construct two compound attributes* $\varphi := \bigwedge\{\texttt{human}, \texttt{unstuck}\}$ *and* $\psi := \neg\texttt{violent}$*, marked in light grey.*

Slaughterhouse 5	human	unstuck	soldier	pacifist	traumatized	violent	φ	ψ
Billy Pilgrim	×	×	×	×	×		×	×
Roland Weary	×		×			×		
Mary O'Hare	×			×				×
Paul Lazzaro	×		×			×		
Tralfamadorians		×				×		

Figure 2.5: A context describing characters and some of their qualities, from Slaughterhouse 5

Each of these compound attributes have the extension consisting solely of {`Billy Pilgrim`}*, and so at least they are locally equivalent. It may come as no surprise that their equivalence is not global: suppose Tralfamadorians were not violent. If we consider a third compound attribute,* $\neg\neg$`violent`*, we would find that it is globally equivalent to the attribute* `violent`.

A somewhat blunt approach to determining if two compound attributes, over a set M of attributes, are globally equivalent is to test for extensional equivalence in the formal context $(\mathfrak{P}(M), M, \ni)$, such a context is called the *test context* over M.

Definition 31. Let M be a set of attributes, then the language of (finite) compound attributes is inductively as

$$\phi ::= m \mid (\neg\phi) \mid (\phi_1 \vee \phi_2) \mid (\phi_1 \wedge \phi_2) \mid (\phi_1 \rightarrow \phi_2) \tag{2.6}$$

Classical Music	well-rounded	well-balanced	dramatic	transparent	structured	strong	lively	sprightly	rhythmizing	fast	playful
Beethoven: Romance for violin and orchestra F-major	×	×		×	×						×
Bach: Contrapunctus I	×	×		×	×	×					
Chajkovskij: Piano concerto b flat minor, 1st movement			×			×					
Mahler: 2nd Symphony, 2nd movement					×		×	×	×	×	×
Bartok: Concert for ochestra						×	×	×	×		×
Beethoven: 9th symphony, 4th movement (presto)			×			×			×	×	
Bach: WTP 1, prelude c minor			×		×	×	×			×	
Bach: 3rd Brandenburg Concerto, 3rd movement	×	×		×	×		×	×		×	×
Ligeti: Continuum				×			×			×	×
Mahler: 9th symphony, 2nd movement (Ländler)					×	×	×		×		
Beethoven: Moonlight sonata, 3rd movement			×	×		×	×		×	×	
Hindemith: Chamber music No.1, finale				×		×	×	×	×	×	×
Bizet: Suite arlesienne	×	×		×	×	×			×		×
Mozart: Figaro, overture	×	×		×	×	×	×	×		×	×
Schubert: Wayfarer fantasy			×			×	×		×	×	
Beethoven: Spring sonata, 1st movement	×	×		×	×		×	×			×
Bach: WTP 1, fuge c minor	×	×		×	×		×	×	×		×
Shostakovich: 15th symphony, 1st movement					×	×	×	×			×
Wagner: Mastersinger, overture	×	×		×	×	×	×	×	×		×
Beethoven: String quartet op.131, final movement			×	×	×	×			×	×	
Johann Strauß: Spring voice waltz	×	×		×	×		×	×	×	×	×
Mozart: Il Seragio, “O, how I will triumph...”			×	×	×	×	×			×	×
Bach: Mathew’s passion No.5 (choir)			×	×	×	×	×				
Brahms: Intermezzo op. 117, No.2	×	×		×	×	×	×		×	×	
Wagner: Ride of the valkyries			×	×	×	×	×		×		
Mozart: Magic flute, “The hell revenge rages ...”			×	×	×	×	×			×	
Mendelsohn: 4th symphony, 4th movement	×	×		×	×	×	×	×	×	×	×
Brahms: 4th symphony, 4th movement	×	×	×	×	×	×					
Beethoven: Great fuge op. 133	×	×	×	×	×	×	×		×		
Goretzky: Lament symphony	×	×	×	×	×	×	×				
Verdi: Requiem, dies irae	×	×	×	×	×	×	×			×	

Figure 2.4: Characteristics of pieces of classical music

where $m \in M$ is a "plain" attribute. Then, the satisfaction relation $\Vdash$ is

- $g \Vdash \varphi$ if and only if $g \in \varphi^{\downarrow}$
- $g \Vdash \neg\varphi$ if and only if $g \in G \setminus \varphi^{\Vdash}$
- $g \Vdash \varphi \vee \gamma$ if and only if $g \in \varphi^{\Vdash} \cup \gamma^{\Vdash}$
- $g \Vdash \varphi \wedge \gamma$ if and only if $g \in \varphi^{\Vdash} \cap \gamma^{\Vdash}$
- $g \Vdash \varphi \rightarrow \gamma$ if and only if $g \in \neg\varphi^{\Vdash} \vee \gamma^{\Vdash}$

Example 8. *Consider the context below, which represents a portion of the* 1984 *United States Congressional voting records taken from the UC Irvine Machine Learning repository [con, 1987].*

Congressional Voting Records	mx-missile	crime	immigration	satellite ban	education	republican	democrat
Representative 1		×	×		×	×	
Representative 4							×
Representative 9		×			×	×	
Representative 17			×		×		×

Figure 2.6: A context describing a portion of congressional voting records for 1984

We can construct meaningful compound attributes such as `education` $\wedge$ $\neg$`immigration`*, the extension of which consists of all representatives who voted "yay" with respect to the education bill, and either abstained or voted "nay" to the immigration bill. This compound attribute is extensionally equivalent to* `republican` $\wedge$ $\neg$`immigration`.

We may be tempted to say that `republican` *and* $\neg$`democrat` *are globally equivalent compound attributes; but this would be an error. Globally equivalent attributes have the same extension in any conceivable context, not only those which align with background assumptions about the nature of the attributes. For an example of global equivalence, consider* `democrat` *and* $\neg\neg$`democrat` .

Chapter 3

Defeasible Reasoning

Non-monotonicity in logical systems has been the focus of study for decades, and several distinct formalisms have been developed. The motivation is to expand the inference power beyond that of the classical, to a more credulous one where inferences may, upon learning new information, be retracted.

This work is interested in the kind of non-monotonic reasoning put forward by Kraus, Lehmann, and Magidor [Kraus *et al.*, 1990; Lehmann and Magidor, 1992], frequently initialised to the *KLM framework*.

This could be a bit more discursive I think

3.1 Background on Non-monotonic Reasoning

Near the end of the previous chapter, the matter of consequence was discussed in a very formal sense. It is perhaps helpful to distinguish this formal subject—classical consequence—from the common concept, as the former yields some surprising results which do not appear at all congruent with how a person, or otherwise intelligent agent should reason [Tarski, 1956b; Kraus *et al.*, 1990]. As a demonstration of a result, which may be surprising in this way, consider the following propositions which state that *humans experience chronological time* , *soldiers are human*, and *Billy Pilgrim experiences non-chronological time*.

1. `human` $\rightarrow$ `chronological time`
2. `soldier` $\rightarrow$ `human`
3. `Billy Pilgrim` $\rightarrow$ $\neg$`chronological time`

If we were to encounter a soldier, we might find it sensible

Knowing these propositions, ~~if we were to encounter an individual in combat fatigues we might find it~~ sensible—by propositions 2 and 1—to deduce that the individual experienced time chronologically. If we were to later learn that the individual were, in fact, Billy Pilgrim, ~~given proposition 3~~, we should like to retract our prior inference and replace it with the knowledge that the individual does not experience time chronologically.

Such recourse is not, as it stands, possible. When we see the that the individual is a soldier, the possible worlds satisfying our existing knowledge are reduced to the single model: $\{\overline{b}, c, h, s\}$ (where b: `Billy Pilgrim`, c: `chronological time`, h: `human`, and s: `soldier`). Should we later learn that the individual is indeed Billy Pilgrim, the theory becomes inconsistent. And, by the principle of explosion, discussed in Subsection 1.3.3, our theory now entails that the individual experiences time both chronologically and non-chronologically. Indeed, our theory now entails everything and is accordingly worthless.

This property of classical logic—that adding new information never results in retraction of pre-existing knowledge—is called monotonicity. The presence of monotonicity requires that when we make a claim such as "humans experience time chronologically", we must be absolutely sure of ourselves, so as to never worry about needing to retract an inference. This is, of course, too strict a requirement as we cannot determine for all future, present, and past humans if it were the case that they experienced time chronologically. If we remain in the classical realm, it seems our only options are to abandon our original claim or risk explosion.

At this point it is a good idea to provide some clarification on how we might begin to approach solutions to this issue. Continuing with the same example—and continuing to allow ourselves to entertain the possibility of experiencing non-chronological time—it would certainly be agreed that typically soldiers are human, and also that typically humans experience time chronologically. To resolve that Billy Pilgrim is a soldier, and therefore a human, who does not experience chronological time, we need only to point out that he may be an atypical human, and so the qualities we associate with typical humans need not apply to him.

To make the previous paragraph more formal, we remind the reader of the discussion held around Definition 13: A formula ϕ is a logical consequence of a set Γ thereof if every model of Γ is also a model of ϕ. Put differently, there is no valuation (or, possible world) where Γ is true and ϕ is false. It follows directly that ϕ remains a logical consequence of $\Gamma \cup \{\psi\}$, since Γ is true in any world where $\Gamma \cup \{\psi\}$ is true. It was pointed out in [Shoham, 1987] that we may "bend the rules" and restrict semantic consideration to a privileged subset of models deemed "preferable". We call these selected models the *minimal models*—the reasons for this name should become clearer as the chapter progresses.

3.2 Preferential Reasoning and the KLM Framework

The KLM framework for non-monotonic reasoning was initially described by a collection of consequence relations satisfying certain axioms—frequently called the *rationality postulates*, thought to describe a reasonable account of non-monotonic consequence. We borrow a nice story from [Gabbay, 1985], in which he motivates why consequence relations are a good starting point for the study of non-monotonic systems.

Paraphrasing, he begins by asking the reader to imagine a machine that does non-monotonic inference in some domain. The machine represents knowledge as formulae and so we pose queries of the form "Does ψ non-monotonically follow from ϕ?". Something goes awry (suppose some coffee was spilled), calling into question whether the logic of the machine still functions correctly. Even worse, the interface, which tells us what real-world instance each formula maps to, is destroyed and so function cannot be evaluated based on the meaning of the formulae the machine reasons on. How might we then evaluate the machine's function?

If we were interested in classical consequence, we would be well-equipped to assess the correctness of the machine by determining if it satisfied reflexivity, monotonicity, and cut (we point to Definition 16 as a reminder). This is precisely the starting point that Kraus, Lehmann, and Magidor took up in [1990], suggesting that before getting to the semantics of a non-monotonic system, it is a good idea to formalise axiomatise the system as a consequence relation satisfying certain properties.

The rationality postulates are precisely this axiomatisation, characterising a sensible pattern of reasoning for non-monotonic systems. We use '$\mid\sim$' (pronounced "twiddle") instead of '$\vdash$' to denote a non-monotonic consequence relation. As we may expect, $\phi \mid\sim \psi$ has the same meaning as $(\phi, \psi) \in \mid\sim$, and $\phi \not\mid\sim \psi$ as $(\phi, \psi) \notin \mid\sim$. We may, at times of potential confusion, use a subscript to disambiguate which consequence relation is being referred to, and so $\mid\sim_p$ would refer to a cumulative relation, as defined below.

3.2.1 System P

We will begin the exposition of the KLM framework by looking at *system P*. The pattern of reasoning associated with this system is captured by *preferential consequence relations*. Either we need to justify the choice of omitting cumulative relations, or include them.

Definition 32. The consequence relation $\mid\!\sim$ is a *preferential consequence relation* if and only if it satisfies the properties of *Reflexivity*, *Left Logical Equivalence*, *Right Weakening*, *And*, *Or*, and *Cautious Monotony*.

The first axiom, *Reflexivity*, is largely self justifying. It makes little sense to speak about a notion of consequence that does not satisfy this property.

$$\text{(Reflexivity)} \quad \frac{}{\phi \mid\!\sim \phi} \tag{3.1}$$

The justification for *Left Logical Equivalence* is a bit more opaque. The principle it describes is that if two scenarios represent the same state of affairs, and in one of these scenarios it we typically expect some consequence, then we should expect the same in the other scenario. Invariant to substitution of logically equivalent formulas?

$$\text{(Left Logical Equivalence)} \quad \frac{\vdash \phi \leftrightarrow \psi, \quad \phi \mid\!\sim \gamma}{\psi \mid\!\sim \gamma} \tag{3.2}$$

Right Weakening allows the preservation of classical consequence within preferential logic. It says that, if from ϕ we normally expect ψ, but from ψ we *always* see γ, then we are entitled to think that from ϕ we normally expect γ as well.

$$\text{(Right Weakning)} \quad \frac{\vdash \psi \rightarrow \gamma, \quad \phi \mid\!\sim \psi}{\phi \mid\!\sim \gamma} \tag{3.3}$$

As a justification for *Or*, consider that *Normally, if Billy were abducted by aliens he would be traumatised*, but also *Normally, if Billy witnessed the fire-bombing of Dresden he would be traumatised*. If we learned that either of these events took place, we should find it plausible to infer that Billy were traumatised.

$$\text{(Or)} \quad \frac{\phi \mid\!\sim \gamma, \quad \psi \mid\!\sim \gamma}{\phi \vee \psi \mid\!\sim \gamma} \tag{3.4}$$

The *And* postulate suggests that if ψ and γ are both expected consequence of ϕ, then their conjunction is also expected. This postulate fails in probabilistic systems, such as *association rules* [Gabbay, 1985].

$$\text{(And)} \quad \frac{\phi \mid\!\sim \psi, \quad \phi \mid\!\sim \gamma}{\phi \mid\!\sim \psi \wedge \gamma} \tag{3.5}$$

Cautious Monotony (which has also be called *Cumulative Monotony* by [Makinson, 2003], and *Restricted Monotony* by [Gabbay, 1985]) corresponds to the notion that if we are in an epistemic state ϕ where one expectation, among others, is that ψ holds. Learning that ψ indeed holds should not alter the epistemic state in such a way that the *other* expectations are abandoned, and so the new state, $\phi \wedge \psi$, we should expect everything that was expected when all we knew was ϕ. In other words, we reason monotonically with respect to expected information.

$$\text{(Cautious Monotony)} \quad \frac{\phi \mathrel{|\!\sim} \psi, \quad \phi \mathrel{|\!\sim} \gamma}{\phi \wedge \psi \mathrel{|\!\sim} \gamma} \tag{3.6}$$

The above postulates capture the essence of preferential (consequence) relations. Much like we saw with Hilbert systems, however, certain rules, which reveal interesting properties of system P, may be derived. A version of *Cut*:

$$\text{(Cut)} \quad \frac{\phi \wedge \psi \mathrel{|\!\sim} \gamma, \quad \phi \mathrel{|\!\sim} \psi}{\phi \mathrel{|\!\sim} \gamma} \tag{3.7}$$

The original version due to Gentzen [Ben-Ari, 2012], which is presented as:

$$\text{(Monotonic Cut)} \quad \frac{\phi \wedge \psi \mathrel{|\!\sim} \gamma, \quad \alpha \mathrel{|\!\sim} \psi}{\phi \wedge \alpha \mathrel{|\!\sim} \gamma} \tag{3.8}$$

implies monotonicity, as it requires that if ψ is a typical consequence of α, then it must remain a consequence of $\alpha \wedge \phi$: ergo, monotonicity. The former variation does not enforce this, and rather says "Suppose I have certain knowledge of ϕ, and that if I were to assume ψ I should expect to conclude γ. Then if I can show that infact ψ was already an expected consequence of knowing ϕ, I should expect that γ follows from ϕ". When considered alongside the argument for Cautious Monotony, Cut seems obviously acceptable.

The rule S, which is analogous to one half of the deduction theorem (cf. Equation (1.27)), expresses the notion that we retain plausible inferences when conditioned on a hypothesis.

$$\text{(S)} \quad \frac{\phi \wedge \psi \mathrel{|\!\sim} \gamma}{\phi \mathrel{|\!\sim} \psi \to \gamma} \tag{3.9}$$

The following Lemma is a helpful intuition pump: it suggests that we can use the consequences of plausible inferences to make further plausible inferences.

Lemma 2. *We can cover the properties of* cut *and* cautious monotonicity *with the following principle: "If $\phi \mathrel{|\!\sim} \psi$, then the typical consequences of ϕ and $\phi \wedge \psi$ coincide".*

It is, however, worth reminding ourselves of the goal that was set out at the start of this chapter: To formalise a system which facilitates a more credulous style of reasoning. That is, we are interested in making *more* inferences than can be made under classical notions of consequence. One property which has not been discussed—and which may be quite easy to assume—is whether the proposed system be *supraclassical* [Makinson, 2003].

$$\text{(Supraclassical)} \quad \frac{\phi \to \psi}{\phi \mathrel{|\!\sim} \psi} \tag{3.10}$$

Supraclassicality is the question of whether classical inferences be preserved in a non-classical system. It seems reasonable to answer this question in the affirmative. Defeasible inferences are useful as they tolerate reasoning with exceptions, allowing useful inferences that could not be made in classical systems. It seems that the dual of this, where some non-defeasible (that is, classical) inference is made, it is because no exceptions apply. In this case, we have an even stronger claim to make this inference.

3.2.1.1 Preferential Interpretations

The semantics of system P, provided by Kraus, Lehmann, and Magidor [Kraus *et al.*, 1990], are based on the *preference logics* introduced by Shoham [Shoham, 1987]. The fundamental idea is that valuations, or *worlds*, can be ordered by a *preference relation* so that one world being preferred to another is a normative claim that we should consider deductions which hold in the preferred world—but may not in other worlds—as plausible.

It will be useful to recall some definitions from propositional logic: a valuation $u \in \mathcal{U}$ is a *model* of a formula $\phi \in \mathcal{L}$ if it satisfies ϕ, ϕ is *satisfiable* if it has a model. Another formula ψ is a *logical consequence* of ϕ if the models of ϕ are a subset of the models of ψ.

We now introduce analogous definitions in the setting of preferential logic. The first change is that we consider a richer language, $\mathcal{L}_+$, which is simply $\mathcal{L}$ extended with the new connective '$\mid\!\sim$' such that $\phi \mid\!\sim \psi$ is a sentence in $\mathcal{L}_+$ when $\phi, \psi \in \mathcal{L}$. It is easy to skip over, but usage of the '$\mid\!\sim$' has now shifted from an element of the metalanguage, where it refers to a kind of consequence relation, to an object level connective, used analogously to the classical Boolean operator '$\rightarrow$'. Certain restrictions on the usage of '$\mid\!\sim$' do apply. In particular, the nesting of defeasible statements is not permitted, and so $\phi \mid\!\sim (\psi \mid\!\sim \gamma)$ is not a statement in the language $\mathcal{L}_+$. At times, we may wish to distinguish between formulae in $\mathcal{L}_+$ which do not use '$\mid\!\sim$', and so we refer to these as classical, and the alternative as defeasible formulae.

Definition 33. A *preferential interpretation* $\mathcal{W} = \langle S, l, \prec \rangle$ is a triple where S is a set of *states*, $l : S \rightarrow \mathcal{U}$ is a function mapping states to valuations, and $\prec$ is a strict partial-order on S.

The preference relation provides a basis for restricting ourselves to consider only those preferred worlds, which represent a normal state of affairs. This idea is formalised as *minimal states*:

Definition 34. Given a preferential interpretation $\mathcal{W} = \langle S, l, \prec \rangle$ and some $\phi \in \mathcal{L}_+$, we write $\underline{\hat{\phi}}$ to denote the set $\{s \in \hat{\phi} \mid \nexists s' \in \hat{\phi}$ such that $s' \prec s\}$, which is the set of *preferred states* that satisfy ϕ. We may also call these the *ϕ-minimal* states.

It is necessary to stress the distinction between states in a preferential interpretation and valuations. There is no requirement that the mapping from states to valuations be injective, and so the same valuation may appear many times in the preference relation under distinct states. A corollary is that, given a finite set of valuations, the set of states may be infinite. In turn, it is often required that the preference relation of a preferential interpretation satisfy the *smoothness* condition (which has also been called *bounded* in [Shoham, 1987], or *stoppering* [Makinson, 2003]):

Definition 35. A preferential interpretation $\mathcal{W} = \langle S, l, \prec \rangle$ is called *smooth* if and only if for each $\phi \in \mathcal{L}_+$, and each state $s \in \hat{\phi}$ satisfying ϕ, either $s \in \underline{\hat{\phi}}$ or there exists another state $s' \in \hat{\phi}$ where $s' \in \underline{\hat{\phi}}$.

The smoothness condition is violated by preference relations which fail to satisfy transitivity, or where there are infinite descending chains of more preferred states [Schlechta, 1996]. The consequence relations of preferential interpretations with non-smooth preference relations may fail to satisfy cautious monotony [Kraus *et al.*, 1990; Makinson, 2003]. give a demonstration of why

The language $\mathcal{L}_+$ was introduced as the union of the standard propositional language $\mathcal{L}$ and the language of defeasible conditionals. There are, regrettably, distinct formulations of what it means for a preferential interpretation to satisfy a classical or defeasible formula. For the classical case, we have:

Definition 36. Given a preferential interpretation $\mathcal{W}$, we say that a state $s \in S$ *satisfies* a classical formula $\phi \in \mathcal{L}_+$ if and only if the valuation $l(s) \vDash \phi$. In this case, we write $\mathcal{W}, s \vDash \phi$, and use $\hat{\phi}$ to denote the set $\{s \in S \mid \mathcal{W}, s \vDash \phi\}$. Then, $\mathcal{W}$ *satisfies* ϕ if and only if $\mathcal{W}, s \vDash \phi$ for all $s \in S$.

What it means for a preferential interpretation to satisfy a classical formula is unsurprising, and corresponds to the idea that the formula holds in every possible world. We are more interested in what it means for defeasible formulae to be satisfied, and so we have:

Definition 37. A defeasible formula $\phi \mathrel{|\!\sim} \psi \in \mathcal{L}_+$ is *satisfied* by a preferential interpretation $\mathcal{W} = \langle S, l, \prec \rangle$ if and only if the every minimal state $s \in \underline{\hat{\phi}}$ also satisfies ψ. In this case we write $\mathcal{W} \Vdash \phi \mathrel{|\!\sim} \psi$ and say that $\mathcal{W}$ is a *preferential model* of $\phi \mathrel{|\!\sim} \psi$.

One pleasing perspective on the semantics of preferential consequence relations is that we require little in the way of "new machinery". Outside of the preference relation, satisfaction of a defeasible conditional behaves similarly to satisfaction of classical formulae, only with a restricted consideration on states.

Example 9. *Consider the knowledge base*

$$\Delta = \{h \mathrel{|\!\sim} c,\ s \mathrel{|\!\sim} h,\ b \rightarrow \neg c\}$$

containing both defeasible conditionals, as well as classical propositional formula. The knowledge base encodes the setting where "Humans normally experience time chronologically", "Soldiers are normally human", and "Billy Pilgrim experiences non-chronological time."

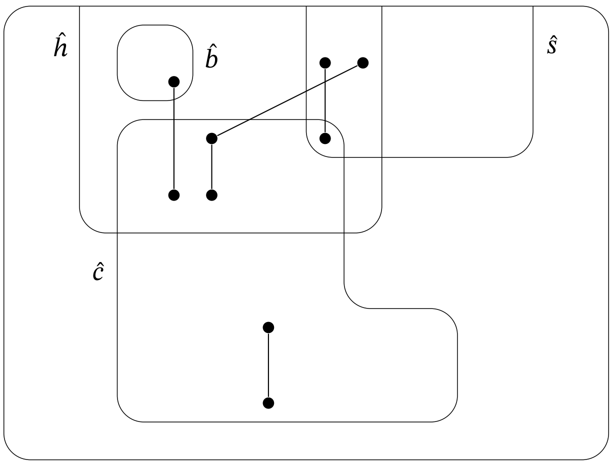


Figure 3.1: A preferential model of Δ

Figure 3.1 demonstrates one model of Δ. It is relatively easy to verify that each statement (classical and defeasible) in Δ is satisfied by the preferential interpretation. As a means of providing some intuition for the preceding discussion, we remind the reader of Lemma 2, which says that "If $\phi \mathrel{|\!\sim} \psi$, then the plausible consequences of $\phi \wedge \psi$ coincide with those of just ϕ." As we have a model of $h \mathrel{|\!\sim} c$, the plausible consequences of h and $h \wedge c$ should coincide. This Lemma becomes quite apparent when it is recognised that the two sets $\underline{\hat{h}}$ and $\underline{\widehat{h \wedge c}}$ are precisely the same sets.

Within the context of preferential models, there is another treatment of classical formulae which allows for the discarding of Definition 36 and a single notion of what it means for a preferential interpretation to be a model of a formula in $\mathcal{L}_+$. The idea is to translate the hard-constraints of classical formulae to defeasible syntax.

Lemma 3. *If ϕ is a classical formula in the language $\mathcal{L}_+$, then it can be expressed as the defeasible conditional $\neg\phi \mid\sim \bot$. Then, any preferential interpretation $\mathcal{W}$ is a model of ϕ if and only if it is a model of the defeasible conditional $\neg\phi \mid\sim \bot$.*

To explain the mechanics behind Lemma 3, consider that a preferential interpretation $\mathcal{W}$ being a model of the conditional $\neg\phi \mid\sim \bot$ is equivalent to the $\neg\phi$-minimal states being a subset of the states which satisfy $\bot$. Of course, $\bot$ represents logical falshood, and the set of states which satisfy it is the emptyset. In turn, if $\mathcal{W}$ is a model of $\neg\phi \mid\sim \bot$, there can be no states which satisfy $\neg\phi$, which is equivalent to the condition that all states satisfy ϕ. If ϕ is thought of as the proposition "All men are mortal", then the defeasible counterpart reads as "If not all men are mortal, then it is normal to infer anything, including a contradiction" [Kraus *et al.*, 1990; Lehmann and Magidor, 1992].

We assume this treatment of classical formulae from this point onwards, and consolidate the notation as $\mathcal{W} \Vdash \phi$ where $\phi \in \mathcal{L}_+$ is a defeasible conditional.

maybe something about how without smoothness, we don't have CM

Theorem 4 (Soundness)**.** *If $\mathcal{W} = \langle S, l, \prec \rangle$ is a preferential interpretation and $\phi, \psi \in \mathcal{L}_+$, then $\mathcal{W}$ defines the consequence relation $\mid\sim_{\mathcal{W}}$ given by the set $\{\phi \mid\sim_{\mathcal{W}} \psi \mid \mathcal{W} \Vdash \phi \mid\sim \psi\}$, which satisfies Reflexivity, Left Logical Equivalence, Right Weakening, Or, and Cautious Monotony and is thus a preferential consequence relation.*

Theorem 5 (Completeness)**.** *If $\mid\sim_P$ is a preferential consequence relation, then there exists a preferential interpretation W which induces the consequence relation $\mid\sim_W$ such that $\mid\sim_P$ is precisely $\mid\sim_W$.*

The decision to construct the preference relation on states which map to valuations, rather than valuations directly, may seem arbitrary. Certainly, Shoham [Shoham, 1987] makes no such distinction in his preferential logic. However, Kraus, Lehmann, and Magidor [Kraus *et al.*, 1990] provide an example, apty described as "en pessant" by [Bezzazi *et al.*, 1997], which shows that it is necessary in the setting of preferential consequence relations. The example shows a preferential consequence relation generated by a preferential interpretation where the function mapping states to valuations is not injective. It is then shown that there is no injective counterpart which gives rise to an equivalent consequence relation.

Lemma 4. *There exists a non-injective preferential interpretation $\mathcal{W}$ which defines the consequence relation $\mid\sim_{\mathcal{W}}$ such that no injective preferential interpretation defines the same consequence relation.*

Figure 3.2 shows a portion of the preferential interpretation in Example 9. The states s_1 and s_2 are duplicate states, specifically they both map to the valuation $\{s, h, \overline{c}\}$. As before, this prefrential interpretation is a model of $s \mid\sim h$ as the set of minimal s-states, $\{s_2, s_3\}$, is a subset of the h-states.

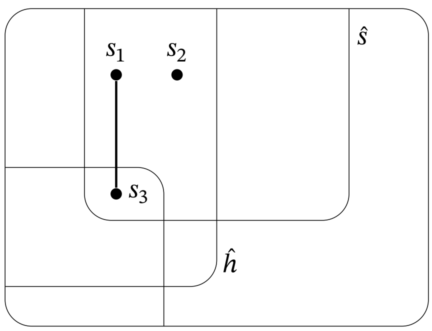


Figure 3.2: The *en pessant* argument for allowing non-injective interpretations

The interpretation is not a model of the conditional $s \mid\sim c$, due to the counter-example of s_2 being a typical soldier who does not experience chronological time. Should s_1 and s_2 be merged, the result would be an interpretation which is a model of $s \mid\sim c$; that is, a distinct consequence relation.

3.2.2 Preferential Entailment

Equipped with definitions for the class of preferential relations and a representation theorem that relates these to a corresponding semantics in the form of preferential interpretations, it is appropriate to discuss the matter of *preferential entailment*, or *p-entailment*.

Definition 38. A defeasible knowledge base Δ *preferentially entails* a defeasible conditional $\phi \in \mathcal{L}_+$ if and only if ϕ is satisfied by every preferential model of Δ, in this case we write $\Delta \mid\approx_p \phi$. The set of all defeasible conditionals that can be preferentially entailed from Δ is denoted by $\mathcal{C}n_p(\Delta)$ and is called the *preferential closure* of Δ.

The definition provided for p-entailment is worryingly similar to the Tarskian idea of logical consequence, which was described in Subsection 1.3.3. It is "worrying" in the sense that Tarskian consequence is monotonic, and we are interested in non-monotonic entailment relations. Indeed, our fear is justified: If we consider the set of preferential models of a defeasible knowledge base Δ, then the set of conditionals they all agree on is exactly the preferential closure, $\mathcal{C}n_p(\Delta)$. The addition of another conditional to our knowledge base, resulting in $\Delta \cup \{\phi\}$, only serves to reduce consideration to those models of Δ that also satisfy ϕ. Within this restricted set of models, the consensus on $\mathcal{C}n_p(\Delta)$ remains. Formally, we have:

$$\Delta \subseteq \Delta \cup \{\phi\} \quad \textit{implies} \quad \mathcal{C}n_p(\Delta) \subseteq \mathcal{C}n_p(\Delta \cup \{\phi\}) \tag{3.11}$$

which confirms our fear that p-entailment is monotonic.

There is room for some (justified) confusion here. After all, we have carefully avoided satisfying otherwise useful properties because they imply monotonicity, only to end up with precisely that. To clarify this, we recognise that a system can display non-monotonic behaviour in different areas. Indeed the entailment relation of system P—which reasons on the metalevel—is monotonic, while the object level expresses non-monotonic behaviour, facilitating the expression of statements like *Normally Humans experience chronological time*, *Billy Pilgrim is a Human*, *Billy Pilgrim experiences non-chronological time*.

There are arguments, in fact we make one in chapter:defeasible-reasoning-in-fca, which suggest that object level non-monotonicity is sufficient.

3.2.3 Rational Consequence Relations

System P provides a characterisation, in the form of its postulates, of a particular style of non-monotonic reasoning. There are other kinds of reasoning, not valid in system P, for which there are good reasons think of as reasonable properties of non-monotonic reasoning [Kraus *et al.*, 1990; Lehmann and Magidor, 1992].

Negation Rationality suggests that our defeasible inferences should withstand piercing the veil of ignorance. Suppose Alice is a university student, if she holds the belief that *Normally, she will pass her exams*, then it would not be reasonable for her to simultaneously think *Normally, if she studies, she will not pass her exams* and also *Normally, if she does not study, she will not pass her exams*.

$$\text{(Negation Rationality)} \quad \frac{\phi \wedge \psi \mathrel{|\!\!\not\sim} \gamma, \quad \phi \wedge \neg\psi \mathrel{|\!\!\not\sim} \gamma}{\phi \mathrel{|\!\!\not\sim} \gamma} \tag{3.12}$$

Either Alice studies or she does not. If, in either case, she will not pass her exams, then it is difficult to comprehend how it might be normal for her to pass her exams.

Disjunctive Rationality says that a plausible inference made from a disjunction should be plausible from at least one of the disjuncts.

$$\text{(Disjunctive Rationality)} \quad \frac{\phi \mathrel{|\!\!\not\sim} \gamma, \quad \psi \mathrel{|\!\!\not\sim} \gamma}{\phi \vee \psi \mathrel{|\!\!\not\sim} \gamma} \tag{3.13}$$

Rational monotony is a much stronger form of restricted monotony than cautious monotony. Where cautious monotony holds the position that we should only reason monotonically with the addition of new information that was already expected given our beliefs, rational monotony suggests that we should reason monotonically with the addition of new information if it is merely *consistent* with existing beliefs; that is, we did not expect the negation of the new information.

$$\text{(Rational Monotony)} \quad \frac{\phi \wedge \psi \mathrel{|\!\!\not\sim} \gamma, \quad \phi \mathrel{|\!\!\not\sim} \neg\psi}{\phi \mathrel{|\!\!\not\sim} \gamma} \tag{3.14}$$

Rational monotony can be equivalently stated as the following, which makes the intuition a bit clearer.

$$\text{(Rational Monotony)} \quad \frac{\phi \mathrel{|\!\sim} \psi, \quad \phi \mathrel{|\!\!\not\sim} \neg\gamma}{\phi \wedge \gamma \mathrel{|\!\sim} \psi} \tag{3.15}$$

At a university, it might be normal for students to graduate. At the same time, it would be too great a requirement to suggest that it is normal for students to be exceptionally clever. It seems completely appropriate to maintain our belief that it is normal for students, who are not exceptionally clever, to graduate.

Stalnaker [Stalnaker, 1994; Strasser and Antonelli, 2024], and separately Ginsberg [1986], provide an argument against accepting rational monotony. The objection is that rational monotony commits us to make inferences when the more reasonable position is to remain undecided. Paraphrasing, we are asked to consider three composers: *Verdi*, who is believed to be Italian, and *Satie* and *Bizet*, who are believed to be French. If it were learned that Verdi and Bizet were compatriots, then it seems reasonable that the conclusion that either of them are French or Italian is indeterminate. What is reasonable, however, is to maintain the belief that Satie is French.[1]

$$\texttt{Compatriots(Bizet, Verdi)} \mathrel{|\!\sim} \texttt{French(Satie)} \tag{3.16}$$

The argument proceeds by considering that Verdi and Satie may too be compatriots. Since Verdi's nationality is now unknown, we are no longer equipped to reject the plausibility of the new compatriotship, and so

$$\texttt{Compatriots(Bizet, Verdi)} \mathrel{|\!\!\not\sim} \neg\texttt{Compatriots(Verdi,Satie)} \tag{3.17}$$

[1]We will use some syntax from predicate logic, this is merely an aid to make Stalnaker's example clearer.

The two conditions of rational monotony have been met, and so the following inference should continue to hold

$$\texttt{Compatriots(Bizet, Verdi)} \land \texttt{Compatriots(Verdi,Satie)} \mid\!\sim \texttt{French(Satie)} \tag{3.18}$$

The issue, as Stalnaker sees it, is that rational monotony forces the inference Satie is still French, and by their compatriotship, so are Bizet and Verdi. The "correct" thing to do in this scenario is to remain uncertain about which nationality they all share.

However, the result Stalnaker objects to is dependent his particular construction of this problem, rather than being intrinsic to rational monotony. The second condition for rational monotony, that $\phi \mid\!\not\sim \neg\gamma$, makes no requirement about the syntactic structure of γ outside of it being a formula in the language. In the example, Stalnaker could just as easily have written

$$\texttt{Compatriots(Bizet,Verdi)} \mid\!\not\sim \neg(\neg\texttt{Compatriots(Verdi,Satie)}) \tag{3.19}$$

meaning that it is not expected that Verdi and Satie are compatriots, to which we can apply an instance of rational monotony, from which we are not forced to believe that all three composers are French.

$$\texttt{Compatriots(Bizet, Verdi)} \land \neg(\texttt{Compatriots(Verdi,Satie)}) \mid\!\sim \texttt{French(Satie)} \tag{3.20}$$

There is another objection to rational monotony, which will be considered later on. For now, we accept it as a valid property of non-monotonic reasoning, and thus introduce *rational relations*.

Definition 39. The consequence relation $\mid\!\sim$ is a *rational consequence relation* if and only if it satisfies the properties of *Reflexivity*, *Left Logical Equivalence*, *Right Weakening*, *And*, *Or*, *Cautious Monotony*, and *Rational Monotony*.

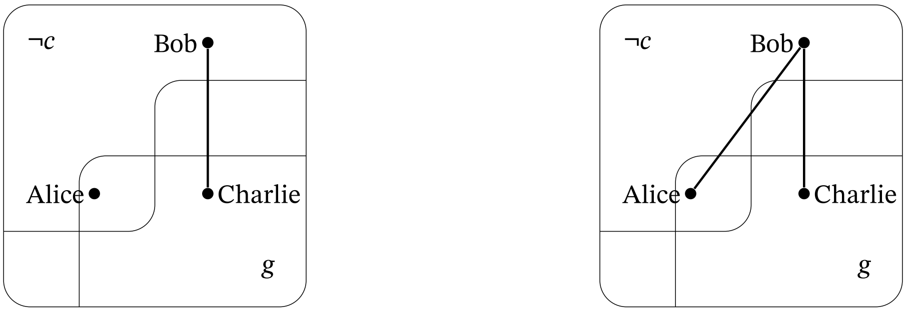
Quite evidently, they are a particular kind of preferential relation.

3.2.3.1 Ranked Interpretations

From the knowledge that rational relations are a particular kind of preferential relation, it seems a reasonable jump to think that the semantics we give to rational relations may be a particular kind of preferential interpretation. Before discussing this jump, a useful exercise is to recognise how preferential interpretations may fail to satisfy rational monotony.

Consider the two preferential interpretations shown in Figure 3.3, which model the scenario which was discussed in the example near the end of the last subsection.

Example 10. *Suppose that neither Alice nor Bob is a clever student. In spite of her status, Alice is considered by the preference relation to be a normal student, alongside Charlie. This situation is modelled in each of the preferential interpretations below.*



(a) Non-modular preference relation (b) Modular preference relation

Figure 3.3: Two preferential models of the above scenario

We may verify that both are models of `student` $\mid\!\sim$ `graduate`*, and neither are models of* `student` $\mid\!\sim$ `clever`*. By rational monotony, we should expect that* `student` $\wedge$ `¬clever` $\mid\!\sim$ `graduate` *be satisfied, which is not the case in Subfigure 3.3a.*

Fortunately, the culprit is quite easy to spot. Bob, who is not under consideration in the context of normal students, appears when we discuss normal students who are not clever. In isolation, this seems acceptable: if all the normal students were also clever, then we would be fine accepting the appearance of a normal, non-clever student who was not a normal student. However, we have explicitly ruled out this line of argument—by the second antecedent of rational monotony—as Alice is a normal student who is also not clever.

In order to ensure rational monotony, it is required that the preference relation commit to Alice being preferred to Bob; more generally, all normal students should be explicitly preferred to non-normal students. This notion is formalised by the following property called *modualarity*:

Definition 40. A partial order $\prec$ on the set X is *modular* if and only if for any incomparable $x, y \in X$ when $z \prec x$ then also $z \prec y$.

One perspective on *modular* partial orders is that they partition the set of states into equivalence classes of preference, so that all states in the same class are incomparable to one another [Ginsberg, 1986]. Then, if the preference relation is defined as the equivalence relation itself, such that one state is preferred to another if the class it belongs to is preferred to the other's, we get a modular order.

Definition 41. A *ranked interpretation* is a preferential interpretation $\mathcal{W} = \langle S, l, \prec \rangle$ where the preference relation $\prec$ is *modular*.

A somewhat superficial result of ranked interpretations is that they admit a new representation as a stratified equivalence relation, depicted in Figure 3.4. Each strata, or *rank*, has an associated ordinal.

Lemma 5. *If* $\prec$ *is a modular partial order on* X *then there is a* ranking function $\mathrm{r} : X \to \Omega$*, with* Ω *being a totally ordered set for which the strict order is denoted* $<$*, such that if* $x \prec y$ *then* $\mathrm{r}(x) < \mathrm{r}(y)$ *for any* $x, y \in X$*.*

1	$\{c, s\}$
0	$\{c, g, s\}$ $\{g, s\}$

Figure 3.4: A depiction of a ranked interpretation

A second, less superficial, departure is that Lemma 4 does not hold in the context of ranked interpretations and rational consequence relations. This follows directly from modularity, as the en pessant style arguments, discussed earlier, are avoided by modularity. This allows the elements which populate the ranks of a ranked interpretation to be valuations directly, rather than states.

Definition 42. A *ranked interpretation* is a function $\mathsf{r} : \mathcal{U} \to \mathbb{N}$

Talk about smoothness with ranked models

3.2.4 Ranked Entailment

3.3 Rational Closure

Part II

Rational Concept Analysis

Chapter 4

Defeasible Reasoning in Formal Concept Analysis

The conclusion of Part I marks the end of the scaffolding component of this work. The remaining chapters are devoted to the unification of these ideas. That is, to develop an approach to defeasible reasoning in FCA in the style of the KLM framework. We begin this construction by introducing an FCA-counterpart to the defeasible conditionals that were discussed in Chapter 3.

Fortunately, the context switch is quite natural. As we progress, the reasons behind the selection of the KLM framework as an ideal candidate for defeasible reasoning in FCA should become apparent. Indeed, understanding the proceeding chapter(s) is benefitted by frequently "pattern-matching" between the definitions presented here, and their counterparts from prior chapters. So we encourage—and will attempt to facilitate—these frequent backwards-facing references.

4.1 Motivation

There are multiple views one may consider when thinking about non-monotonic reasoning in FCA, each with its own implicit justification for violating monotonicity. The most familiar one, which has already been discussed in Chapter 3, takes up the position that *usually* there are exceptional and typical instances, and that there should be a clear way to reason prototypically. Certainly, there is good reason for this argument in FCA; but this will be the subject of the next chapter.

The perspective we consider here views FCA as a tool which facilitates the extraction of information from data. Of course, data is frequently polluted with errors or noise, which inhibit the discovery of useful information. In this case *exceptions* are the erroneous datum

Exceptions in this case are erroneous datum [Sacco *et al.*, 2024]. The point, then, is to facilitate reasoning from

Represents a kind of counter factual: If these implications had been true, then these would follow

4.1.1 Association Rules

4.2 Ranked Contexts

4.2.1 Finding Order

Algorithm 2 OBJECTRANK

Input: A set Δ of defeasible conditionals over an attribute set M
Input: A formal context (G, M, I)
Output: A ranking $\mathsf{R} : G \to \mathbb{N}$ such that Δ holds in (G, M, I, R) if (G, M, I) is Δ-compatible; $\bot$, otherwise.

```
i := 0
initialise R(g) := |G| for all g ∈ G
Γ := Δ
while ∃ g ∈ G : R(g) = |G| do
    for all g ∈ G with R(g) = |G| do
        if g ⊨ Γ then update R(g) = i
        end if
        if ∀ g ∈ G : R(g) ≠ i then
            return ⊥
        end if
        Γ := {(φ |~ ψ) ∈ Γ | g ⊭ φ for all g ∈ G with R(g) = i}
        i := i + 1
    end for
end while
return R
```

4.2.1.1 Complexity

4.3 Entailment

4.3.1 Contextual Rational Closure

Definition 43. A conditional $A \mid\!\sim B$ *rationally follows* from a set $\mathcal{T}$ of conditionals if and only if $\mathbb{T}, \mathcal{T} \mid\!\approx A \mid\!\sim B$, where $\mathbb{T}$ is the *test context*. It *contextually follows* when $\mathbb{R}, \mathcal{T} \mid\!\approx A \to B$.

Definition 44. A set $\mathcal{T}$ is *closed* with respect to a pair if and only if it contains every conditional that is rationally entailed. That is, if $\mathcal{T} = \{A \mid\!\sim B \mid A, B \subseteq M \text{ and } \mathbb{T}, \mathcal{T} \mid\!\approx A \mid\!\sim B\}$. $\mathcal{T}$ is *contextually closed* if it contains every conditional that is contextually entailed, i.e. $\mathcal{T} = \{A \mid\!\sim B \mid A, B \subseteq M \text{ and } \mathbb{R}, \mathcal{T} \mid\!\approx A \to B\}$.

Definition 45. A set $\mathcal{T}$ of conditionals is *rationally complete* with respect to $\mathbb{T}, \Delta$ if and only if every conditional that is in the rational closure of $\mathbb{T}, \Delta$ rationally follows from $\mathcal{T}$, i.e., $\mathbb{T}, \mathcal{T}$ and $\mathbb{T}, \Delta$ define the same entailment relation.

Is it possible to find a $\mathcal{T}$ such that $\mathcal{T}$ is a compact form of Δ. I.e., it should produce the same ranking on $\mathbb{T}$.

Definition 46. A set $\mathcal{T}$ of conditionals is *contextually complete* with respect to $\mathbb{R}, \Delta$ if and only if every implication that is in the contextual closure of $\mathbb{R}, \Delta$ rationally follows from $\mathcal{T}$.

A set of conditionals that is contextually complete to a ranked context $\mathbb{R}$ and constraint set Δ provides abstracted perspective, in terms of defeasible conditionals, on the information contained in the pair $(\mathbb{R}, \Delta)$. It is now the goal, as was done in Subsection 2.2.1.2, to find such a set that is non-redundant, or the *rational basis* of $(\mathbb{R}, \Delta)$.

Definition 47. Let $\mathcal{T}$ be a set of defeasible conditionals over M. The operator $X \mapsto \mathcal{C}n_{\mid\sim}(X)$ given by $X \cup \{Y \subseteq M \mid X \mid\sim Y\}$ is called the *defeasible closure* of X. It is idempotent, extensive, and non-monotonic.

Remark 2. If $\mathcal{T}$ were the defeasible basis we would want $\mathcal{C}n_{\mid\sim}(X) = (\underline{X^{\downarrow}})^{\uparrow}$

4.3.2 Rational Closure

4.3.3 A Basis for Ranked Contexts

4.3.4 Lexicographic Closure

Chapter 5

Rational Concepts

5.1 Motivation

5.2 Background

5.3 Preconcepts

5.4 Naive Rational Concepts

5.5 Version II

5.6 Relation to Contextual Rational Closure

Chapter 6

Conclusions and Future Work

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