Rational Concept Analysis

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Contents

Chapter 1

Mathematical Preliminaries

This chapter is intended to serve as a reference point, clarifying ideas and notation for the more fundamental concepts that will be used throughout the remainder of this dissertation. In the first section we discuss order and lattice theory. The third and final section introduces propositional logic, as well more general ideas in logic.

1.1 Order and Lattice Theory

This section refers extensively to the fundamental text by Davey and Priestley [1], as well as Ganter and Wille [2].

1.1.1 Orders

A *binary relation* R over the sets X and Y is a set of ordered pairs (x, y) where $x \in X$ and $y \in Y$. We may choose to express $(x, y) \in R$ using infix notation and write xRy, which tells us that R relates x to y.

Certain binary relations, satisfying specific properties, occur frequently enough to warrant their own denomination. One such relation, which will be used in almost every section of this dissertation, is called a *partial order*.

Definition 1. A *partial-order* is a binary relation $\preceq \subseteq X \times X$ that satisfies the following properties:

$$\text{(Reflexivity)} \quad x \preceq x \tag{1.1}$$

$$\text{(Antisymmetry)} \quad x \preceq y \text{ and } y \preceq x \text{ implies } x = y \tag{1.2}$$

$$\text{(Transitivity)} \quad x \preceq y \text{ and } y \preceq z \text{ implies } x \preceq z \tag{1.3}$$

for all $x, y, z \in X$.

Frequently, 'preference' is used as a metonymy for an order, and so in this context "element x is preferred to y" should be taken to mean that $(x, y) \in \preceq$, or simply $x \preceq y$.

We write $x \not\preceq y$ to indicate that (x, y) is not in the relation, and $x \prec y$ for the case where $x \preceq y$ and $x \neq y$. In the scenario where $x \not\preceq y$ and $y \not\preceq x$—i.e., that x and y are incomparable—we write $x \| y$. From a partial-order we can quite easily induce the notion of a *strict partial-order*.

Definition 2. A *strict partial-order* is a binary relation $\prec \subseteq X \times X$ that satisfies:

$$\text{(Irreflexivity)} \qquad x \nprec x \tag{1.4}$$

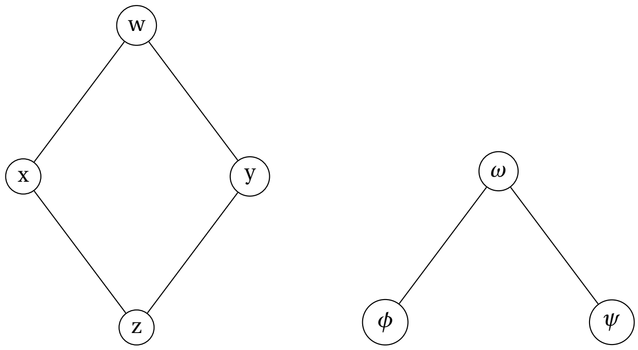
$$\text{(Asymmetry)} \qquad x \prec y \text{ implies } y \nprec x \tag{1.5}$$

$$\text{(Transitivity)} \qquad x \prec y \text{ and } y \prec z \text{ implies } x \prec z \tag{1.6}$$

for all $x, y, z \in X$.

An *ordered set* is a pair $(X, \preceq)$ with X being a set and $\preceq$ being an ordering on the elements of X. We make notation easier, and use $\mathbf{X}$ to denote the pair; moreover, the order relation associated with $\mathbf{X}$ may be written $\preceq_X$ in settings where ambiguity arises. If $\mathbf{Y}$ is a subset of $\mathbf{X}$, then $\mathbf{Y}$ inherits the order relation from $\mathbf{X}$; and so, for $x, y \in \mathbf{Y}$, $x \preceq_Y y$ if and only if $x \preceq_X y$.

We can visualise ordered sets through the use of *Hasse* diagrams.



(a) **A** (b) **B**

Figure 1.1: Hasse diagrams of two partially ordered sets

As an illustrative example, from the ordered set in Sub-figure 1.1a we read that $z \preceq x$, as there is a (strictly) upward path from z to x. In fact, it is clear that $z \preceq w, x, y, z$, or that "z is preferred to every other element in $\mathbf{A}$". We say such an element is *minimal*.

More formally, an element $x \in \mathbf{X}$ is *minimal* with respect to the ordering if there exists no distinct element $y \in \mathbf{X}$ such that $y \preceq x$. Conversely, we say x is *maximal* if there exists no distinct $y \in \mathbf{X}$ where $x \preceq y$. Then x is the *minimum* element if $x \preceq y$ for all $y \in \mathbf{X}$; the dual notion of a *maximum* is defined as we might expect.

Definition 3. Let $\mathbf{X}$ and $\mathbf{Y}$ be ordered sets with a mapping $\varphi : \mathbf{X} \to \mathbf{Y}$. We call φ an *order-preserving* (or, isotone) map if $x \preceq_X y$ implies $\varphi(x) \preceq_Y \varphi(y)$. It is an *order-embedding* if it is injective, and $x \preceq_X y$ if and only if $\varphi(x) \preceq_Y \varphi(y)$ for all $x, y \in X$. Finally, φ is an *order-isomorphism* if it is an order-embedding that is also *surjective*. The dual notion to an order-preserving map is an *order-reversing* (or, antitone) map.

1.1.2 Lattice Theory

Lattice theory studies partially ordered sets that behave well with respect to upper and lower bounds. Informally, a lattice is a partially ordered set where any two elements have both a smallest element above them and a largest element below them in the order.

Definition 4. Let $\mathbf{X}$ be a partially ordered set, and $\mathbf{Y} \subseteq \mathbf{X}$ a subset. A *lower bound* of $\mathbf{Y}$ is an element $x \in \mathbf{X}$ such that $x \preceq y$ for all $y \in \mathbf{Y}$. The set of lower bounds of $\mathbf{Y}$ is defined as

$$\mathbf{Y}^l := \{x \in \mathbf{X} \mid x \preceq y \text{ for all } y \in \mathbf{Y}\}.$$

The notion of an *upper bound* (and the set of upper bounds) of $\mathbf{Y}$ is defined dually. If $\mathbf{Y}^l$ has a greatest element, this element is called the *infimum* (or, meet) of $\mathbf{Y}$ and is often denoted $\bigwedge \mathbf{Y}$. Dually, if the set of upper bounds of $\mathbf{Y}$ has a least element, this element is the *supremum* (or, join) of $\mathbf{Y}$ and is written $\bigvee \mathbf{Y}$.

Formally, a *lattice* is a partially ordered set $\mathbf{L}$ where the meet and join exist for any pair of elements. A *complete lattice* is a stronger condition, where the meet and join exist for all subsets of $\mathbf{L}$.

1.1.2.1 Lattices as algebraic structures

We test this now

Another perspective views a lattice as an algebraic structure; that is $(\mathbf{L}, \vee, \wedge)$ where $\vee$ and $\wedge$ are binary operations that are associative, commutative, idempotent, and satisfy the property of absorbtion. That meets and joins are order-preserving; and in fact induce a partial order, is a direct result of each of these properties.

Lemma 1 (The Connecting Lemma)**.** If $\mathbf{L}$ is a lattice with $x, y \in \mathbf{L}$, then the following are equivalent:

1. $x \preceq y$;
2. $x \vee y = y$;
3. $x \wedge y = x$.

Chapter 2

Formal Concept Analysis

Formal Concept Analysis (FCA) provides a simple, and yet mathematically rigorous, framework for identifying and reasoning about "concepts" and their corresponding hierarchies in data [2, 3]. The central view of concepts as a dual between *extension*—what one refers to as instances of a concept—and *intension*—what meaning is ascribed to a concept—is supported by a rich philosophical backing.

2.1 Basic Notions

The universe of discourse in FCA is made-up of sets of *objects* and *attributes*. Objects are extensional, they are the things pointed to as instances of some more general concept. In turn, attributes construct the intensional component of a concept.

We collect these extensional and intensional building blocks in a structure called a *formal context*, which includes a binary relation that allows traversal from extension to intension, and vice verse.

Definition 5. A *formal context* $\mathbb{K} = (G, M, I)$ is a triple comprised of a set of objects G, a set of attributes M, and a binary relation $I \subseteq G \times M$ referred to as an 'incidence' relation. For an object-attribute pair $(g, m) \in I$ we might say that "object g *has* the attribute m".

A formal context in some sense describes an open-world interpretation, and so $(g, m) \notin I$ is not usually interpreted as saying that "object g has the negation of the attribute m".

Example 1. Finite formal contexts of a reasonable size can be described entirely by a tabular representation. Each object corresponds to a row, and each attribute to a column.

	closure	associativity	identity	divisibility	commutativity
magma	×				
semigroup	×	×			
monoid	×	×	×		
group	×	×	×	×	
abelian group	×	×	×	×	×
loop	×		×	×	
quasigroup	×			×	
groupoid		×	×	×	
category		×	×		
semicategory		×			

Figure 2.1: A formal context showing necessary properties of group-like structures.

The tabular representation of a formal context allows for easy identification of the set of attributes that a given object satisfies: one need only scan across the respective row in the table and note where an '×' symbol appears. This set of attributes is called the *object intent*. The dual notion of an *attribute extent* can similarly be found by scanning down the column of a given attribute. The utility of this visual metaphor diminishes when, as is often the case, we consider larger sets of objects or attributes. Instead, we opt for a more formal approach to determining the intents and extents for (sets of) objects and attributes, respectively.

Definition 6. Given a formal context (G, M, I), the *derivation operators* are two order-reversing maps $(\cdot)^{\uparrow} : 2^G \to 2^M$ and $(\cdot)^{\downarrow} : 2^M \to 2^G$ where the order is given by subset inclusion. Then, for any subsets $A \subseteq G$ and $B \subseteq M$,

$$A^{\uparrow} := \{m \in M \mid \forall g \in A,\ \langle g, m\rangle \in I\}$$
$$B^{\downarrow} := \{g \in G \mid \forall m \in B,\ \langle g, m\rangle \in I\}$$

The derivation operators provide a clear way of describing, for a given set $A \subseteq G$ of objects, the set of attributes which every object in A satisfies, denoted $A^{\uparrow}$. As an illustration, given the set of objects {`semigroup, monoid`} from Figure 2.1, its derivation would be {`closure, associativity`}. It is quite easy to spot that this is just the intersection of the object intents of `semigroup` and `monoid`.

Of course, these two functions can be composed; and so, $A^{\uparrow\downarrow}$ would yield the set of objects {`semigroup, monoid, group, abelian group`}, which can rather cumbersomely be described as "the set of all objects that satisfy all the attributes satisfied by `semigroup` and `monoid`".

In fact, this double-application of derivation operators satisfies very specific properties,

$$\text{(monotonicity)} \qquad A \subseteq A_1 \textit{ implies } A^{\uparrow\downarrow} \subseteq A_1^{\uparrow\downarrow} \tag{2.1}$$
$$\text{(extensivity)} \qquad A \subseteq A^{\uparrow\downarrow} \tag{2.2}$$
$$\text{(idempotency)} \qquad A^{\uparrow\downarrow} = (A^{\uparrow\downarrow})^{\uparrow\downarrow} \tag{2.3}$$

for all $A, A_1 \subseteq G$. Thus, $(\cdot)^{\uparrow\downarrow}$ describes a closure operator on 2^G. The dual notion holds for attributes, and $(\cdot)^{\downarrow\uparrow}$ describes a closure operator on 2^M.

Proposition 1. Let (G, M, I) be a formal context with subsets $A_0, A_1, A_2 \subseteq G$ and $B_0, B_1, B_2 \subseteq M$ of attributes. Then,

1. $A_0 \subseteq A_1 \Rightarrow A_1^{\uparrow} \subseteq A_0^{\uparrow}$
2. $A_0 \subseteq A_0^{\uparrow\downarrow}$
3. $A_0^{\uparrow} = A_0^{\uparrow\downarrow\uparrow}$

1. $B_0 \subseteq B_1 \Rightarrow B_1^{\downarrow} \subseteq B_0^{\downarrow}$
2. $B_0 \subseteq B_0^{\downarrow\uparrow}$
3. $B_0^{\downarrow} = B_0^{\downarrow\uparrow\downarrow}$

Definition 7. A *formal concept* of a formal context (G, M, I) is a pair (A, B) of subsets $A \subseteq G$ and $B \subseteq M$ that satisfies $A^{\uparrow} = B$ and $B^{\downarrow} = A$. Then, A is the concept *extent* and B is the *intent*. We write $\mathfrak{B}(G, M, I)$ to denote the set of all concepts of (G, M, I).

Now explain why derivation of set of objects is always a concept intent by proposition 1. And the dual. and Galois connections

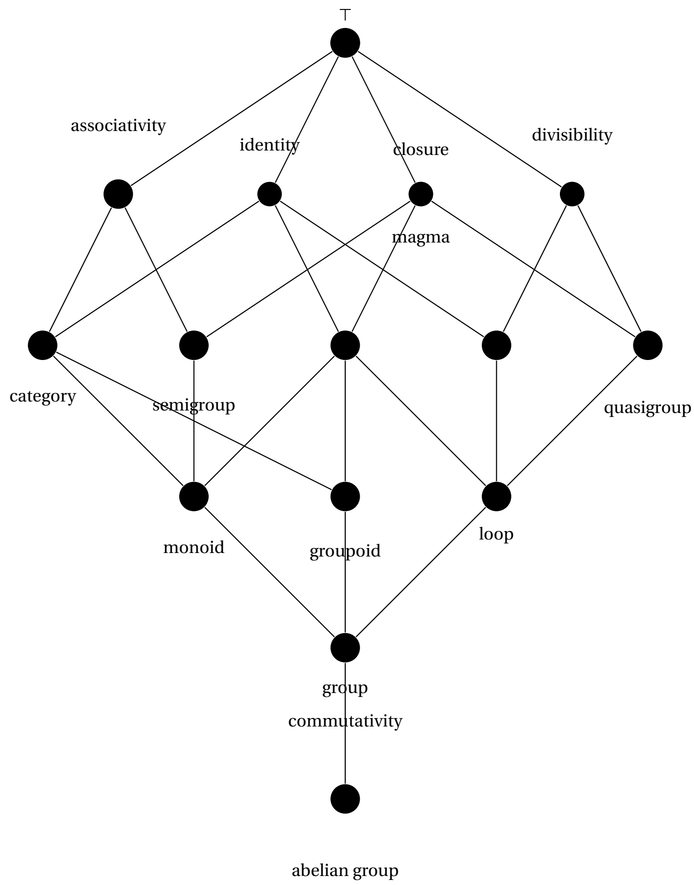


Figure 2.2: The concept lattice associated with the formal context in Figure 2.1

Bibliography

[1] B. A. Davey and H. A. Priestley, *Introduction to Lattices and Order*. Cambridge University Press, 2002.

[2] B. Ganter and R. Wille, *Formal Concept Analysis: Mathematical Foundations*. Springer Berlin, Heidelberg, 1999.

[3] B. Ganter and S. Obiedkov, *Conceptual Exploration*. Springer Berlin Heidelberg, 2016, pp. 1–315.

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