Rational Concept Analysis

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Contents

Chapter 1

Mathematical Preliminaries

This chapter is intended to serve as a reference point, clarifying ideas and notation, for the more fundamental concepts that will be used throughout the remainder of this dissertation. In the first section we discuss order and lattice theory. The third and final section introduces propositional logic, as well more general ideas in logic.

1.1 Order and Lattice Theory

This section refers extensively to the fundamental text by Davey and Priestley [1], as well as Ganter and Wille [2].

1.1.1 Orders

A *binary relation* R over the sets X and Y is a set of ordered pairs $\langle x, y\rangle$ with $x \in X$ and $y \in Y$. Alternatively, we can view R as a subset of the Cartesian product of these sets, and write $R \subseteq X \times Y$. A pair $\langle x, y\rangle \in R$ tells us that R relates x to y, and we may instead write xRy.

Certain binary relations, satisfying specific properties, occur frequently enough to warrant their own denomination. One such relation, which will be used in almost every section of this dissertation, is called a *partial order*.

Definition 1. A *partial-order* is a binary relation $\preceq\, \subseteq X \times X$ that satisfies the following properties:

$$\text{(Reflexivity)} \qquad x \preceq x \tag{1.1}$$

$$\text{(Antisymmetry)} \qquad x \preceq y \text{ and } y \preceq x \text{ implies } x = y \tag{1.2}$$

$$\text{(Transitivity)} \qquad x \preceq y \text{ and } y \preceq z \text{ implies } x \preceq z \tag{1.3}$$

for all $x, y, z \in X$.

Frequently, we use 'preference' as a metonymy for an order; and so in the context of an order, "element x is preferred to y" should be taken to mean that $\langle x, y \rangle \in \preceq$, or simply $x \preceq y$.

We write $x \npreceq y$ to indicate that $\langle x, y \rangle$ is not in the relation, and $x \prec y$ for the case where $x \preceq y$ and $x \neq y$. When $x \npreceq y$ and $y \npreceq x$—i.e., that x and y are incomparable—we write $x \| y$. From a partial-order we can quite easily induce the notion of a *strict partial-order*.

Definition 2. A *strict partial-order* is a binary relation $\prec \subseteq X \times X$ that satisfies:

$$\text{(Irreflexivity)} \qquad x \nprec x \tag{1.4}$$

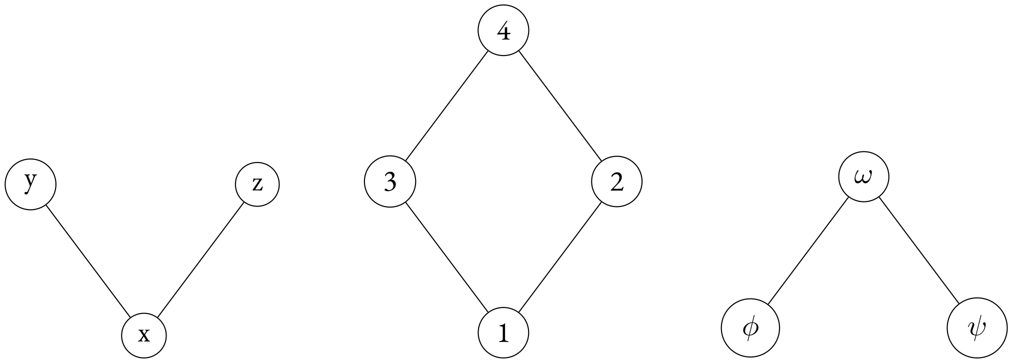
$$\text{(Asymmetry)} \qquad x \prec y \text{ implies } y \nprec x \tag{1.5}$$

$$\text{(Transitivity)} \qquad x \prec y \text{ and } y \prec z \text{ implies } x \prec z \tag{1.6}$$

for all $x, y, z \in X$.

An *ordered set* is a pair $(X, \preceq)$ with X being a set and $\preceq$ being an ordering on X, we use bold-face as a shorthand, and so $\mathbf{X}$ is an ordered set, and the order relation associated with $\mathbf{X}$ may be written $\preceq_X$ if there is ambiguity. If $\mathbf{Y}$ is a subset of $\mathbf{X}$, then $\mathbf{Y}$ inherits the order relation from $\mathbf{X}$; and so, for $x, y \in \mathbf{Y}$, $x \preceq_Y y$ if and only if $x \preceq_X y$.

We can visualise ordered sets through the use of *Hasse* diagrams.



(a) **A** (b) **B** (c) **C**

Figure 1.1: Three partial-orders over a set P

Then, a pair $\langle x, y \rangle$ is in a given order relation $\preceq \subseteq X \times X$ if there exists a strictly upward path connecting x to y (or, if $x = y$). From **C** we can read-off that $\phi || \psi$, and that $\phi \preceq_C \omega$, for instance.

Definition 3. Let **X** and **Y** be ordered sets with a map $\varphi : \mathbf{X} \to \mathbf{Y}$. We say that φ is an *order-preserving* (or, isotone) map if $x \preceq_X y$ implies $\varphi(x) \preceq_Y \varphi(y)$. It is an *order-embedding* if it is injective, and $x \preceq_X y$ if and only if $\varphi(x) \preceq_Y \varphi(y)$ for all $x, y \in X$. Finally, φ is an *order-isomorphism* if it is an order-embedding that is also *onto*.

An order-mapping is called *antitone* if it reverses the order and is the dual to isotone mappings, where *antitone order-embeddings* and *antitone order-isomorphisms* are defined as one might expect. In Figure 1.1 it is quite simply to find an antitone order-isomorphism between **A** and **C**.

If **X** is an ordered set, then an element $x \in X$ is *minimal* in **X** if there exists no distinct element $y \in X$ such that $y \preceq x$. Conversely, x is *maximal* in **X** if there exists no distinct $y \in X$ with $x \preceq y$. Continuing this example, x is the *minimum* of **X** if it is minimal in **X** and there are no elements to which x is incomparable. In other words, for every element $y \in X$ it is the case that $x \preceq y$. Naturally, x is the *maximum* element of **X** if for all $y \in X$ it is the case that $y \preceq x$.

Definition 4. Let $(X, \preceq)$ be a partially ordered set, and **Y** a subset of **X**. The *lower bound* of **Y** is an element $x \in X$ with $x \preceq y$ for all $y \in Y$. The *upper bound* of Y is defined dually. If the set of lower bounds of Y has a maximum (greatest) element then this element is called the *infimum* of Y. Dually, if there is a minimum (least) element in the set of upper bounds, then this element is the *supremum* of Y.

Remark 1. For any set X, the power-set of X equipped with an order determined by subset inclusion will form a partial order. Then, $(2^X, \preceq)$ is a partially ordered set where for two sets $A, B \in 2^X$ where $A \preceq B$ if and only if $A \subseteq B$. It follows that **X** will always have a minimum: $\emptyset$, and a maximum: X.

1.1.2 Lattices

One perspective, which we adopt, views a *lattice* as a particular kind of partially ordered set satisfying specific properties involving upper and lower bounds. In particular,

Definition 5. A partially ordered set $\mathbf{X} = (X, \preceq)$ is a *lattice* if and only if, for any two elements $x, y \in \mathbf{X}$, both the supremum $x \vee y$ and the infimum $x \wedge y$ exist. The set **X** is a *complete lattice* if and only if the meet and join exist for every subset of **X**.

In Figure 1.1 only (b) is a lattice (and a complete lattice too). In fact, every finite lattice is a complete lattice [2].

1.2 Propositional Logic

Chapter 2

Formal Concept Analysis

Formal Concept Analysis (FCA) provides a simple, and yet mathematically rigorous, framework for identifying and reasoning about "concepts" and their corresponding hierarchies in data [2, 3]. The central view of concepts as a dual between *extension*—what one refers to as instances of a concept—and *intension*—what meaning is ascribed to a concept—is supported by a rich philosophical backing.

2.1 Basic Notions

The universe of discourse in FCA is made-up of sets of *objects* and *attributes*. Objects are extensional, they are the things pointed to as instances of some more general concept. In turn, attributes construct the intensional component of a concept.

We collect these extensional and intensional building blocks in a structure called a *formal context*, which includes a binary relation that allows traversal from extension to intension, and vice verse.

Definition 6. A *formal context* $\mathbb{K} = (G, M, I)$ is a triple comprised of a set of objects G, a set of attributes M, and a binary relation $I \subseteq G \times M$ referred to as an 'incidence' relation. For an object-attribute pair $(g, m) \in I$ we might say that "object g *has* the attribute m".

A formal context in some sense describes an open-world interpretation, and so $(g, m) \notin I$ is not usually interpreted as saying that "object g has the negation of the attribute m".

Example 2.1.1. Finite formal contexts of a reasonable size can be described entirely by a tabular representation. Each object corresponds to a row, and each attribute to a column.

	closure	associativity	identity	divisibility	commutativity
magma	×				
semigroup	×	×			
monoid	×	×	×		
group	×	×	×	×	
abelian group	×	×	×	×	×
loop	×		×	×	
quasigroup	×			×	
groupoid		×	×	×	
category		×	×		
semicategory		×			

Figure 2.1: A formal context showing necessary properties of group-like structures.

The tabular representation of a formal context allows for easy identification of the set of attributes that a given object satisfies: one need only scan across the respective row in the table and note where an '$\times$' symbol appears. This set of attributes is called the *object intent*. The dual notion of an *attribute extent* can similarly be found by scanning down the column of a given attribute. The utility of this visual metaphor diminishes when, as is often the case, we consider larger sets of objects or attributes. Instead, we opt for a more formal approach to determining the intents and extents for (sets of) objects and attributes, respectively.

Definition 7. Given a formal context (G, M, I), the *derivation operators* are two order-reversing maps $(\cdot)^{\uparrow} : 2^G \to 2^M$ and $(\cdot)^{\downarrow} : 2^M \to 2^G$ where the order is given by subset inclusion. Then, for any subsets $A \subseteq G$ and $B \subseteq M$,

$$A^{\uparrow} := \{m \in M \mid \forall g \in A,\ \langle g, m\rangle \in I\}$$
$$B^{\downarrow} := \{g \in G \mid \forall m \in B,\ \langle g, m\rangle \in I\}$$

The derivation operators provide a clear way of describing, for a given set $A \subseteq G$ of objects, the set of attributes which every object in A satisfies, denoted $A^{\uparrow}$. As an illustration, given the set of objects {`semigroup, monoid`} from Figure 2.1, its derivation would be {`closure, associativity`}. It is quite easy to spot that this is just the intersection of the object intents of `semigroup` and `monoid`.

Of course, these two functions can be composed; and so, $A^{\uparrow\downarrow}$ would yield the set of objects {`semigroup, monoid, group, abelian group`}, which can rather cumbersomely be described as "the set of all objects that satisfy all the attributes satisfied by `semigroup` and `monoid`".

In fact, this double-application of derivation operators satisfies very specific properties,

$$\text{(monotonicity)} \qquad A \subseteq A_1 \textit{ implies } A^{\uparrow\downarrow} \subseteq A_1^{\uparrow\downarrow} \tag{2.1}$$

$$\text{(extensivity)} \qquad A \subseteq A^{\uparrow\downarrow} \tag{2.2}$$

$$\text{(idempotency)} \qquad A^{\uparrow\downarrow} = (A^{\uparrow\downarrow})^{\uparrow\downarrow} \tag{2.3}$$

for all $A, A_1 \subseteq G$. Thus, $(\cdot)^{\uparrow\downarrow}$ describes a closure operator on 2^G. The dual notion holds for attributes, and $(\cdot)^{\downarrow\uparrow}$ describes a closure operator on 2^M.

Proposition 1. Let (G, M, I) be a formal context with subsets $A_0, A_1, A_2 \subseteq G$ of objects and $B_0, B_1, B_2 \subseteq M$ of attributes. Then,

1. $A_0 \subseteq A_1 \Rightarrow A_1^{\uparrow} \subseteq A_0^{\uparrow}$
2. $A_0 \subseteq A_0^{\uparrow\downarrow}$
3. $A_0^{\uparrow} = A_0^{\uparrow\downarrow\uparrow}$

1. $B_0 \subseteq B_1 \Rightarrow B_1^{\downarrow} \subseteq B_0^{\downarrow}$
2. $B_0 \subseteq B_0^{\downarrow\uparrow}$
3. $B_0^{\downarrow} = B_0^{\downarrow\uparrow\downarrow}$

Definition 8. A *formal concept* of a formal context (G, M, I) is a pair (A, B) of subsets $A \subseteq G$ and $B \subseteq M$ that satisfies $A^{\uparrow} = B$ and $B^{\downarrow} = A$. Then, A is the concept *extent* and B is the *intent*. We write $\mathfrak{B}(G, M, I)$ to denote the set of all concepts of (G, M, I).

Now explain why derivation of set of objects is always a concept intent by proposition 1. And the dual. and Galois connections

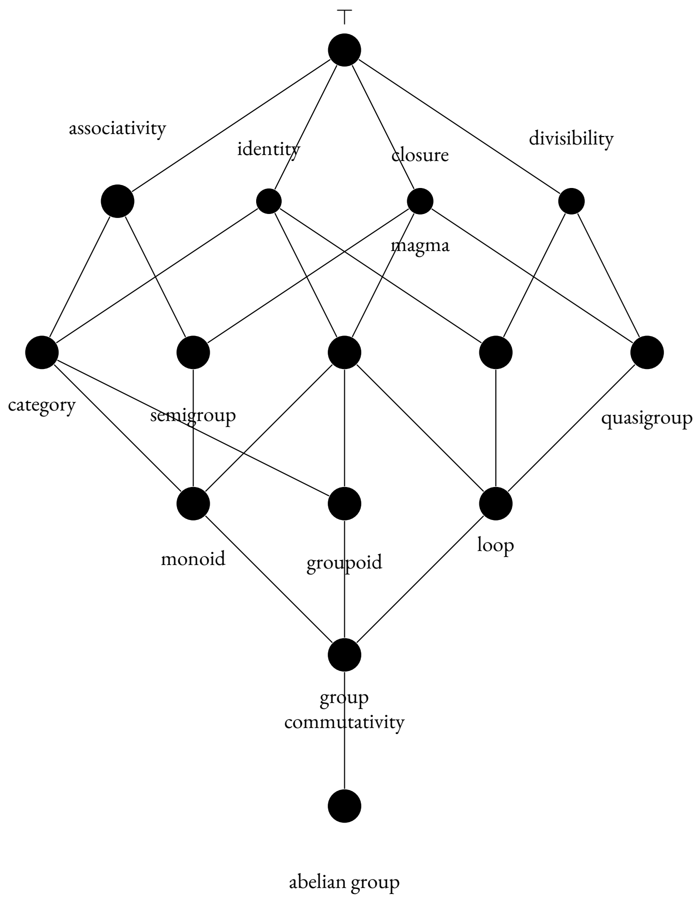


Figure 2.2: The concept lattice associated with the formal context in Figure 2.1

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[3] B. Ganter and S. Obiedkov, *Conceptual Exploration*. Springer Berlin Heidelberg, 2016, pp. 1–315.

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