Rational Concept Analysis

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Contents

Part I

Foundations

Chapter 1

Mathematical Preliminaries

This chapter is intended to serve as a reference point, clarifying ideas and notation for the more fundamental concepts that will be used throughout the remainder of this dissertation. In the first section we discuss order and lattice theory. The third and final section introduces propositional logic, as well more general ideas in logic.

1.1 Order and Lattice Theory

This section refers extensively to the fundamental text by Davey and Priestley [1], as well as Ganter and Wille [2].

1.1.1 Orders

A *binary relation* R over the sets X and Y is a set of ordered pairs (x, y) where $x \in X$ and $y \in Y$. We may choose to express that $(x, y) \in R$ using infix notation and write xRy, which tells us that R relates x to y.

Certain binary relations, satisfying specific properties, occur frequently enough to warrant their own denomination. One such relation, which will be used in almost every section of this dissertation, is called a *partial order*.

Definition 1. A *partial-order* is a binary relation $\preceq \subseteq X \times X$ that satisfies the following properties:

$$\text{(Reflexivity)} \quad x \preceq x \tag{1.1}$$

$$\text{(Antisymmetry)} \quad x \preceq y \text{ and } y \preceq x \text{ implies } x = y \tag{1.2}$$

$$\text{(Transitivity)} \quad x \preceq y \text{ and } y \preceq z \text{ implies } x \preceq z \tag{1.3}$$

for all $x, y, z \in X$.

Frequently, 'preference' is used as a metonymy for an order, in this context writing "element x is preferred to y" should be interpreted to mean that $(x, y) \in \preceq$, or simply $x \preceq y$. The use of the metaphor will become more apparent in Chapter 3.

We write $x \npreceq y$ to indicate that (x, y) is not in the relation, and $x \prec y$ for the case where $x \preceq y$ and $x \neq y$. In the scenario where $x \npreceq y$ and $y \npreceq x$—i.e., that x and y are incomparable—we may write $x || y$. From a partial-order we can quite easily induce the notion of a *strict partial-order*.

Definition 2. A *strict partial-order* is a binary relation $\prec \subseteq X \times X$ that satisfies:

$$\text{(Irreflexivity)} \quad x \nprec x \tag{1.4}$$

$$\text{(Asymmetry)} \quad x \prec y \text{ implies } y \nprec x \tag{1.5}$$

$$\text{(Transitivity)} \quad x \prec y \text{ and } y \prec z \text{ implies } x \prec z \tag{1.6}$$

for all $x, y, z \in X$.

An *ordered set* is a pair $(X, \preceq)$ with X being a set and $\preceq$ being an ordering on the elements of X. We make notation easier, and use $\mathbf{X}$ to denote the pair; moreover, we may reference the order relation associated with $\mathbf{X}$ by writing $\preceq_X$ in settings where there is ambiguity. If $\mathbf{Y}$ is a subset of $\mathbf{X}$, then $\mathbf{Y}$ inherits the order relation from $\mathbf{X}$; and so, for $x, y \in \mathbf{Y}$, $x \preceq_Y y$ if and only if $x \preceq_X y$.

We can visually describe ordered sets through the use of *Hasse* diagrams.

(a) **A** (b) **B**

Figure 1.1: Hasse diagrams of two partially ordered sets

As an illustrative example, from the ordered set in Subfigure 1.1a we read that $z \preceq x$, as there is a (strictly) upward path from z to x. In fact, it is clear that $z \preceq w, x, y, z$, or that "z is preferred to every other element in $\mathbf{A}$". We say such an element is *minimal*.

More formally, an element $x \in \mathbf{X}$ is *minimal* with respect to the ordering if there exists no distinct element $y \in \mathbf{X}$ such that $y \preceq x$. Conversely, we say x is *maximal* if there exists no distinct $y \in \mathbf{X}$ where $x \preceq y$. Then x is the *minimum* element if $x \preceq y$ for all $y \in \mathbf{X}$; the dual notion of a *maximum* is defined as we might expect.

Definition 3. Let $\mathbf{X}$ and $\mathbf{Y}$ be ordered sets with a mapping $\varphi : \mathbf{X} \to \mathbf{Y}$. We call φ an *order-preserving* (or, isotone) map if $x \preceq_X y$ implies $\varphi(x) \preceq_Y \varphi(y)$. It is an *order-embedding* if it is injective, and $x \preceq_X y$ if and only if $\varphi(x) \preceq_Y \varphi(y)$ for all $x, y \in X$. Finally, φ is an *order-isomorphism* if it is an order-embedding that is also *surjective*. The dual notion to an order-preserving map is an *order-reversing* (or, antitone) map.

1.1.2 Lattice Theory

Lattice theory studies partially ordered sets that behave well with respect to certain properties involving upper and lower bounds. Given an ordered set $\mathbf{X}$ and a subset $\mathbf{Y} \subseteq \mathbf{X}$, the set of upper bounds of $\mathbf{Y}$ is defined as

$$\mathbf{Y}^u := \{x \in \mathbf{X} \mid \forall y \in \mathbf{Y} : y \preceq x\},$$

and the set of lower bounds, $\mathbf{Y}^l$, is defined dually. If $\mathbf{Y}^u$ has a minimum element, then we call that element the *supremum* of $\mathbf{Y}$; dually, if $\mathbf{Y}^l$ has a maximum element, then we call that element the *infimum* of $\mathbf{Y}$ (the supremum and infinum are also referred to as the *least upper bound* and *greatest lower bound*, respectively).

Instead of talking about the supremum of two elements $x, y \in \mathbf{X}$, we opt for the term *join* and write $x \vee y$, or $\bigvee \mathbf{Y}$. Instead of infimum, we say *meet* and write $x \wedge y$, or $\bigwedge \mathbf{Y}$ [1, p. 33].

With these definitions of meets and joins in mind, we are able to define a lattice as

Definition 4. Given a partially ordered set $\mathbf{L}$, we say that $\mathbf{L}$ is a *lattice* if for every pair $x, y \in \mathbf{L}$ the join $x \vee y$ and meet $x \wedge y$ exist, and are unique. Then, we call $\mathbf{L}$ a *complete lattice* if for every subset $\mathbf{M} \subseteq \mathbf{L}$ both $\bigvee \mathbf{M}$ and $\bigwedge \mathbf{M}$ exist in $\mathbf{L}$.

We say that a lattice $\mathbf{L}$ is *bounded* if there exists an element $x \in \mathbf{L}$ such that $x \vee y = x$ for all $y \in \mathbf{L}$, and there exists an element $z \in \mathbf{L}$ with $z \wedge y = z$ for all $y \in \mathbf{L}$. Frequently we refer to such an x and y as a top and bottom element and denote them by $\top$ and $\bot$, respectively. It is not difficult to spot that every complete lattice is bounded—in fact this is true by definition of a complete lattice—as a corollary every finite lattice is also bounded [1, p. 36].

Remark 1. The lattice $(\mathbb{Z}, <)$, constructed by imposing the natural order over the integers, is an example of an lattice that is not bounded.

Example 1. The set of natural numbers $\mathbb{N}$ forms a complete lattice when ordered by divisibility, sometimes called the *division lattice*. In the division lattice, the join operation corresponds to the *least common multiple*, and the meet operation to the greatest common divisor. The bottom element of this lattice is 1 as it divides every other natural number, while the top element is 0 since it is divisible by all other naturals. Although the division lattice is infinite, it is indeed an example of a complete lattice.

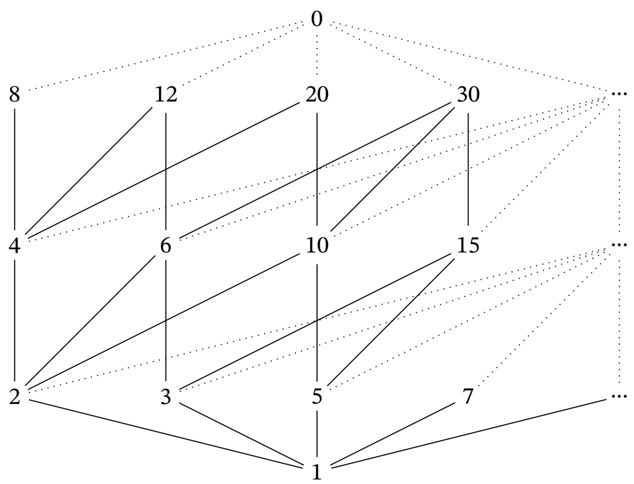


Figure 1.2: The division lattice $(\mathbb{N}, \leq)$

1.1.2.1 Lattices as algebraic structures

Lattices can also be considered from an algebraic perspective; although, as we will soon show, the algebraic and order-theroetic perspectives coincide it is often useful to switch consideration between these perspectives.

Consider a set L equipped with a binary operation $\vee$ that satisfies the following properties for all $x, y, z \in L$,

$$\text{(idempotence)} \quad x \vee x = x \tag{1.7}$$

$$\text{(commutativity)} \quad x \vee y = y \vee x \tag{1.8}$$

$$\text{(associativity)} \quad (x \vee y) \vee z = x \vee (y \vee z) \tag{1.9}$$

From the algebraic structure $(L, \vee)$ one can induce a *unique* partial order $\preceq$ on L by construction of the relation $\{(x, x \vee y) \mid x, y \in L\}$ [3, pp. 173]. The structure $(L, \vee)$ is called an algebraic *join semilattice*, and $\preceq$ its *underlying partial order*. The relation can equivalently be described as "$x \preceq y$ if and only if $x \vee y = y$" [3, pp.173].

Example 2. Consider the set $S = \{1, 2, 3\}$ and the union operation $\cup$. From $(S, \cup)$ we can induce the order relation characterised by the set $\{(X, X \cup Y) \mid X, Y \subseteq S\}$ (frequently, this kind of ordering is called the *set inclusion order*). From the Hasse diagram in Figure 1.3 representing this order, we observe that the $\cup$ binary relation corresponds to the $\vee$ described in Definition 4.

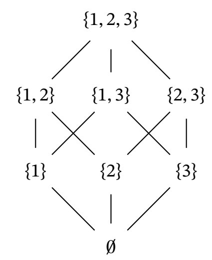


Figure 1.3: The underlying order of $(S, \cup)$

If we instead consider the structure $(L, \wedge)$ where $\wedge$ is a different binary operation on L satisfying the same properties eq. (1.7)–eq. (1.9) we can construct a partial order characterised by the set $\{(x, y) \mid x \wedge y = x \text{ and } x, y \in L\}$. The structure $(L, \wedge)$ is called a *meet semilattice*.

It is an obvious next step to wonder how these two algebraic structures fit together: fix a set non-empty set L with the two binary operations, $\vee$ and $\wedge$, introduced in the prior paragraphs. We have already seen that the underlying order of the join semilattice can be described as $x \preceq y$ if and only if $x \vee y = y$, while the underlying order of the meet semilattice is described by $x \preceq y$ if and only if $x \wedge y = x$. These two partial orders are *compatible* under the following condition:

$$\text{(compatibility)} \quad x \vee (x \wedge y) = x \qquad\qquad \text{(resp.)} \quad x \wedge (x \vee y) = x \tag{1.10}$$

Compatibility (which is sometimes also called *absorption*) is satisfied when the underlying orders of both semilattices refer to the same partial order. The property can be rewritten as $x \vee y = y$ if and only if $x \wedge y = x$. The formal definition of an algebraic lattice is given below:

Definition 5. The algebraic structure $(L, \vee, \wedge)$, where L is a set and $\vee, \wedge$ are two binary operations on L, is a *lattice* if both binary operations satisfy *idempotence, commutativity,* and *associativity* as well well as *absorbtion*.

Lemma 1 (The Connecting Lemma)**.** If $\mathbf{L}$ is a lattice with $x, y \in \mathbf{L}$, then the following are equivalent:

1. $x \preceq y$;
2. $x \vee y = y$;
3. $x \wedge y = x$.

Moving forward, we will no longer maintain a rigid distinction between these two perspectives on lattices; rather, we will tacitly adopt whichever viewpoint best suits the surrounding context. Lattices are central to Formal Concept Analysis, and will be revisited in Chapter 2.

1.2 Closure Systems & Galois Connections

When studying a collection of certain *things* that satisfy a certain property, it is often beneficial to know if the collection considers every item satisfying the specified properties. We aliken this to the more formal notion of *closure* and *closed sets*.

Definition 6. A subset S of X is *closed* with respect to an operation f if and only if the application of f to elements of S always results in an element of S. Formally, $\mathsf{f}(S) = S$

Closely related is the notion of *closure operators*

Definition 7. A *closure operator* on a set S is a function $\mathsf{cl} : 2^S \to 2^S$ that satisfies the following properties for all $X, Y \subseteq S$.

$$\text{(Monotony)} \quad X \subseteq Y \text{ implies } \mathsf{cl}(X) \subseteq \mathsf{cl}(Y) \tag{1.11}$$

$$\text{(Extensivity)} \quad X \subseteq \mathsf{cl}(X) \tag{1.12}$$

$$\text{(Idempotency)} \quad \mathsf{cl}(X) = \mathsf{cl}(\mathsf{cl}(X)) \tag{1.13}$$

The subsets $X \subseteq S$ where $X = \mathsf{cl}(X)$ are called *closed* subsets of S with respect to cl.

More generally, we say that a set S is closed with respect to an operation f if applying f to members of S always produces another member of S. A simple example is that the the natural numbers are closed under multiplication, but not division, as the product of two natural numbers is always another natural number, while the quotient is not.

Definition 8. Given a set S, a *closure system* on S is a set of subsets $\mathcal{C} \subseteq 2^S$ that is closed under intersection. In other words, if $X \subseteq \mathcal{C}$ then $\bigcap X \in \mathcal{C}$.

Given a set X the closure of X can be constructed by taking $\bigcap\{Y \in 2^S \mid X \subseteq Y\}$

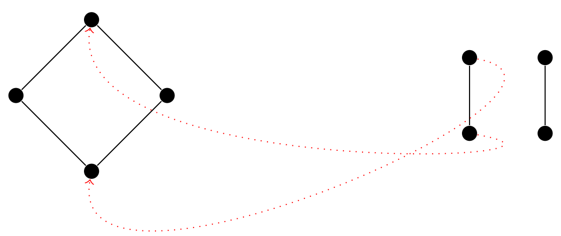
Definition 9. Let $\mathbf{X}, \mathbf{Y}$ be ordered sets, then a pair of maps $\mathsf{f} : 2^{\mathbf{X}} \to 2^{\mathbf{Y}}$ and $\mathsf{g} : 2^{\mathbf{Y}} \to 2^{\mathbf{X}}$ constitute a *Galois connection* between $\mathbf{X}$ and $\mathbf{Y}$ if and only if they satisfy

$$x \preceq_X x' \Rightarrow \mathsf{f}(x') \preceq_Y \mathsf{f}(x) \tag{1.14}$$

$$y \preceq_Y y' \Rightarrow \mathsf{g}(y') \preceq_X \mathsf{g}(y) \tag{1.15}$$

$$x \preceq_X \mathsf{g}(\mathsf{f}(x)) \text{ and } y \preceq_Y \mathsf{f}(\mathsf{g}(y)) \tag{1.16}$$

We say that such a pair of maps are *dually adjoint* to eachother.



1.3 Propositional Logic

Propositional logic is a system for abstracting reasoning away from natural language. A *propositional statement* is a sentence like "Tralfamadorians have one eye". That is, something that can be assigned a value of *true* or *false* [4, p. 7]. More complex propositions may be constructed recursively from simpler ones. The main interest is to then discover which true propositions follow—under some agreeable sense of what it means to 'follow'—from others.

1.3.1 Syntax

Propositional atoms are the fundamental building blocks of a propositional language. They are indivisible statements that can either be true or false, and may be be combined with the boolean connectives $\{\neg, \vee, \wedge, \rightarrow\}$ to construct more complex statements, or *formulae*. We denote propositional atoms with lower-case Latin letters p, q, r, s, and t, and the set of all atoms will usually be denoted by $\mathcal{P}$. Lower-case Greek letters $\alpha, \beta, \gamma, \phi, \psi$, and φ will be used to denote formulae, and corresponding upper-case Greek letters will usually represent sets of formulae.

Not all combinations of atoms and boolean connectives result in meaningful expressions. For example, '$\wedge \rightarrow \phi$', which can be parsed as "conjunction materially implies ϕ", has no discernible meaning; so we make a distinction between any formulae and so-called *well-formed formulae*. The latter being constructions that are valid with respect to the rules of the grammar defined in Backus-Naur form below [5, p. 33].

$$\phi ::= p \mid (\neg\phi) \mid (\phi_1 \vee \phi_2) \mid (\phi_1 \wedge \phi_2) \mid (\phi_1 \rightarrow \phi_2) \mid (\phi_1 \leftrightarrow \phi_2) \tag{1.17}$$

We are entirely uninterested in formulae that are not well-formed formulae, and so we drop the 'well-formed' suffix with the recognition that we shall never again mention the alternative [5, p. 33]. Any formula accepted by this grammar is said to be in the language $\mathcal{L}^{\mathcal{P}}$, although we may also drop the superscript where possible. To make reading this dissertation slightly more enjoyable, we may construct examples where we denote propositional atoms by monospaced text, enabling the expression of formulae suggesting a more interesting domain, such as `tralfamadorians` $\rightarrow \neg$`human`, which should be interpreted as proposing that "Tralfamadorians are not human".

1.3.2 Semantics

In the previous section we described the syntax of propositional logic and how the boolean connectives enable the construction of arbitrary formulae from atoms. The semantics of propositional logic are concerned with how meaning may be ascribed to these formulae. The aim is to provide a method of answer questions like "when is this formula true?" or, "if ϕ, ψ, φ are true, what else must be true?".

Propositional atoms were described as indivisible statements that can be assigned values of *true* or *false*. We now define a function that assigns a truth value to each atom in the set $\mathcal{P}$ of atoms, sometimes called the *universe of discourse*.

Definition 10. A *valuation* is a function $u : \mathcal{P} \rightarrow \{ \textit{true}, \textit{false} \}$ that assigns a truth value to each propositional atom.

Given a set of atoms $\mathcal{P} = \{p, q, r\}$, we write $\mathcal{U}$ to denote the set of all possible valuations. A valuation $u \in \mathcal{U}$ *satisfies* an atom $p \in \mathcal{P}$ if u maps p to *true*, and so we write $u \Vdash p$. Otherwise, we write $u \nVdash p$ to indicate that u does not satisfy p (in this context, meaning u maps p to *false*) [4, p. 16]. We might represent the valuation that maps p and q to true but r to false by $\{p, q, \overline{r}\}$.

This satisfaction relation can be extended beyond propositional atoms to include more complex formulae, as described in Subsection 1.3.1. For any $\phi, \psi \in \mathcal{L}$

- $u \Vdash \neg\phi$ if and only if $u \nVdash \phi$ (negation)
- $u \Vdash \phi \vee \psi$ if and only if $u \Vdash \phi$ or $u \Vdash \psi$ (disjunction)
- $u \Vdash \phi \wedge \psi$ if and only if $u \Vdash \phi$ and $u \Vdash \psi$ (conjunction)
- $u \Vdash \phi \to \psi$ if and only if $u \Vdash \neg\phi$ or $u \Vdash \psi$ (material implication)
- $u \Vdash \phi \leftrightarrow \psi$ if and only if $u \Vdash \phi \to \psi$ and $u \Vdash \psi \to \phi$ (material equivalence)

Definition 11. For a valuation $u \in \mathcal{U}$ and formula $\phi \in \mathcal{L}$ we call u a *model* of ϕ if and only if u satisfies ϕ. The set of all models of ϕ is constructed by $\hat{\phi} := \{u \in \mathcal{U} \mid u \Vdash \phi\}$.

Satisfiability can be extended to sets of of formulae so that a valuation $u \in \mathcal{U}$ *satisfies* the set $\Phi := \{\phi_0, \dots, \phi_n\}$ when $u \Vdash \phi_i$ for all $0 \leq i \leq n$, and we write $u \Vdash \Phi$. If no such valuation exists then Φ is unsatisfiable [4, p. 31].

Definition 12. INSERT COMPACTNESS

1.3.3 Logical Consequence

The introduction of models at the end of Section 1.3 leads quite naturally into a discussion on the matter of *logical consequence*, which provides an answer—in terms of the semantics—to the question of when it is appropriate to infer one formula from another [6, p. 408].

For example, if we know that "*Billy Pilgrim lived in Slaughterhouse 5*" and that "*The inhabitants of Slaughterhouse 5 survived the bombing of Dresden*". We may then hold the view that, as a consequence of these two pieces of knowledge, it would be sensible to infer that "*Billy Pilgrim survived the bombing of Dresden*". Of course, this inference being *sensible*—under some common concept of consequence—gives little insight into how logical consequence may be appropriately be formalised. Informally, it might help ones' intuition to try imagine a world where the first propositions are true while the final one false.

We place a brief moratorium on this discussion to (formally) introduce the notions of the *object* and *metalanguage*. In the scenario we have just described, the italicised sentences form a part of the object language: the language we use to model the world and represent information. The metalanguage facilitates reasoning about elements in the object language; that is, we can use the metalanguage to describe one sentence being a consequence of another (where these sentences are elements in the object language). Here, *consequence* is an element of the metalanguage [4, p 22].

In this case, both the object and metalanguage are comprised of English, which can certainly lead to confusion. The distinction is clearer in Propositional logic: constructions derived from combinations of atoms and the boolean operators result in elements of the object language; while the metalanguage uses symbols like $\vDash$ and $\vdash$, which are introduced below.

Definition 13. Let Γ be a set of formulae and φ a formula in the language $\mathcal{L}$. We say that φ is a *logical consequence* of Γ, and write $\Gamma \vDash \varphi$, if and only if every model of Γ is a model of φ, or equivalently if $\hat{\Gamma} \subseteq \hat{\varphi}$.

This semantic account of consequence, due to Tarski [6, p. 417], makes implicit the view that if φ is in fact a consequence of Γ, then it should not be possible for all the sentences (formulae) in Γ to be true while φ be false.

Example 3. If we modelled the earlier example in propositional logic, we might initialise *Billy Pilgrim* with $\mathtt{b}$, *slaughterhouse 5* with $\mathtt{h}$, and *survived* with $\mathtt{s}$. It is then our aim to determine whether $\{\mathtt{b} \to \mathtt{h},\ \mathtt{h} \to \mathtt{s}\} \vDash \mathtt{b} \to \mathtt{s}$. From the satisfaction relation in Subsection 1.3.2 it can be derived that the models of $\{\mathtt{b} \to \mathtt{h},\ \mathtt{h} \to \mathtt{s}\}$ are precisely

$$\big\{\{\overline{\mathtt{b}},\ \overline{\mathtt{h}},\ \overline{\mathtt{s}}\}, \{\overline{\mathtt{b}},\ \overline{\mathtt{h}},\ \mathtt{s}\}, \{\overline{\mathtt{b}},\ \mathtt{h},\ \mathtt{s}\}, \{\mathtt{b},\ \mathtt{h},\ \mathtt{s}\}\big\}.$$

which are indeed all models of $\mathtt{b} \to \mathtt{s}$, and so we answer the question in the affirmative.

Consequence operators are functions over a given language that map sets of formulae to their consequences. They were introduced by Tarski [7, p. 84] as the symbol $\mathcal{C}n$ representing a *general* idea that sets can be closed under a specified notion consequence. In future, we use $\mathcal{C}n$ to refer specifically to closure under logical (Tarskian) consequence; and so, application of the $\mathcal{C}n$ operator to a set Γ would yield $\mathcal{C}n(\Gamma) := \{\varphi \mid \Gamma \vDash \varphi\}$. Where we wish to reference closure under a different notion of consequence, suppose that of a Hilbert system, we will use a subscript to denote the system $\mathcal{C}n_{\mathcal{H}}$. To refer to the general notion of closure under *some* notion of consequence, we write $\mathcal{C}n_X$ [8, p. 4].

A *theory* (also called a *deductive system*, but we avoid this terminology) is a set of formulae Γ that equals its closure $\mathcal{C}n_X(\Gamma)$. The study of such operators is useful in its ability to reveal properties about the respective notion of consequence. For instance, it was observed by Tarski [7, p. 84] that $\mathcal{C}n$ satisfies the following properties:

$$\text{(inclusion)} \quad \Gamma \subseteq \mathcal{C}n(\Gamma) \tag{1.18}$$

$$\text{(idempotency)} \quad \mathcal{C}n(\Gamma) = \mathcal{C}n(\mathcal{C}n(\Gamma)) \tag{1.19}$$

$$\text{(monotonicity)} \quad \Gamma \subseteq \Gamma' \Rightarrow \mathcal{C}n(\Gamma) \subseteq \mathcal{C}n(\Gamma') \tag{1.20}$$

Inclusion is a relatively easy property to justify: all that is required is that the consequences of some information includes at least that starting information. *Idempotency* requires that the supposed set of all consequences is in fact the set of *all* consequences.

1.3.4 Deductive Systems

Logical consequence, described in Subsection 1.3.3, offers a purely semantic account of how it might be inferred that one formula follows (logically) from another set thereof. In contrast, deductive systems answer this question syntactically by describing a system of *axiomata* and *rules of inference* that affirm the truth of one formula from the truth of others [4, p. 49].

Definition 14. A *deductive system* is a collection of axiomata and rules of inference. A *proof* in such a system is a sequence of formulae where each formula is either an axiom, or has been inferred by application of an inference rule to previous formulae in the sequence. The final formula in the sequence, ϕ, is called the *theorem* and is then *provable*, and so we write $\vdash \phi$.

A *theory* (or, *deductively closed theory*) in a deduction system is a set of formulae closed under application of axioms and inference rules of the system; and so a set of formulae Γ is a theory if it is equal to its deductive closure $\mathcal{C}n_{\mathcal{H}}(\Gamma) := \{\varphi \mid \Gamma \vdash \varphi\}$. As before, elements of a theory are called theorems.

Deductive systems offer some advantages over their semantic counterparts; particularly, when reasoning over large—possibly infinite—domains, logical consequence can become difficult. Moreover, semantic consequence provides little insight into the relationships between pieces of information that lead to the inferences we make; while the sequential nature of deduction systems trace a path describing this relationship [4, p. 55].

1.3.4.1 Hilbert Systems

A propositional Hilbert system $\mathcal{H}$ is characterised by three axiom schemata,

$$\text{(Axiom 1)} \quad \vdash (\phi \to (\psi \to \phi)), \tag{1.21}$$

$$\text{(Axiom 2)} \quad \vdash (\phi \to (\psi \to \gamma)) \to ((\phi \to \psi) \to (\phi \to \gamma)), \tag{1.22}$$

$$\text{(Axiom 3)} \quad \vdash (\neg\phi \to \neg\psi) \to (\phi \to \psi), \tag{1.23}$$

and a single rule of inference:

$$\text{(Modus Ponens)} \quad \frac{\vdash \phi, \qquad \vdash \phi \to \psi}{\vdash \psi}. \tag{1.24}$$

The axiom schemata themselves are not axioms, but rather patterns containing meta-variables that, when uniformly substituted for formulae, result in an instantiated axiom. The turnstile symbol ($\vdash$) is the syntactic counterpart to the double-turnstile ($\vDash$) used for logical consequence. We frequently express that ϕ is a theorem by writing $\vdash \phi$ [4, p. 55].

As it stands, constructing proofs from instances of axiom schema and applications of modus ponens is a challenging ordeal. As an illustration, we provide the following Hilbert-style proof for the inference made in Example 3.

Example 4. We demonstrate a Hilbert-style derivation of $\mathsf{b} \to \mathsf{s}$ from the assumptions $\mathsf{b} \to \mathsf{h}$ and $\mathsf{h} \to \mathsf{s}$.

Proof.

1. $\vdash (\mathsf{b} \to \mathsf{h})$ (Premise)

2. $\vdash (\mathsf{h} \to \mathsf{s})$ (Premise)

3. $\vdash (\mathsf{h} \to \mathsf{s}) \to (\mathsf{b} \to (\mathsf{h} \to \mathsf{s}))$ (Axiom 1)

4. $\vdash (\mathsf{b} \to (\mathsf{h} \to \mathsf{s}))$ (MP 2,3)

5. $\vdash (\mathsf{b} \to (\mathsf{h} \to \mathsf{s})) \to ((\mathsf{b} \to \mathsf{h}) \to (\mathsf{b} \to \mathsf{s}))$ (Axiom 2)

6. $\vdash (\mathsf{b} \to \mathsf{h}) \to (\mathsf{b} \to \mathsf{s})$ (MP 4,5)

7. $\vdash (\mathsf{b} \to \mathsf{s})$ (MP 1,6)

□

Derived rules are introduced as a means to make it easier to spot the next step in a proof sequence. Of particular importance is the so called *deduction theorem*, which facilitates the construction of proofs that are conditioned on a hypothesis, without requiring that the hypothesis be an axiom.

$$\text{(Deduction Theorem)} \quad \frac{\Delta \cup \phi \vdash \psi}{\Delta \vdash \phi \to \psi} \tag{1.25}$$

As an illustration, the proof in Example 4 can be restated using the *transitivity* derived rule:

$$\text{(Transitivity)} \quad \frac{\Delta \vdash \varphi \to \psi, \qquad \Delta \vdash \psi \to \gamma}{\Delta \vdash \varphi \to \gamma} \tag{1.26}$$

Derived rules must be sound with respect to what can be proved by the three axioms and applications of modus ponens. That is, a derived rule should not enable one to make an inference that would not be possible without such a rule.

The fact that it was able to be shown that "Billy Pilgrim survived the bombing of Dresden" through both semantic and syntactic notions of consequence is not a coincidence. This correspondence between logical consequence and Hilbert-style deduction systems holds in general for propositional logic (and, in fact for many other systems with relatively limited expressive power). We capture this correspondence more formally through the notions of *soundness* and *completeness*.

Definition 15. Given a set of formulae Γ and one γ in $\mathcal{L}$, $\Gamma \vdash \gamma$ if and only if $\Gamma \vDash \gamma$. Where $\vDash$ and $\vdash$ are used in the context with which they have been introduced.

The proofs for Definition 15 are well-known, and can be found in [4, p. 64].

1.3.5 Consequence Relations

In the preceding sections the discussions on consequence—be they either semantic or syntactic—have been concrete realisations of the more general idea of *consequence relations*. When discussing consequence relations we will abuse notation and denote the relation with '$\vdash$' (relying on the surrounding context to distinguish between consequence relations and syntactic derivations). Any particular consequence relation will be denoted by a subscript referencing the system.

Definition 16. A *consequence relation* $\vdash \subseteq \mathcal{L} \times \mathcal{L}$ is a binary relation over a formal language that satisfies

$$\text{(Reflexivity)} \quad \varphi \in \Gamma \text{ implies } \Gamma \vdash \varphi \tag{1.27}$$

$$\text{(Monotonicity)} \quad \Gamma \vdash \varphi \text{ and } \Gamma \subseteq \Psi \text{ imply } \Psi \vdash \varphi \tag{1.28}$$

$$\text{(Cut)} \quad \Gamma \vdash \Delta, \Psi \text{ and } \Psi, \Phi \vdash \Omega \text{ imply } \Gamma, \Phi \vdash \Delta, \Omega \tag{1.29}$$

We can view a consequence relation as a set of ordered pairs $\{(\Gamma_0, \varphi_0), \ldots, (\Gamma_n, \varphi_n), \ldots\}$; in a pair (Γ, φ) Γ represents a premise, and φ a conclusion [8, p. 16]. Then, the presence of said pair describes that φ is a consequence of Γ. The absence of such a pair is understood to mean that φ is not a consequence of Γ.

This is a much more significant abstraction on the matter of consequence, requiring no concrete proof system or semantics.

As we will see in Chapter 3, consequence relations provide a useful abstraction which allow a logic to be described in terms of certain properties, or *postulates*, corresponding to a certain desired behaviour. The properties are able to provide an intuition for the logic without the construction of a semantics, or deduction-system.

Chapter 2

Formal Concept Analysis

Formal Concept Analysis (henceforth initialised to FCA) is an approach to reasoning about *concepts* and corresponding *conceptual structures* in terms of lattice theory. At the foundation of FCA is a philosophical viewpoint, tracing back to Aristotle, which describes concepts as a unit consisting of two parts, the dual between *extension*: those things that exist as an instance of the concept; and *intension*: the properties which give meaning to the concept [9, p. 1] [10, p. 414]. As we will discuss in the following section, this perspective on concepts can be modelled quite elegantly by relations on sets of extensional and intensional elements.

2.1 Basic Notions in Formal Concept Analysis

The starting point in FCA is a structure called a *formal context*, or simply a *context*. A context represents the relational aspect between the extensional and intensional sets.

Definition 17. A *formal context* (G, M, I) is a triple comprised of a set of objects G, a set of attributes M, and a binary relation $I \subseteq G \times M$ referred to as an 'incidence' relation. For an object-attribute pair $(g, m) \in I$ we might say that "object g *has* the attribute m".

We regard the set G of objects as being the extensional dimension of the context, while the set M of attributes is intensional. Although, there is no strict requirement around what these sets are made up of, or that they be distinct—it is perfectly fine to have a context where G and M are the same sets.

It should be noted that, while the presence of an object–attribute pair (g, m) in the incidence relation is interpreted as the object having the respective attribute, FCA is concerned with only *positive* information, and so the absence of a pair in the relation is not usually interpreted to mean the object has the negation of the attribute.

When the cardinalities of G and M are small, contexts may be represented as a cross-table like Figure 2.1 where each object is represented by a row in the table, each attribute by a column, and each element in the incidence relation is marked with an '×' at the appropriate position [2, pp. 17]. Given a context is presented in this form, it is easy to identify all the attributes that a particular object satisfies: one need only scan across the respective row and note where the marks appear. The resulting set of attributes is called the *object's intent*. The dual notion of an *attribute extent* can be found by traversing down a column in the table.

Example 5. Finite formal contexts of a reasonable size can be described entirely by a tabular representation. Each object corresponds to a row, and each attribute to a column.

Algebraic Structures	closure	associativity	identity	divisibility	commutativity
magma	×				
semigroup	×	×			
monoid	×	×	×		
group	×	×	×	×	
abelian group	×	×	×	×	×
loop	×		×	×	
quasiagroup	×			×	
groupoid		×	×	×	
category		×	×		
semicategory		×			

Figure 2.1: A formal context showing necessary properties of group-like structures.

The efficacy of this approach obviously diminishes when we are interested in non-trivial contexts, or determining the intents or extents for sets of objects or attributes, respectively. We can instead formalise the procedure by defining *derivation operators*:

Definition 18. Given a formal context (G, M, I), the *derivation operators* are two order-reversing maps $(\cdot)^{\uparrow} : 2^G \to 2^M$ and $(\cdot)^{\downarrow} : 2^M \to 2^G$ where the order is given by subset inclusion. Then, for any subsets $A \subseteq G$ and $B \subseteq M$,

$$A^{\uparrow} := \{m \in M \mid \forall g \in A,\ (g, m) \in I\}$$
$$B^{\downarrow} := \{g \in G \mid \forall m \in B,\ (g, m) \in I\}$$

It is common to denote either derivation operator with a prime and use the surrounding context to resolve ambiguity, so $A^{\uparrow}$ and $B^{\downarrow}$ would become A' and B', respectively. We avoid this notation as later on it will become increasingly challenging to avoid ambiguity while maintaining pleasing notation. In cases where a derivation operator is applied to a singleton of objects $\{g\}$ (*resp.* attributes $\{m\}$) we omit the parenthesis and write $g^{\uparrow}$ (*resp.* $m^{\downarrow}$).

The derivation operators provide a clear way of describing, for a given set $A \subseteq G$ of objects, the set of attributes which every object in A satisfies, denoted $A^{\uparrow}$. As an illustration, given the set of objects from Figure 2.1 {`semigroup, monoid`}, its' derivation would be {`closure, associativity`}. It is quite easy to spot that this is just the intersection of the object intents of `semigroup` and `monoid`.

2.1.1 Galois Connections

The derivation operators introduced in Definition 18 are not unique to FCA, and in fact constitute a *Galois connection* between the set of objects and attributes of a formal context. Galois connections are a generalisation of *The Fundemental Theorem of Galois Theory* (see [3, pp. 205]), and are useful ways of describing correspondence between two sets and with a relation. We will spend a bit of time describing Galois connections, as they are fundemental to FCA, and closely related to ideas around closures.

Proposition 1. We recall Definition 18, but imagine that the sets G and M are any arbitrary set, and I is any relation on $G \times M$. Then, for $X, X_1 \subseteq G$ and $Y, Y_1 \subseteq M$ we have

$$X \subseteq X_1 \Rightarrow X_1^{\uparrow} \subseteq X^{\uparrow} \qquad \textit{(resp.)} \qquad Y \subseteq Y_1 \Rightarrow Y_1^{\downarrow} \subseteq Y^{\downarrow} \tag{2.1}$$

$$X \subseteq X^{\uparrow\downarrow} \qquad \textit{(resp.)} \qquad Y \subseteq Y^{\downarrow\uparrow} \tag{2.2}$$

$$X^{\uparrow} = X^{\uparrow\downarrow\uparrow} \qquad \textit{(resp.)} \qquad Y^{\downarrow} = Y^{\downarrow\uparrow\downarrow} \tag{2.3}$$

As presented, these derivation operators are not unique to FCA; they are simply an instantiation of operators from Galois connections. Galois connections appear frequently, and are a generelatisation of *The Fundemental Theorem of Galois Theory* (see [3, pp. 205]). They are used to express a correspondence between two sets and a relation, and are closely related to ideas around closure systems.

Of course, these two functions can be composed; and so, $A^{\uparrow\downarrow}$—which can rather cumbersomely be described as "The set of all objects which satisfy all the attributes satisfied by `semigroup` and `monoid`"—would yield the set {`semigroup, monoid, group, abelian group`}. In fact, this composition of derivation operators satisfies very specific properties.

$$\text{(monotonicity)} \qquad A \subseteq A_1 \textit{ implies } A^{\uparrow\downarrow} \subseteq A_1^{\uparrow\downarrow} \tag{2.4}$$

$$\text{(extensivity)} \qquad A \subseteq A^{\uparrow\downarrow} \tag{2.5}$$

$$\text{(idempotency)} \qquad A^{\uparrow\downarrow} = (A^{\uparrow\downarrow})^{\uparrow\downarrow} \tag{2.6}$$

for all $A, A_1 \subseteq G$. Operators of this kind have already been discussed in Section 1.2. Thus, $(\cdot)^{\uparrow\downarrow}$ describes a closure operator on the powerset 2^G of objects. The dual notion holds for attributes, and $(\cdot)^{\downarrow\uparrow}$ describes a closure operator on 2^M [2, pp. 18].

Proposition 2. Let (G, M, I) be a formal context with subsets $A_0, A_1, A_2 \subseteq G$ and $B_0, B_1, B_2 \subseteq M$ of attributes. Then,

$$A_0 \subseteq B_0^{\downarrow} \quad \Longleftrightarrow B_0 \subseteq A_0^{\uparrow} \quad \Longleftrightarrow A_0 \times B_0 \subseteq I \tag{2.7}$$

Equation (2.1) describes the process where, as the size of a set of objects grows, the size of the set of attributes satisfied by this set of objects is reduced. The dual holds with respect to sets of attributes.

Let us temporarily speak imprecisely and consider the fairly general concept of a 'human'. The size of this concept's extension—those things which may be considered instances of 'human'—is quite large. Consider another concept, that of an 'optometrist'; presumably, the latter concept's extension is contained within that of the former. While the intension—those properties we associate with an optometrist—would certainly be larger than, and more precisely a superset of, those associated with the concept of 'human'.

We have, quite informally, described what constitutes a *Galois connection* between the sets $(2^G, \subseteq)$ and $(2^M, \subseteq)$ induced by the two derivation operators.

Definition 19. Given two partially ordered sets $\mathbf{X}$ and $\mathbf{Y}$, an *antitone Galois connection* is a pair of functions $f : \mathbf{X} \rightarrow \mathbf{Y}$ and $g : \mathbf{Y} \rightarrow \mathbf{X}$ such that $y \leq_Y f(x)$ if and only if $x \leq_X g(y)$ for all $x \in X$ and $y \in Y$.

With the above relationship between sets of objets and attributes in mind, it is appropriate to introduce a *formal concept*.

Definition 20. A *formal concept* of a formal context (G, M, I) is a pair (A, B) of subsets $A \subseteq G$ and $B \subseteq M$ that satisfies $A^{\uparrow} = B$ and $B^{\downarrow} = A$. Then, we say that A is the concept *extent* and that B is the *intent*. We use $\mathfrak{B}(G, M, I)$ to denote the set of all concepts of (G, M, I).

Example 6. If we consider the `groupoid` object from Figure 2.1, we can derive the concept:

$$\big(\{\texttt{groupoid,group,abelian group}\}, \{\texttt{associativity,identity,closure}\}\big)$$

The derivation of `groupoid` yields {`associativity,identity,closure`}, by Equation (2.3) this set is closed under its Galois connection, and forms a concept intent.

To begin, the object intent $\texttt{groupoid}^{\uparrow}$ is determined, which yields the set {`associativity`, `identity`, `closure`}. From Proposition 2 this set is closed, and represents a concept intent. The concept extent is determined by application of a derivation operator to the concept intent.

A direct consequence of Proposition 2 is that

The set of all concepts has a natural ordering to it, explained as the *subconcept–superconcept* relation: If (A_0, B_0) and (A_1, B_1) are two concepts in a context, then (A_0, B_0) is a *subconcept* of (A_1, B_1) if and only if $A_0 \subseteq A_1$, equivalently if $B_1 \subseteq B_0$. Otherwise, (A_0, B_0) is a *superconcept* of (A_1, B_1). We denote set of all concepts ordered in this way by $\underline{\mathfrak{B}}(G, M, I)$, and call this set the *concept lattice* of (G, M, I).

Intuitively, one concept is a subconcept of another, if every instance of the first concept is also an instance of the second. From the Galois connection, this is equivalently, albeit less intuitively, explained by every attribute of the second concept being included in the first.

We can prescribe a rather intuitive ordering over concepts induced by the *subconceptsuperconcept* relation.

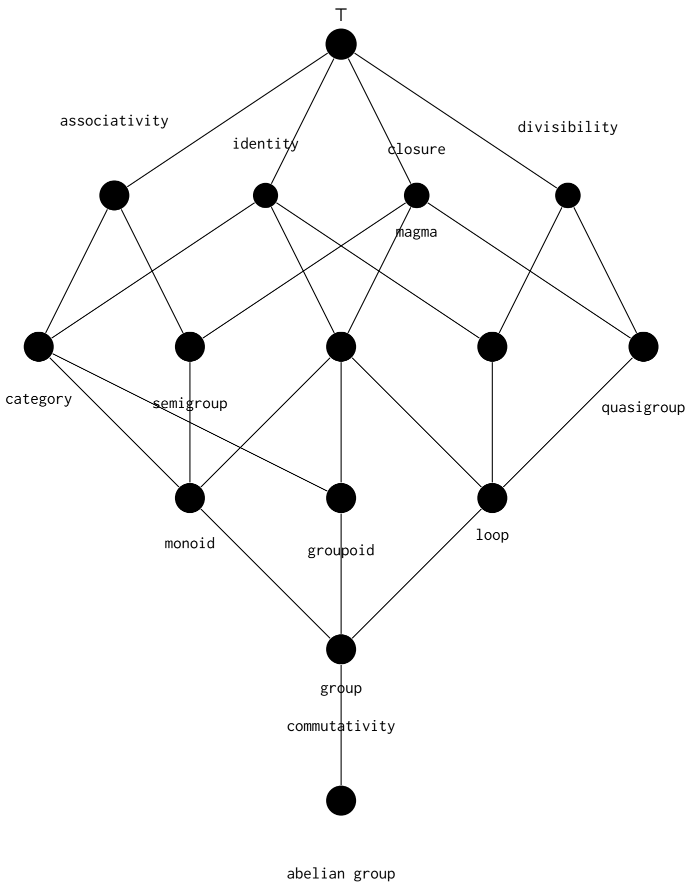


Figure 2.2: The concept lattice associated with the formal context in Figure 2.1

2.1.2 Attribute Implications

In FCA, attribute implications represent dependencies that exist between attributes in a context. If M is a non-empty set of attributes with $A, B \subseteq M$, then we denote an attribute implication over M as $A \to B$.

Definition 21. Let M be a non-empty set of attributes, then an *attribute implication* over M is a

If we examine the context in Figure 2.3 it is apparent that every representative who voted in favour of the proposed crime bill also voted in favour of the immigration bill; we write this dependency as `crime` → `immigration`.

Congressional Voting Records	mx-missile	crime	immigration	satellite ban	education	republican	democrat
Representative 1		×	×		×	×	
Representative 4							×
Representative 9		×			×	×	
Representative 17			×		×		×

Figure 2.3: A context describing a portion of congressional voting records for 1984

2.2 Contextual Attribute Logic

The attribute logic that usually underlies any discussion on FCA is analagous to Horn logic. Attribute implications are quite easily shown to be analagous to propositional Horn clauses: If we consider the (attribute) implication $A \to B$, with $A, B \subseteq M$ it can be phrased, with respect to holding in a context (G, M, I), as "all objects which satisfy every attribute in A also satisfy every attribute in B".

The underlying logic we have just described for FCA can quite easily be shown to be analagous to propositional Horn logic. Horn rules are syntactically similar to material implications, and share the same '→' operator. However, none of the premise nor conclusion of a Horn rule may be negative, or contain disjunction. They are not able to express what is *not* a consequence of a premise.

Chapter 3

Defeasible Reasoning

Non-monotonicity in logical systems has been the focus of study for decades, and several distinct formalisms have been developed. The motivation is to expand the inference power beyond that of the classical, to a more credulous one. This work is primarily interested in non-monotonic reasoning following the preferential semantics introduced by Shoham [11], for which there is a well-developed model theory which further benefits from an obvious analogue to notions central to FCA. In particular, we are interested in the framework put forward by Kraus, Lehmann, and Magidor [12, 13], frequently initialised to the 'KLM framework'.

In the proceeding, we provide a background on non-monotonic reasoning in general, before introducing the KLM framework. We assume all the same conventions introduced in Section 1.3 for the technical discussions which occur in this chapter.

3.1 Background on Non-monotonic Reasoning

Near the end of the previous chapter, the matter of consequence was discussed in a very formal sense. It is perhaps helpful to distinguish this subject—classical consequence—from the common concept, as the former yields some surprising results which do not appear at all congruent with how a person, or otherwise intelligent agent should reason [6, 12]. For a demonstration of a result which may be *surprising* in this way, consider the following propositions:

1. `human` → `chronological time`
2. `soldier` → `human`
3. `billy pilgrim` → ¬`chronological time`

Knowing these propositions, if we were to encounter an individual in fatigues we might find it sensible—by propositions 2 and 1—to infer that the individual experienced time chronologically. If we were to later learn that the individual was, in fact, Billy Pilgrim, given proposition 3, we should like to retract our prior inference, replacing it with the knowledge that the individual does not experience time chronologically.

Such recourse is not, as it stands, possible: When we see the that the individual is a soldier, the possible worlds satisfying our knowledge are reduced to the single model: $\{\overline{b}, c, h, s\}$ (where b: `billy pilgrim`, c: `chronological time`, h: `human`, and s: `soldier`). Should we later learn that the individual is indeed Billy Pilgrim, the theory becomes inconsistent. And, by the principle of explosion, discussed in Subsection 1.3.3, our theory now entails that the individual experiences time both chronologically and non-chronologically; indeed our theory entails everything and is accordingly worthless.

This property of classical logic—that adding new information never results in retraction of pre-existing knowledge—is called monotonicity. Monotonicity requires that when we make a claim like "humans experience time chronologically", we must be absolutely sure of ourselves, so as to never worry about needing to retract an inference. This is, of course, too strict a requirement as we cannot determine for all future, present, and past humans if it were the case that they experienced time chronologically. If we remain in the classical realm, it seems our only options are to abandon our original claim or risk explosion.

At this point it is a good idea to provide some clarification on how we might begin to approach this issue. Continuing with the same example—and continuing to allow ourselves to entertain the possibility of experiencing non-chronological time—it would certainly be agreed that typically soldiers are human, and also that typically humans experience time chronologically. To resolve that Billy Pilgrim is a soldier, and therefore a human, who does not experience chronological time, we need only to point out that he is an atypical human.

To make the previous paragraph more formal, we remind the reader of the discussion held around Definition 13: A formula ϕ is a logical consequence of a set Γ thereof if every model of Γ is also a model of ϕ. Put differently, there is no valuation (or, possible world) where Γ is true and ϕ is false. It follows directly that ϕ remains a logical consequence of $\Gamma \cup \{\psi\}$, since Γ is true in any world where $\Gamma \cup \{\psi\}$ is true.

It was pointed out by Shoham [11] that we may "bend the rules" and restrict semantic consideration to a privileged subset of models deemed "preferable". We call these selected models the "minimal" models—a choice that will become clearer as this chapter progresses.

3.2 The KLM Framework

The KLM framework for non-monotonic reasoning was initially described by a collection of consequence relations satisfying certain axioms—frequently called the *rationality postulates*—with each successive system being stronger than its predecessor. We borrow a nice story from Dov Gabbay [14] which motivates why consequence relations are a good starting point for the study of a non-monotonic system.

It begins by asking the reader to imagine a machine that does non-monotonic inference in some domain; the machine represents knowledge as formulae and so we pose queries of the form "Does ψ non-monotonically follow from ϕ?". Something goes awry (suppose some coffee was spilled), calling into question whether the logic of the machine still functions correctly. Even worse, the interface, which tells us what real-world instance each formula maps to, is destroyed and so function cannot be evaluated based on the meaning of the formulae the machine reasons on. How might we then evaluate the machine's function?

If we were interested in classical consequence, we would be well-equipped to assess the correctness of the machine by determining if it satisfied reflexivity, monotonicity, and cut (we point to Definition 16 as a reminder). This is precisely the starting point that Kraus, Lehmann, and Magidor took up in [12], suggesting that before getting to the semantics of a non-monotonic system, it is pertinent to formalise how exactly that system should behave.

The argument put forward is that the rationality postulates characterise a sensible pattern of reasoning for a non-monotonic system. We use '$\mid\sim$' (pronounced "twiddle") instead of '$\vdash$', to denote a consequence relation in the KLM framework, and as we may expect $\phi \mid\sim \psi$ means the same thing as $(\phi, \psi) \in \mid\sim$, while $\phi \not\mid\sim \psi$ means $(\phi, \psi) \notin \mid\sim$.

An expression like $\phi \mid\sim \psi$ is to be understood as saying "If ϕ holds, then ψ is a typical consequence", we call expressions of this nature *conditional assertions*. The properties that a particular consequence relation satisfies are represented as rules of inference in the style of conditional assertions.

We may, at times of potential confusion, use a subscript to disambiguate which consequence relation is being referred to, and so $\mid\!\sim_C$ would refer to a cumulative relation, as defined below.

3.2.1 System C

Cumulative consequence relations (shortened to 'System C') [12] represent the weakest of the systems in the KLM framework, and so the notions introduced in here will carry through to each successive stronger system, making cumulative relations a helpful starting point.

Definition 22. The consequence relation $\mid\!\sim$ is a *cumulative consequence relation* if and only if it satisfies the properties of *Reflexivity, Left Logical Equivalence, Right Weakening, Cut* and *Cautious Monotony*.

The first axiom, *Reflexivity*, is largely self justifying: It makes little sense to speak about a notion of consequence that does not satisfy this property.

$$\text{(Reflexivity)} \quad \frac{}{\phi \mid\!\sim \phi} \tag{3.1}$$

The justification for *Left Logical Equivalence* is a bit more opaque; the principle is that if two scenarios represent the same state of affairs, and in one of these scenarios it we typically expect some consequence, then we should expect the same in the other scenario.

$$\text{(Left Logical Equivalence)} \quad \frac{\vdash \phi \leftrightarrow \psi, \quad \phi \mid\!\sim \gamma}{\psi \mid\!\sim \gamma} \tag{3.2}$$

Right Weakening allows the preservation of classical consequence within the logic. It says that if it is always the case that ψ-worlds are also γ-worlds, and that in the presence of ϕ we typically expect ψ also, then we should expect γ as well.

$$\text{(Right Weakning)} \quad \frac{\vdash \psi \rightarrow \gamma, \quad \phi \mid\!\sim \psi}{\phi \mid\!\sim \gamma} \tag{3.3}$$

$$\text{(Cautious Monotony)} \quad \frac{\phi \mid\!\sim \psi, \quad \phi \mid\!\sim \gamma}{\phi \wedge \psi \mid\!\sim \gamma} \tag{3.4}$$

Certain other rules may derived from the presence of already discussed postulates. A version of *Cut*

$$\text{(Cut)} \quad \frac{\phi \wedge \psi \mid\!\sim \gamma, \quad \phi \mid\!\sim \psi}{\phi \mid\!\sim \gamma} \tag{3.5}$$

The original version due to Gentzen [4] is presented as:

$$\text{(Monotonic Cut)} \quad \frac{\phi \wedge \psi \mid\!\sim \gamma, \quad \alpha \mid\!\sim \psi}{\phi \wedge \alpha \mid\!\sim \gamma} \tag{3.6}$$

The latter version implies monotonicity, as it requires that if ψ is a typical consequence of α, then it must remain a consequence of $\alpha \wedge \phi$: ergo, monotonicity. The former variation does not enforce this, and rather says "Suppose I have certain knowledge of ϕ, and that if I were to assume ψ I should expect to conclude γ. Then if I can show that infact ψ was already an expected consequence of knowing ϕ, I should expect that γ follows from ϕ".

The following Lemma is a helpful intuition pump, it is largely why the term *cumulative* is used, as using the conclusion of a premise as another premise does not [14]

Lemma 2. We can cover the properties of *Cut* and *Cautious Monotonicity* with the following principle: "If $\phi \mid\sim \psi$, then the typical consequences of ϕ and $\phi \wedge \psi$ coincide".

We will skip over any discussion of semantics for cumulative consequence relations, and rather opt to introduce these in the next section where we discuss preferential consequence relations.

3.2.2 System P

The jump from cumulative to preferential consequence merely requires us to consider a single additional postulate, *Or*:

$$\text{(Or)} \quad \frac{\phi \mid\sim \gamma, \psi \mid\sim \gamma}{\phi \vee \psi \mid\sim \gamma} \tag{3.7}$$

The addition of *Or*

Or gives us deduction rule

$$\text{(S)} \quad \frac{\phi \wedge \psi \mid\sim \gamma}{\phi \mid\sim \psi \rightarrow \gamma} \tag{3.8}$$

Definition 23. The consequence relation $\mid\sim$ is a *preferential consequence relation* if and only if it satisfies the properties of *Reflexivity, Left Logical Equivalence, Right Weakening, Cut, Or,* and *Cautious Monotony*.

Lemma 3 ([11])**.** For $\phi, \psi \in \mathcal{L}$ and some valuation $u \in \mathcal{U}$, if $u \vdash \psi$ and $u \mid\sim \phi$, then $u \mid\sim \phi \wedge \psi$.

3.2.2.1 Model theory

Definition 24.

3.2.3 Preferential Entailment

3.3 Rational Closure

3.3.1 Rational Consequence Relations

Part II

Rational Concept Analysis

Chapter 4

Defeasible Reasoning in Formal Concept Analysis

This phenomenon was already known to the ancient Greeks, who used the term enthymeme to refer to an argument in which one or more premises are left implicit. That is the idea that we develop in this section [15]. Also called *expectations*

Chapter 5

Rational Concepts

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