Rational Concept Analysis

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Acknowledgements

There is a dog

Abstract

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Introduction

Part I

Foundations

Chapter 1

Mathematical Preliminaries

This chapter is intended to serve as a reference point for the more fundamental concepts that are used throughout this dissertation. In particular, this dissertation contains a significant amount of discussion on *Formal Concept Analysis* and *Preference Relations*; which, in turn, are based on order theory, which is where we begin. The second half of this chapter provides an introduction to basic notions in logic, given in the setting of propositional logic.

1.1 Order and Lattice Theory

1.1.1 Orders

Definition 1. A **binary relation** R over two sets X and Y is a set of ordered pairs $\langle x, y \rangle$ with $x \in X$ and $y \in Y$; and so $R \subseteq X \times Y$. In many cases we express this pair using infix notation, and we write xRy.

Binary relations are not particularly interesting until they satisfy certain properties. We now discuss certain binary relations which occur frequently enough to deserve a distinct name.

Definition 2. A **partial-order** is a binary relation $\preceq \subseteq X \times X$ that satisfies the following properties:

$$\text{(Reflexivity)} \qquad x \preceq x \tag{1.1}$$

$$\text{(Antisymmetry)} \qquad x \preceq y \text{ and } y \preceq x \text{ implies } x = y \tag{1.2}$$

$$\text{(Transitivity)} \qquad x \preceq y \text{ and } y \preceq z \text{ implies } x \preceq z \tag{1.3}$$

for all $x, y, z \in X$.

We write $x \npreceq y$ to indicate that $x \preceq y$ does not hold, and $x \prec y$ for the case where $x \preceq y$ and $x \neq y$. When $x \npreceq y$ and $y \npreceq x$—i.e., that x and y are incomparable—we write $x \| y$ [1]. From a partial-order we can quite easily induce the notion of a *strict partial-order*.

Definition 3. A **strict partial-order** is a binary relation $\prec \subseteq X \times X$ that satisfies:

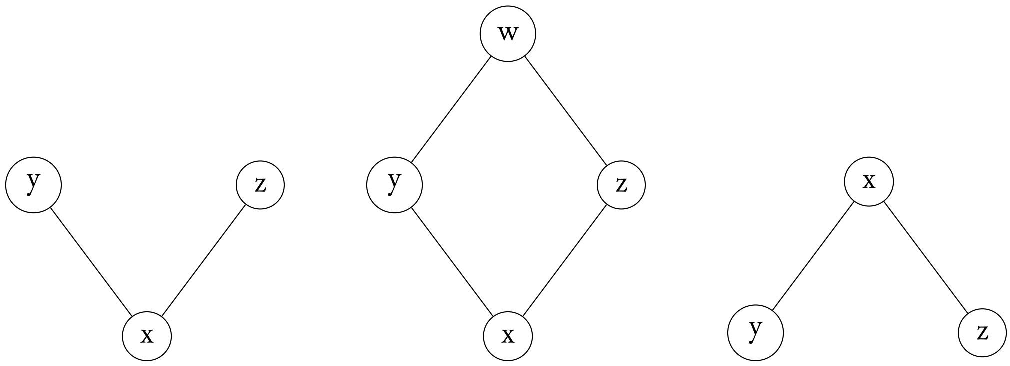
$$\text{(Irreflexivity)} \qquad x \not\prec x \tag{1.4}$$

$$\text{(Asymmetry)} \qquad x \prec y \text{ implies } y \not\prec x \tag{1.5}$$

$$\text{(Transitivity)} \qquad x \prec y \text{ and } y \prec z \text{ implies } x \prec z \tag{1.6}$$

for all $x, y, z \in X$.

An ordered set is a pair $(X, \preceq)$ with X being a set and $\preceq$ being an ordering on X. If $\preceq$ is a partial-ordering, we might then refer to X as a *poset*. We can describe ordered sets diagrammatically through the use of *Hasse* diagrams.



(a) $\preceq_a$ (b) $\preceq_b$ (c) $\preceq_a$

Figure 1.1: Three partial-orders over a set P

Then, a pair $\langle x, y \rangle$ is in a given binary relation $\preceq \subseteq X \times X$ if there exists a strictly upward path connecting x to y (or, if $x = y$). From Sub-figure 1.1b we can infer that $x \preceq_b y$, $x \preceq_b w$, and $y \|_b z$. For the inverse of an order $\preceq$ we write $\preceq^{-1}$, and so in Figure 1.1 $\preceq_a^{-1} = \preceq_c$ [2].

If $(X, \preceq)$ is an ordered set, then an element $x \in X$ is *minimal* in X if there exists no distinct element $y \in X$ such that $y \preceq x$. Conversely, x is *maximal* in X if there exists no distinct $y \in X$ with $x \preceq y$. Continuing this example, x is the *minimum* of X if it is minimal in X and there are no elements to which x is incomparable. In other words, for every element $y \in X$ it is the case that $x \preceq y$. Naturally, x is the *maximum* element of X if for all $y \in X$ it is the case that $y \preceq x$.

Definition 4. Let $(X, \preceq)$ be a partially ordered set, and Y a subset of X. The **lower bound** of Y is an element $x \in X$ with $x \preceq y$ for all $y \in Y$. The **upper bound** of Y is defined dually. If the set of lower bounds of Y has a maximum (greatest) element then this element is called the **infimum** of Y. Dually, if there is a minimum (least) element in the set of upper bounds, then this element is the **supremum** of Y.

1.1.2 Lattices

Certain, particularly interesting, posets satisfy some nice properties involving infimums and supremums. Frequently, we call the supremum of elements $A = \{x, y, z\}$ of a poset the *join*, and write $x \vee y$, or $\bigvee A$. The dual notion of an infimum corresponds to the *meet*, and is written $x \wedge y$ or $\bigwedge A$.

Definition 5. A partially ordered set $\mathbf{X} = (X, \preceq)$ is a **lattice** if and only if, for any two elements $x, y \in \mathbf{X}$, both the supremum $x \vee y$ and the infimum $x \wedge y$ exist. The set $\mathbf{X}$ is a **complete lattice** if and only if the meet and join exist for every subset of $\mathbf{X}$.

In Figure 1.1 only (b) is a lattice (and a complete lattice too). In fact, every finite lattice is a complete lattice [2].

Chapter 2

Formal Concept Analysis

Formal Concept Analysis (FCA) provides a simple, and yet mathematically rigorous, framework for identifying and reasoning about "concepts" and their corresponding hierarchies in data [2, 3]. Its central view of concepts as a dual between *extension*—what one refers to as instances of a concept—and *intension*—what meaning is ascribed to a concept—is supported by a rich philosophical backing.

2.1 Basic Notions

The 'atoms' in FCA are given by *objects* and *attributes*; objects relate to extension, and attributes to intension. These are unified under a structure called a *formal context*.

Definition 6. A **formal context** $\mathbb{K} = (G, M, I)$ is a triple comprised of a set of objects G, a set of attributes M, and a binary relation $I \subseteq G \times M$ referred to as an 'incidence' relation. For an object-attribute pair $(g, m) \in I$ we say that "object g *has* the attribute m".

A formal context in some sense describes an open-world interpretation, and so $(g, m) \notin I$ is not usually interpreted as saying that "object g has the negation of the attribute m".

Example 2.1.1. Formal contexts of reasonable size can be described entirely by a matrix-like representation. Each object corresponds to a row, and each attribute to a column.

	closure	associativity	identity	divisibility	commutativity
magma	×				
semigroup	×	×			
monoid	×	×	×		
group	×	×	×	×	
abelian group	×	×	×	×	×
loop	×		×	×	
quasigroup	×			×	
groupoid		×	×	×	
category		×	×		
semicategory		×			

Figure 2.1: A formal context showing necessary properties of group-like structures.

A formal

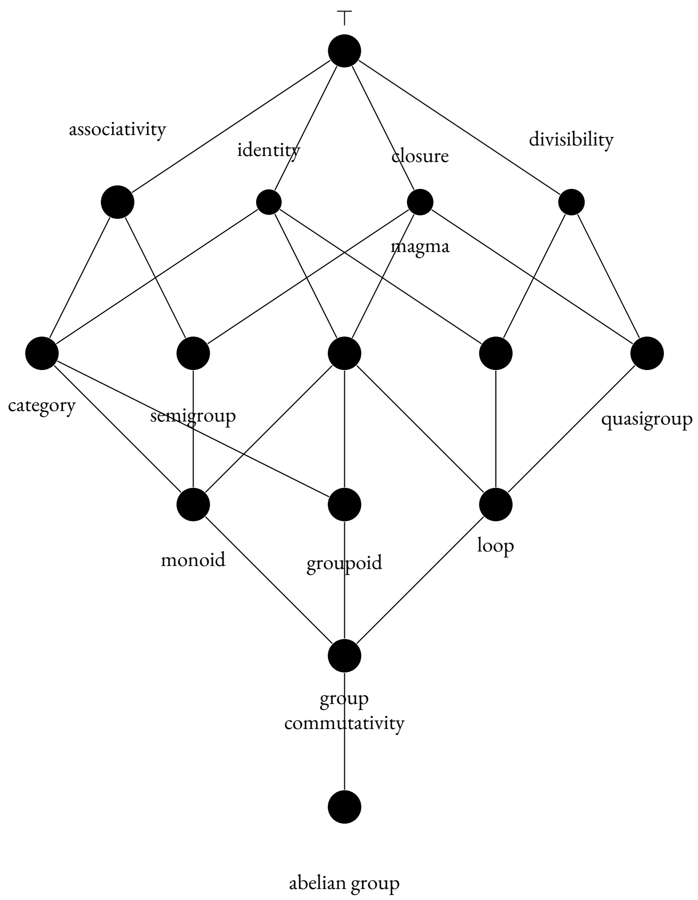


Figure 2.2: The concept lattice associated with the formal context in Figure 2.1

Chapter 3

Non-Monotonic Reasoning

Hello there my dog A wfaew

Part II

Rational Concept Analysis

Chapter 4

Defeasible Reasoning in Formal Concept Analysis

Chapter 5

Rational Concepts

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