Derivation of Rocket Motion Equations

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May 21, 2023

Abstract

Algebraic derivation of the equations of instantaneous velocity and altitude as a function of time for a rocket during launch until main engine shutoff.

1 Summary

The following is a step-by-step derivation of the equations used to graph velocity and altitude as a function of time during rocket launch. While many online calculators and simulators use **rocket thrust** and **instantaneous impulse**, these equations use **exhaust speed** and **exhaust rate**, allowing us to derive an equation that we can plot more easily.

I originally learned to derive these equations in Grade 12 Enriched Mechanics and my goal through this project was to take the derivations further by incorporating air resistance as another force affecting the system. This proved more difficult than expected as I was faced with a differential equation that was impossible to solve with my current knowledge of mathematics. I have included this unsuccessful derivation with air resistance at the end of the document.

I have included more steps than necessary in illustrating the derivation of the equations to hopefully make the math easier to follow for readers that may only have a working knowledge of integrals.

2 Rocket Equation Derivation

2.1 Variable Definition

Symbol	Definition
v	Instantaneous velocity
y	Instantaneous altitude
m	Instantaneous mass
t	Time

Table 1: Variables used in the following equations.

Symbol	Definition
g	Gravity
u	Exhaust speed
α	Exhaust rate
m_i	Initial mass
m_{dry}	Dry mass
m_{fuel}	Fuel mass
m_{lift}	Mass at liftoff
t_{lift}	Time at liftoff
t_{burn}	Engine burn time
v_{burn}	Velocity at engine shutoff

Table 2: Constants used in the following equations.

2.2 Main Derivation

Take a rocket system where the **upwards direction is positive and the downwards direction is negative**.The following derivation is common for both velocity and altitude as a function of time.

$$
\begin{aligned}
F_{\text{net}} &= ma \\
-F_{\text{g}} &= \frac{dP_{\text{system}}}{dt} \\
-mg &= \frac{P_{\text{sys}}(t+dt) - P_{\text{sys}}(t)}{dt} && (1) \\
-mgdt &= (m+dm)(v+dv) + (-dm)(v+dv-u) - mv && (2) \\
-mgdt &= mv + mdv + vdm + dmdv - vdm - dmdv + udm - mv \\
-mgdt &= mdv + udm && (3) \\
mdv &= -udm - mgdt \\
dv &= -\frac{u}{m}dm - gdt && (4) \\
\int dv &= -\int \frac{u}{m}\,dm - \int g\,dt && (5)
\end{aligned}
$$

A rocket can be launched thanks to Newton's Third Law which states that every action has an equal and opposite reaction. As the exhaust exits the vehicle at speed u, the rocket gains momentum equal to the momentum of the exhaust. From Newton's Second Law $F_{\text{net}} = ma = \frac{dP_{\text{system}}}{dt}$, we evaluate using momentum as shown in equations (1) and (2). By expanding and simplifying, equation (3) shows how the momentum of the rocket at time dt increases by udm. Further algebraic manipulation yields equation (5)

2.3 Calculating Liftoff

When watching a rocket launch, we notice that the rocket does not leave the ground immediately. In fact, the rocket does not leave the ground until thrust ($u\alpha$) is larger than the gravitational force. Thus, we can calculate the time at which the rocket will achieve liftoff and it's mass at that moment.

$$
\begin{aligned}
dv &= -\frac{u}{m}dm - gdt && && (4) \\
\frac{dv}{dt} &= -\frac{udm}{mdt} - g \\
a &= \frac{u\alpha}{m} - g && \text{where } a = \frac{dv}{dt},\ \alpha = -\frac{dm}{dt} \\
ma &= u\alpha - mg \\
0 &= u\alpha - mg && \text{liftoff occurs when } ma = 0 \\
u\alpha &= (m_i - \alpha t)g && m_{\text{lift}} = m_i - \alpha t \\
t_{\text{lift}} &= \frac{m_i}{\alpha} - \frac{u}{g} && && (6)
\end{aligned}
$$

Using t_{lift}, we can derive the mass of the rocket at liftoff:

$$
\begin{aligned}
m_{\text{lift}} &= m_i - \alpha t_{\text{lift}} \\
m_{\text{lift}} &= m_i - \alpha\left(\frac{m_i}{\alpha} - \frac{u}{g}\right) \\
m_{\text{lift}} &= \frac{\alpha u}{g} && (7)
\end{aligned}
$$

Miscellaneous equation calculations related to liftoff:

$$
\begin{aligned}
t_{\text{burn}} &= \frac{m_{\text{fuel}}}{\alpha} && (8) \\
m_{\text{dry}} &= m_i - m_{\text{fuel}} && (9)
\end{aligned}
$$

2.4 Rocket Velocity vs Time Equation Derivation

Use equation (5) and integrate within the correct bounds to solve for instantaneous velocity as a function of time. Note that this equation only graphs velocity up until rocket engine cutoff (the point where the rocket runs out of fuel).

$$\int_0^{v_{\text{burn}}} dv = -\int_{m_{\text{lift}}}^{m_{\text{dry}}} \frac{u}{m}\, dm - \int_{t_{\text{lift}}}^{t_{\text{burn}}} g\, dt \tag{5}$$

$$v_{\text{burn}} = -u \ln |\frac{m_{\text{dry}}}{m_{\text{lift}}}| - g(t_{\text{burn}} - t_{\text{lift}}) \tag{10}$$

Rewriting equation (10) with the original variables yields:

$$v = -u \ln |\frac{m_i - m_{\text{fuel}}}{\frac{\alpha u}{g}}| - g(\frac{m_{\text{fuel}}}{\alpha} - (\frac{m_i}{\alpha} - \frac{u}{g})) \tag{11}$$

2.5 Rocket Altitude vs Time Equation Derivation

Start from equation (5) again but integrate using more general bounds to solve for instantaneous altitude as a function of time. Note that this equation only graphs altitude until rocket engine cutoff.

$$\int_0^{v} dv = -\int_{m_{\text{lift}}}^{m_f} \frac{u}{m}\, dm - \int_{t_{\text{lift}}}^{t} g\, dt \tag{5}$$

$$v = -u \ln |\frac{m_f}{m_{\text{lift}}}| - g(t - t_{\text{lift}})^*$$

$$v = -u \ln |\frac{m_i - \alpha t}{\frac{\alpha u}{g}}| - g(t - (\frac{m_i}{\alpha} - \frac{u}{g}))^{**}$$

$$\frac{dy}{dt} = -u \ln \frac{m_i - \alpha t}{\frac{\alpha u}{g}} - gt + \frac{m_i g}{\alpha} - u$$

$$\int_0^{y} dy = -u \int_{t_{\text{lift}}}^{t} \ln \frac{m_i - \alpha t}{\frac{\alpha u}{g}}\, dt - g \int_{t_{\text{lift}}}^{t} t\, dt + \frac{m_i g}{\alpha} \int_{t_{\text{lift}}}^{t} dt - u \int_{t_{\text{lift}}}^{t} dt \tag{12}$$

*Note: $m_f = m_i - \alpha t$
**Note: integrating over the correct bounds will always yield a positive result inside the absolute value.

Perform a u-substitution to simplify $\ln \frac{m_i - \alpha t}{\frac{\alpha u}{g}}$ and integration by parts with $w = \ln x$ and $dv = dx$ where x is the placeholder variable in the u-sub. Integrating the differential equation (12) yields equation (13) describing instantaneous altitude as a function of time up until main engine cutoff:

$$y = \frac{u^2}{g}(x \ln x - x)|_{t_{\text{lift}}}^{t} - gt^2|_{t_{\text{lift}}}^{t} + \frac{m_i g}{\alpha} t|_{t_{\text{lift}}}^{t} - ut|_{t_{\text{lift}}}^{t}$$

$$y = [\frac{u^2}{g}(\frac{(m_i - \alpha t)g}{\alpha u} \ln \frac{(m_i - \alpha t)g}{\alpha u} - \frac{(m_i - \alpha t)g}{\alpha u}) - \frac{1}{2}gt^2 + \frac{m_i g t}{\alpha} - ut]|_{t_{\text{lift}}}^{t}$$

$$\begin{aligned} y = {} & [\frac{u^2}{g}(\frac{(m_i - \alpha t)g}{\alpha u} \ln \frac{(m_i - \alpha t)g}{\alpha u} - \frac{(m_i - \alpha t)g}{\alpha u}) - \frac{1}{2}gt^2 + \frac{m_i g t}{\alpha} - ut] \\ & - [\frac{u^2}{g}(\frac{(m_i - \alpha t_{\text{lift}})g}{\alpha u} \ln \frac{(m_i - \alpha t_{\text{lift}})g}{\alpha u} - \frac{(m_i - \alpha t_{\text{lift}})g}{\alpha u}) - \frac{1}{2}g(t_{\text{lift}})^2 + \frac{m_i g t_{\text{lift}}}{\alpha} - ut_{\text{lift}}] \end{aligned}$$

$$\begin{aligned} y = {} & [\frac{u^2}{g}(\frac{(m_i - \alpha t)g}{\alpha u} \ln \frac{(m_i - \alpha t)g}{\alpha u} - \frac{(m_i - \alpha t)g}{\alpha u}) - \frac{1}{2}gt^2 + \frac{m_i g t}{\alpha} - ut] \\ & - [\frac{u^2}{g}(\frac{\alpha u}{g} \ln \frac{\alpha u}{g} - \frac{\alpha u}{g}) - \frac{1}{2}g(\frac{m_i}{\alpha} - \frac{u}{g})^2 + \frac{m_i g(\frac{m_i}{\alpha} - \frac{u}{g})}{\alpha} - u(\frac{m_i}{\alpha} - \frac{u}{g})]^* \end{aligned} \tag{13}$$

*Note: this equation is only valid until t_{burn} after which the rocket runs out of fuel and becomes a projectile under the sole influence of gravity (and air resistance).

2.6 Main Derivation Including Air Resistance

The following derivation is my attempt to include air resistance into the equation in an attempt to make the simulation more accurate. However, not only does the derivation produce a differential equation that is impossible to solve algebraically, but there are several limitations that would make this equation (if successfully derived) inaccurate.

For one, rocket's generally achieve speeds greater than mach 5. In the following derivation, I simulate air resistance using the formula $F_{\text{air}} = cv^2$. Since this formula is generally applied to free-falling objects and rockets are incredibly aerodynamic, I am unaware of this formula's accuracy at speeds much larger than mach 1.

$$\begin{aligned}
F_{\text{net}} &= ma \\
-F_{\text{g}} - F_{\text{air}} &= \frac{dP_{\text{system}}}{dt} \\
-mg - cv^2 &= \frac{P_{\text{sys}}(t+dt) - P_{\text{sys}}(t)}{dt} \\
-mgdt - cv^2dt &= (m+dm)(v+dv) + (-dm)(v+dv-u) - mv \\
-mgdt - cv^2dt &= mv + mdv + vdm + dmdv - vdm - dmdv + udm - mv \\
-mgdt - cv^2dt &= mdv + udm \\
mdv &= -udm - mgdt - cv^2dt \\
dv &= -\frac{u}{m}dm - gdt - \frac{cv^2}{m}dt
\end{aligned}$$

As per my current knowledge of calculus, I am unable to solve this equation since air resistance is being integrated by the wrong variable.