PROBLEM SET 2

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Problems: Rudin Chapter 8 Problems 12-17

Solutions:

Problem 12. (a) We solve

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$$
$$= \frac{1}{2\pi}\int_{-\delta}^{\delta} e^{-inx}\,dx.$$
$$= \frac{1}{2\pi}\left(-\frac{e^{-inx}}{in}\right)\Bigg|_{\delta}^{\delta} = \frac{\sin(n\delta)}{\pi n}$$

For the case $n = 0$ we have $C_0 = \frac{1}{2\pi}\int_{-\delta}^{\delta} 1\,dx = \frac{\delta}{\pi}$.

(b) We apply the coefficients to the Fourier series:

$$\sum_{n=-\infty}^{\infty} C_n e^{inx} = \sum_{n=-\infty}^{\infty} \frac{\sin(n\delta)}{\pi n} e^{inx}$$

Since f is constant and therefore Lipschitz continuous at 0 its Fourier series converges to it at 0, thus by plugging in 0 we achieve

$$\sum C_n = \frac{\delta}{\pi} + 2\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{\pi n} = 1$$

and

$$\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} = \frac{\pi - \delta}{2}.$$

(c) Since the constants and f are all real, by Parseval's theorem we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)^2\,dx = \sum_{n=-\infty}^{\infty} C_n^2$$
$$\frac{\delta}{\pi} = \frac{\delta^2}{\pi^2} + 2\sum_{n=1}^{\infty} \frac{\sin(n\delta)^2}{\pi^2 n^2}$$
$$\sum_{n=1}^{\infty} \frac{\sin(n\delta)^2}{\delta n^2} = \frac{\pi - \delta}{2}.$$

(d) The answer to (c) can be written as $\sum_{n=1}^{\infty}\left(\frac{\sin(n\delta)}{n\delta}\right)^2\delta$, which we recognize as an infinite series of Riemann sums of our desired integrand, with rectangle width

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δ. We must prove that the integral converges, and that our limit converges to the improper integral.

We know the integral converges absolutely since it is bounded above by $1/x^2$. Since $\frac{\pi-\delta}{2}$ is monotonic we need only consider values of δ of the form $\pi/2^m$: if the sums converge correctly to the integral there then we know the limit overall must also converge. Take $S_m = \sum_{n=1}^{\infty} \left(\frac{\sin(n\pi/2^m)}{n\pi/2^m}\right)^2 \pi/2^m$.

Take $\epsilon > 0$. Then pick N such that $\int_a^\infty \frac{1}{x^2} < \epsilon/3$ for $a > N$. Take a a multiple of π. Then we have

$$\int_a^\infty \left(\frac{\sin(x)}{x}\right)^2 dx \leq \int_a^\infty \frac{1}{x^2} < \epsilon/3.$$

We pick m such that

$$d\left(\sum_{n=1}^{2^m a} \left(\frac{\sin(n\pi/2^m)}{n\pi/2^m}\right)^2 \pi/2^m, \int_1^a \left(\frac{\sin(x)}{x}\right)^2 dx\right) < \epsilon/3$$

which exists since the left side is part of a series of successively refined Riemann sums whose partitions have consecutive distances equal to $\pi/2^m$.

Finally, we have

$$\sum_{n=2^m a}^{\infty} \left(\frac{\sin(n\pi/2^m)}{n\pi/2^m}\right)^2 \pi/2^m \leq \int_a^\infty \frac{1}{x^2}\, dx < \epsilon/3$$

since the sum is a right Riemann sum of a function bounded by the monotonically decreasing $\frac{1}{x^2}$ (its area is therefore bounded by the integral).

By these inequalities and the triangle inequality we have $d\left(S_m, \int_1^\infty \left(\frac{\sin(x)}{x}\right)^2 dx\right) < \epsilon$ and since S_m decreases with increase of m, the limit converges to the integral and we have

$$\int_1^\infty \left(\frac{\sin(x)}{x}\right)^2 dx = \lim_{\delta\to 0} \frac{\pi-\delta}{2} = \frac{\pi}{2}.$$

(e) We have

$$\sum_{n=1}^{\infty} \frac{\sin(\frac{\pi}{2}n)^2}{\frac{\pi}{2}n^2} = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \frac{\pi}{4}$$

Problem 13. We may treat f as a 2π-periodic function and take the Fourier coefficients as

$$C_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\, dx$$

since both factors of the integrand are 2π-periodic and the integral is the same as the standard formula over $[-\pi, \pi]$.

We thus solve the integral via integration by parts:

$$\int_0^{2\pi} xe^{-inx}\, dx = \left(-\frac{xe^{-inx}}{in}\right)\Bigg|_0^{2\pi} + \int_0^{2\pi} \frac{e^{-inx}}{in}\, dx.$$

$$= \frac{-2\pi}{in}$$

and thus $C_n = \frac{-1}{in}$. For the case $n = 0$ we have $C_0 = \pi$. Parseval's theorem yields

$$\frac{1}{2\pi}\int_0^{2\pi} x^2\,dx = \sum_{n=-\infty}^{\infty} \frac{1}{n^2}$$

$$\frac{4\pi^2}{3} = \pi^2 + 2\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

Problem 14. We solve for Fourier coefficients.

$$2\pi C_n = \int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = \int_{-\pi}^{0}(\pi+x)^2 e^{-inx}\,dx + \int_0^{\pi}(\pi-x)^2e^{-inx}\,dx$$

$$= \left(-\frac{(\pi+x)^2}{in} + \frac{2(\pi+x)}{n^2} + \frac{2}{in^3}\right)(e^{-inx})\Big|_{-\pi}^{0}$$

$$+\left(-\frac{(\pi-x)^2}{in} - \frac{2(\pi-x)}{n^2} + \frac{2}{in^3}\right)(e^{-inx})\Big|_{0}^{\pi}$$

$$= \frac{4\pi}{n^2}$$

and

$$C_0 = 2\int_0^{\pi}(\pi-x)^2\,dx = \frac{\pi^2}{3}$$

Then since f is continuous with bounded derivative we have

$$f(x) = \sum_{n=-\infty}^{0}\frac{2}{n^2}e^{inx} + C_0 + \sum_{n=1}^{\infty}\frac{2}{n^2}e^{inx}$$

$$= \sum_{n=-\infty}^{0}\frac{2}{n^2}(\cos(nx) + i\sin(nx)) + \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{2}{n^2}(\cos(nx)+i\sin(nx))$$

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos(nx).$$

We take $x = 0$ which yields

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}$$

and

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

Then we apply Parseval's Theorem to the constants:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)^2\,dx = \frac{\pi^4}{9} + \sum_{n=1}^{\infty}\frac{8}{n^4}$$

$$= \frac{1}{2\pi}\left(\int_{-\pi}^{0}(\pi+x)^4\,dx + \int_0^{\pi}(\pi-x)^4\,dx\right) = \frac{1}{2\pi}(\frac{2\pi^5}{5})$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

Problem 15. Take $k_N = (N+1)K_N$. We notice that $2 - 2\cos x = 2 - e^{ix} - e^{-ix}$. Then we separate $(2 - 2\cos x)k_N$:

$$(1 - e^{-ix})k_N - (e^{ix})(1 - e^{-ix})k_N$$

where

$$k_N = \sum_{n=0}^{N} D_n$$

Then

$$(1 - e^{-ix})k_N = \sum_{n=0}^{N} (1 - e^{-ix})D_n = \sum_{n=0}^{N} e^{inx} - e^{-ix(n+1)}$$

$$(2 - 2\cos x)k_N = (1 - e^i x)\left(\sum_{n=0}^{N} e^{inx} - \sum_{n=0}^{N} e^{-ix(n+1)}\right)$$

$$= -e^{ix(n+1)} - e^{-ix(n+1)} + 2 = 2(1 - \cos(x(n+1))$$

$$\implies K_N = \left(\frac{1}{N+1}\right) \frac{1 - \cos(x(n+1))}{1 - \cos x}$$

(a) We prove $K_N \geq 0$. Since $\cos \leq 1$ we have $K_N \geq 0$ for all x outside of multiples of 2π. We consider $K_N(0)$: since the K_N is a sum of variations of e^{inx} $K_N(0)$ is a sum of 1s and is therefore positive.

(b) For $n \neq 0$ we have $\int_{-\pi}^{\pi} e^{inx}\, dx = 0$. Thus since $(N+1)K_N$ is the sum of $N+1$ D-kernels, each of which containing one e^{0ix}, it integrates to 1 over $[-\pi, \pi]$.

(c) We have $1 - \cos(x(n+1)) \leq 2$ and $\cos\delta \geq \cos x$. Then $\frac{1}{1-\cos\delta} \leq \frac{1}{1-\cos x}$ and the statement is proven.

Since $K_N = \frac{\sum_{n=0}^{N} D_n}{N+1}$ we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_N(t)\, dt = \frac{1}{N+1}\sum_{n=0}^{N}\int_{-\pi}^{\pi} f(x-t)D_n(t)\, dt = \sigma_N$$

Proof of Fejer's Theorem.

Note that f is continuous over $[-\pi, \pi]$ and therefore uniformly continuous over $\mathbb{R}$. Take $\epsilon > 0$ and consider $f(x) - \sigma_N(x)$:

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(t)(f(x) - f(x-t))\, dt$$

Since f is uniformly continuous on a compact set it must be bounded: take $M \geq |f|$. Take $\delta \in (0, \pi)$ where $d(x_1, x_2) < \delta \implies d(f(x_1), f(x_2)) < \epsilon/2$. Then we have

$$\frac{1}{2\pi}\left(\int_{-\pi}^{-\delta} K_N(t)(f(x) - f(x-t))\, dt + \int_{-\delta}^{\delta} K_N(t)(f(x) - f(x-t))\, dt \right.$$
$$\left. + \int_{\delta}^{\pi} K_N(t)(f(x) - f(x-t))\, dt\right)$$

$$\leq \frac{M}{\pi}\left(\int_{-\pi}^{-\delta} \frac{2}{(N+1)(1-\cos\delta)}\,dt + \int_{\delta}^{\pi} \frac{2}{(N+1)(1-\cos\delta)}\,dt\right) + \frac{\epsilon}{4\pi}\int_{-\delta}^{\delta} K_N(t)\,dt$$
$$\leq \frac{4M}{(N+1)(1-\cos\delta)} + \frac{\epsilon}{2}$$

Since $1-\cos\delta$ is bounded (and $\delta \neq \pi$) we pick N such that the first term is less than $\epsilon/2$ and the sum is thus less than ϵ for all x, completing the proof.

Problem 16. We take

$$\frac{1}{2}(f(x+)+f(x-))-\sigma_N(x) = \frac{1}{2\pi}\left(\int_{-\pi}^{0} f(x+)K_N(t)\,dt + \int_{0}^{\pi} f(x-)K_N(t)\,dt\right) - \sigma_N(x)$$

since K_N is even as the sin terms of Dirichlet kernels cancel. This evaluates to

$$\frac{1}{2\pi}\left(\int_{-\pi}^{0} (f(x+)-f(x-t))K_N(t)\,dt + \int_{0}^{\pi} (f(x-)-f(x-t))K_N(t)\,dt\right)$$

For $\epsilon/3 > 0$ we now take δ satisfying both the left and right limit conditions. Then since f is bounded we proceed as in 15 and our expression is bounded above by

$$\frac{4M}{(N+1)(1-\cos\delta)} + \int_{-\delta}^{0} (f(x+)-f(x-t))K_N\,dt$$
$$+ \int_{0}^{\delta} (f(x+)-f(x-t))K_N\,dt$$
$$\leq \frac{4M}{(N+1)(1-\cos\delta)} + \frac{\epsilon}{6\pi}\int_{-\delta}^{0} K_N\,dt + \frac{\epsilon}{6\pi}\int_{0}^{\delta} K_N\,dt$$
$$\leq \frac{3\epsilon}{3}$$

if we select N so the first term is less than $\epsilon/3$, and the proof is completed.

Problem 17. (a) We take $G(x) = e^{-inx}$ and $\alpha(x) = f(x)$, replacing $f(\pi)$ with an upper/lower bound M of f since f is only monotonic on $[-\pi,\pi)$. Then 6.17 yields:

$$\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = \frac{i}{n}\left((-1)^n(M-f(-\pi)) - \int_{-\pi}^{\pi} e^{-inx}\,df(x)\right)$$

Then:

$$|nc_n| \leq \frac{1}{\pi}(M-f(-\pi))$$

by the triangle inequality (since the second integral is bounded by $M-f(-\pi)$ as well) and we are done.

(b) f bounded and monotonic implies f has only jump discontinuities and therefore its left and right limits at each point x exist. Then 16 implies $\lim_{N\to\infty}\sigma_N(x) = \frac{1}{2}(f(x+)+f(x-))$ for all x, and exercise 3.14e yields the desired result.

(c) WLOG assume increasing. Define $f_1(x)$ with $f_1(x) = f(x)$ on (α,β), and $f_1(x) = \inf_{(\alpha,\beta)}(f(x))$ on $[-\pi,\alpha]$ and $f_1(x) = \sup_{(\alpha,\beta)}(f(x))$ on $[\beta,\pi]$. Then f_1 is monotonic on $[-\pi,\pi]$. By the localization theorem $\sigma_N(f;x) - \sigma_N(f_1;x) \to 0$ on (α,β) so $\sigma_N(f;x) \to \frac{1}{2}(f_1(x+)+f_1(x-)) = \frac{1}{2}(f(x+)+f(x-))$ and the proof is done.