PROBLEM SET 2

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Problems: Rudin Chapter 8 Problems 12-17

Solutions:

Problem 12. (a) We solve

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$$
$$= \frac{1}{2\pi}\int_{-\delta}^{\delta} e^{-inx}\,dx.$$
$$= \frac{1}{2\pi}\left(-\frac{e^{-inx}}{in}\right)\Bigg|_{\delta}^{\delta} = \frac{\sin(n\delta)}{\pi n}$$

For the case $n = 0$ we have $C_0 = \frac{1}{2\pi}\int_{-\delta}^{\delta} 1\,dx = \frac{\delta}{\pi}$.

(b) We apply the coefficients to the Fourier series:

$$\sum_{n=-\infty}^{\infty} C_n e^{inx} = \sum_{n=-\infty}^{\infty} \frac{\sin(n\delta)}{\pi n} e^{inx}$$

Since f is constant and therefore Lipschitz continuous at 0 its Fourier series converges to it at 0, thus by plugging in 0 we achieve

$$\sum C_n = \frac{\delta}{\pi} + 2\sum_{n=1}^{\infty}\frac{\sin(n\delta)}{\pi n} = 1$$

and

$$\sum_{n=1}^{\infty}\frac{\sin(n\delta)}{n} = \frac{\pi-\delta}{2}.$$

(c) Since the constants and f are all real, by Parseval's theorem we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)^2\,dx = \sum_{n=-\infty}^{\infty} C_n^2$$
$$\frac{\delta}{\pi} = \frac{\delta^2}{\pi^2} + 2\sum_{n=1}^{\infty}\frac{\sin(n\delta)^2}{\pi^2 n^2}$$
$$\sum_{n=1}^{\infty}\frac{\sin(n\delta)^2}{\delta n^2} = \frac{\pi-\delta}{2}.$$

(d) The answer to (c) can be written as $\sum_{n=1}^{\infty}\left(\frac{\sin(n\delta)}{n\delta}\right)^2\delta$, which we recognize as an infinite series of Riemann sums of our desired integrand, with rectangle width

Date: January 19, 2025.

δ. We must prove that the integral converges, and that our limit converges to the improper integral.

We know the integral converges absolutely since it is bounded above by $1/x^2$. Since $\frac{\pi-\delta}{2}$ is monotonic we need only consider values of δ of the form $\pi/2^m$: if the sums converge correctly to the integral there then we know the limit overall must also converge. Take $S_m = \sum_{n=1}^{\infty} \left(\frac{\sin(n\pi/2^m)}{n\pi/2^m}\right)^2 \pi/2^m$.

Take $\epsilon > 0$. Then pick N such that $\int_a^\infty \frac{1}{x^2} < \epsilon/3$ for $a > N$. Take a a multiple of π. Then we have

$$\int_a^\infty \left(\frac{\sin(x)}{x}\right)^2 dx \leq \int_a^\infty \frac{1}{x^2} < \epsilon/3.$$

We pick m such that

$$d\left(\sum_{n=1}^{2^m a} \left(\frac{\sin(n\pi/2^m)}{n\pi/2^m}\right)^2 \pi/2^m, \int_1^a \left(\frac{\sin(x)}{x}\right)^2 dx\right) < \epsilon/3$$

which exists since the left side is part of a series of successively refined Riemann sums whose partitions have consecutive distances equal to $\pi/2^m$.

Finally, we have

$$\sum_{n=2^m a}^{\infty} \left(\frac{\sin(n\pi/2^m)}{n\pi/2^m}\right)^2 \pi/2^m \leq \int_a^\infty \frac{1}{x^2}\, dx < \epsilon/3$$

since the sum is a right Riemann sum of a function bounded by the monotonically decreasing $\frac{1}{x^2}$ (its area is therefore bounded by the integral).

By these inequalities and the triangle inequality we have $d\left(S_m, \int_1^\infty \left(\frac{\sin(x)}{x}\right)^2 dx\right) < \epsilon$ and since S_m decreases with increase of m, the limit converges to the integral and we have

$$\int_1^\infty \left(\frac{\sin(x)}{x}\right)^2 dx = \lim_{\delta\to 0} \frac{\pi-\delta}{2} = \frac{\pi}{2}.$$

(e) We have

$$\sum_{n=1}^{\infty} \frac{\sin(\frac{\pi}{2}n)^2}{\frac{\pi}{2}n^2} = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \frac{\pi}{4}$$

Problem 13. We may treat f as a 2π-periodic function and take the Fourier coefficients as

$$C_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\, dx$$

since both factors of the integrand are 2π-periodic and the integral is the same as the standard formula over $[-\pi, \pi]$.

We thus solve the integral via integration by parts:

$$\int_0^{2\pi} xe^{-inx}\, dx = \left(-\frac{xe^{-inx}}{in}\right)\Bigg|_0^{2\pi} + \int_0^{2\pi} \frac{e^{-inx}}{in}\, dx.$$

$$= \frac{-2\pi}{in}$$

and thus $C_n = \frac{-1}{in}$. For the case $n = 0$ we have $C_0 = \pi$. Parseval's theorem yields

$$\frac{1}{2\pi}\int_0^{2\pi} x^2\,dx = \sum_{n=-\infty}^{\infty}\frac{1}{n^2}$$

$$\frac{4\pi^2}{3} = \pi^2 + 2\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

Problem 14. We solve for Fourier coefficients.

$$2\pi C_n = \int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = \int_{-\pi}^{0}(\pi+x)^2 e^{-inx}\,dx + \int_0^{\pi}(\pi-x)^2 e^{-inx}\,dx$$

$$= \left(-\frac{(\pi+x)^2}{in} + \frac{2(\pi+x)}{n^2} + \frac{2}{in^3}\right)(e^{-inx})\Bigg|_{-\pi}^{0}$$

$$+\left(-\frac{(\pi-x)^2}{in} - \frac{2(\pi-x)}{n^2} + \frac{2}{in^3}\right)(e^{-inx})\Bigg|_{0}^{\pi}$$

$$= \frac{4\pi}{n^2}$$

and

$$C_0 = 2\int_0^{\pi}(\pi-x)^2\,dx = \frac{\pi^2}{3}$$

Then since f is continuous with bounded derivative we have

$$f(x) = \sum_{n=-\infty}^{0}\frac{2}{n^2}e^{inx} + C_0 + \sum_{n=1}^{\infty}\frac{2}{n^2}e^{inx}$$

$$= \sum_{n=-\infty}^{0}\frac{2}{n^2}(\cos(nx) + i\sin(nx)) + \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{2}{n^2}(\cos(nx) + i\sin(nx))$$

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos(nx).$$

We take $x = 0$ which yields

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}$$

and

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

Then we apply Parseval's Theorem to the constants:

Problem 15. Take $k_N = (N+1)K_N$. We notice that $2 - 2\cos x = 2 - e^{ix} - e^{-ix}$. Then we separate $(2 - 2\cos x)k_N$:

$$(1 - e^{-ix})k_N - (e^{ix})(1 - e^{-ix})k_N$$

where

$$k_N = \sum_{n=0}^{N} D_n$$

Then

$$(1 - e^{-ix})k_N = \sum_{n=0}^{N}(1 - e^{-ix})D_n = \sum_{n=0}^{N} e^{inx} - e^{-ix(n+1)}$$

$$(2 - 2\cos x)k_N = (1 - e^ix)\left(\sum_{n=0}^{N} e^{inx} - \sum_{n=0}^{N} e^{-ix(n+1)}\right)$$

$$= -e^{ix(n+1)} - e^{-ix(n+1)} + 2 = 2(1 - \cos(x(n+1))$$

$$\implies K_N = \left(\frac{1}{N+1}\right)\frac{1 - \cos(x(n+1))}{1 - \cos x}$$

(a) We prove $K_N \geq 0$. Since $\cos \leq 1$ we have $K_N \geq 0$ for all x outside of multiples of 2π. We consider $K_N(0)$: since the K_N is a sum of variations of e^{inx} $K_N(0)$ is a sum of 1s and is therefore positive.

(b) For $n \neq 0$ we have $\int_{-\pi}^{\pi} e^{inx}\,dx = 0$. Thus since $(N+1)K_N$ is the sum of $N+1$ D-kernels, each of which containing one e^{0ix}, it integrates to 1 over $[-\pi, \pi]$.

(c) We have $1 - \cos(x(n+1)) \leq 2$ and $\cos\delta \geq \cos x$. Then $\frac{1}{1-\cos\delta} \leq \frac{1}{1-\cos x}$ and the statement is proven.

Since $K_N = \frac{\sum_{n=0}^{N} D_n}{N+1}$ we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_N(t)\,dt = \frac{1}{N+1}\sum_{n=0}^{N}\int_{-\pi}^{\pi} f(x-t)D_n(t)\,dt = \sigma_N$$

Proof of Fejer's Theorem.

Note that f is continuous over $[-\pi, \pi]$ and therefore uniformly continuous over $\mathbb{R}$. Take $\epsilon > 0$ and consider $f(x) - \sigma_N(x)$:

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(t)(f(x) - f(x-t))\,dt$$

Since f is uniformly continuous on a compact set it must be bounded: take $M \geq |f|$. Take $\delta \in (0, \pi)$ where $d(x_1, x_2) < \delta \implies d(f(x_1), f(x_2)) < \epsilon/2$. Then we have

$$\frac{1}{2\pi}\left(\int_{-\pi}^{-\delta} K_N(t)(f(x) - f(x-t))\,dt + \int_{-\delta}^{\delta} K_N(t)(f(x) - f(x-t))\,dt\right.$$

$$\left. + \int_{\delta}^{\pi} K_N(t)(f(x) - f(x-t))\,dt\right)$$

$$\leq \frac{M}{\pi}\left(\int_{-\pi}^{-\delta}\frac{2}{(N+1)(1-\cos\delta)}\,dt + \int_{\delta}^{\pi}\frac{2}{(N+1)(1-\cos\delta)}\,dt\right) + \frac{\epsilon}{4\pi}\int_{-\delta}^{\delta} K_N(t)\,dt$$

$$\leq \frac{4M}{(N+1)(1-\cos\delta)} + \frac{\epsilon}{2}$$

Since $1 - \cos\delta$ is bounded (and $\delta \neq \pi$) we pick N such that the first term is less than $\epsilon/2$ and the sum is thus less than ϵ, completing the proof.

Problem 16.

Problem 17.