HONORS B FINAL PART B

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1. PSET 1 Problem 3

Problem: Let $I = [0, 1]$ and let $f : I \to I$ such that $f(f(x)) = x$ and $f(x) \neq x$ for any $x \in I$. Show that f has an infinite number of points of discontinuity.
Proof:

Assume there are n points of discontinuity in f, i.e. that the number of discontinuities is finite. We proceed with a proof by contradiction.

We list the x-values of the points of discontinuity in increasing order:

$$\{x_1, x_2, \dots, x_n\}$$

We know that for any x-value x_j that $f(x_j) = x_k$, $f(x_k) = x_j$ for some $k \in 1, \dots, n$. If $f(x_j)$ is not a point of discontinuity, then it must be within an interval A on which f is continuous, and thus $f(f(x_j)) = x_j \in f(A)$, where $f|_A$ is necessarily homeomorphic since every closed set in A is compact and therefore $f|_A$ is continuous and closed. Then $f|_{f(A)}$ is continuous and contains x_j which is a contradiction.

Thus, we have that the points of discontinuity must come in pairs. If n is odd, then necessarily we have $f(x_j) = x_j$ for some x_j and we arrive at a contradiction. If n is even, we consider the continuous intervals between the x-values:

$$\{A_1, \dots, A_{n+1}\}$$

where A_m is a continuous open interval between x_{k-1} and x_k, or between 0 and x_1, or x_n and 1. We similarly argue that $f(A_m) = A_p$ for some $p \in \{1, \dots, n-1\}$: $f|_{A_m}$ is a homeomorphism by the previous argument, so $f(A_m) = U \subset A_p$ for some p. However, if $U \neq A_p$ then f cannot be continuous on A_p since A_m is surrounded by discontinuous points and is fully accounted for (and f is injective) and therefore $f(A_p)$ would not be connected — thus $f(A_m) = f(A_p)$ for some p.

Then if n is even, $n + 1$ is odd and we have that $f(A_m) = A_m$ for some m. We can now use the result of PSET 1 Problem 2 to find that $f|_{A_m}$ must cross $y = x$ by shrinking the domain and codomain of its function from $[0, 1]$ to A_m and substituting its function with f_{A_m}. This yields us yet another contradiction for both cases, so f cannot have finitely many points of discontinuity.

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