PROBLEM SET 4

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Problems: 3.11, 3.17, 3.19, 3.36, 3.37, 3.45, 3.46, 3.57, 3.60, 3.67

Problem 11. Let $f : (a,b) \to R$ be given.

(a) If $f''(x)$ exists, prove that

$$\lim_{h \to 0} \frac{f(x-h) - 2f(x) + f(x+h)}{h^2} = f''(x)$$

Take the first-order Taylor polynomial of $f(x+h)$:

$$f(x+h) = f(x) + f'(x)h + \frac{f''(c)}{2}h^2$$

and of $f(x-h)$:

$$f(x-h) = f(x) - f'(x)h + \frac{f''(c)}{2}h^2$$

Since $c \in [x, x+h]$ we have if $h \to 0$ then $c \to x$. This yields $f''(x) = \lim_{h\to 0} \frac{f(x-h)-2f(x)+f(x+h)}{h^2} = f''(x)$ as desired.

(b) Find an example that this limit can exist even when $f''(x)$ fails to exist.

Example: $f(x) = x^{4/3}$. On $\mathbb{R}$ the second derivative of this function is undefined at $x = 0$, but the limit for $x = 0$ is 0 at this point since the function is odd.

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Problem 17. Define $e : \mathbb{R} \to \mathbb{R}$ by

$$e(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \le 0. \end{cases}$$

(a) Prove that e is smooth; that is, e has derivatives of all orders at all points x.[Hint: L'Hôpital and induction. Feel free to use the standard differentiation formulas about e^x from calculus.]

Claim: Every derivative of e is of the form

$$\begin{cases} \frac{p(x)e^{-1/x}}{x^n} & \text{if } x > 0 \\ 0 & \text{if } x \le 0. \end{cases}$$

where $p(x)$ is a polynomial in $\mathbb{R}$ and $n \in \mathbb{N}$.

Base case: $f^{(0)}$ obvious.

Inductive case: We have

$$\left(\frac{p(x)e^{-1/x}}{x^n}\right)' = \frac{(x^2p'(x) + p(x) - nxp(x))e^{-1/x}}{x^{n+2}}$$

$x^2p'(x)+p(x)-nxp(x))$ is a polynomial and $n+2 \in \mathbb{N}$. We prove that $\lim\limits_{x\to 0^+} \frac{p(x)e^{-1/x}}{x^n} = 0$. We note that $p(x)$ converges, so

$$\lim_{x\to 0^+} \frac{p(x)e^{-1/x}}{x^n} = \lim_{x\to 0^+} p(x) \lim_{x\to 0^+} \frac{e^{-1/x}}{x^n}$$

if $\lim_{x\to 0^+} \frac{e^{-1/x}}{x^n}$ exists. We take $y = 1/x$. For $\delta > 0$, we have $\delta < y$ for some y implies $0 < x < \frac{1}{\delta}$. Thus if we prove $\lim\limits_{y\to\infty} \frac{e^{-y}}{y^{-n}} = 0$ we have $\lim\limits_{x\to 0^+} \frac{e^{-1/x}}{x^n} = 0$, since we can apply any ϵ needed to prove the second limit to the first. We prove this using L'Hopital's rule:

$$\begin{aligned} \lim_{y\to\infty} \frac{e^{-y}}{y^{-n}} = \lim_{y\to\infty} \frac{y^n}{e^y} &= \lim_{y\to\infty} \frac{(y^n)^{(n)}}{(e^y)^{(n)}} \\ &= \lim_{y\to\infty} \frac{n!}{e^y}. \end{aligned}$$

Then for any $\epsilon > 0$ we have $\delta = \log(\frac{n!}{\epsilon})$. For $y > \delta$ we have $y = \delta + a$ for positive a such that $y = \log(\frac{n!e^a}{\epsilon})$: this yields us $\frac{n!}{e^y} = \frac{\epsilon}{e^a} < \epsilon$ since $e^a > 1$. Thus

$$\lim_{y\to\infty} \frac{n!}{e^y} = 0$$

$$\implies \lim_{x\to 0^+} \frac{p(x)e^{-1/x}}{x^n} = 0.$$

Thus $f^{(k)}$ is continuous since $\lim_{x\to 0^-} 0 = 0$ and differentiable since $\lim_{x\to 0^-} (0)' = 0$ and $\lim_{x\to 0^+} f^{(k+1)} = 0$ and $p(x), e^{-1/x}$, and $\frac{1}{x^n}$ are all continuous on $(0,\infty)$, and 0 on $(-\infty, 0)$. Thus e is smooth.

(b) Is e analytic?

e cannot be analytic. We prove if a power series $\sum a_n h^n$ converges pointwise to $f(x+h)$ then the power series must be identically 0, a contradiction since e is nonzero at $x > 0$. (Shitty proof:) A nonzero power series has at most countably many zeros by the Fundamental Theorem of Algebra. e has uncountably many. Contradiction.

(Better proof that I'm not sure I'm allowed to use): if $\exists$ a power series $p(x) = \sum a_n h^n$ then at 0 its n^{th} derivative must match that of e. However at 0, each derivative must be 0, and $p^{(k)}(0) = a_k k! \implies a_k = 0$ for each k. This is an identically 0 power series that clearly cannot converge to any point $e(x)$ at $x > 0$ since it never comes within $e(x)$ of $e(x)$.

(c) Show that the **bump function**

$$\beta(x) = e^2 e(1-x) \cdot e(x+1)$$

is smooth, identically zero outside the interval $(-1, 1)$, positive inside the interval $(-1, 1)$, and takes value 1 at $x = 0$. (e^2 is the square of the base of the natural logarithms, while $e(x)$ is the function just defined. Apologies to the abused notation.)

We separate out the constant e^2. From here we know that since $e(x)$ is smooth and $1-x$ and $x+1$ are smooth (with constant derivative), we can use the chain rule and product rule repeatedly with a constant derivative for $1-x$ and $x+1$ to get continuous derivatives of order n, since sums and products of continuous functions are continuous. Thus $\beta(x)$ is smooth.

$x \geq 1$ implies $e(1-x) = 0$ and $x \leq -1$ implies $e(x+1) = 0$, so the function is identically 0 for $|x| \geq 1$. For $|x| < 1$ both $x+1$ and $1-x$ are positive, and $e(1-x), e(1+x)$ are also positive. Thus $\beta(x)$ is positive. For $x = 0 \in (-1, 1)$ we have $\beta(x) = e^2(e(1))^2 = e^2(\frac{1}{e^2}) = 1$.

(d) For $|x| < 1$ show that

$$\beta(x) = e^{2x^2/(x^2-1)}$$

Bump functions have wide use in smooth function theory and differential topology. The graph of β looks like a bump. See Figure 86.

$|x| < 1 \implies 1 > x$ and $x > -1$. Then

$$\begin{aligned} e^2 e(1-x) \cdot e(x+1) &= e^{(2 - \frac{1}{1-x} - \frac{1}{1+x})} \\ &= e^{\frac{-2x^2}{1-x^2}}. \end{aligned}$$

for $x \in (-1, 1)$.

Problem 19. Recall that the oscillation of an arbitrary function $f : [a, b] \to R$ at x is

$$\mathrm{osc}_x f = \limsup_{t \to x} f(t) - \liminf_{t \to x} f(t)$$

In the proof of the Riemann-Lebesgue Theorem D_k refers to the set of points with oscillation $\geq 1/k$.

(a) Prove that D_k is closed.

We take an infinite convergent sequence in D_k (d_n) and take $\lim_{n\to\infty} d_n = d$. Take sequences

$$L_n = \liminf_{t \to d_n} f(t),\ U_n = \limsup_{t \to d_n} f(t),$$
$$L = \liminf_{t \to d} f(t),\ U = \limsup_{t \to d} f(t).$$

Take $\epsilon > 0$. $\exists \delta$ such that for $|d-t| < \delta$ we have $L - \inf f(B_\delta(d)) < \epsilon$. (Note that for $A \subset B_\delta(d)$, $\inf f(A) \geq \inf f(B_\delta(d))$. Additionally note that $L \geq \inf f(B_\delta(d))$ since if L is less than it by ϵ' for some δ, then nesting of the balls implies that each $\delta_1 < \delta$ is further than ϵ away from L.) Take some $d_n \in B_\delta(d)$, which we know exists by convergence. Then since $B_\delta(d)$ is an open set $\exists$ a neighborhood $B_{\delta_n}(d_n) \subset B_\delta(d)$. We have

$$L - \epsilon < \inf f(B_\delta(d)) \leq \inf f(B_{\delta_n}(d_n)) \leq L_n,$$

implying $L \leq L_n$ by ϵ-principle.

By similar inequalities (existence of δ_2 such that $\sup f(B_{\delta_2}(d)) - U < \epsilon$, existence) we achieve the analogous inequality

$$U + \epsilon > \sup f(B_{\delta_2}(d)) \geq \sup f(B_{\delta_{2n}}) \geq U_n \implies U \geq U_n.$$

Then by combining the two inequalities we have $\mathrm{osc}_d f = U - L \geq U_n - L_n \geq 1/k$. Thus D_k is closed.

(b) Infer that the discontinuity set of f is a countable union of closed sets. (This is called an $\mathbf{F}_{\boldsymbol{\sigma}}$**-set**.)

Since $k \in \mathbb{N}$ we have

$$\bigcup_{k=1}^{\infty} D_k = D$$

is a countable union by definition (If $\mathrm{osc}_x f > 0$ then $k > \frac{1}{\mathrm{osc}_x f} \implies 1/k < \mathrm{osc}_x f$ and $x \in D_k$: thus every point of discontinuity is in this set).

(c) Infer from (b) that the set of continuity points is a countable intersection of open sets. (This is called a $\mathbf{G}_{\boldsymbol{\delta}}$**-set**).

The set of continuity points is the complement of the set of discontinuity points. We have

$$(\bigcup_{k=1}^{\infty} D_k)^C = \bigcap_{k=1}^{\infty} D_k^C$$

and each D_k^C is the complement of a closed set and therefore open. Thus the set of continuity points is a countable intersection of open sets.

Problem 36. Generalizing Exercise 1.31, we say that $f : (a, b) \to \mathbb{R}$ has a **jump discontinuity** (or a discontinuity of the **first kind**) at $c \in (a, b)$ if

$$f(c^-) = \lim_{x \to c^-} f(x) \text{ and } f(c^+) = \lim_{x \to c^+} f(x)$$

exist, but are either unequal or are unequal to $f(c)$. (The three quantities exist and are equal if and only if f is continuous at c.) An **oscillating discontinuity** (or a discontinuity of the **second kind**) is any nonjump discontinuity.

(a) Show that $f : \mathbb{R} \to \mathbb{R}$ has at most countably many jump discontinuities.

We start by separating the set J of jump discontinuities into the union

$$\bigcup_{k=1}^{\infty} J_k$$

of jump discontinuities where $|f(c^+) - f(c^-)| > 1/k$. We prove J_k is a set of isolated points. For some $c \in J_k$ assume there exists an element c_r of J_k in every neighborhood $B_r(c)$. Assume WLOG that $f(c_r^+) > f(c_r^-)$. Then for $\epsilon > 0$ we have c_{r1}, c_{r2} such that $|f(c_{r1}) - f(c_r^+)| < \epsilon/2$ and $|f(c_{r2}) - f(c_r^-)| < \epsilon/2$. Then c_{r1}, c_{r2} are at least $f(c_r^+) - f(c_r^-) - \epsilon$ away from each other.

Since $f(c_r^+) - f(c_r^-) > 1/k$ $\exists$ an ϵ where $f(c_r^+) - f(c_r^-) - \epsilon > 1/k$ and thus if we select $2\epsilon' < 1/k$ there exists no neighborhood of c where all elements of the image of the neighborhood are within 2ϵ of each other, and therefore no neighborhood where all elements of its image are within ϵ of either $f(c^+)$ or $f(c^-)$. This violates the limit definition and we conclude that J_k is isolated.

We now prove that isolated points are countable. Start with an arbitrary point in J_k j. Then there exists a neighborhood $B_{r_j}(j)$around j without any other element of J_k that contains a rational number q greater than j. We prove this rational number is unique to j. If q is in $B_{r_j}(j)$ and $B_{r_j 2}(j_2)$ then WLOG assume $j < j_2$. Then $j_2 > j + r_j > q$ is a contradiction. Then since there is a rational number for each element of J_k we conclude J_k is countable, and since J is a countable union of countable sets it must also be countable.

(b) What about the function

$$f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0? \end{cases}$$

This function is continuous everywhere except 0, where the left and right limits do not exist since the $\limsup$ does not equal the $\liminf$. Thus it has no jump discontinuities.

(c) What about the characteristic function of the rationals?

This function has no jump discontinuities since it has no limit anywhere: the $\limsup$ is different from the $\liminf$ at every point since the rationals and the irrationals are both dense in $\mathbb{R}$.

Problem 37. Suppose that $f : \mathbb{R} \to [-M, M]$ has no jump discontinuities. Does f have the intermediate value property? (Proof or counterexample.)

Take

$$f(x) = \begin{cases} M/3(\sin(1/x)) + 2M/3 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ M/3(\sin(1/x)) - 2M/3 & \text{if } x > 0. \end{cases}$$

Then $f(x)$ does not cross $M/6$ despite this number being an intermediate value. However, the function has no jump discontinuities since $\lim_{x \to 0^+} f(x)$ and $\lim_{x \to 0^-} f(x)$ do not exist.

Problem 45. (a) Define the oscillation for a function from one metric space to another, $f : M \to N$.

$$\mathrm{osc}_x f := \limsup_{r \to 0} \text{ diam } f(B_r(x))$$

for $f : M \to N$.

(b) Is it true that f is continuous at a point if and only if its oscillation is zero there? Prove or disprove.

Assume the oscillation is $a > 0$. Then for $r_k = 1/k$ we take the points $x_{k1}, x_{k2} \in B_{r_k}(x)$ where $d(f(x_{k1}), f(x_{k2})) \geq a$. Consider the convergent sequence $x_{11}, x_{12}, x_{21}, x_{22}, \dots$. No matter how high a k we choose, $d(f(x_{k1}), f(x_{k2}))$ will always be greater than or equal to a, so the sequence $(f(x_{k1}), f(x_{k2})$ is not Cauchy and therefore not convergent. By definition this makes f not continuous at x.

(c) Is the set of points at which the oscillation of f is $\geq 1/k$ closed in M? Prove or disprove.

Take a sequence (x_n) in M that converges to x. Consider $\mathrm{osc}_x f$. For any $r > 0$ we have $x_k \in B_r(x)$ and $B_{r_k}(x_k) \subset B_r(x)$. For sets A, B with $A \subset B$ we have diam $A \leq$ diam B since every point in A is in B and therefore the pairwise supremum of distances of B is an upper bound for distances of A: then diam $f(B_a(x_r)) \leq$ diam $f(B_b(x_r))$ for $a \leq b$. We posit that $\mathrm{osc}_{x_n} f \leq$ diam $f(B_{r_k}(x_k))$ since if otherwise for $a < r_k$,

$$\mathrm{osc}_{x_n} f - \text{diam } f(B_a(x_k)) \geq \mathrm{osc}_{x_n} f - \text{diam } f(B_{r_k}(x_k)) = \epsilon > 0.$$

Then

$$1/k < \mathrm{osc}_{x_n} f \leq \text{diam } f(B_{r_k}(x_k)) \leq \text{diam } f(B_r(x)) \leq \sup\{\text{diam } f(B_a(x)) : a \leq r\}.$$

From here we note that for any ϵ we can pick an r where $\sup\{\text{diam } f(B_a(x)) : a \leq r\} - \mathrm{osc}_x f < \epsilon$. This yields $\mathrm{osc}_x f + \epsilon > 1/k$ and by ϵ-principle x is in the set and we are done.

Problem 46. (a) Prove that the integral of the Zeno's staircase function defined on page 174 is $2/3$.

Take partition $P_k = P_{k-1} \cup \{1/2^k\}$ with $P_1 = \{1/2\}$.

$$L_{P_k}(f) = (1/2^k)(1 - 1/2^{k+1}) + \sum_{n=1}^{k} \frac{2^n - 1}{2^{2n}}$$

The first term approaches 0 as $k \to \infty$. We separate the second into

$$\sum_{n=1}^{k} \frac{1}{2^n} - \sum_{n=1}^{k} \frac{1}{2^{2n}} = \frac{\frac{1}{2}(1 - (\frac{1}{2})^k)}{\frac{1}{2}} - \frac{\frac{1}{4}(1 - (\frac{1}{4})^k)}{\frac{3}{4}}$$

As k approaches ∞ this approaches $1 - 1/3 = 2/3$.

On the other side,

$$U_{P_k}(f) = (1/2^k) + \sum_{n=1}^{k} \frac{2^n - 1}{2^{2n}}$$

which approaches $2/3$ by the same argument. As a result, $\sup L_P(f) = \inf U_P(f) = 2/3$ since all upper sums must be greater than all lower sums so $2/3$ is an upper bound for lower sums and a lower bound for upper sums.

(b) What about the Devil's staircase?

For $x \in [0, 1]$ with base-3 expansion $.\omega_1\omega_2\omega_3\ldots$ we have that the base-3 expansion of $1 - x$ is $.(2 - \omega_1)(2 - \omega_2)(2 - \omega_3)\ldots$ since $\sum 2/3^k$ converges to 1. We aim to prove that $H(1 - x) = 1 - H(x)$.

$$H(1 - x) = \sum_{i=1}^{\infty} \frac{(2 - \omega_i)/2}{2^i} = \sum_{i=1}^{\infty} \frac{1}{2^i} - \sum_{j=1}^{\infty} \frac{\omega_j/2}{2^j} = 1 - H(x)$$

For any partition P of $G(x) = H(1 - x)$ we have $P' = \{a : 1 - a \in P\}$ a partition of H. For $a, b \in P'$ we have $|a - b| = |(1 - a) - (1 - b)|$ and so the upper and lower sums are the same for P on G. as they are for P' on H. Thus $\int_{[0,1]} H(1 - x) = \int_{[0,1]} H(x)$. This yields

$$\int_{[0,1]} H(x) = 1 - \int_{[0,1]} H(x) \implies \int_{[0,1]} H(x) = 1/2.$$

Problem 57. Construct a function $f : [-1, 1] \to \mathbb{R}$ such that

$$\lim_{r\to 0}\Big(\int_{-1}^{-r} f(x)\,dx + \int_{r}^{1} f(x)\,dx\Big)$$

exists (and is a finite real number) but the improper integral $\int_{-1}^{1} f(x)\,dx$ does not exist. Do the same for a function $g : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{R\to\infty}\int_{-R}^{R} g(x)\,dx$$

exists but the improper integral $\int_{-\infty}^{\infty} g(x)\,dx$ fails to exist. [Hint: The functions are not symmetric across 0.]

For the first function we take $f(x) = 1/x$ with $f(0) = 0$. Since $f(-x) = -f(x)$, $\int_{-1}^{-r} f(x)\,dx + \int_{r}^{1} f(x)\,dx = 0$ for all r and the limit is also 0. The improper integral does not exist since no integral on an interval containing 0 converges: WLOG take the interval $[0, a]$ and choose k such that $1/2^k < a$. Then partition the interval into

$$\bigcup_{i=k}^{n}\{1/2^i\}.$$

For each interval $[1/2^i, 1/2^{i+1}]$ the lower-bound area is $1/2$. Thus for each partition P_n we have

$$L_{P_n}(f) = (n-k)(1/2) + (a - 1/2^k)\cdot 1/a + 1$$

which diverges as n increases.

For the second function we take $g(x) = x$.

Since once again $g(-x) = -g(x)$ we have

$$\int_{-R}^{R} g(x)\,dx = \int_{0}^{R} g(x)\,dx + \int_{-R}^{0} g(x)\,dx = \int_{0}^{R} g(x)\,dx + \int_{0}^{R} -g(x)\,dx = 0$$

The improper integral here is

$$\lim_{a\to-\infty}\lim_{b\to\infty}\int_{a}^{b} g(x)\,dx = \lim_{a\to-\infty}\lim_{b\to\infty} b^2 - a^2$$

by the Fundamental Theorem of Calculus, but since the first limit diverges the entire integral must diverge.

Problem 60. (a) If $\sum a_n$ converges and (b_n) is monotonic and bounded, prove that $\sum a_n b_n$ converges.

WLOG assume (b_n) is increasing. Since (b_n) is increasing and bounded it has a least upper bound b which is the limit of its sequence (if $\exists$ an ϵ where there is no N where b_n is within ϵ of b for $n > N$, then there exists no N where $b - b_N < \epsilon$ and we can set $b - \epsilon$ as another upper bound.) Thus we can define $b_n - b = c_n$, a sequence that converges to 0 and is monotonic. If $\sum a_n c_n$ converges then $b \sum a_n + \sum a_n c_n = \sum a_n b_n$ converges.

Take the convergent sequence (A_n) with $A_n = \sum_{i=0}^{n} a_i$. Then $a_n = A_n - A_{n-1}$ $(A_0 = 0)$. We have

$$\sum_{i=1}^{n} a_i c_i = \sum_{i=1}^{n} (A_i c_i - A_{i-1} c_i) = \sum_{i=1}^{n} A_i c_i - \sum_{i=1}^{n} A_i c_{i+1} = \sum_{i=1}^{n} A_i (c_i - c_{i+1})$$

We note that $c_i - c_{i+1}$ is always negative, so if it converges it is absolutely convergent: $\sum_{i=1}^{\infty} = c_i$.

Since A_i converges it is bounded: $|A_i| < M_A$. Then since for $d_i = (c_i - c_{i+1})$ $\sum |d_i| = M_d$ we have $\sum |A_i d_i| \leq \sum |d_i M_A| = M_A M_d$. Since $\sum |A_i d_i|$ is an increasing bounded sequence it converges to a least upper bound L, and thus $\sum a_i c_i = \sum A_i d_i$ is absolutely convergent.

(b) If the monotonicity condition is dropped, or replaced by the assumption that $\lim_{n \to \infty} b_n = 0$, find a counterexample to convergence of $\sum a_n b_n$.

For dropped monotonicity, we take $\sum a_n$ the alternating harmonic series and $(b_n) = (-1)^n$. Then $\sum a_n b_n$ becomes the single-sign harmonic series which diverges.

Consider $\sum a_n = \sum \frac{(-1)^n}{n^{1/2}}$ and $(b_n) = \frac{(-1)^n}{n^{1/2}}$. Then $\sum a_n b_n$ is the single-sign harmonic series again. $\sum a_n$ converges by the alternating series test.

Problem 67. An **infinite product** is an expression $\prod c_k$ where $c_k > 0$. The n^{th} **partial product** is $C_n = c_1 \dots c_n$. If C_n converges to a limit $C \neq 0$ then the product converges to C. Write $c_k = 1 + a_k$. If each $a_k \geq 0$ or each $a_k \leq 0$ prove that $\sum a_k$ converges if and only if $\prod c_k$ converges. [Hint: Take logarithms.]

Assume $\sum a_k$ converges absolutely (as stated in the problem). (I'm out of time so I'm only going to prove forward case for positive a_k and backward case for negative a_k.) Assume a_k positive. Then for some partial product $C_n = \prod_{k=1}^n c_k$ we have

$$\log C_n = \sum_{k=1}^{n} \log(a_k + 1)$$

We prove $\log(x+1) \leq x$ for all $x \in \mathbb{R}$. Take the function $f(x) = x - \log(x+1)$. We have $f'(x) = \frac{x}{x+1}$. $f(x)$ is defined from $(-1, \infty)$, so we have $f'(x)$ negative on $(-1, 0)$ and positive on $(0, \infty)$ with global minimum at 0, since anything lower implies positive derivative for $x' < 0$ or negative derivative for $x' > 0$ by mean value theorem, which is impossible. $f(0) = 0$, so $f(x) \geq 0$ and $\log(x+1) \leq x$. Then for positive a_k we have $\log(a_k + 1) \leq a_k$, meaning $\log C_k$ is bounded above by A the limit of $\sum a_k$. Since a_k is positive $\log(a_k + 1)$ is also positive, meaning $(\log C_k)$ is an increasing bounded sequence with a supremum that it must converge to.

Since e^x is a continuous function we have that if $\log C_k$ converges to $\log C$ then C_k converges to C.

Take a_k negative and assume $\prod c_k$ converges. Then $a_k + 1 < 1$ and $\log(c_k)$ is also negative. If $\sum -a_k$ converges then a_k is absolutely convergent. We have that C_n converges to a nonzero C so $\log C_n$ converges to $\log C$. Again $\log C_n = \sum_{k=1}^n \log(a_k + 1)$, so

$$-\log C_n = \sum_{k=1}^{n} -\log(a_k + 1) \geq \sum_{k=1}^{n} -a_k$$

. Then since $-x$ is continuous $-\log C_n$ converges to $\log C$ which is an upper bound for $\sum -a_k$ which must increase since a_k are negative. Again $\sum -a_k$ must converge to $-A$ by the least upper bound property etc. and since $-x$ is continuous $\sum a_k$ converges to A.