PROBLEM SET 1

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Lemma 1. (General lemma I will use): For a cut $x = A|B$ if $a' \in \mathbb{Q}$ and $a' < a$ for $a \in A$ then $a' \in A$:

Proof: Since $a' \in \mathbb{Q}$ and $A \cup B = \mathbb{Q}$ and $A \cap B = \emptyset$ then if $a' \notin B$ it must be in A. Then $a' \in A$ since if $a' \in B$ by definition $a' > a$, which is impossible.

Problem 9. Let $x = A|B$, $x' = A'|B'$ be cuts in $\mathbb{Q}$. We defined

$$x + x' = (A + A') \mid \text{rest of } \mathbb{Q}$$

(a) Show that although $B+B'$ is disjoint from $A+A'$, it may happen in degenerate cases that $\mathbb{Q}$ is not the union of $A + A'$ and $B + B'$.

Sol: For a degenerate case, we consider x, x' for which $A = \{x : x < 0 \text{ or } x^2 < 2\}$, and $A' = \{x' : (x' - 1)^2 > 2 \text{ and } x' < 0\}$.

Intuitively, these cuts represent real numbers $\sqrt{2}$ and $1 - \sqrt{2}$, but since they are irrational their corresponding B-sets do not contain their greatest lower bounds. $B + B'$ therefore also does not include 1, which means 1 is excluded from the sets $A + A'$ and $B + B'$. This means $(A + A') \mid (B + B')$ is by definition not a cut.

Rigorously, we prove that $\nexists a \in A, a' \in A'$ where $a + a' = 1$, and $\nexists b \in B, b' \in B'$ where $b + b' = 1$, so $1 \notin A + A' \cup B + B'$. We note that the conditions for B and B' are $B = \{y : y \geq 0 \text{ and } y^2 \geq 2\}$ $B' = \{y' : (y' - 1)^2 \leq 2 \text{ or } y' \geq 0\}$.

We first assume the negative of our desired result for the A-sets. Then given a s.t. $a < 0$ or $a^2 < 2$, we plug in $a' = 1 - a$ to the condition for A':

$$a^2 > 2 \text{ and } a > 1$$

which yields us a contradiction since $a \in A$. Now we test the corresponding hypothesis for B, B': Given $b \in B'$ we plug in $1 - b$ to B's conditions: $b \leq 1$ and $(1 - b)^2 = (b - 1)^2 \geq 2$ yields a contradiction. Then $A + A' \cup B + B'$ excludes 1.

(b) Infer that the definition of $x + x'$ as $(A + A') \mid (B + B')$ would be incorrect.

Sol: As stated above $A + A' \cup B + B'$ excludes 1 and $\neq \mathbb{Q}$: thus $(A + A') \mid (B + B')$ is not a cut.

(c) Why did we not define $x \cdot x' = (A \cdot A') \mid \text{rest of } \mathbb{Q}$?

Sol: Given two positive cuts $A \mid B$ and $A' \mid B'$ $A \cdot A'$ includes all of $\mathbb{Q}^+$ since $\mathbb{Q}^- \subset A$ and $\mathbb{Q}^- \subset A'$. Thus the cut $(A \cdot A') \mid \text{rest of } \mathbb{Q}$ does not make sense because every product of positive numbers would correspond to ∞ or be undefined.

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Problem 10. Prove that for each cut x we have $x + (-x) = 0^*$

Sol: From the definition of the cut $-x$ we have $x + (-x) = A \mid B$ where $A = \{x : x = a + b, a \in A', -b \in B', -b \text{ not a lower bound of } B'\}$, for $x = A' \mid B'$. We note that by definition of a cut $-b > a$, so $a + b < 0$ and A is bounded above by 0.

We now prove that $\forall\ c < 0$, $c = a + b \in A$ for some $a \in A'$, $-b \in B'$. To do this, we pick an arbitrary a' in A' and an arbitrary b' in B', since neither can be the empty set. Given c we can find the multiple of c by the smallest natural d where $d \in \mathbb{N}$, and $cd < a' - b'$, since every rational is in an interval of integers — $a' - cd > b' \implies a' - cd \in B'$.

From here we note that $a' - cn$ for $n \in \mathbb{N} \cap [0, d]$ is a rational and therefore $a' - cn \in A'$ or $a' - cn \in B'$. As such $\exists n$ s.t. $a' - cn \in A'$ and $a' - c(n+1) \in B'$, since $a' \in A'$ and $a' - cd \in B'$. Thus we set $a = a' - cn$ and $-b = a' - c(n+1) \implies a + b = c$.

In the case that $a' - c(n+1)$ is the l.u.b. of A', we may add a rational f less than c to both values for satisfactory elements of A' and B' — since $a' - c(n+1)$ is the l.u.b., $a' - cn + f < a' - c(n+1) \implies a' - cn + f \notin B'$ but in A'. Then in all cases $\exists$ elements a and b that satisfy $a + b = c$ for $c < 0$ and the cut is the zero cut.

Lemma 2. Multiplication by a positive real preserves inequalities and equalities.

Proof: Given $a, b \in \mathbb{R}$ with $a < b$ and $r \in \mathbb{R}^+$ we aim to prove $ar < br$. For $a < b$ we have $a + (-a) < b + (-a)$ by translation or $0 < b - a$. Then $b - a$ is a positive real number. Then we have $r(b - a) > 0$ as a property of multiplication, and we use the distributive property for $br - ar > 0$. By translation $br > ar$.

Equality is proven since the lower sets of the equal cuts are the same and so the product cuts have the same definition. Equality-inclusive inequalities are consistent by considering the case of inequality and the case of equality.

Problem 11. A multiplicative inverse of a nonzero cut $x = A \mid B$ is a cut $y = C \mid D$ such that $x \cdot y = 1*$.

(a) If $x > 0*$, what are C and D?

Sol: We posit that $C = \{x \in \mathbb{Q} : x \leq 0 \text{ or } \frac{1}{x} \in B \text{ but not g.l.b. of } B\}$. (Note: we define g.l.b. later but here it only matters if it is the smallest element of B, so we use it in place of that term, i.e. I do not have the time to replace "g.l.b." with smallest element everywhere.)

Proof: $x > 0$ and $1 > 0 \implies y > 0$, so we have for $x = A|B$ and $y = C|D$ $xy = E'|F'$ where

$$E' = \{r \in \mathbb{Q} : r \leq 0 \text{ or } \exists a \in A \text{ and } \exists c \in C \text{ such that } a > 0, c > 0, \text{ and } r = ac\}$$

We prove for $1 = E|F$ that $E \subset E'$ and $E' \subset E$. First we prove that E' is bounded above at 1, which implies it is a subset of E. Assume $\exists r \geq 1$ where $r = ac$ for $a > 0, c > 0, a \in A, c \in C$. We have $a \in A$, $\frac{1}{c} \in B$, so $a < \frac{1}{c}$: however, $r = ac \implies a = \frac{r}{c}$ for $r \geq 1$ meaning $a \geq c$ which is a contradiction.

Now we prove that $r \in E'$ $\forall$ $r < 1, r \in \mathbb{Q}$. If $r \leq 0$, $r \in E'$ by definition. We aim to, for $r \in (0, 1)$, find $a \in A$ and $b \in B(b \neq \text{l.u.b } B)$ such that $\frac{a}{b} = r$.

As in Problem 10 we select arbitrary $a' \in A$ and $b' \in B$. Since a', b' are positive rationals with $b' > a'$ and $1/r > 1$, we have $\frac{b'}{a'} < (1/r)^d$ for some $d \in \mathbb{N}$. Then we have $a'(1/r)^d \in B$ and $\exists$ $n \in \mathbb{N}$ s.t. $a'(1/r)^n \in A$ and $a'(1/r)^{n+1} \in B$. Setting $a = a'(1/r)^n$ and $b = a'(1/r)^{n+1}$ yields us $a \cdot \frac{1}{b} = r$.

If incidentally b is the lower bound of B we pick $s \in (1, 1/r)$ and choose $a = a'(1/r)^n s$ and $b = a'(1/r)^{n+1} s$ where $a \in A$ since $s < 1/r$. Thus, $E \subset E'$ and we have $E = E'$ and $y = \frac{1}{x}$.

(b) If $x < 0*$, what are they?

Sol: If $x < 0$ then $y < 0$, and $x \cdot y$ is defined by $(-x) \cdot (-y)$. We write $-x = A'|B'$ and $-y = C'|D'$. We have

$$A' = \{a' \in \mathbb{Q} : a' = -b \text{ for some } b \in B, b \text{ not a lower bound}\}$$

Then

$$C' = \{c' \in \mathbb{Q} : c' \leq 0 \text{ or } \frac{1}{c'} \in B', \frac{1}{c'} \text{ not a lower bound.}\}$$

Two cases: If B includes a lower bound x, then $-x \in B'$ since it is excluded from A'. Note here that $-x$ becomes the lower bound of B' since x is an upper bound for A (whose negatives are excluded from A'). Then $-\frac{1}{x}$ is excluded from C'. If B does not include a lower bound then naturally C' does not include a function of the lower bound of B' since it does not exist.

From here, we combine the two sets:

$$B' = \{b' \in \mathbb{Q} : b' = -a \text{ for some } a \in A \text{ or } a \text{ lower bound of } B\}$$

$$C' = \{c' \in \mathbb{Q} : c' \leq 0 \text{ or } -\frac{1}{c'} \in A\}$$

We can then take the negative of this cut:

$$D' = \{d' \in \mathbb{Q} : d' > 0 \text{ and } -\frac{1}{d'} \in B\}$$

$$C = \{c \in \mathbb{Q} : -c = d' > 0 \text{ and } \frac{1}{c} \in B \text{ but not g.l.b. of} B\}$$

(note: the g.l.b. of D' is $-\frac{1}{b}$ for b the g.l.b. of B) and we have

$$C = \{c \in \mathbb{Q} : c < 0 \text{ and } \frac{1}{c} \in B \text{ but not g.l.b of } B\}$$

(c) Prove that x uniquely determines y.

Sol: We must consider the cases of positive and negative x. For positive x assume there exists $y \neq y'$: we will prove the contrapositive, that $xy \neq xy'$ and thus $\exists$ a unique y for $xy = 1$.

We have $y = C|D$ and $y' = C'|D'$: we will assume WLOG that $y > y'$ (trichotomy) so that $\exists s \in C, s \notin C'$. Then we write $xy = E|F$ with $E = \{r \in \mathbb{Q} : r \leq 0$ or $r = ac$ for $a \in A \cap \mathbb{Q}^+$ and $c \in C \cap \mathbb{Q}^+\}$ and $xy' = E'|F'$ with $E' = \{r'$: defined similarly as above.$\}$

We plug s into E to get $r = as$: since $a \in \mathbb{Q}^+$ $r > ac'$ $\forall$ $c' \in C'$ so $\exists$ $r \in E$, $r \notin E'$. Then $E \neq E'$.

For the negative case, we define y and y' similarly as above and assume $y > y'$. Then since $\exists s \in c, s \notin C'$, $s \in D', s \notin D$. (We note that we can pick s not the lower bound of D' since if only the lower bound is in D' but not D, then s is necessarily the least upper bound of C and in C, which is impossible). Then for $-y = C^-|D^-$ and $-y' = C^{-'}|D^{-'}$ we have $-s \in C^{-'}$, $-s \notin C^-$. From here, we compare $C^{-'}$ and C^- to $-x$ as in the positive case to achieve $E \neq E'$.

Problem 13. Let $b = \text{l.u.b } S$, where S is a bounded nonempty subset of $\mathbb{R}$.

(a) Given $\epsilon > 0$ show that there exists an $s \in S$ with

$$b - \epsilon \leq s \leq b.$$

Sol: Since b is the l.u.b., $\forall\ s \in S\ b \geq s$. Thus if $\nexists s$ for $b-\epsilon \leq s \leq b$ then $s < b - \epsilon\ \forall\ s \in S$. Then $b - \epsilon$ is an upper bound of S, and since $b - \epsilon < b$, this is a contradiction.

(b) Can $s \in S$ always be found so that $b - \epsilon < s < b$?

Sol: No, because of the counterexample $S = b$, which b is an upper bound for.

(c) If $x = A \mid B$ is a cut in $\mathbb{Q}$, show that $x = \text{l.u.b.}A$.

Sol: For $a \in A$ we have $a^* \in \mathbb{R}$ with $a^* = A'|B'$ $A' = \{a' < a\}$ the rational cut in $\mathbb{R}$ corresponding to a. Then we have $A' \subset A$ since $\forall\ a' \in A'$ we have $a' < a \in A$: if $a' \notin A$ then $a \in B$ and $a' > a$, contradiction. Then x is an upper bound of $A^* = \{a^*\}$. To prove x is the least upper bound assume $\exists$ an upper bound y of A^*. Then for a in A we have $a^* \leq y$. If $a^* = y$ then y is a rational cut and therefore in A, but since $a^* < y\ \forall\ a \in A$ (y is an upper bound of rational set A) this contradicts the definition of the cut x. Thus $a^* < y$ implies $x \leq y$, and x is the l.u.b..

Problem 14. Prove that $\sqrt{2} \in \mathbb{R}$ by showing that $x \cdot x = 2$ where $x = A|B$ is the cut in $\mathbb{Q}$ with $A = \{r \in \mathbb{Q} : r \leq 0 \text{ or } r^2 < 2\}$.

Sol: We note that x is a positive cut, so $x^2 = y = E|F$ where $E = \{r \in \mathbb{Q} : r \leq 0$ or $r = ab$ for $a \in A \cap \mathbb{Q}^+$ and $b \in A \cap \mathbb{Q}^+\}$. We also note that if an element r of one set ≤ 0 it must be in the other by definition.

Lemma: Given $0 < r < 2$ $\exists a \in \mathbb{Q}$ s.t. $a^2 \in [r, 2)$.
Proof: By problem 13 we have that $x = \sup A^*$ with $A^* = \{a^* \in \mathbb{R} : a^*$ for a in $A\}$. Then we define $\epsilon = \frac{x^2 - r}{2x}$. Then $\exists a \in A$ s.t. $x - \epsilon \leq a^* \leq x$. We note that $x - \epsilon = \frac{x^2 + r}{2x} > 0$. Then

$$x - \epsilon \leq a^* \implies x^2 - 2x\epsilon \leq (x - \epsilon)^2 \leq a^{*2} < 2$$

since $a > 0$ and $a \in A$. (Note $(x - \epsilon)^2 \leq a^*(x - \epsilon) \leq a^{*2}$ by Lemma 2, and$(x - \epsilon)^2 \leq a^{*2}$ by transitivity.) This yields us $r \leq a^2 < 2$ since an inequality in $\mathbb{R}$ of rational cuts implies the corresponding inequality in the rationals.

From here, the proof follows: $r \leq a^2 \implies \frac{r}{a} \leq a$, and $\frac{r}{a} > 0 \implies (\frac{r}{a})^2 \leq a^2 < 2$, and $\frac{r}{a} \in A$. Thus we define $b = \frac{r}{a}$ and we have achieved $ab = r$ $\forall r > 0, r \in E$. Then $2 \leq x^2$.

For the opposite direction we assume $r \in E$ and we prove $r < 2$. Assume $r \geq 2$. Then $r = ab$ with $a, b > 0 \implies a^2b^2 \geq 4$, but $a^2, b^2 < 2 \implies a^2b^2 < 4$: contradiction. Thus $x^2 \leq 2$ and $x \cdot x = 2$.

Problem 18. Prove that real numbers correspond bijectively to decimal expansions not terminating in an infinite string of nines, as follows. The decimal expansion of $x \in \mathbb{R}$ is $N.x_1x_2\ldots$, where N is the largest integer $\leq x$, x_1 is the largest integer $\leq 10(x-N)$, x_2 is the largest integer $\leq 100(x-(N+x_1/10))$, and so on.

(a) Show that each x_k is a digit between 0 and 9.

Sol: Describe x_k the greatest integer $\leq 10^k(x-(N+\sum_{j=1}^{k-1} x_j/10^j))$. Thus, $10^k(x-(N+\sum_{j=1}^{k-1} x_j/10^j))-x_k < 1$ and

$$10^{k+1}(x-(N+\sum_{j=1}^{k} x_j/10^j)) < 10$$

implies for $k \geq 2$ $x_k < 10$ and is an digit from $0-9$. For x_1, since $x-N<1$ and $10(x-N)<10$, x_1 is a digit.

(b) Show that for each k there is an $l \geq k$ such that $x_l \neq 9$.

Sol: Assume $\exists\ k$ where $x_l = 9\ \forall l > k$. Then $\frac{9}{10^l} \leq x-(N+\sum_{j=1}^{l-1} x_j/10^j)\ \forall l > k$ and

$$x \geq N + \sum_{j=1}^{k} \frac{x_j}{10^j} + \sum_{m=k+1}^{l} \frac{9}{10^m}$$

We aim to prove that $\forall\ a \geq \sum_{m=k+1}^{l} \frac{9}{10^m}\ \forall\ l > k$, $a \geq \frac{1}{10^k}$ For this, we aim to use ϵ-principle. Define $y = \frac{1}{10^k}$. Then for any upper bound of our sequence b, and any given $\epsilon > 0$, we can take the least $c \in \mathbb{N}$ so that $10^c \geq 1/\epsilon$. Then if $c > k$, $\epsilon \geq 1/10^c \implies \frac{1}{10^k} - \epsilon \leq \frac{1}{10^k} - \frac{1}{10^c} = \sum_{m=k+1}^{l} \frac{9}{10^m} \leq y$ implies $\frac{1}{10^k} - \epsilon \leq y$ and $\frac{1}{10^k} \leq y$ by ϵ-principle. Then

$$x \geq N + \sum_{j=1}^{k} \frac{x_j}{10^j} + \frac{1}{10^k}$$

$$x_k + 1 \leq 10^k(x-(N+\sum_{j=1}^{k-1} \frac{x_j}{10^j}))$$

which contradicts the definition of x_j. By proof by contradiction, there must $\exists$ some $l \geq k\ \forall k$ s.t. $x_l \neq 9$.

(c) Conversely, show that for each such expansion $N.x_1x_2\ldots$ not terminating in an infinite string of nines, the set

$$\{N, N+\frac{x_1}{10}, N+\frac{x_1}{10}+\frac{x_2}{100}, \ldots\}$$

is bounded and its least upper bound is a real number x with decimal expansion $N.x_1x_2\ldots$.

We prove first that x is an upper bound of the given set. By the definition of each x-term, we have

$$x_k \leq 10^k(x - (N + \sum_{j=1}^{k-1} x_j/10^j)) \implies x \geq N + \sum_{j=1}^{k} x_j/10^j$$

for each $j \in \mathbb{N}$. Thus x is an upper bound by definition.

We then prove that x is the least upper bound. Assume $\exists y < x$ with y an upper bound of our sequence. Then $\exists\ r_1, r_2, r_1 \neq r_2$, By definition r_1, r_2 are upper bounds of the set.

We have: for $s_k = N + \sum_{j=1}^{k} x_j/10^j$ $s_k < r_1\ \forall\ k \in \mathbb{N}$. Then $s_k + (r_2 - r_1) < r_2$ $\forall k \in \mathbb{N}$. We consider the least k such that $1/10^k < (r_2 - r_1)$, which exists since s_k and $r_2 - r_1$ are rationals: then $s_k + 1/10^k < s_k + (r_2 - r_1) < r_2 < x$. This yields us

$$x_k + 1 < 10^k(x - (N + \sum_{j=1}^{k-1} \frac{x_j}{10^j}))$$

which contradicts the definition of x_k. Therefore, $\nexists\ y < x$ a lower bound of our set $\{s_k\}$.

Thus, the decimal expansion is well-defined, and the real number x may be derived as the l.u.b. of $N, x_1, \ldots$. By definition this makes the decimal expansion invertible and a bijection.

(d) Repeat the exercise with a general base in place of 10.

We copy the entire proof again and replace 10 with 10_b for an arbitrary base b, and 9 with $b - 1$.
Prove that real numbers correspond bijectively to decimal expansions not terminating in an infinite string of nines, as follows. The decimal expansion of $x \in \mathbb{R}$ is $N.x_1x_2\ldots$, where N is the largest integer $\leq x$, x_1 is the largest integer $\leq 10(x - N)$, x_2 is the largest integer $\leq 100(x - (N + x_1/10))$, and so on.
(d-a) Show that each x_k is a digit between 0 and $b - 1$.

Sol: (Note that $10_b = b_{10}$.) Describe x_k the greatest integer $\leq 10_b^k(x - (N + \sum_{j=1}^{k-1} x_j/10_b^j))$. Thus, $10_b^k(x - (N + \sum_{j=1}^{k-1} x_j/10_b^j)) - x_k < 1$ and

$$10_b^{k+1}(x - (N + \sum_{j=1}^{k} x_j/10_b^j)) < 10_b$$

implies for $k \geq 2$ $x_k < 10_b$ and is an digit from $0 - (b - 1)$. For x_1, since $x - N < 1$ and $10_b(x - N) < 10_b$, x_1 is a digit.

(d-b) Show that for each k there is an $l \geq k$ such that $x_l \neq b - 1$.

Sol: Assume $\exists\ k$ where $x_l = b - 1\ \forall l > k$. Then $\frac{b-1}{10_b^l} \leq x - (N + \sum_{j=1}^{l-1} x_j/10_b^j)\ \forall l > k$ and

$$x \geq N + \sum_{j=1}^{k} \frac{x_j}{10_b^j} + \sum_{m=k+1}^{l} \frac{b-1}{10_b^m}$$

We aim to prove that $\forall\ a \geq \sum\limits_{m=k+1}^{l} \frac{b-1}{10_b^m}\ \forall\ l > k,\ a \geq \frac{1}{10_b^k}$ For this, we aim to use ϵ-principle. Define $y = \frac{1}{10_b^k}$. Then for any upper bound of our sequence d, and any given $\epsilon > 0$, we can take the least $c \in \mathbb{N}$ so that $10_b^c \geq 1/\epsilon$. Then if $c > k$, $\epsilon \geq 1/10_b^c \implies \frac{1}{10_b^k} - \epsilon \leq \frac{1}{10_b^k} - \frac{1}{10_b^c} = \sum\limits_{m=k+1}^{l} \frac{b-1}{10^m} \leq y$ implies $\frac{1}{10_b^k} - \epsilon \leq y$ and $\frac{1}{10_b^k} \leq y$ by ϵ-principle. Then

$$x \geq N + \sum_{j=1}^{k} \frac{x_j}{10_b^j} + \frac{1}{10_b^k}$$

$$x_k + 1 \leq 10_b^k(x - (N + \sum_{j=1}^{k-1} \frac{x_j}{10_b^j}))$$

which contradicts the definition of x_j. By proof by contradiction, there must $\exists$ some $l \geq k\ \forall k$ s.t. $x_l \neq b-1$.

(d-c) Conversely, show that for each such expansion $N.x_1x_2\ldots$ not terminating in an infinite string of $b-1$s, the set

$$\{N, N + \frac{x_1}{10_b}, N + \frac{x_1}{10_b} + \frac{x_2}{100_b}, \ldots\}$$

is bounded and its least upper bound is a real number x with decimal expansion $N.x_1x_2\ldots$.

We prove first that x is an upper bound of the given set. By the definition of each x-term, we have

$$x_k \leq 10_b^k(x - (N + \sum_{j=1}^{k-1} x_j/10_b^j)) \implies x \geq N + \sum_{j=1}^{k} x_j/10_b^j$$

for each $j \in \mathbb{N}$. Thus x is an upper bound by definition.

We then prove that x is the least upper bound. Assume $\exists y < x$ with y an upper bound of our sequence. Then $\exists\ r_1, r_2, r_1 \neq r_2$, By definition r_1, r_2 are upper bounds of the set.

We have: for $s_k = N + \sum_{j=1}^{k} x_j/10_b^j\ s_k < r_1\ \forall\ k \in \mathbb{N}$. Then $s_k + (r_2 - r_1) < r_2$ $\forall k \in \mathbb{N}$. We consider the least k such that $1/10_b^k < (r_2 - r_1)$, which exists since s_k and $r_2 - r_1$ are rationals: then $s_k + 1/10_b^k < s_k + (r_2 - r_1) < r_2 < x$. This yields us

$$x_k + 1 < 10_b^k(x - (N + \sum_{j=1}^{k-1} \frac{x_j}{10_b^j}))$$

which contradicts the definition of x_k. Therefore, $\nexists\ y < x$ a lower bound of our set $\{s_k\}$.

Thus, the decimal expansion is well-defined, and the real number x may be derived as the l.u.b. of $N, x_1, \ldots$. By definition this makes the decimal expansion invertible and a bijection.

Problem 19. Formulate the definition of the **greatest lower bound** (g.l.b.) of a set of real numbers. State a g.l.b. property of $\mathbb{R}$ and show it is equivalent to the l.u.b. property of $\mathbb{R}$.

First, definitions:

Definition. $M \in \mathbb{R}$ is a **lower bound** for a set $S \subset \mathbb{R}$ if each $s \in S$ satisfies $s \geq M$. The **greatest lower bound** is the lower bound M such that $M \geq M'$ for each M' a lower bound of S.

The **Greatest Lower Bound Property** of the complete set $\mathbb{R}$ states: If S is a nonempty subset of $\mathbb{R}$ and is bounded above then in $\mathbb{R}$ there exists a greatest lower bound of S.

Proof of equivalence to l.u.b. property: Assume the greatest lower bound property. Then for a set S we define $S' = s : -s \in S$. S' follows the greatest lower bound property, so $\exists M$ where $s \geq M \ \forall \ s \in S'$. Then we argue that $-M$ is the least upper bound of S. For each $s' \in S$ we have $-s' \in S'$: $(s+(-s) = 0 = (-s)+(-(-(s))) \implies s = -(-(s)))$. Then $-s' \geq M \implies -M \geq s$ by translation.

For each M' an upper bound of S we have $M' \geq s'$ for each $s' \in S$: by translation this gives us $-M' \leq s$ for $s \in S'$ since $-s \in S$. Then $-M'$ is a lower bound of S' which gives us $-M' \leq M \implies M' \geq -M$ and $-M$ is the least upper bound of S for each bounded $S \subset \mathbb{R}$.

The proof in the other direction (l.u.b property $\implies$ g.l.b. property) is the same with the properties, but I'm copy-pasting it below anyway out of anxiety :(

Assume the least upper bound property. Then for a set S we define $S' = s : -s \in S$. S' follows the least upper bound property, so $\exists M$ where $s \geq M \ \forall \ s \in S'$. Then we argue that $-M$ is the least upper bound of S. For each $s' \in S$ we have $-s' \in S'$: $(s+(-s) = 0 = (-s)+(-(-(s))) \implies s = -(-(s)))$. Then $-s' \leq M \implies -M \leq s$ by translation.

For each M' a lower bound of S we have $M' \leq s'$ for each $s' \in S$: by translation this gives us $-M' \geq s$ for $s \in S'$ since $-s \in S$. Then $-M'$ is an upper bound of S' which gives us $-M' \geq M \implies M' \leq -M$ and $-M$ is the greatest lower bound of S for each bounded $S \subset \mathbb{R}$.

Problem 20. Prove that limits are unique, i.e., if (a_n) is a sequence of real numbers that converges to a real number b and also converges to a real number b', then $b = b'$.

Sol: Assume $\exists\ b \neq b'$ where (a_n) converges to both b and b'. WLOG assume $b < b'$. Then $\exists\ c \in \mathbb{R}$ where $c \in (b, b')$. Take $\epsilon = c - b$. Then by the definition of convergence $\exists N \in \mathbb{N}$ where for $n \geq N$, $|a_n - b| < \epsilon$. If $a_n - b$ is positive, $|a_n - b| = a_n - b < c - b \implies a_n < c$. If $a_n - b$ is negative, nevertheless we have $a_n - b < 0 < c - b \implies a_n < c$. From here we take $\epsilon' = b' - c$: ϵ' is positive. Then $a_n < c \implies b' - a_n > b' - c = \epsilon'$ for all $n \geq N$, and since $b' - a_n$ is positive it cannot satisfy the definition of convergence to b'. Thus, $b = b'$ for any two limits of (a_n).

Problem 26. Let $b(R)$ and $s(R)$ be the number of integer unit cubes in $\mathbb{R}^m$ that intersect the ball and sphere of radius R, centered at the origin.

(a) Let $m = 2$ and calculate the limits

$$\lim_{R\to\infty} \frac{s(R)}{b(R)} \text{ and } \lim_{R\to\infty} \frac{s(R)^2}{b(R)}.$$

Sol: We'll calculate $s(R)$ exactly and $b(R)$ using an upper and lower bound, then use the squeeze theorem. We know $s(R) = b(R) - c(R)$, so we can define sums for $b(R)/4$ and $c(R)/4$ (quadrants) without evaluating them:

$$b(R)/4 = \sum_{n=0}^{\lfloor R\rfloor} \left\lceil \sqrt{R^2 - n^2} \right\rceil$$

$$c(R)/4 = \sum_{n=1}^{\lfloor R\rfloor} \left\lceil \sqrt{R^2 - n^2} \right\rceil - 1$$

Note that we can use the ceiling function here with half-open boxes only. Then $b(R) - c(R) = \lfloor R\rfloor + \lceil R\rceil$. We define upper and lower bounds for $b(R)$:

$$\pi R^2 < b(R) < \pi(R + \sqrt{2})^2$$

We note that the circle with radius $R + \sqrt{2}$ is an outer bound for $b(R)$ since the greatest distance between points in a unit cube in $\mathbb{R}^2$ is $\sqrt{2}$.

Then we have

$$\frac{8R}{\pi(R+\sqrt{2})^2} \le \frac{4(\lfloor R\rfloor + \lceil R\rceil)}{\pi(R+\sqrt{2})^2} \le \frac{s(R)}{b(R)} \le \frac{4(\lfloor R\rfloor + \lceil R\rceil)}{\pi R^2} \le \frac{8(R+1)}{\pi R^2}$$

$$\frac{8}{\pi(R+\sqrt{2})} - \frac{8\sqrt{2}}{\pi(R+\sqrt{2})^2} \le \frac{s(R)}{b(R)} \le \frac{8}{\pi R} + \frac{8}{\pi R^2}$$

$$\lim_{R\to\infty} \frac{8}{\pi(R+\sqrt{2})} - \frac{8\sqrt{2}}{\pi(R+\sqrt{2})^2} \le \lim_{R\to\infty} \frac{s(R)}{b(R)} \le \lim_{R\to\infty} \frac{8}{\pi R} + \frac{8}{\pi R^2}$$

$$0 \le \lim_{R\to\infty} \frac{s(R)}{b(R)} \le 0 \implies \lim_{R\to\infty} \frac{s(R)}{b(R)} = 0$$

We can now consider $\frac{s(R)^2}{b(R)}$, noting once again $2R \le \lfloor R\rfloor + \lceil R\rceil \le 2(R+1)$:

$$\frac{64R^2}{\pi(R+\sqrt{2})^2} \le \frac{16(\lfloor R\rfloor + \lceil R\rceil)^2}{\pi(R+\sqrt{2})^2} \le \frac{s(R)^2}{b(R)} \le \frac{16(\lfloor R\rfloor + \lceil R\rceil)^2}{\pi R^2} \le \frac{64(R+1)^2}{\pi R^2}$$

$$\frac{64(R+\sqrt{2})^2}{\pi(R+\sqrt{2})^2} - \frac{64(2\sqrt{2}R+2)}{\pi(R+\sqrt{2})^2} \le \frac{s(R)^2}{b(R)} \le \frac{64R^2}{\pi R^2} + \frac{64(2R+1)}{\pi R^2}$$

$$\frac{64}{\pi} \le \lim_{R\to\infty} \frac{s(R)^2}{b(R)} \le \frac{64}{\pi} \implies \lim_{R\to\infty} \frac{s(R)^2}{b(R)} = \frac{64}{\pi}$$

(b) Take $m \ge 3$. What exponent k makes the limit

$$\lim_{R\to\infty} \frac{s(R)^k}{b(R)}$$

interesting?

Sol: For $k = m/m-1$ the limit is interesting: for any $m \in \mathbb{N}$ we slice the sphere into $\lceil R \rceil$ sections, each of which we can then take the largest and smallest $m-1$-sphere slices and compare $b(R)$ and $c(R)$. In this case $s(R)$ becomes a degree-$m-1$ function of $\lfloor R \rfloor$ and $\lceil R \rceil$. Then $b(R)$ is bounded by R^m and $(R+\sqrt{m})^m$, and so the degrees of the numerator and denominator are the same if $k = \frac{m}{m-1}$, which yields us nonzero finite bounds for the limit.

(c) Let $c(R)$ be the number of integer unit cubes that are contained in the ball of radius R, centered at the origin. Calculate

$$\lim_{R\to\infty} \frac{c(R)}{b(R)}.$$

Sol: $\lim\limits_{R\to\infty} \frac{c(R)}{b(R)} = \lim\limits_{R\to\infty} 1 - \frac{s(R)}{b(R)} = 1 - 0 = 1.$

(d) Shift the ball to a new, arbitrary center (not on the integer lattice) and re-calculate the limits.

We note that it does not matter in what cube the center is but rather the location of the center within the cube. We can still divide the ball into four quadrants and exclude the cubes that contain the y- or x-value of the center of the ball, (x_0, y_0) (WLOG assume $x_0, y_0 \in [0,1)^2$). From there we have

$$b(R) - c(R) = 4 + 2\lceil R + y_0 \rceil + 2\lceil R - y_0 \rceil + 2\lfloor R + x_0 \rfloor + 2\lfloor R - x_0 \rfloor$$

which is once again bounded between $8R+4$ and $8R+12$. We take the same bounds for $b(R)$ to get

$$\frac{8R+4}{\pi(R+\sqrt{2})^2} \le \frac{s(R)}{b(R)} \le \frac{8R+12}{\pi R^2}$$

and

$$\frac{(8R+4)^2}{\pi(R+\sqrt{2})^2} \le \frac{s(R)}{b(R)} \le \frac{(8R+12)^2}{\pi R^2}$$

After some calculation we achieve $\lim_{R\to\infty} \frac{s(R)}{b(R)} = 0$ and $\lim_{R\to\infty} \frac{s(R)^2}{b(R)} = \frac{64}{\pi}$.

(In general, since the difference between interior and exterior cube numbers remains a function of a rounded R-value through the height difference of interior/exterior of adjacent columns, the ratios and R-degrees remain the same regardless of the center's location.)