PROBLEM SET 7

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Problems: 5.21, 5.23, 5.44, 5.51, 5.62

Problem 5.21. For U as described above, assume that f is second-differentiable everywhere, and $(D^2 f)_p = 0$ for all p. What can you say about f? Generalize to higher-order differentiability.

Assuming the n^{th} total derivative is 0 implies that each n^{th} order partial derivative is 0. Thus for component f_k we have $\frac{\partial}{\partial x_{i_1}} \dots \frac{\partial}{\partial x_{i_n}} f_k = 0$ for each $i_j \in 1, \dots, m$. Then we have $\frac{\partial}{\partial x_{i_2}} \dots \frac{\partial}{\partial x_{i_n}} f_k = C_{i_2 \dots i_n}$ a constant, since $\frac{\partial}{\partial x_{i_2}} \dots \frac{\partial}{\partial x_{i_n}} f_k$ must be independent of each x_{i_1} (and by 20).

From here we will prove by induction that f must be an at most degree $n-1$ polynomial for $D^n f = 0$. Assume that for $D^{n-1} f = 0$ we have f a degree at most $n-2$ polynomial. We know that the order of the is in $C_{i_2 \dots i_n}$ can be rearranged since rearranging partial variable order results in the same derivative. Take $g = f - \sum C_? \frac{x_1^{p_1} \dots x_m^{p_m}}{p_1! \dots p_m!}$, where the coefficient corresponds to the constant partial $(n-1)$th derivative with respect to each x_j taken p_j times ($\sum p_j = n-1$). (? represents the code corresponding to p_j number of x_js each.) Then every $(n-1)^{\text{th}}$ partial derivative of g is 0 (since mismatched partial counts lead to the term being differentiated to 0, and the $(n-1)$th partial of f is $C_?$) and we know g is a degree at most $n-2$ polynomial.

Since $\sum C_? \frac{x_1^{p_1} \dots x_m^{p_m}}{p_1! \dots p_m!}$ is a degree $n-1$ polynomial, we have that $f = g + \sum C_? \frac{x_1^{p_1} \dots x_m^{p_m}}{p_1! \dots p_m!}$ must be a degree $n-1$ polynomial and our inductive hypothesis holds. Our base case is $n = 1$, since we know if $Df = 0$ then f is constant (or degree 0). Thus by induction our hypothesis holds for all n.

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Problem 5.23. Assume that $f : [a,b] \times Y \to \mathbb{R}^m$ is continuous, where Y is an open subset of $\mathbb{R}^n$, the partial derivatives $\partial f_i(x,y)/\partial y_j$ exist, and they are continuous. Let $D_y f$ be the linear transformation $\mathbb{R}^n \to \mathbb{R}^m$ represented by the $m \times n$ matrix of partials.

(a) Show that

$$F(y) = \int_a^b f(x,y)\,dx$$

is of class C^1 and

$$(DF)_y = \int_a^b (D_y f)\,dx.$$

This generalizes Theorem 14 to higher dimensions.

We use the C^1 mean value theorem on $f(x,y)$ with respect to y. Since Y is open, for some y_1 we take the ϵ-neighborhood around y_1 and choose h where $|h| < \epsilon$. Then the segment between y_1 and $y_1 + h$ is in Y and we have

$$f(x, y_1 + h) - f(x, y_1) = \left(\int_0^1 (D_y f)_{(x, y_1 + th)}\,dt \right) h.$$

Then we take the integral of both sides:

$$F(y_1 + h) - F(y_1) = \int_a^b \left(\int_0^1 (D_y f)_{(x, y_1 + th)}\,dt \right) h\,dx$$

Take $D_y F = \int_a^b (D_y f)\,dx$. If

$$\lim_{h \to 0} \frac{F(y_1 + h) - F(y_1) - (D_y F)_{y_1} h}{|h|} = 0$$

then we have verified the derivative. Since both $F(y_1 + h) - F(y_1)$ and $(D_y F)_{y_1}$ are linear transformations we can distribute the scalar divisor $|h|$ and merge the transformations:

$$\lim_{h \to 0} \left(\int_a^b \left(\int_0^1 (D_y f)_{(x, y_1 + th)}\,dt \right) h\,dx - \int_a^b (D_y f)_{y_1}\,dx \right) \frac{h}{|h|}$$

Then if the transformation is identically 0 we have proven the derivative. The inner integral is the average derivative over the segment from y_1 to $y_1 + h$, which approaches $(D_y f)_{y_1}$ as $h \to 0$, which yields the transformation as the zero transformation. We have $D_y F$ continuous since the integral of a continuous function is continuous.

(b) Generalize (a) to higher-order differentiability.

We take an inductive step of the same proof. We must assume that f is C^n and take $D_y^{n-1} f$ as the function whose integral is $D_y^{n-1} F$. Since these derivatives are $(n-1)$-linear transformations we can fix their inputs and then redo the proof with an additional derivative with respect to h_n: the proof then follows the exact same way replacing f with $D_y^{n-1} f(h_1, \ldots h_{n-1})$ since nothing we did required f to map to R^m necessarily (please don't make me write it out please please please :'( )

Problem 5.44. Let $S \subset M$ be given.

(a) Define the characteristic function $\chi_S : M \to \mathbb{R}$.

$$\chi_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{otherwise.} \end{cases}$$

(b) If M is a metric space, show that $\chi_S(x)$ is discontinuous at x if and only if x is a boundary point of S.

Assume first that $x \in S$. If x is a boundary point of S, then for each $r > 0$ there exists a point p where $p \notin \overline{S}$ but $p \in B_r(x)$. Take $r_n = 1/n$ for $n \in \mathbb{N}$, and the corresponding point p_n. Then (p_n) is a sequence that by definition converges to $x \in S$, but every point of which is mapped to 0 by χ_S: thus the sequence $\chi_S(p_n)$ converges to 0 but $\chi_S(x) = 1$ implies discontinuity.

Now assume that $x \notin S$. Then if x is a boundary point of S it is in $\overline{S}$, and is therefore a limit point of a sequence (x_n) in S. Once again $\chi_S(x_n) = 1$ for all n, and thus the limit of $\chi_S(x_n)$ is 1: however, (x_n) converges to x and $\chi_S(x) = 0$, implying discontinuity.

In the other direction, we assume χ_S is discontinuous at x. Then there exists some sequence x_n that converges to x where $\chi_S(x_n)$ does not converge to $\chi_S(x)$, or where $\chi_S(x_n)$ does not converge. Once again consider $\chi_S(x) = 1$: then since $\chi_S(a) = 0$ or $\chi_S(a) = 1$ for all $a \in M$, we must have infinitely many x_k where $\chi_S(x_k) = 0$, since otherwise we have a maximum N past which all x_n for $n > N$ must be mapped to 1. As such, for any $\epsilon > 0$ we can pick an M where $k > M$ implies $d(x_k, x) < \epsilon$: since $\chi_S(x_k) = 0$ we have $x_k \notin S$ for all k. Thus x is a boundary point.

Now assume $\chi_S(x) = 0$. We take the same argument with a convergent sequence (x_n) and infinitely many k for which $\chi_S(x_k) = 1$: then since $x_k \in S$ for all k $(x_k) \to x$ (since it is a subsequence) implies x is a limit point of S and therefore in the closure of S. Thus x is a boundary point.

Problem 5.51. A region R in the plane is of type 1 if there are smooth functions $g_1 : [a, b] \to \mathbb{R}$, $g_2 : [a, b] \to \mathbb{R}$ such that $g_1(x) \leq g_2(x)$ and

$$R = \{(x, y) : a \leq x \leq b \text{ and } g_1(x) \leq y \leq g_2(x)\}.$$

R is of type 2 if the roles of x and y can be reversed, and it is a **simple region** if it is of both type 1 and type 2.

(a) Give an example of a region that is type 1 but not type 2.

Take $g_1, g_2 : [-2\pi, 2\pi] \to \mathbb{R}$ with $g_1(x) = \sin(x) - 2$ and $g_2(x) = \sin(x) + 2$. The points $(\pi/4, 2)$ and $(-3\pi/4, 2)$ are in this set but the horizontal line between them is not (for instance, $(-\pi/4, 2)$.) Thus the region is not type 2.

(b) Give an example of a region that is neither type 1 nor type 2.

Just take a disconnected set of two points x_1, y_1, x_2, y_2 with $x_1 \neq x_2$ and $y_1 \neq y_2$. Then the set is not type 1 since there is no horizontal interval of values of x which correspond to points in the set, and the same reason with vertical intervals for type 2. (Or, take a donut.)

(c) Is every simple region starlike? Convex?

Not convex: take the region bounded in y by $y = 0$ and $y = x^2$, and in x by $0, 2$. Then the points $(0, 0)$ and $(2, 4)$ are in the set but the point $(1, 2)$ is not despite being on the segment between them.

Not every simple region is starlike :(tried to prove it for twenty minutes

Take the simple region bounded by x^3 and x^2: this is horizontally bounded by $\sqrt{y}$ and $\sqrt[3]{y}$, each of which is smooth as they are polynomials. The point $(0, 0)$ has derivative 0 at both points, so any segment with derivative > 0 necessarily exceeds x^3 initially, and $y = 0$ is less than x^2 immediately after 0. Thus no point is connected by segment to $(0, 0)$ and the region is not starlike.

(d) If a convex region is bounded by a smooth simple closed curve, is it simple?

Since a smooth simple closed curve is a parametrization γ from a compact interval, we can use the extreme value theorem on the continuous x- and y-components of the curve to find the t-values that achieve maximum and minimum of x and y.

We show via convexity that the curve has a vertical or horizontal tangent line on at most two mutually disconnected intervals for each. If there are three vertical tangent lines, then two must have the same orientation with regard to the parametrization. We assume these two tangent lines are distinct (because if they are the same with a disconnection between the intervals, then mean value and intermediate value theorems imply a third distinct vertical tangent line).

Then the line between these points must cross the curve, since both points start in a vertical direction and begin on opposite sides of the segment. Contradiction. As such there are at most two vertical and horizontal tangent lines each (due to opposite orientations of the parametrization.) These two must be the minimum and maximum of x and y respectively. Since the parametrization is smooth, the intermediate value theorem implies on the intervals between the extrema that the components of the parametrization are monotonic. If we redo the parametrization with respect to distance (i'm grasping at straws here) then they are strictly monotonic since the parametrization can't just stop. This implies homeomorphism since the parametrization's components are continuous bijections (by monotonicity) on compact intervals: since the distance parametrization's components are smooth homeomorphisms and strictly monotonic (derivative zero nowhere) by the inverse function theorem they are smooth diffeomorphisms. As a result the inverse of the y-component of the parametrization composed under the x-component of the parametrization on the interval between y-extrema is a smooth function (note that it is not a homeomorphism since x is not monotonic between y-extrema!) and we have the conditions for a simple region satisfied (holy garbage.)

(Continuous bijection on a compact interval is a homeomorphism: will use later.)

(e) Give an example of a region that divides into three simple subregions but not into two.

Take the union of three mutually disconnected simple regions. You are done.

If annoying, instead take the union of three rectangles that share only parts of their boundary. Then this region is connected but any cut of the set into two subregions yields too many corners for both of the subregions to be simple.

(f) If a region is bounded by a smooth simple closed curve C then it need not divide into a finite number of simple subregions. Find an example.

Take any smooth simple closed curve with a smooth transition to the function $f(x) = e^{-\frac{1}{x}} \sin(\frac{1}{x})$. This function has infinitely many zeros and crosses the x-axist across each one, since $e^{-\frac{1}{x}}$ is positive for all $x > 0$. Then any attempt to cut the region into finitely many simple subregions will yield a subregion with more than three zeros, which excludes that subregion from type 2 status.

(g) Infer that the standard proof of Green's Formulas for simple regions (as, for example, in J. Stewart's *Calculus* does not immediately carry over to the general planar region R with smooth boundary; i.e., cutting R into simple regions can fail.

R cannot always be cut into simple regions and therefore Stewart's proof of Green's Theorem does not always have a valid split into useable boundaries and regions. (Essentially, this follows exactly from (f).)

(h) Is there a planar region bounded by a smooth simple closed curve such that for every linear coordinate system (i.e., a new pair of axes), the region does not divine into finitely many simple subregions? In other words, is Stewart's proof of Green's Theorem doomed?

(i) Show that if the curve in (f) is analytic, then no such example exists. [Hint: C is analytic if it is locally the graph of a function defined by convergent power series. A nonconstant analytic function has the property that for each x, there is some derivative of f which is nonzero, $f^{(r)}(x) \neq 0$.]

We prove it is impossible for an analytic function to cross a point a infinitely many times on a compact set. Assume the opposite. Then since the set is compact there is a convergent subsequence whose limit c also is mapped to a by continuity. By mean value theorem, for each point x_n there is a point between x_n and c whose derivative is 0: these points must also converge to c since they are closer than the original sequence, and therefore since the function is analytic its derivative at c must also be 0. We continue using the mean value theorem in this way to conclude that each derivative of the function at c is 0, which leads to a contradiction since the function is analytic. Thus the curve cannot cross an x or y value infinitely many times, or on any axis, meaning that the region is able to be subdivided into finitely many type 1 and type 2 regions, the intersection of which should be finitely many simple regions.

Problem 5.62. Does there exist a continuous mapping from the circle to itself that has no fixed-point? What about the 2-torus? The 2-sphere?

Yes, yes, and yes: for the circle, simply take a rotation of the circle about the origin by some angle $\neq 2\pi$. For the 2-torus, take the plane parametrization of the torus and translate the plane by some vector $\neq$ the vector difference between two corners and modulate the outside vectors back into the plane (this representation is homeomorphic to the torus and thus any fixed points should be preserved.) For the 2-sphere, map each point to the opposite pole: then no point can be mapped to itself, but the map is continous (we take $f(x) = -x$.)