PROBLEM SET 6

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Problems: 4.27, 4.29, 4.41, 5.1, 5.2, 5.7, 5.8, 5.17, 5.20

Problem 4.27. Suppose that $f : M \to M$ and for all $x, y \in M$, if $x \neq y$ then $d(fx, fy) < d(x, y)$. Such an f is a weak contraction.

(a) Is a weak contraction a contraction? (Proof or counterexample.)

No. Consider $f(x) = -\arctan x + x$. The derivative of this function is $\frac{x^2}{1+x^2}$ which is bounded between 0 and 1. Thus the function is a weak contraction by mean value theorem. However, the function is not a contraction

(b) If M is compact, is a weak contraction a contraction? (Proof or counterexample.)

No. Note $f(x) = x^2$ on $[0, 1]$. Consider $d(a, 1)$ and $d(f(a), f(1))$:

(c) If M is compact, prove that a weak contraction has a unique fixed-point.

We prove there is a fixed point. Take d the infinum of $d(x, f(x))$ for each $x \in M$. Then $\exists$ a sequence (x_n) such that $d(x_n, f(x_n)) - d < 1/n$. Since M is compact we take a convergent subsequence (x_{n_k}) with limit x. We know $\exists$ N such that $d(x_m, f(x_m)) < d + \epsilon/3$ for each $m > N$ and we can take M such that $d(f(x_{n_k}), f(x)) \leq d(x_{n_k}, x) < \epsilon/3$ for $k > M$. Then pick $n_k > k > \max(N, M)$. Then $d(x, f(x)) \leq d(f(x_{n_k}), f(x)) + d(f(x_{n_k}), x_{n_k}) + d(x_{n_k}, x) < \epsilon + d$ for each ϵ: by ϵ-principle we have $d(f(x), x) \leq d$ and $d(f(x), x) = d$ by definition. If $d > 0$, $f(x) \neq x$ implies $d(f(f(x)), f(x)) < d(f(x), x) = d$ which is a contradiction. Thus $d = 0$ and x is the fixed point.

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Problem 4.29. Give an example to show that the fixed-point in Brouwer's Theorem need not be unique.

$f(x) = x$ on $[-1, 1]$ is a function on B^1 and each point in the graph of the function is a fixed-point.

Problem 4.41. (a) Give an example of a function $f : [0,1] \times [0,1] \to \mathbb{R}$ such that for each fixed x, the function $y \mapsto f(x,y)$ is a continuous function of y, and for each fixed y, the function $x \mapsto f(x,y)$ is a continuous function of x, but f is not continuous.

$f(x) = \frac{xy}{x^2+y^2}$.

(b) Suppose in addition that the set of functions

$$\mathcal{E} = \{x \mapsto f(x,y) : y \in [0,1]\}$$

is equicontinuous. Prove that f is continuous.

Take some $x_1, y_1 \in [0,1]$. We prove that for each ϵ $\exists$ a δ such that $d((x,y),(x_1,y_1)) < \delta \implies d(f(x,y), f(x_1,y_1)) < \epsilon$. For a given ϵ take δ_y such that $d(y,y_1) < \delta_y$ implies $d(f(x_1,y), f(x_1,y_1)) < \epsilon/2$. Then take δ_x such that $d(x,x_1) < \delta_x$ implies $d(f(x,y), f(x_1,y)) < \epsilon/2$ for all y: this exists due to the equicontinuity condition. Then take $\delta = \min(\delta_x, \delta_y)$. We have if $d((x_1,y_1),(x,y)) < \delta$ then $d(x_1,x), d(y_1,y) < \delta$ (using Euclidean, taxicab, and maximum metrics) and $d(f(x_1,y_1), f(x,y)) \leq d(f(x,y), f(x_1,y)) + d(f(x_1,y), f(x_1,y_1)) < \epsilon$ by triangle inequality. Thus continuity holds.

Problem 5.1. Let $T : V \to W$ be a linear transformation, and let $p \in V$ be given. Prove that the following are equivalent.

(a) T is continuous at the origin.
(b) T is continuous at p.
(c) T is continuous at at least one point of V.

We prove (a) implies (b) first. Given ϵ we pick δ such that $||v - 0|| = ||v|| < \delta$ implies $||Tv - T(0)|| = ||Tv|| < \epsilon$. Then consider x_1 such that $d(p, x_1) < \delta$: since V is a vector space we have $d(p, x_1) = ||p - x_1|| = ||(p - x_1) - 0|| = d(p - x_1, 0)$. Then $||T(p - x_1)|| = ||T(p) - T(x_1)|| < \epsilon$.

(b) $\implies$ (c) is obvious since $p \in V$.

We prove (c) implies (a). Take this point x. Then given ϵ we have δ where $||x - x_1|| < \delta$ implies $||T(x_1) - T(x)|| < \epsilon$. Take $||v|| < \delta$. Then $||x - (x - v)|| = ||v|| < \delta$ implies $||T(x) - T(x - v)|| = ||T(x) - T(x) + T(v)|| = ||T(v)|| < \epsilon$, implying continuity at the origin.

Problem 5.2. Let $\mathcal{L}$ be the vector space of continuous linear transformations from a normed space V to a normed space W. Show that the operator norm makes $\mathcal{L}$ a normed space.

We prove the three properties. Take $T, T_1, T_2 \in \mathcal{L}$ and the operator norm $||\cdot||$. Since $|Tv|_W$ and $|v|_V$ are nonnegative so is $||T||$. If $||T||$ is 0 then

$$\frac{|Tv|_W}{|v|_V} \leq 0 \implies |T_v|_W = T_v = 0$$

for all v, so $T = 0$. If $T = 0$ then clearly $\frac{|Tv|_W}{|v|_V} = 0$ for all $v \neq 0$: then $||T|| = 0$.

We now prove the triangle inequality. We have

$$||T_1|| \geq \frac{|T_1 v|_W}{|v|_V}, ||T_2|| \geq \frac{|T_2 v|_W}{|v|_V}$$

for each $v \neq 0$ in V. Then

$$||T_1|| + ||T_2|| \geq \frac{|T_1 v|_W + |T_2 v|_W}{|v|_V} \geq \frac{|(T_1 + T_2)v|_W}{|v|_V}.$$

Since $||T_1 + T_2|| = \sup\{\frac{|(T_1+T_2)v|_W}{|v|_V}\}$ we have $||T_1|| + ||T_2|| \geq ||T_1 + T_2||$.

We prove the third property. Assume λ a positive scalar. Then $||\lambda T|| = \sup\{\frac{\lambda|T|_W}{|v|_V}\}$ We claim $||\lambda T|| = \lambda||T||$: $||T|| \geq \frac{|T|_W}{|v|_V}$ implies $\lambda||T|| \geq \frac{\lambda|T|_W}{|v|_V}$, and if $\exists s$ where $s \geq \frac{\lambda|T|_W}{|v|_V}$ with $s < \lambda||T||$ then $s/\lambda \geq \frac{|T|_W}{|v|_V}$ implies s/λ is an lower upper bound for $\frac{|T|_W}{|v|_V}$ which is impossible.

Problem 5.7. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space are comparable if there are positive constants c and C such that for all nonzero vectors in V we have

$$c \leq \frac{\|v\|_1}{\|v\|_2} \leq C.$$

(a) Prove that comparability is an equivalence relation on norms.

Reflexivity: $\frac{||v||_1}{||v||_1} = 1$ which is between $1/2$ and $3/2$.

Symmetry: $\frac{\|v\|_1}{\|v\|_2} \leq C \implies \frac{\|v\|_2}{\|v\|_1} \geq 1/C$, $\frac{\|v\|_1}{\|v\|_2} \geq c \implies \frac{\|v\|_2}{\|v\|_1} \leq 1/c$ and since $c, C > 0$, we know $1/c, 1/C > 0$.

Transitivity: $c_1 \leq \frac{\|v\|_1}{\|v\|_2} \leq C_2$ and $c_2 \leq \frac{\|v\|_2}{\|v\|_3} \leq C_2$ implies $c_1c_2 \leq \frac{\|v\|_1}{\|v\|_3} \leq C_1C_2$ which are both greater than 0 since each of the factors are.

(b) Prove that any two norms on a finite-dimensional vector space are comparable. [Hint: Use Theorem 3.]

Let V be of dimension n and take norms $|\cdot|_1$ and $|\cdot|_2$. Take T an isomorphism (bijective operator) from $\mathbb{R}^n$ to V: T is a homeomorphism by theorem 3. For any x in $\mathbb{R}^n$, we have c_1, c_2 where $\frac{|Tx|_1}{c_1} \leq |x| \leq c_1|Tx|_1$, and $\frac{|Tx|_2}{c_2} \leq |x| \leq c_2|Tx|_2$. Then $\frac{1}{c_1c_2} \leq \frac{|T|_1}{|T|_2} \leq c_1c_2$ and we have the two norms comparable.

(c) Consider the norms

$$\|f\|_{L_1} = \int_0^1 |f(t)|\, dt \quad \text{and} \quad \|f\|_{C^0} = \max\{|f(t)| : t \in [0,1]\},$$

defined on the infinite-dimensional vector space C^0 of continuous functions $f : [0,1] \to \mathbb{R}$. Show that the norms are not comparable by finding functions $f \in C^0$ whose integral norm is small but whose C^0 norm is 1.

Take the set $\{f_n = t^n : n \in \mathbb{N}\}$ on $[0,1]$. We have $||f_n||_{L_1} = 1/(n+1)$ and $||f_n||_{C^0} = 1$. Then the infinum of $\frac{||f_n||_{L_1}}{||f_n||_{C^0}}$ is 0 and the norms are not comparable.

Problem 5.8. Let $\| \cdot \| = \| \cdot \|_{C^0}$ be the supremum norm on C^0 as in the previous exercise. Define an integral transformation $T : C^0 \to C^0$ by

$$T : f \mapsto \int_0^x f(t)\, dt.$$

(a) Show that T is linear, continuous, and find its norm.

$T(\lambda f) = \int_0^x \lambda f(t)\, dt = \lambda \int_0^x f(t)\, dt = \lambda T f$ and $T(f_1 + f_2) = \int_0^x f_1 + f_2\, dt = \int_0^x f_1\, dt + \int_0^x f_2\, dt = Tf_1 + Tf_2$. Thus T is linear.

T is continuous since it is a constant away from the indefinite integral of $f(x)$ and the uniform convergence (convergence under sup norm) of fs yields the uniform convergence of indefinite integrals of f and therefore the uniform convergence of T (since the indefinite integrals of f at 0 must also converge by uniform convergence; the sum of convergent sequences is convergent).

(b) Let $f_n(t) = \cos(nt)$, $n = 1, 2, \ldots$. What is $T(f_n)$?

$\sin(nx)/n$.

(c) Is the set of functions $K = \{f_n : n \in \mathbb{N}\}$ closed? Bounded? Compact?

It's bounded for obvious reasons. It is closed but not compact. (Out of time to justify closed, but it's not equicontinuous and so isn't compact.)

(d) Is $T(K)$ compact? How about its closure?

$T(K)$ is not compact, since its uniform limit converges to 0, which is not in the set. Its closure is compact, however, since the set $\sin(nx)/n$ is equicontinuous (its derivative is bounded between -1 and 1) and so the closure is closed, equicontinuous, and trivially bounded.

Problem 5.17. Let $f : U \to \mathbb{R}^m$ be differentiable, $[p, q] \subset U \subset \mathbb{R}^n$, and ask whether the direct analog of the one-dimensional Mean Value Theorem is true: Does there exist a point $\theta \in [p, q]$ such that

$$f(q) - f(p) = (Df)_\theta(q - p)?$$

(a) Take $n = 1$, $m = 2$, and examine the function

$$f(t) = (\cos t, \sin t)$$

for $\pi \leq t \leq 2\pi$. Take $p = \pi$ and $q = 2\pi$. Show that there is no $\theta \in [p, q]$ which satisfies (28).

We have $f(q) - f(p) = (2, 0)$, $Df = (-\sin t, \cos t)$, and $q - p = \pi$. If $\cos t \neq 0$ then $\theta = 0$ leads to a contradiction, and if $\cos t = 0$ we have $-\sin t = 1$ does not satisfy (28).

(b) Assume that the set of derivatives

$$\{(Df)_x \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) : x \in [p, q]\}$$

is convex. Prove there exists $\theta \in [p, q]$ which satisfies (28). [Hint: Google "support plane."]

(c) How does (b) imply the one-dimensional Mean Value Theorem?

We have in $\mathbb{R}$ that each interval is connected and each connected set is convex, so we must prove that the set of derivatives $\{f'(x) : x \in [p, q]\}$ is an interval. The MVT requres that f is C^1, so we have f' continuous: since $[p, q]$ is connected and compact (an interval), $f'([p, q])$ is connected and compact and therefore an interval.

Problem 5.20. Assume that U is a connected open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^m$ is differentiable everywhere on U. If $(Df)_p = 0$ for all $p \in U$, show that f is constant.

By the Mean Value Theorem (Theorem 11 of Chapter 5), $|f(q) - f(p)| \leq 0|q - p|$ for all q, p with the segment between them in U. Since U is a connected open subset of $\mathbb{R}^n$ it is path connected and therefore any two points x_1 and x_2 of U can be connected via segments (take a finite closed neighborhood of a point on the path and select a further point on the sphere, on the path between x_1 and x_2, since the path is compact these neighborhoods cannot converge to a point along the middle of the path) and the function must be constant.