Assignment **HW 1**

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This document contains my solutions to the Gradescope assignment named on the top of this page. Specifically, my solutions to the following problems are included:

- 3.35 (page 2)
- 3.38 (a)(b) (page 3)
- 3.41 (page 4)
- 3.44 (page 5)
- 3.47 (a)(b) (page 6)
- 3.50 (page 7)

I did not forget

- to REFRESH my browser for the latest information about each problem
- to link problems to pages.
  This page is linked to the problems I did not solve.
- to update the items marked *** in the template (my name, email, the Gradescope title of the assignment, the list of problems solved, the `\lhead` statements (left page headers: list of (sub)problems solved on each page)
- to make sure no subproblem solution spills over to the next page (except when this is unavoidable, i.e., when the solution to a subproblem does not fit on a page)
- if a problem takes more than one page, I linked each of those pages to the problem
- I took care not to defeat the mechanisms provided by this template.

With each problem, **I stated my sources and collaborations**.
By submitting this solution *I certify* that *my statement of sources and collaborations is accurate and complete.* I understand that without this certification, my solutions will not be accepted.
In case I am giving a link to a source, *I am also sending this link to the instructor by email.*

3.35 Question.
Let F_n denote the n-th Fibonacci number (see Def. 2.11). Give a very simple proof of the following fact.
If the quotients F_{n+1}/F_n converge then their limit is the golden ratio.
Do not use the explicit formula for Fibonacci numbers.
This is a case when it is easier to compute the limit assuming it exists, than proving the existence of the limit.

Sources and collaborations.
Worked with Guan Chen on this solution; he suggested solving the quadratic.

Answer.
We assume there exists a limit. Then for sufficiently high n we have $d(\frac{F_{n+1}}{F_n}, \frac{F_{n+2}}{F_{n+1}}) < \epsilon$.

Take $\frac{F_{n+1}}{F_n} = \frac{F_{n+2}}{F_{n+1}} + a_n$. Then:

$$\frac{F_{n+1}}{F_n} = \frac{F_{n+1} + F_n}{F_{n+1}} + a_n.$$

Take $r_n = \frac{F_{n+1}}{F_n}$. Then we have $r_n = 1 + \frac{1}{r_n} + a_n$ with $a_n < \epsilon$ for $n > N_\epsilon$ which yields a quadratic $r_n^2 - r_n - 1 - a_n = 0$ which evaluates to

$$\frac{1 + a_n \pm \sqrt{(1 + a_n)^2 + 4}}{2}$$

and since $\lim_{n\to\infty} a_n = 0$, and the ratio must be positive since the Fibonacci sequence is always positive, this approaches $\frac{1+\sqrt{5}}{2}$ and we are done.

3.38(a) Question.
Find two bounded sequences, (a_n) and (b_n), such that $\limsup(a_n+b_n) < \limsup(a_n) + \limsup(b_n)$.

Sources and collaborations. Worked with Guan Chen.

Answer.
We define $(a_n) = \frac{(-1)^n}{2}+3/2$ and $(b_n) = \frac{(-1)^{n+1}}{2}+5/2$. Then $\limsup a_n = 2$ and $\limsup b_n = 3$ but $a_n + b_n = 4\ \forall n$ and the condition is satisfied. $\square$

3.38(b) Question.
Find two sequences, (a_n) and (b_n), such that $\limsup(a_n + b_n) = -\infty$ while $\limsup a_n = \limsup b_n = \infty$.

Sources and collaborations. None.

Answer.
Take the sequences $(a_n) = (1, -2, 2, -4, 3, -6, \dots)$ and $(b_n) = (-2, 1, -4, 2, -6, 3, \dots)$. Then $\sup(a_n)_{n>N} = \infty$ and $\sup(b_n)_{n>N} = \infty$ but $\sup(a_n + b_n)_{n>N} = \lfloor -N/2 \rfloor$. $\square$

3.41 Question.
Prove $\sqrt{n^2+1} - n \sim 1/(2n)$.

Sources and collaborations. None.

Answer.
Consider the fraction

$$\frac{1/2n}{\sqrt{n^2+1} - n} = \frac{\sqrt{n^2+1} + n}{2n} = \frac{\sqrt{n^2+1}}{2n} + 1/2$$

Then

$$\lim_{n\to\infty} \frac{\sqrt{n^2+1}}{2n} + 1/2 = \sqrt{\lim_{n\to\infty} \frac{n^2+1}{4n^2}} + 1/2$$

by positive continuity of $\sqrt{x}$

$$= \sqrt{\lim_{n\to\infty} \frac{n^2}{4n^2} + \lim_{n\to\infty} \frac{1}{4n^2}} + 1/2 = 1/2 + 1/2 = 1$$

3.44 Question.
Prove: there exist real numbers a, b, c such that $\binom{2n}{n} \sim a \cdot n^b \cdot c^n$. Find a, b, c.

Sources and collaborations. Isaac Chang implored me to use my ability to read to see the formula immediately above the problem.

Answer.
We apply Stirling's Formula.

$$\binom{2n}{n} = \frac{2n!}{n!n!}$$

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \text{ and } 2n! \sim \left(\frac{2n}{e}\right)^{2n} \sqrt{4\pi n}.$$

Then by Exercise 5.3

$$\binom{2n}{n} \sim \frac{\left(\frac{2n}{e}\right)^{2n} \text{‘} \sqrt{4\pi n}}{\left(\frac{n}{e}\right)^{2n} 2\pi n}$$
$$= \frac{4^n}{\sqrt{\pi n}}$$

and we have $a = \frac{1}{\sqrt{\pi}}$, $b = -1/2$, and $c = 4$.‘

3.47(a) Question.
Assume $a_n, b_n > 1$. Consider the following statements:
(A) $a_n \sim b_n$;
(B) $\ln a_n \sim \ln b_n$.
Prove (A) does not imply (B).

Sources and collaborations. None.

Answer.
Take $a_n = \exp(\frac{1}{2^n})$ and $b_n = \exp(\frac{1}{2^{n+1}})$. Then $\frac{a_n}{b_n} = \exp(\frac{1}{2^{n+1}})$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = \exp(0) = 1$, but $\frac{\ln a_n}{\ln b_n} = 2 \neq 1$. Thus (A) cannot imply (B).

3.47(b) Question.
Prove (A) does imply (B) under the stronger assumption that $a_n \geq 1.01$.

Sources and collaborations. None.

Answer.
Assume (A). Then $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$, and since ln is a continuous function it preserves limits, yielding

$$\lim_{n\to\infty} (\ln a_n - \ln b_n) = 0.$$

Since $a_n \geq 1.01$ $\ln(a_n) \geq \ln(1.01) > 0$. Then

$$\frac{|\ln a_n - \ln b_n|}{\ln a_n} \leq \frac{|\ln a_n - \ln b_n|}{\ln(1.01)}$$

and

$$\lim_{n\to\infty} \frac{\ln a_n - \ln b_n}{\ln b_n} = \lim_{n\to\infty} \frac{\ln a_n - \ln b_n}{\ln 1.01} = 0 \implies \lim_{n\to\infty} \frac{\ln a_n}{\ln b_n} = 1$$

3.50 Question.
For the positive integer m, let $\nu(m)$ denote the number of distinct prime divisors of m. Prove:

$$\nu(m) \leq \log_2(m)$$

Sources and collaborations. None.

Answer.
The number of prime divisors of m can be multiplied to some divisor of m less than m, call it $\prod p_j$. Since all primes ≥ 2, we have $m \geq \prod p_j \geq \prod_{j=1}^{\nu(m)} 2$. Since $\log_2$ is an increasing function, $\log_2 m \geq \nu(m)$ and we are done.