PROBLEM SET 2

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Problems: 5, 6, 7, 9, 11, 12, 13, 17, 22, 28 and 101.

Problem 5. For $p, q \in S^1$, the unit circle in the plane, let

$$d_a(p,q) = \min\{|\measuredangle(p) - \measuredangle(q)|, 2\pi - |\measuredangle(p) - \measuredangle(q)|\}$$

where $\measuredangle(z) \in [0, 2\pi)$ refers to the angle that z makes with the positive x-axis. Use your geometric talent to prove that d_a is a metric on S^1.

Problem 6. For $p, q \in [0, \pi/2)$ let

$$d_s(p,q) = \sin|p - q|.$$

Use your calculus talent to decide whether d_s is a metric.

Problem 7. Prove that every convergent sequence (p_n) in a metric space M is bounded, i.e., that for some $r > 0$, some $q \in M$, and all $n \in \mathbb{N}$, we have $p_n \in M_r q$.

Problem 9. A sequence (x_n) in $\mathbb{R}$ **increases** if $n < m$ implies $x_n \leq x_m$. It **strictly increases** if $n < m$ implies $x_n < x_m$. It decreases or strictly decreases if $n < m$ always implies $x_n \geq x_m$ or always implies $x_n > x_m$. A sequence is monotone if it increases or it decreases. Prove that every sequence in $\mathbb{R}$ which is monotone and bounded converges in $\mathbb{R}$.

Problem 11. Let (x_n) be a sequence in $\mathbb{R}$.

*(a) Prove that (x_n) has a monotone subsequence.

(b) How can you deduce that every bounded sequence in $\mathbb{R}$ has a convergent subsequence?

(c) Infer that you have a second proof of the Bolzano-Weierstrass Theorem in $\mathbb{R}$.

(d) What about the Heine-Borel Theorem?

Problem 12. Let (p_n) be a sequence and $f : \mathbb{N} \to \mathbb{N}$ be a bijection. The sequence $(q_k)_{k \in N}$ with $q_k = p_{f(k)}$ is a rearrangement of (p_n).

(a) Are limits of a sequence unaffected by rearrangement?

(b) What if f is an injection?

(c) A surjection?

Date: October 13, 2024.

Problem 13. Assume that $f : M \to N$ is a function from one metric space to another which satisfies the following condition: If a sequence (p_n) in M converges then the sequence $(f(p_n))$ in N converges. Prove that f is continuous. [This result improves Theorem 4.]

Problem 17. 17. Which capital letters of the Roman alphabet are homeomorphic? Are any isometric? Explain.

Problem 22. If every closed and bounded subset of a metric space M is compact, does it follow that M is complete?

Problem 28. A map $f : M \to N$ is open if for each open set $U \subset M$, the image set $f(U)$ is open in N.

(a) If f is open, is it continuous?
(b) If f is a homeomorphism, is it open?
(c) If f is an open, continuous bijection, is it a homeomorphism?
(d) If $f : R \to R$ is a continuous surjection, must it be open?
(e) If $f : R \to R$ is a continuous, open surjection, must it be a homeomorphism?
(f) What happens in (e) if R is replaced by the unit circle S^1?

Problem 101. Let Σ be the set of all infinite sequences of zeroes and ones. For example, $(100111000011111...) \in \Sigma$. Define the metric

$$d(a, b) = \sum \frac{|a_n - b_n|}{2^n}$$

where $a = (a_n)$ and $b = (b_n)$ are points in Σ. (a) Prove that Σ is compact. (b) Prove that Σ is homeomorphic to the Cantor set.