PROBLEM SET 3

LUCAS CHEN

Problems: 2.44, 2.53, 2.55, 2.56, 2.57, 3.2, 3.14

Problem 44. Consider a function $f : M \to \mathbb{R}$. Its graph is the set
$$\{(p, y) \in M \times \mathbb{R} : y = fp\}.$$

(a) Prove that if f is continuous then its graph is closed (as a subset of $M \times \mathbb{R}$.

Take f continuous. Then for some convergent sequence (x_n) in the graph of f we have a limit point x. We aim to prove that x is also in the graph of f. For x_n we take p_n the component of x_n in M. Since (x_n) converges p_n must also converge: call the limit p. By the definition of continuity $f(p_n)$ converges to $f(p)$. Then $x_n = (p_n, f(p_n))$ must converge to $(p, f(p))$ which is in the graph of f. Thus by definition the graph of f is closed.

(b) Prove that if f is continuous and M is compact then its graph is compact.

The image of M by f is compact since M is compact. Then a sequence (x_n) in the graph has component sequence (p_n) with a convergent subsequence (p_{n_k}) in M, with limit p. Then by continuity this subsequence is mapped to $f(p_{n_k})$ with limit $f(p)$. Thus the subsequence x_{n_k} must converge to $(p, f(p))$ in the graph of f and the graph is compact.

(c) Prove that if the graph of f is compact then f is continuous.

Take some convergent sequence (p_n) in M with limit p. We aim to prove that $(f(p_n))$ converges to $f(p)$. We know that $((p_n, f(p_n)))$ has a convergent subsequence since the graph of f is compact. We call this sequence (x_{n_k}), and its limit x. Since (p_{n_k}) is a subsequence of (p_n) it converges to p. x_{n_k} converges if both its component sequences converge to the components of its limit: thus p is the component of x in M. Since x is in the graph of f this means $x = (p, f(p))$ and $(f(p_{n_k}))$ converges to $f(p)$.

Assume now that $f(p_n)$ does not converge. Then $\exists$ some ϵ where there are infinitely many $n \in \mathbb{N}$ where $d(f(p_n), f(p)) \geq \epsilon$. We take this as a subsequence $f(p_{n_j})$. Then x_{n_j} is a sequence in the graph of f and we can redo the above proof to find a convergent subsequence of $f(p_{n_j})$, whose limit must be the image of its corresponding subsequence limit in M: p_{n_j} is a subsequence of p_n and must converge to p. Then $f(p_{n_j})$ must converge to $f(p)$ since x_{n_j} converges: this is a contradiction and thus $f(p_n)$ must converge, and since $f(p_{n_k})$ converges to p so too must $f(p_n)$; f is thus continuous by definition.

(d) What if the graph is merely closed? Give an example of a discontinuous function $f : \mathbb{R} \to \mathbb{R}$ whose graph is closed.

Date: October 21, 2024.

The function $f : \mathbb{R} \to \mathbb{R}$ where

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

In this function the preimage of any open set containing 0 contains the singleton set $\{0\}$ (no open neighborhood) and is therefore discontinuous.

Problem 53. Suppose that (K_n) is a nested sequence of compact nonempty sets, $K_1 \supset K_2 \supset \dots$ and $K = \bigcap K_n$. If for some $\mu > 0$, diam $K_n \geq \mu$ for all n, is it true that diam $K \geq \mu$?

For each K_n take k_{n1}, k_{n2} so that $d(k_{n1}, k_{n2}) > \mu$. These exist by the definition of diameter. Then we have sequences $(k_{n1}), (k_{n2})$. Since K_1 is compact we have convergent subsequences $(k_{n_p1}), (k_{n_p2})$ for each sequence with limits k_1, k_2. k_1, k_2 are in K_n for any finite n since only finitely many elements of the subsequences can be excluded from K_n and so by definition $k_1, k_2 \in K$. Given any ϵ we can find p such that $d(k_{n_p1}, k_1) < \epsilon/2$ and $d(k_{n_p2}, k_2) < \epsilon/2$.

From here, by twice applying the triangle inequality we find $d(k_1, k_2) + \epsilon > d(k_{n_p1}, k_1) + d(k_1, k_2) + d(k_{n_p2}, k_2) \geq d(k_{n_p1}, k_{n_p2}) \geq \mu$. By ϵ-principle we have $d(k_1, k_2) \geq \mu$. Thus diam $K \geq \mu$.

Problem 55. The **distance** from a point p in a metric space M to a nonempty subset $S \subset M$ is defined to be $\text{dist}(p, S) = \inf\{d(p, s) : s \in S\}$.

(a) Show that p is a limit of S if and only if dist $(p, S) = 0$.

If dist $(p, S) = 0$ then $\nexists \epsilon$ where $d(p, s) \geq \epsilon$ for $s \in S$. Then for every $\epsilon > 0$ there is a point $s \in S$ where $d(p, s) < \epsilon$. Take $\epsilon_n = \frac{1}{n}$: then take $s_n \in S$ with the prior condition for ϵ_n. s_n is a sequence that converges to p because for any ϵ we can take $n > 1/\epsilon$ so $d(s_m, p) < \epsilon$ for $m \geq n$. If p is a limit of s then necessarily there $\nexists \epsilon$ where $d(p, s) > \epsilon$ for every $s \in S$: since distance satisfies positive definiteness, dist $(p, S) = 0$.

(b) Show that $p \mapsto$ dist (p, S) is a uniformly continuous function of $p \in M$.

Given two points $p, q \in M$ we have dist $(p, S) < d(p, s)$ for every $s \in S$. This gives used

$$\text{dist }(p, S) < d(p, s) \leq d(p, q) + d(q, s)$$

Since dist (q, S) is the inf of $d(q, s)$ over $s \in S$ we have that $\exists s \in S$ for each $\epsilon > 0$ where dist $(q, S)+\epsilon \geq d(q, s)$: if not dist $(q, S)+\epsilon$ is a lower bound. Thus we can select s for each ϵ such that dist $(p, S) \leq d(p, q)+$dist $(q, S)+\epsilon$. Then by ϵ-principle we have dist $(p, S) \leq d(p, q) +$ dist (q, S), implying dist $(p, S) -$ dist $(q, S) \leq d(p, q)$. We can reverse the proof starting with dist (q, S) to achieve $|$dist $(p, S)-$dist $(q, S)| \leq d(p, q)$.

Then for any ϵ and $\delta = \epsilon$, $d(p, q) < \delta$ implies $|$dist $(p, S) -$ dist $(q, S)| < \epsilon$ and the function is uniformly continuous.

Problem 56. Prove that the 2-sphere is not homeomorphic to the plane.

We solve the problem for the infinite plane $\mathbb{R}^2$ and a finite plane $A \times B$ for closed intervals A, B. We prove first that the 2-sphere is compact. The 2-sphere is bounded by the 2-ball around the origin in 3-space. The complement of the 2-sphere is the set $\{\mathbf{x} : d(\mathbf{x}, \vec{0}) \neq 1\} = \{\mathbf{x} : d(\mathbf{x}, \vec{0}) > 1\} \cup \{\mathbf{x} : d(\mathbf{x}, \vec{0}) < 1\}$. The second set is the open 1-ball and the first the complement of the closed 1-ball (which is open). Thus the sphere is closed and compact.

The 2-sphere is compact and so cannot be homeomorphic to the infinite plane. For a finite plane, remove a line across the plane. This disconnects the plane. We note that the line cannot be mapped homeomorphically to any path across the sphere that meets itself (since removing a point from the line would disconnect the line but not the path), so removal of the continuous image of the line onto the sphere cannot disconnect the sphere — as such, the sphere cannot be mapped homeomorphically to a finite plane.

Problem 57. If S is connected, is the interior of S connected? Prove this or give a counterexample.

Take two path-connected (but disconnected in union) sets in $\mathbb{R}^2$ and connect them by union with a line between them. Then the set is path-connected and therefore connected. However, at no point along the line between the two sets is there a point with an open neighborhood in $\mathbb{R}^n$ between them. As such, the interior of the set excludes the line and is disconnected.

Problem 2. A function $f : (a, b) \to \mathbb{R}$ satisfies a **Hölder condition** of order $\boldsymbol{\alpha}$ if $\alpha > 0$, and for some constant H and all $u, x \in (a, b)$ we have

$$|f(u) - f(x)| \leq H|u - x|^{\alpha}$$

The function is said to be α-Hölder, with α-Hölder constant H. (The terms "Lipschitz function of order α" and "α-Lipschitz function" are sometimes used with the same meaning.)

(a) Prove that an α-Hölder function defined on (a, b) is uniformly continuous and infer that it extends uniquely to a continuous function defined on $[a, b]$. Is the extended function α-Hölder?

Given an ϵ pick $\delta = (\frac{\epsilon}{H})^{\frac{1}{\alpha}}$. Then by the Hölder condition we have

$$|u - x| < \delta \implies H|u - x|^{\alpha} < \epsilon$$

for positive H and positive α. Then $|f(u) - f(x)| < \epsilon$ implies uniform continuity.

We prove that there exist unique values of $f(a)$, $f(b)$ such that the extension is continuous. Since we know that for continuous functions convergent sequences get mapped to convergent sequences and limits to limits, the value of $f(a)$ and $f(b)$ can be defined as the limits of $(f(a + 1/n))$ and $(f(b - 1/n))$ for $n \in \mathbb{N}$. Then if these limits exist they must be unique. We prove the limits exist. Given a δ we have n_1, n_2 where $\max(1/n_1, 1/n_2) < \delta$, so $|1/n_1 - 1/n_2| < \delta$ implies that for every ϵ, we can pick the corresponding $\delta = (\frac{\epsilon}{H})^{\frac{1}{\alpha}}$ so $|f(a + 1/n_1) - f(a + 1/n_2)| < \epsilon$. (This process can be mirrored for $f(b - 1/n)$.) Thus $(f(a + 1/n))$ is a Cauchy sequence and must converge as $f(a + 1/n)$ are elements of $\mathbb{R}$.

The extended function is α-Hölder.

(b) What does α-Hölder continuity mean when $\alpha = 1$?

When $\alpha = 1$ the function is Lipschitz continuous. We prove first that Lipschitz continuity implies 1-Hölder: if there is no H that satisfies the Hölder condition then for any Lipschitz continuity constant we can use the mean value theorem to prove the existence of a derivative of the function greater than the constant at some point.

Now assume the 1-Hölder condition. Then we have

$$\frac{|f(u) - f(x)|}{|u - x|} \leq H$$

. We note that for any $x' \in (a, b)$ if $|f'(x')| > H$ then we have $\lim_{h \to 0} \frac{|f(x'+h) - f(x')|}{|h|} = H + \epsilon$ and no δ exists that bounds $\frac{|f(x'+h) - f(x')|}{|h|}$ within ϵ of the limit due to the Hölder condition. By contradiction this means the Lipschitz continuity condition is satisfied.

(c) Prove that α-Hölder continuity when $\alpha > 1$ implies that f is constant.

Lemma: Squeeze Theorem. Continuous f, g, h. If $f(x)$ is bounded between $g(x)$ below and $h(x)$ above then $f(x) - g(x) \geq 0$ implies the limit $L \geq 0$ since if otherwise then $f(x) - g(x)$ cannot get within $|L|$ of the limit. We distribute the limits since each function is continuous. Same for opposite direction implies $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \lim_{x \to a} h(x)$. (Not sure if proved in textbook so lazy panicked proof included.)

Proof of (c): We have

$$\frac{|f(u) - f(x)|}{|u-x|} \leq H|u-x|^\alpha \implies -H|u-x|^\alpha \leq \frac{f(u)-f(x)}{u-x} \leq H|u-x|^\alpha$$

By squeeze theorem this implies that $f'(x) = 0$ since $\lim_{h \to 0} H|h|^\alpha = 0$. Then if f is not constant since f is uniformly continuous we can use the Mean Value Theorem to find that $f'(x) \neq 0$ at some point, which yields a contradiction.

Problem 14. For each $r \geq 1$, find a function that is C^r but not C^{r+1}.

We use the function $f_r(x) = |x|x^r$. Define

$$g(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ -1 & \text{otherwise.} \end{cases}$$

By Leibniz' rule we have $f_r'(x) = rx^{r-1}|x| + x^r g(x)$. $g(x)x = |x|$ implies $f_r'(x) = (r+1)f_{r-1}(x)$. Thus since $|x|$ is continuous f_r is C^r. However, $g(x)$ is not continuous because the preimage of -1 is not closed. Thus $f_r(x)$ is not C^{r+1} since $|x|' = g(x)$.