NOTES 10/2

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1. Continuity

Definition: We say a function f is continuous at $x \in [a, b]$ if $\forall\ \epsilon > 0\ \exists\ \delta > 0$ s.t. if $y \in [a, b]$ with $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$.

Theorem. For $f : [a, b] \to \mathbb{R}$ continous, f is bounded and it achieves its maximum value

Proof. Bounded: $S = \{x \in [a, b] : f \text{ is bounded in } [a, x]\}$. Let $x_0 \in \sup S$. For $\epsilon = 1$, $\exists\delta$ s.t. $|f(y) - f(x_0)| < 1$ if $|x_0 - y| < \delta$.
Achieves its maximum: Redefine $S = \{x \in [a, b] : \sup\{f(y) : a \leq y \leq x\} < \sup\{f(y) : a \leq y \leq b\}\}$. Let $x_0 \in \sup S$. Claim $f(x_0) = \sup f$. If $f(x_0) < \sup f$, $\exists$ $\delta > 0$ s.t. $f(x) < \sup f$, $x \in [x_0 - \delta, x_0 + \delta]$. However, then $x_0 + \delta/2 > x_0$ and is in the set S: contradiction. $f(x_0) \not> \sup f$ for obvious reasons.

Theorem. If $f(a) < y < f(b)$, $\exists\ x_0 \in (a, b)$ s.t. $f(x_0) = y$.

Proof. Define set $S = \{x \in [a, b] :\ \text{l.u.b. } f([a, x]) \leq y\}$. Take $c = \text{l.u.b } S$.

Cauchy-Schwarz Inequality. $\langle x, y\rangle < |x||y|\ \forall\ x, y \in \mathbb{R}^n$.

Proof: Consider $\langle w, w\rangle$ for $w = x + ty$ for some vectors x, y and $t \in \mathbb{R}$. $\langle w, w\rangle = \langle x, x\rangle + 2t\langle x, y\rangle + t^2\langle y, y\rangle$. Since $\langle w, w\rangle$ is positive we have $4\langle x, y\rangle < 4\langle x, x\rangle\langle y, y\rangle$ by the property of the discriminant.

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